Network Flows

This is to be read in conjunction with section 4.3. We recall that an $a-z$ *cut* $P,\bar{P}$ is simply a division of all vertices of the network so that $a \in P$ and $z \in \bar{P}$. The capacity $k\left(P,\bar{P}\right)$ of such a cut is the sum of all capacities of edges going from P to $\bar{P}$

$$k\left(P,\bar{P}\right) = \sum_{e\in\left(P,\bar{P}\right)} k(e).$$

The simplest case occurs when $P=\{a\}$. Given that the strength of a flow is defined as the value of the flow on the edges emmanating from a we clearly have

$$|f| \leq k\left(\{a\},\bar{P}\right).$$

This generalizes to

$$|f| \leq k\left(P,\bar{P}\right)$$

for any $a-z$ cut of the network as observed in Theorem 2. A different way of seeing this, without resorting to tricks, is by first observing that what f flows from P to $\bar{P}$ can't exceed capacity, i.e.,

$$\sum_{e\in\left(P,\bar{P}\right)} f(e) \leq \sum_{e\in\left(P,\bar{P}\right)} k(e).$$

On the other hand the strength of the flow must equal what flows from P to $\bar{P}$ if we also subtract what flows back from $\bar{P}$ to P, i.e.,

$$|f| = \sum_{e\in\left(P,\bar{P}\right)} f(e) - \sum_{e\in\left(\bar{P},P\right)} f(e).$$

To give a rigorous proof of this generalized conservation law we introduce the function $\alpha(x,e)$, where $x \in V$ is a vertex and $e \in E$ is an edge,

$$\alpha(x,e) = \begin{cases} 1 & \text{if } e \text{ points away from } x, \\ -1 & \text{if } e \text{ points into } x, \\ 0 & \text{if } e \text{ does not have } x \text{ as an edge.} \end{cases}$$

The conservation law says that for any vertex $x \in V - \{a,z\}$ we have

$$\sum_{e\in E} \alpha(x,e) f(e) = 0,$$

while the strength is

$$|f| = \sum_{e\in E} \alpha(a,e) f(e).$$

If we add these sums over all vertices in P we get

$$\begin{aligned} |f| &= \sum_{e\in E} \alpha(a,e) f(e) + \sum_{x\in P-\{a\}} \sum_{e\in E} \alpha(x,e) f(e) \\ &= \sum_{x\in P} \sum_{e\in E} \alpha(x,e) f(e) \\ &= \sum_{e\in E} \sum_{x\in P} f(e) \alpha(x,e) \\ &= \sum_{e\in E} f(e) \sum_{x\in P} \alpha(x,e). \end{aligned}$$

So for a fixed edge we see that

$$\sum_{x\in P} \alpha(x,e) = \begin{cases} 1 & \text{if } e \text{ starts in } P \text{ and ends in } \bar{P}, \\ -1 & \text{if } e \text{ starts in } \bar{P} \text{ and ends in } P, \\ 0 & \text{if both endpoints of } e \text{ are in } P, \\ 0 & \text{if both endpoints of } e \text{ are in } \bar{P}. \end{cases}$$

This shows that

$$|f| = \sum_{e\in E} f(e) \sum_{x\in P} \alpha(x,e) = \sum_{e\in\left(P,\bar{P}\right)} f(e) - \sum_{e\in\left(\bar{P},P\right)} f(e).$$

The fact that $|f|$ can be calculated by adding the amount that flows into z is a consequence of this fundamental formula. We simply use $P = V - \{z\}$, $\bar{P} = \{z\}$ and note that there are no edges that begin at z, to see that

$$\begin{aligned} |f| &= \sum_{e\in(V-\{z\},\{z\})} f(e) - \sum_{e\in(V-\{z\},\{z\})} f(e) \\ &= \sum_{e\in(V-\{z\},\{z\})} f(e). \end{aligned}$$

These observations also establish corollary 2a. Namely, if

$$\begin{aligned} \sum_{e\in\left(P,\bar{P}\right)} f(e) &= \sum_{e\in\left(P,\bar{P}\right)} k(e), \\ \sum_{e\in\left(\bar{P},P\right)} f(e) &= 0, \end{aligned}$$

then

$$\begin{aligned} |f| &= \sum_{e\in\left(P,\bar{P}\right)} f(e) - \sum_{e\in\left(\bar{P},P\right)} f(e) \\ &= \sum_{e\in\left(P,\bar{P}\right)} k(e) - 0 \\ &= k\left(P,\bar{P}\right). \end{aligned}$$

A *simple path* in a graph is a path or trail from one vertex to another which never repeats an edge. If we have such a path in a network with a flow f, then we say that it is α *flexible* if $k(e) - f(e) \geq \alpha$ for all edges that are directed in the same direction as the path is traveled, while $f(e) \geq \alpha$ on all edges that are directed against the direction of the way we travel along the path. For each flow f we define P_f as the set of vertices in the network that we can reach starting at a by traveling along α flexible simple paths with $\alpha \geq 1$.

The key observation is that if $z \in P_f$, then the strength of f can be improved to $|f| + \alpha$ by adding α to f along the edges that flow with the path, while *subtracting* α from f along edges that are directed against the path. Having made such a change we can repeat the procedure. Since we add at least 1 to the strength each time we make such a change and the strength of a flow can't exceed any capacity this procedure will terminate in a finite number of steps. When this happens we have found a flow f such that $z \notin P_f$. Thus we have found an $a - z$ cut $P_f, \bar{P}_f$. We now claim that if this happens then

$$|f| = k\left(P_f, \bar{P}_f\right).$$

In other words, we have found a flow whose strength equals the capacity of a cut. This proves that a maximum flow has strength that is equal to the minimal capacity of an $a - z$ cut. In other words it proves the max flow/min cut theorem.

To prove the assertion note that if e is an edge from P_f to $\bar{P}_f$ then $f(e) = k(e)$, because otherwise the endpoint would be in P_f as it would be the end point for a path with positive flexibility from a. Likewise if e is an edge from $\bar{P}_f$ to P_f, then $f(e) = 0$ because otherwise we could travel against the arrow of the edge and have a path with positive flexibility ending up in $\bar{P}_f$. This means that our assertion follows (see also corollary 2a).