WARPED PRODUCTS

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1. Definitions

We shall define as few concepts as possible. A tangent vector always has the local coordinate expansion

$$v = dx^i(v)\frac{\partial}{\partial x^i}$$

and a function the differential

$$df = \frac{\partial f}{\partial x^i}dx^i$$

We start with a Riemannian metric (M, g). In local coordinates we obtain the metric coefficients g_{ij}

$$g = g_{ij}dx^i dx^j = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)dx^i dx^j$$

1.1. **The Gradient.**

The gradient of a function $f : M \to \mathbb{R}$ is a vector field dual to the differential df

$$g(\nabla f, v) = df(v)$$

In local coordinates this reads

$$g_{ij}dx^i(\nabla f)\,dx^j(v) = \frac{\partial f}{\partial x^j}dx^j(v)$$

showing that

$$g_{ij}dx^i(\nabla f) = \frac{\partial f}{\partial x^j}$$

Using g^{ij} for the inverse of g_{ij} we then obtain

$$dx^i(\nabla f) = g^{ij}\frac{\partial f}{\partial x^j}$$

and

$$\nabla f = g^{ij}\frac{\partial f}{\partial x^j}\frac{\partial}{\partial x^i}.$$

One can easily show that:

$$g^{ij} = g\left(dx^i, dx^j\right) = g\left(\nabla x^i, \nabla x^j\right).$$

Note that

$$g^{ik}g_{kj} = \delta^i_j$$

implies

$$\begin{aligned}\frac{\partial g^{ik}}{\partial x^l}g_{kj} + g^{ik}\frac{\partial g_{kj}}{\partial x^l} &= 0,\\ \frac{\partial g^{ik}}{\partial x^l}g_{kj} &= -g^{ik}\frac{\partial g_{kj}}{\partial x^l}\end{aligned}$$

and

$$\frac{\partial g^{ij}}{\partial x^s} = -g^{ik}\frac{\partial g_{kl}}{\partial x^s}g^{lj}.$$

A slightly more involved calculation shows that

$$\frac{\partial \det g_{ij}}{\partial x^s} = g^{kl}\frac{\partial g_{kl}}{\partial x^s}\det g_{ij}$$

1.2. **The Divergence.** To motivate our definition of the Hessian of a function we first define the *divergence* of a vector field using a dynamic approach. Specifically, by checking how the volume form changes along the flow of the vector field:

$$L_X d\mathrm{vol} = \mathrm{div}\,(X)\,d\mathrm{vol}$$

The volume form is a metric concept just like the metric and in positively oriented local coordinates it has the form

$$d\mathrm{vol} = \sqrt{\det g_{kl}}dx^1 \wedge \cdots \wedge dx^n$$

Thus

$$\begin{aligned}
L_X d\mathrm{vol} &= L_X\left(\sqrt{\det g_{kl}}dx^1 \wedge \cdots \wedge dx^n\right) \\
&= D_X\left(\sqrt{\det g_{kl}}\right)dx^1 \wedge \cdots \wedge dx^n \\
&\quad +\sqrt{\det g_{kl}}\sum dx^1 \wedge \cdots \wedge L_X dx^i \wedge \cdots \wedge dx^n \\
&= \frac{D_X\left(\sqrt{\det g_{kl}}\right)}{\sqrt{\det g_{kl}}}d\mathrm{vol} \\
&\quad +\sqrt{\det g_{kl}}\sum dx^1 \wedge \cdots \wedge dD_X x^i \wedge \cdots \wedge dx^n \\
&= \frac{D_X\left(\sqrt{\det g_{kl}}\right)}{\sqrt{\det g_{kl}}}d\mathrm{vol} + \sqrt{\det g_{kl}}\sum dx^1 \wedge \cdots \wedge d\left(dx^i\left(X\right)\right) \wedge \cdots \wedge dx^n \\
&= \frac{D_X\left(\sqrt{\det g_{kl}}\right)}{\sqrt{\det g_{kl}}}d\mathrm{vol} + \sqrt{\det g_{kl}}\sum dx^1 \wedge \cdots \wedge \frac{\partial dx^i\left(X\right)}{\partial x^i}dx^i \wedge \cdots \wedge dx^n \\
&= \left(\frac{D_X\left(\sqrt{\det g_{kl}}\right)}{\sqrt{\det g_{kl}}} + \frac{\partial dx^i\left(X\right)}{\partial x^i}\right)d\mathrm{vol} \\
&= \frac{1}{\sqrt{\det g_{kl}}}\frac{\partial\left(\sqrt{\det g_{kl}}dx^i\left(X\right)\right)}{\partial x^i}d\mathrm{vol}
\end{aligned}$$

