Matrix Algebra Worksheet 2 Solutions

1. Find the inverse of the following matrix
\[
\begin{bmatrix}
1 & -2 & 3 \\
0 & 4 & -5 \\
0 & 0 & 6
\end{bmatrix}
\]

Solution: We will use Gaussian elimination. The matrix is already upper triangular, but recall that for finding inverses, this is not enough.

\[
\begin{bmatrix}
1 & -2 & 3 & 1 & 0 & 0 \\
0 & 4 & -5 & 0 & 1 & 0 \\
0 & 0 & 6 & 0 & 0 & 1
\end{bmatrix}
\]

\[
\begin{array}{c}
\frac{1}{2}R_3 \rightarrow R_3 \\
R_2 + 5R_3 \rightarrow R_2 \\
R_1 - 3R_3 \rightarrow R_1 \\
\frac{1}{2}R_2 \rightarrow R_2 \\
R_1 + 2R_2 \rightarrow R_1
\end{array}
\]

So the inverse is
\[
\begin{bmatrix}
1 & 0 & -1/2 \\
0 & 1/4 & 5/24 \\
0 & 0 & 1/6
\end{bmatrix}
\]

2. (a) Find the inverse of the following matrix
\[
\begin{bmatrix}
1 & 2 & 3 \\
0 & 2 & 5 \\
0 & 6 & 7
\end{bmatrix}
\]
Solution:

\[
\begin{bmatrix}
1 & 2 & 3 & 1 & 0 & 0 \\
0 & 2 & 5 & 0 & 1 & 0 \\
0 & 6 & 7 & 0 & 0 & 1
\end{bmatrix}
\xrightarrow{\frac{1}{2}R_2 \rightarrow R_2}
\begin{bmatrix}
1 & 2 & 3 & 1 & 0 & 0 \\
0 & 1 & 5/2 & 0 & 1/2 & 0 \\
0 & 6 & 7 & 0 & 0 & 1
\end{bmatrix}
\]

\[R_3 - 6R_2 \rightarrow R_3 \]

\[\begin{bmatrix}
1 & 2 & 3 & 1 & 0 & 0 \\
0 & 1 & 5/2 & 0 & 1/2 & 0 \\
0 & 0 & -8 & 0 & -3 & 1
\end{bmatrix}
\]

\[-\frac{1}{2}R_3 \rightarrow R_3 \]

\[\begin{bmatrix}
1 & 2 & 3 & 1 & 0 & 0 \\
0 & 1 & 5/2 & 0 & 1/2 & 0 \\
0 & 0 & 1 & 0 & 3/8 & -1/8
\end{bmatrix}
\]

\[R_2 - \frac{5}{2}R_3 \rightarrow R_2 \]

\[\begin{bmatrix}
1 & 2 & 3 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & -7/16 & 5/16 \\
0 & 0 & 1 & 0 & 3/8 & -1/8
\end{bmatrix}
\]

\[R_1 - 3R_3 \rightarrow R_1 \]

\[\begin{bmatrix}
1 & 2 & 0 & 1 & -9/8 & 3/8 \\
0 & 1 & 0 & 0 & -7/16 & 5/16 \\
0 & 0 & 1 & 0 & 3/8 & -1/8
\end{bmatrix}
\]

\[R_1 - 2R_2 \rightarrow R_1 \]

\[\begin{bmatrix}
1 & 0 & 0 & 1 & -1/4 & -1/4 \\
0 & 1 & 0 & 0 & -7/16 & 5/16 \\
0 & 0 & 1 & 0 & 3/8 & -1/8
\end{bmatrix}
\]

So the inverse is

\[
\begin{bmatrix}
1 & -1/4 & -1/4 \\
0 & -7/16 & 5/16 \\
0 & 3/8 & -1/8
\end{bmatrix}
\]

(b) Let \( A \) be the matrix from the previous question and suppose that \( BC = A \) where \( B \) and \( C \) are both \( 3 \times 3 \) matrices and \( B \) is as shown below. Find \( C^{-1} \).

\[
B = \begin{bmatrix}
3 & 2 & 1 \\
0 & 1 & 0 \\
-1 & 2 & 1
\end{bmatrix}
\]

Solution: It is possible to solve this problem by first finding the inverse of \( B \), multiplying \( BC = A \) by \( B^{-1} \) on both sides to get \( C = B^{-1}A \) and then finding the inverse of \( B^{-1}A \). But that’s a lot of effort. Instead, let’s first observe that if we multiply \( BC = A \) on both sides by \( A^{-1} \) then we get \( A^{-1}(BC) = A^{-1}A = I \). So \( (A^{-1}B)C = I \) and by the definition of the matrix inverse, that means that \( A^{-1}B \) is the inverse of \( C \). We already found \( A^{-1} \) in part (a) and we know what \( B \) is. So we can calculate

\[
C^{-1} = A^{-1}B = \begin{bmatrix}
1 & -1/4 & -1/4 \\
0 & -7/16 & 5/16 \\
0 & 3/8 & -1/8
\end{bmatrix}
\begin{bmatrix}
3 & 2 & 1 \\
0 & 1 & 0 \\
-1 & 2 & 1
\end{bmatrix}
= \begin{bmatrix}
13/4 & 5/4 & 3/4 \\
-5/16 & 3/16 & 5/16 \\
1/8 & 1/8 & -1/8
\end{bmatrix}
\]

3. True or false:
(a) The following vector is an eigenvector of the following matrix.

\[
\begin{bmatrix}
0 \\
1 \\
2
\end{bmatrix}
\begin{bmatrix}
2 & 0 & 0 \\
0 & 3 & 4 \\
0 & 4 & 9
\end{bmatrix}
\]

**Solution:** True. Multiplying the matrix by the vector gives

\[
\begin{bmatrix}
0 \\
11 \\
22
\end{bmatrix}
\]

Since this is equal to 11 times the original vector, the original vector is an eigenvector of the matrix (with eigenvalue 11).