The Laplacian is now naturally defined by

$$\Delta f = \mathrm{div}\left(\nabla f\right)$$

1.3. **The Hessian.** A similar approach can also be used to define the *Hessian*:

$$\mathrm{Hess} f = \frac{1}{2}L_{\nabla f}g$$

The local coordinate calculation is a bit worse, but yields something nice in the end

$$
\begin{aligned}
\mathrm{Hess}f\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right) &= \frac{1}{2}\left(L_{\nabla f}g\right)\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right) \\
&= \frac{1}{2}\left(L_{g^{kl}\frac{\partial f}{\partial x^k}\frac{\partial}{\partial x^l}}g\right)\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right) \\
&= \frac{1}{2}g^{kl}\frac{\partial f}{\partial x^k}\left(L_{\frac{\partial}{\partial x^l}}g\right)\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right)+\frac{1}{2}\frac{\partial\left(g^{kl}\frac{\partial f}{\partial x^k}\right)}{\partial x^i}g_{lj}+\frac{1}{2}\frac{\partial\left(g^{kl}\frac{\partial f}{\partial x^k}\right)}{\partial x^j}g_{li} \\
&= \frac{1}{2}g^{kl}\frac{\partial f}{\partial x^k}\frac{\partial g_{ij}}{\partial x^l}+\frac{1}{2}\frac{\partial\left(g^{kl}\frac{\partial f}{\partial x^k}\right)}{\partial x^i}g_{lj}+\frac{1}{2}\frac{\partial\left(g^{kl}\frac{\partial f}{\partial x^k}\right)}{\partial x^j}g_{li} \\
&= \frac{1}{2}\left(g^{kl}g_{lj}\frac{\partial^2 f}{\partial x^i\partial x^k}+g^{kl}g_{li}\frac{\partial^2 f}{\partial x^j\partial x^k}\right)+\frac{1}{2}\left(g^{kl}\frac{\partial g_{ij}}{\partial x^l}+\frac{\partial g^{kl}}{\partial x^i}g_{lj}+\frac{\partial g^{kl}}{\partial x^j}g_{li}\right)\frac{\partial f}{\partial x^k} \\
&= \frac{\partial^2 f}{\partial x^i\partial x^j}+\frac{1}{2}g^{kl}\left(\frac{\partial g_{ij}}{\partial x^l}-\frac{\partial g_{lj}}{\partial x^i}-\frac{\partial g_{li}}{\partial x^j}\right)\frac{\partial f}{\partial x^k}
\end{aligned}
$$

This is usually rewritten by introducing the *Christoffel symbols* of the first and second kind

$$
\begin{aligned}
\Gamma_{ijl} &= \frac{1}{2}\left(\frac{\partial g_{lj}}{\partial x^i}+\frac{\partial g_{li}}{\partial x^j}-\frac{\partial g_{ij}}{\partial x^l}\right), \\
\Gamma_{ij}^k &= \frac{1}{2}g^{kl}\left(\frac{\partial g_{lj}}{\partial x^i}+\frac{\partial g_{li}}{\partial x^j}-\frac{\partial g_{ij}}{\partial x^l}\right).
\end{aligned}
$$

Observe that this formula gives the expected answer in Cartesian coordinates, and that at a critical point the Hessian does not depend on the metric.

Next we should tie this in with our definition of the Laplacian. Computing the trace in a coordinate system uses the inverse of the metric coefficients

$$
\begin{aligned}
\mathrm{tr}\left(\mathrm{Hess}f\right) &= g^{ij}\mathrm{Hess}f\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right) \\
&= g^{ij}\frac{\partial^2 f}{\partial x^i\partial x^j}+\frac{1}{2}g^{ij}g^{kl}\left(\frac{\partial g_{ij}}{\partial x^l}-\frac{\partial g_{lj}}{\partial x^i}-\frac{\partial g_{li}}{\partial x^j}\right)\frac{\partial f}{\partial x^k}
\end{aligned}
$$

On the other hand

$$\begin{aligned}
\Delta f &= \operatorname{div}(\nabla f) \\
&= \frac{1}{\sqrt{\det g_{kl}}} \frac{\partial\left(\sqrt{\det g_{kl}} g^{ij} \frac{\partial f}{\partial x^j}\right)}{\partial x^i} \\
&= g^{ij} \frac{\partial^2 f}{\partial x^i \partial x^j} + \frac{\partial g^{ij}}{\partial x^i} \frac{\partial f}{\partial x^j} + g^{ij} \frac{\partial f}{\partial x^j} \frac{1}{\sqrt{\det g_{kl}}} \frac{\partial\left(\sqrt{\det g_{kl}}\right)}{\partial x^i} \\
&= g^{ij} \frac{\partial^2 f}{\partial x^i \partial x^j} - g^{ik} g^{lj} \frac{\partial g_{kl}}{\partial x^i} \frac{\partial f}{\partial x^j} + \frac{1}{2} g^{ij} \frac{\partial f}{\partial x^j} \frac{1}{\det g_{kl}} \frac{\partial\left(\det g_{kl}\right)}{\partial x^i} \\
&= g^{ij} \frac{\partial^2 f}{\partial x^i \partial x^j} - g^{ik} g^{lj} \frac{\partial g_{kl}}{\partial x^i} \frac{\partial f}{\partial x^j} + \frac{1}{2} g^{ij} \frac{\partial f}{\partial x^j} g^{kl} \frac{\partial g_{kl}}{\partial x^i} \\
&= g^{ij} \frac{\partial^2 f}{\partial x^i \partial x^j} - g^{ij} g^{kl} \frac{\partial g_{jl}}{\partial x^i} \frac{\partial f}{\partial x^k} + \frac{1}{2} g^{kl} \frac{\partial f}{\partial x^k} g^{ij} \frac{\partial g_{ij}}{\partial x^l} \\
&= g^{ij} \frac{\partial^2 f}{\partial x^i \partial x^j} + g^{ij} g^{kl} \left(\frac{1}{2} \frac{\partial g_{ij}}{\partial x^l} - \frac{\partial g_{jl}}{\partial x^i}\right) \frac{\partial f}{\partial x^k} \\
&= g^{ij} \frac{\partial^2 f}{\partial x^i \partial x^j} + \frac{1}{2} g^{ij} g^{kl} \left(\frac{\partial g_{ij}}{\partial x^l} - \frac{\partial g_{lj}}{\partial x^i} - \frac{\partial g_{li}}{\partial x^j}\right) \frac{\partial f}{\partial x^k} \\
&= \operatorname{tr}(\operatorname{Hess} f)
\end{aligned}$$

2. Examples

Starting with a Riemannian metric (N, h) a *warped product* is defined as a metric on $I \times N$, where $I \subset \mathbb{R}$ is an open interval, by

$$g = dr^2 + \phi^2(r)\, h$$

where $\phi > 0$ on all of I. One could also more generally consider

$$\psi^2(r)\, dr^2 + \phi^2(r)\, h$$

However a change of coordinates defined by relating the differentials $d\rho = \psi(r)\, dr$ allows us to rewrite this as

$$d\rho^2 + \phi^2(r(\rho))\, h.$$

Important special cases are the basic product

$$g = dr^2 + h$$

and polar coordinates

$$dr^2 + r^2 ds_{n-1}^2$$

on $(0, \infty) \times S^{n-1}$ representing the Euclidean metric.