(b) If \(v\) is an eigenvector of \(A\) then \(v\) is also an eigenvector of \(5A\).

**Solution:** True. Suppose \(v\) is an eigenvector of \(A\) with eigenvalue \(\lambda\). So by definition, \(Av = \lambda v\). Thus \((5A)v = 5(Av) = 5(\lambda v) = (5\lambda)v\). So \(v\) is an eigenvector of \(5A\) with eigenvalue \(5\lambda\).

(c) If \(v\) is an eigenvector of \(A\) and of \(B\) then it is also an eigenvector of \(AB\).

**Solution:** True. Suppose \(v\) is an eigenvector of \(A\) with eigenvalue \(\lambda_1\) and also an eigenvector of \(B\) with eigenvalue \(\lambda_2\). So by definition, \(Av = \lambda_1 v\) and \(Bv = \lambda_2 v\). Therefore

\[
(AB)v = A(Bv) = A(\lambda_2 v) = \lambda_2 (Av) = \lambda_2 (\lambda_1 v) = (\lambda_2 \lambda_1) v.
\]

So \(v\) is an eigenvector of \(AB\) with eigenvalue \(\lambda_2 \lambda_1\).

(d) If \(v\) is an eigenvector of an invertible matrix \(A\) then it is also an eigenvector of \(A^{-1}\).

**Solution:** True. Suppose \(v\) is an eigenvector of \(A\) with eigenvalue \(\lambda\). So by definition \(Av = \lambda v\). Multiplying both sides by \(A^{-1}\) gives us

\[
\begin{align*}
A^{-1}(Av) &= A^{-1}(\lambda v) \\
(A^{-1}A)v &= \lambda(A^{-1}v) \\
Iv &= \lambda(A^{-1}v) \\
v &= \lambda(A^{-1}v)
\end{align*}
\]

The main idea at this point is just to divide both sides by \(\lambda\). But to do that, we need to know that \(\lambda\) is not zero. But if \(\lambda\) were zero then by the last line above, \(v\) would have to be the all-zeros vector. Since \(v\) is an eigenvector, it is by definition not the all-zeros vector. So \(\lambda\) is not zero, and dividing the last equation above by \(\lambda\) gives us

\[
\frac{1}{\lambda}v = A^{-1}v
\]

and thus \(v\) is an eigenvector of \(A^{-1}\) with eigenvalue \(1/\lambda\).
(e) If \( v \) is an eigenvector of \( A \) then \( v \) is also an eigenvector of \( A^5 \).

**Solution:** True. This is essentially the same as part (c). If \( v \) is an eigenvector of \( A \) with eigenvalue \( \lambda \) then \( A^5v = \lambda^5v \) and so \( v \) is an eigenvector of \( A^5 \) with eigenvalue \( \lambda^5 \).

4. Find the eigenvalues and eigenvectors of the following matrices.

(a)
\[
\begin{bmatrix}
2 & 1 \\
-2 & 5 \\
\end{bmatrix}
\]

**Solution:** Eigenvalues: 3, 4

Eigenvectors for eigenvalue 3:
\[
\begin{bmatrix}
a \\
a \\
\end{bmatrix}
\]

where \( a \) is any nonzero real number.

Eigenvectors for eigenvalue 3:
\[
\begin{bmatrix}
b \\
2b \\
\end{bmatrix}
\]

where \( b \) is any nonzero real number.

(b)
\[
\begin{bmatrix}
2 & 1 \\
0 & 2 \\
\end{bmatrix}
\]

**Solution:** Eigenvalues: 2

Eigenvectors for eigenvalue 2:
\[
\begin{bmatrix}
a \\
0 \\
\end{bmatrix}
\]

where \( a \) is any nonzero real number.

**Comment:** Note that there is only one eigenvalue even though the matrix is \( 2 \times 2 \). This can happen sometimes, though it is not the typical scenario.

(c)
\[
\begin{bmatrix}
1/2 & -3/5 \\
3/4 & 11/10 \\
\end{bmatrix}
\]

**Solution:** Eigenvalues: \( \frac{4}{5} + \frac{3}{5}i \) and \( \frac{4}{5} - \frac{3}{5}i \).

Eigenvectors for eigenvalue \( \frac{4}{5} + \frac{3}{5}i \):
\[
\begin{bmatrix}
\left( -\frac{2}{5} + \frac{4}{5}i \right) a \\
a \\
\end{bmatrix}
\]

where \( a \) is any nonzero number.
Eigenvalues for eigenvalue $\frac{4}{5} + \frac{2}{5}i$:

$$\begin{pmatrix} -\frac{2}{5} & -\frac{4}{5}i \\ b & \frac{2}{5}i \end{pmatrix} b$$

where $b$ is any nonzero number.

5. **Challenge Question:** When does a $2 \times 2$ matrix whose entries are all integers have an inverse whose entries are all integers? What about for an $n \times n$ matrix?

6. **Challenge Question:** Let $p(x)$ be a degree $n$ polynomial. Can you always find an $n \times n$ matrix whose characteristic polynomial is $p$?