2.1. **Conformal Representations.**

The basic examples are still sufficiently broad that we can reduce almost all problems to these cases by a simple conformal change:

$$\begin{aligned}
dr^2 + \phi^2(r)\, h &= \psi^2(\rho)\left(d\rho^2 + h\right), \\
dr &= \psi(\rho)\, d\rho, \\
\phi(r) &= \psi(\rho)
\end{aligned}$$

or

$$\begin{aligned} dr^2+\phi^2(r)\,h &= \psi^2(\rho)\left(d\rho^2+\rho^2 h\right), \\ dr &= \psi(\rho)\,d\rho, \\ \phi(r) &= \rho\psi(\rho) \end{aligned}$$

The first of these changes has been studied since the time of Mercator. The sphere of radius R can be written as

$$\begin{aligned} R^2 ds_n^2 &= R^2\left(dt^2+\sin^2(t)\,ds_{n-1}^2\right) \\ &= dr^2+R^2\sin^2\left(\frac{r}{R}\right)ds_{n-1}^2 \end{aligned}$$

The conformal change envisioned by Mercator then takes the form

$$R^2 ds_n^2 = R^2\psi^2(\rho)\left(d\rho^2+ds_{n-1}^2\right)$$

As

$$\begin{aligned} \psi(\rho)\,d\rho &= dt, \\ \psi(\rho) &= \sin(t) \end{aligned}$$

this means that we have

$$d\rho = \frac{dt}{\sin(t)}$$

Thus ρ is determined by one of these obnoxious integrals one has to look up.

Switching the spherical metric to being conformal to the polar coordinate representation of Euclidean space took even longer and probably wasn't studied much until the time of Riemann

$$\begin{aligned} R^2 ds_n^2 &= R^2\psi^2(\rho)\left(d\rho^2+\rho^2 ds_{n-1}^2\right) \\ &= R^2\frac{4}{\left(1+\rho^2\right)^2}\left(d\rho^2+\rho^2 ds_{n-1}^2\right) \end{aligned}$$

Hyperbolic space is also conveniently defined using warped products:

$$g = dr^2+\sinh^2(r)\,ds_{n-1}^2,\ r\in(0,\infty)$$

it too has a number of interesting other forms related to warped products:

$$dr^2+\sinh^2(r)\,ds_{n-1}^2 = \frac{4}{\left(1-\rho^2\right)^2}\left(d\rho^2+\rho^2 ds_{n-1}^2\right),\ \rho\in(0,1)$$

And after using inversions to change the unit ball into a half space we can rewrite this as

$$\frac{1}{x^2}\left(dx^2+\mathrm{can}_{\mathbb{R}^{n-1}}\right) = ds^2+e^{2s}\mathrm{can}_{\mathbb{R}^{n-1}}$$

This works as follows. Let the half space model be $H=(-\infty,0)\times\mathbb{R}^{n-1}$ with the metric $\frac{1}{x^2}\left(dx^2+\mathrm{can}_{\mathbb{R}^{n-1}}\right)$. Define

$$\begin{aligned} F(x,z) &= (1,0)+\frac{2(x-1,z)}{|x-1|^2+|z|^2} \\ &= \left(1+\frac{2(x-1)}{|x-1|^2+|z|^2},\frac{2z}{|x-1|^2+|z|^2}\right) \\ &= \left(1+\frac{2(x-1)}{r^2},\frac{2z}{r^2}\right) \end{aligned}$$

as the inversion in the sphere of radius $\sqrt{2}$ centered at $(1,0)\in\mathbb{R}\times\mathbb{R}^{n-1}$. This maps H to the unit ball since

$$|F(x,z)|^2 = 1+\frac{4x}{r^2} = \rho^2.$$

Next we check what F does to the metric. We have

$$\begin{aligned} F^1 &= 1+\frac{2(x-1)}{r^2}, \\ F^i &= \frac{2z^i}{r^2},\ i>1. \end{aligned}$$

and

$$\begin{aligned}
&\frac{4}{(1-\rho^2)^2}\left(\left(dF^1\right)^2+\sum_{k>1}\left(dF^k\right)^2\right) \\
= &\frac{\left(r^2\right)^2}{4x^2}\left(\frac{2dx}{r^2}-\frac{2(x-1)\,2rdr}{\left(r^2\right)^2}\right)^2 \\
&+\sum_{k>1}\frac{\left(r^2\right)^2}{4x^2}\left(\frac{2dz^k}{r^2}-\frac{2z^k2rdr}{\left(r^2\right)^2}\right)^2 \\
= &\frac{1}{x^2}\left(dx-\frac{(x-1)\,2rdr}{r^2}\right)^2+\frac{1}{x^2}\sum_{k>1}\left(dz^k-\frac{z^k2rdr}{r^2}\right)^2 \\
= &\frac{1}{x^2}\left(dx^2+\sum_{k>1}\left(dz^k\right)^2\right)+\frac{1}{x^2}\left(\frac{(x-1)\,2rdr}{r^2}\right)^2+\frac{1}{x^2}\sum_{k>1}\left(\frac{z^k2rdr}{r^2}\right)^2 \\
&-\frac{1}{x^2}dx\frac{(x-1)\,2rdr}{r^2}-\frac{1}{x^2}\sum_{k>1}dz^k\frac{z^k2rdr}{r^2} \\
&-\frac{1}{x^2}\frac{(x-1)\,2rdr}{r^2}dx-\frac{1}{x^2}\sum_{k>1}\frac{z^k2rdr}{r^2}dz^k \\
= &\frac{1}{x^2}\left(dx^2+\mathrm{can}_{\mathbb{R}^{n-1}}\right)+\frac{1}{x^2}r^2\left(\frac{2rdr}{r^2}\right)^2 \\
&-\frac{1}{x^2}rdr\frac{2rdr}{r^2} \\
&-\frac{1}{x^2}\frac{2rdr}{r^2}rdr \\
= &\frac{1}{x^2}\left(dx^2+\mathrm{can}_{\mathbb{R}^{n-1}}\right)
\end{aligned}$$

Showing that F also transforms the conformal unit ball model to the conformal half space model.

2.2. **Singular Points.** The polar coordinate conformal model

$$dr^2+\phi^2(r)\,h = \psi^2(\rho)\left(d\rho^2+\rho^2 h\right)$$

can be used to study smoothness of the metric as we approach a point $r_0\in\partial I$ where $\phi(r_0)=0$. We assume that $\rho(r_0)=0$ in the reparametrization. When $h=ds^2_{n-1}$ we then note that smoothness on the right hand side

$$\psi^2(\rho)\left(d\rho^2+\rho^2 ds^2_{n-1}\right)$$

only depends on $\psi^2(\rho)$ being smooth. Thinking of ρ as being Euclidean distance indicates that this is not entirely trivial. In fact we must assume that

$$\psi(0) > 0$$

and

$$\psi^{(odd)}(0) = 0$$

Translating back to ϕ we see that

$$\begin{aligned}\phi'(0) &= \pm 1, \\ \phi^{(even)}(0) &= 0.\end{aligned}$$

Better yet we can also prove that when h is not ds_{n-1}^2, then it isn't possible to remove the singularity. Assume that g is smooth at the point $p \in M$ corresponding to $\rho = 0$, i.e., we assume that on some neighborhood U of p we have

$$\begin{aligned}U - \{p\} &= (0, b) \times N, \\ g &= d\rho^2 + \rho^2 h\end{aligned}$$

The $\rho = 1$ level corresponds naturally to N. Now observe that each ρ curve in M emanating from p has a unique unit tangent vector at p and also intersects N in a unique point. Thus unit vectors in T_pM and points in N can be identified by moving along ρ curves. This gives the isometry from S^{n-1}, the unit sphere in T_pM, to N.

3. Characterizations

We start by offering yet another version of the warped product representation. Rather than using r and ϕ, the goal is to use only one function f. Starting with a warped product

$$dr^2 + \phi^2(r)\, h$$

we construct the function $f = \int \phi dr$ on M. Since

$$df = \phi dr$$

we immediately see that

$$dr^2 + \phi^2(r)\, h = \frac{1}{\phi^2(r)} df^2 + \phi^2(r)\, h$$

The gradient of f is

$$\nabla f = \phi \nabla r = \phi \frac{\partial}{\partial r}$$

so the Hessian becomes

$$\begin{aligned}(\mathrm{Hess} f)(X, Y) &= \frac{1}{2}\left(L_{\phi \frac{\partial}{\partial r}} g\right)(X, Y) \\ &= \frac{1}{2}\phi\left(L_{\frac{\partial}{\partial r}} g\right)(X, Y) + \frac{1}{2}(D_X \phi)\, g(\nabla r, Y) + \frac{1}{2}(D_Y \phi)\, g(\nabla r, X) \\ &= \frac{1}{2}\phi\left(L_{\frac{\partial}{\partial r}} g\right)(X, Y) + \frac{1}{2}\phi'(D_X r)\, g(\nabla r, Y) + \frac{1}{2}\phi'(D_Y r)\, g(\nabla r, X) \\ &= \frac{1}{2}\phi\left(L_{\frac{\partial}{\partial r}} g\right)(X, Y) + \phi' dr^2(X, Y)\end{aligned}$$

This is further reduced

$$\begin{aligned}
\mathrm{Hess}f &= \frac{1}{2}\phi L_{\frac{\partial}{\partial r}} g + \phi' dr^2 \\
&= \frac{1}{2}\phi L_{\frac{\partial}{\partial r}} \left(dr^2 + \phi^2(r)\,h\right) + \phi' dr^2 \\
&= \frac{1}{2}\phi D_{\frac{\partial}{\partial r}} \left(\phi^2(r)\right) h + \phi' dr^2 \\
&= \phi'\phi^2 h + \phi' dr^2 \\
&= \phi' g
\end{aligned}$$

In other words: it is possible to find a function f whose Hessian is conformal to the metric. Using

$$\phi' = \frac{d\phi}{dr} = \frac{d\phi}{df}\frac{df}{dr} = \frac{d\phi}{df}\phi = \frac{1}{2}\frac{d\left|\nabla f\right|^2}{df}$$

we have obtained a representation that only depends on f and $|\nabla f|$

$$\begin{aligned}
g &= \frac{1}{\left|\nabla f\right|^2} df^2 + \left|\nabla f\right|^2 h, \\
\mathrm{Hess}f &= \frac{1}{2}\frac{d\left|\nabla f\right|^2}{df} g
\end{aligned}$$

Before stating the main theorem we need to establish a general result.

Lemma 3.1. *If f is a smooth function on a Riemannian manifold, then*

$$\mathrm{Hess}f\left(\nabla f, X\right) = \frac{1}{2} D_X \left|\nabla f\right|^2.$$

Proof. At points where ∇f vanishes this is obvious. At other points we can the assume that $f = x^1$ is the first coordinate in a coordinate system. Then

$$\nabla f = g^{1i}\frac{\partial}{\partial x^i}$$

and

$$\begin{aligned}
\left|\nabla f\right|^2 &= g^{1i} g_{ij} g^{1j} \\
&= g^{11}
\end{aligned}$$

With that information the calculation becomes

$$\begin{aligned}
\mathrm{Hess}f\left(\nabla f, \frac{\partial}{\partial x^j}\right) &= g^{1i}\mathrm{Hess}x^1\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \\
&= \frac{1}{2} g^{1i} g^{kl}\left(\frac{\partial g_{ij}}{\partial x^l} - \frac{\partial g_{lj}}{\partial x^i} - \frac{\partial g_{li}}{\partial x^j}\right)\frac{\partial x^1}{\partial x^k} \\
&= \frac{1}{2} g^{1i} g^{1l}\left(\frac{\partial g_{ij}}{\partial x^l} - \frac{\partial g_{lj}}{\partial x^i} - \frac{\partial g_{li}}{\partial x^j}\right) \\
&= -\frac{1}{2} g^{i1} g^{1l}\frac{\partial g_{li}}{\partial x^j} + \frac{1}{2} g^{1i} g^{1l}\frac{\partial g_{ij}}{\partial x^l} - \frac{1}{2} g^{1i} g^{1l}\frac{\partial g_{lj}}{\partial x^i} \\
&= \frac{1}{2}\frac{\partial g^{11}}{\partial x^j} + \frac{1}{2} g^{1i} g^{1l}\frac{\partial g_{ij}}{\partial x^l} - \frac{1}{2} g^{1l} g^{1i}\frac{\partial g_{ij}}{\partial x^l} \\
&= \frac{1}{2}\frac{\partial\left|\nabla f\right|^2}{\partial x^j}
\end{aligned}$$

The formula now follows by expanding X in coordinates. □

We can now state and prove our main result.

Theorem 3.2. *If there are smooth functions f, λ on a Riemannian manifold so that*

$$\mathrm{Hess} f = \lambda g$$

then the Riemannian structure is a warped product around any point where $\nabla f \neq 0$.

Proof. We first need to show that λ only depends on f. This uses the previous lemma:

$$\frac{1}{2} D_X |\nabla f|^2 = \mathrm{Hess} f (\nabla f, X) = \lambda g (\nabla f, X)$$

So we see that $|\nabla f|^2$ is locally constant on level sets of f by taking $X \perp \nabla f$. Thus we can locally assume that

$$|\nabla f|^2 = b(f).$$

If we let $X = \nabla f$, then we obtain

$$\lambda b = \frac{1}{2} D_{\nabla f} b = \frac{1}{2} g (\nabla b, \nabla f) = \frac{1}{2} \frac{db}{df} b$$

or

$$2\lambda = \frac{db}{df}$$

We now claim that

$$g = \frac{1}{b} df^2 + bh$$

where h is defined such that $b(c) h$ is simply the restriction of g to the level set $f = c$. In particular, we have that

$$g|_p = \left(\frac{1}{b} df^2 + bh \right) |_p, \text{ when } f(p) = c$$

Next we observe that

$$L_{\nabla f} g = 2\lambda g$$

and

$$\begin{aligned} L_{\nabla f} \left(\frac{1}{b} df^2 + bh \right) &= \frac{db}{df} \left(\frac{1}{b} df^2 + bh \right) \\ &= 2\lambda \left(\frac{1}{b} df^2 + bh \right) \end{aligned}$$

Thus g and $\frac{1}{b} df^2 + bh$ both solve the same differential equation with the same initial values at $f = c$, showing that the two metrics must agree. □

We can also handle singular points if we assume they are isolated.

Corollary 3.3. *If there are smooth functions f, λ on a Riemannian manifold so that*

$$\begin{aligned} \mathrm{Hess} f &= \lambda g, \\ \nabla f|_p &= 0, \\ \lambda(p) &\neq 0 \end{aligned}$$

then the Riemannian structure is a warped product around p.

Proof. As we've assumed the Hessian to be nondegenerate at p it follows that there are coordinates in a neighborhood around p such that

$$f\left(x^1,...,x^n\right) = f(p) + \lambda(p)\left(\left(x^1\right)^2 + \cdots + (x^n)^2\right).$$

Thus the level sets near p are spheres, and we know as before that

$$\begin{aligned} g &= \frac{1}{b}df^2 + bh, \\ b(f) &= |\nabla f|^2 \end{aligned}$$

Since g is a smooth metric at p we can then also conclude that $h = ds_{n-1}^2$. □

The final goal is to characterize the three standard geometries at least locally.

Corollary 3.4. *Assume that there is a function f on a Riemannian manifold such that*

$$\begin{aligned} f(p) &= 0, \\ \nabla f|_p &= 0. \end{aligned}$$

(1) If

$$\mathrm{Hess} f = g$$

then the metric is Euclidean in a neighborhood of p.

(2) If

$$\mathrm{Hess} f = (1-f)\, g$$

then the metric is the unit sphere metric in a neighborhood of p.

(3) If

$$\mathrm{Hess} f = (1+f)\, g$$

then the metric is hyperbolic in a neighborhood of p.

Proof. Note that in each of three cases λ is already a function of f. Thus we can find

$$b(f) = |\nabla f|^2$$

by solving the initial value problem

$$\begin{aligned} \frac{db}{df} &= 2\lambda(f), \\ b(0) &= 0 \end{aligned}$$

The solution is obviously unique so it actually only remains to be checked that the three standard geometries have functions with the described properties. This works as follows using the standard warped product representations:

Euclidean space

$$\begin{aligned} & dr^2 + r^2 ds_{n-1}^2, \\ f(r) &= \frac{1}{2}r^2 \end{aligned}$$

The unit sphere

$$\begin{aligned} & dr^2 + \sin^2 r\, ds_{n-1}^2, \\ f(r) &= 1 - \cos r \end{aligned}$$

Hyperbolic space

$$\begin{aligned} & dr^2+\sinh^2 r ds_{n-1}^2, \\ f(r) &= -1+\cosh r \end{aligned}$$

In all three cases $r=0$ corresponds to the point p. □

4. Generalizations

In view of what we saw above it is interesting to investigate what generalizations are possible.

Transnormal functions simply satisfy:

$$|\nabla f|^2 = b(f)$$

Note that the equation

$$\mathrm{Hess}f(\nabla f, X) = \frac{1}{2} D_X |\nabla f|^2$$

shows that this is locally equivalent to saying that ∇f is an eigenvector for Hess$f$. Wang proved that such functions further have the property that the only possible critical values are the maximum and minimum values, moreover the corresponding max/min level sets are submanifolds. Such functions can be reparametrized as $f = f(r)$, where r is a distance function, i.e., $|\nabla r|^2 \equiv 1$. Note that it is easy to define r as a function of f

$$r = \int \frac{df}{\sqrt{b}}$$

It is important that b is differentiable for these properties to hold as any strictly increasing function on $\mathbb{R}$ is transnormal for some continuous b. In this case there can certainly be any number of critical points that are merely inflection points.

Isoparametric functions are transnormal functions where in addition

$$\Delta f = a(f).$$

These functions in addition have the property that the min/max sets for r are minimal submanifolds. These functions were introduced and studied by E. Cartan.

Even stricter conditions would be that the function is transnormal and the eigenvalues of Hess$f$ are functions of f. A very good and general model case for this situation is a cohomogeneity 1 manifold with $f = f(r)$ and $r : M \to M/G = I \subset \mathbb{R}$. For this class of functions it is a possibility that the eigenspace distributions for the Hessian are not integrable. Specifically select (M^4, g) with $U(2)$ symmetry, e.g., $\mathbb{C}$ bundles over S^2, $\mathbb{R}^4$ with the Taub-NUT metric, or $\mathbb{C}P^2$. The generic isotropy of $U(2)$ is $U(1)$. The subspace corresponding to $U(1)$ is then a 2-dim distribution that must lie inside an eigenspace for Hess$f$. This distribution is however not integrable as it corresponds to the horizontal space for the Hopf fibration. As long as the metric isn't invariant under the larger group $SO(4)$ the Hopf fiber direction, i.e., the direction tangent to the orbits but perpendicular to this distribution, will generically correspond to an eigenvector with a different eigenvalue.

Finally one can assume that the function is transnormal and satisfies

$$R(X,Y)\nabla f = 0$$

for all $X, Y \perp \nabla f$. The curvature condition is automatic in the three model cases with constant curvature so it doesn't imply that f is isoparametric. But it is quite restrictive as it forces eigenspaces to be integrable distributions. It is also equivalent to saying that $\mathrm{Hess}(r)$ is a Codazzi tensor where r is the distance function so that $f = f(r)$. A good model case is a doubly warped product

$$\begin{aligned} M &= I \times N_1 \times N_2 \\ g &= dr^2 + \phi_1^2(r)\, h_1 + \phi_2^2(r)\, h_2, \\ f &= f(r). \end{aligned}$$

Note that the functions f coming from warped products satisfy all of the conditions mentioned above.

Another interesting observation that might aid in a few calculations is that a general warped product

$$\psi^2 g_B + \phi^2 g_F = \phi^2 \left(\phi^{-2} \psi^2 g_B + g_F \right)$$

on a Riemannian submersion $M \to B$, where g_B is a metric on B and g_F represents the family of metrics on the fibers, and ψ, ϕ functions on B. Here the metric $\phi^{-2}\psi^2 g_B + g_F$ is again a Riemannian submersion with the conformally changed metric $\phi^{-2}\psi^2 g_B$ on the base. Thus general warped products are always conformally changed on the total space of a Riemannian submersion, where the conformal factor has a gradient that is basic horizontal.

5. Geodesics

We obtain formulas for geodesics on warped products that mirror those of geodesics on surfaces of revolution.