

Manifolds, Transversality, and de Rham Cohomology

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Preface

The initial purpose of these notes was to supplement the material in our two quarter sequence on manifold theory as it wasn't possible to find one or even two texts that covered what was felt to be appropriate for those classes. In the meantime they have grown to become a complete, if terse, text covering all relevant topics with the addition of several exercises at the end of each chapter.

The first four chapters correspond to 10 weeks of material and offer a coherent introduction to the basic theory of smooth manifolds, smooth maps, tangent spaces, and vector bundles. The main theorems are the global Frobenius theorem; covering spaces and the fundamental group; submersions and fibrations including Ehresmann's theorem and Godement's characterization of when an equivalence relation yields a quotient space that is a manifold; immersions/embeddings including Whitney's theorem; tubular neighborhoods and their uses for approximating continuous maps with smooth maps; and finally Lie groups and algebras and their correspondence.

The next four chapters likewise correspond to 10 weeks of material. Here the topics change a bit more. To begin, there is a chapter on tensor analysis elaborating on Lie derivatives and the differential and integral calculus of forms. Next comes transversality including intersection theory. This leads to degree theory, fixed point theory, and the Hopf degree theorem. Finally, comes de Rham cohomology: first the basic principles are established and several of the standard examples are calculated; the basic principles are then paired down to a simple set of axioms that are used to characterize de Rham cohomology as well as proving Poincaré duality and Künneth's theorem. This all comes together with intersection theory, offering a new set of formulas for calculating degrees, Euler characteristics, and Lefschetz numbers.

It is worthwhile mentioning that the notes include a little used universal principle for manifolds (UPM). It is hinted at in [Hirsch] for the transversality theorem and enunciated precisely in [Bredon] for the purpose of proving Poincaré duality. We have here significantly expanded the use of the principle to include the construction of bump functions, the Whitney embedding theorem, axiomatic characterization of cohomology, Künneth's theorem and The Leray-Hirsch theorem for calculating the cohomology of certain fibrations. The advantages are that the UPM creates a clear proof strategy for these results that avoids the use of good covers and spectral sequences. It is in essence an elegant way of handling Mayer-Vietoris when dealing with countably many sets. The UPM can be extended to topological manifolds and cohomology with general coefficients. It can also be extended to sheaf cohomology on manifolds. This allows for its use with Dolbeault cohomology where the grounding axiom, that is otherwise for open cubes, is altered to be for polydiscs.

The basic necessary background for this text is abstract linear algebra including diagonalizability of self-adjoint maps; multivariable calculus including the inverse and implicit function theorems; as well as integration; sets of measure zero in Euclidean space; existence and uniqueness of solutions to ordinary differential equations including smoothness

of solutions with respect to initial conditions. Any number of books cover these topics. It is also necessary to have some knowledge of basic topology. The quickest complete introduction is probably [**Hatcher2**].

Manifold theory and differential geometry are in a sense applied pure mathematics, in that, you need a broad knowledge of mathematics, however, many results can be used without a deep understanding of how to prove them.

For the purposes of teaching the material at UCLA, or any place that has a two quarter sequence, it seems reasonable divide the material as follows.

- 225A: Ch 1-4. The proofs of theorems 1.4.26 and 3.3.15 can be skipped, but, there is time to cover them. Section 1.3.2 should be skipped. Section 2.3.3 can also be skipped. The more sheaf theoretic properties characterizing maps can also be skipped
- 225B: Ch 5-8. Section 7.2.3 on invariant cohomology can be mentioned in passing if timing is tight. Section 7.4.2 on relative cohomology can be skipped. In section 7.6 only the Künneth theorem is necessary for chapter 8. Examples 8.3.1,2 can be skipped.

Chapters 1-8 have exercises at the end of the chapters, some exercises are difficult and generally appear later.

There are several texts that expand mathematically or linguistically on the material covered here. The book by [**Guillemin-Pollack**] is a more basic approach for differential topology, while [**Hirsch**] is considerably more demanding. The cohomology aspects covered here correspond to a simplified version of [**Bott-Tu**]. Another text is the well constructed [**Madsen-Tornehave**] where deRham cohomology is extended to cover continuous functions. The comprehensive and sadly underused text [**Spivak**, vol. I] is also worth consulting for many aspects of the theory discussed here. The text [**Serre**] is also worth studying both for its coverage of Lie algebras and Lie groups as well as analytic manifolds over \mathbb{R} , \mathbb{C} , and \mathbb{Q}_p . The short bibliography lists the texts that I have consulted while writing these notes. It is not on purpose if I excluded anyones favorite text!

I wish to acknowledge the numerous students that have suffered through my experimentation and offered many suggestions for improvements. They have remained my main inspiration for finally completing them just as I have taught the material for the last time.

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Manifolds and Smooth Maps

In this chapter we give the basic definitions of manifolds, smooth maps, submanifolds, and ranks of maps. The main results are the preimage theorem, the constant rank theorem, and Sard's theorem.

1.1. Smooth Manifolds

The goal in this section is to give the basic definition of a smooth manifold and a smooth map between manifolds. This is followed in the next section by the most basic examples.

An n -manifold is a topological space, M^n , with a *maximal atlas* or a *maximal smooth structure*. The topological space will eventually be assumed to be Hausdorff and second countable. The importance of these assumptions can be found in section 1.3. These properties are automatically satisfied by all of the examples below as their topologies come from the spaces being subsets of finite dimensional vector spaces.

The standard definition of an atlas is as follows:

DEFINITION 1.1.1. An *atlas* \mathcal{A} consists of a collection of maps $x_\alpha : U_\alpha \rightarrow \mathbb{R}^n$ such that

- (1) U_α is an open covering of M .
- (2) $x_\alpha(U) \subset \mathbb{R}^n$ is open and homeomorphic to U_α via x_α .
- (3) The *transition functions* $x_\alpha \circ x_\beta^{-1} : x_\beta(U_\alpha \cap U_\beta) \rightarrow x_\alpha(U_\alpha \cap U_\beta)$ are diffeomorphisms.

In condition (3) it suffices to show that the transition functions are smooth since $x_\beta \circ x_\alpha^{-1} : x_\alpha(U_\alpha \cap U_\beta) \rightarrow x_\beta(U_\alpha \cap U_\beta)$ is an inverse. The atlas is maximal provided we cannot add a map to it so as to create a larger atlas. The maps $x_\alpha : U_\alpha \rightarrow \mathbb{R}^n$ are called *coordinates* or *charts* or *coordinate charts*. Exercise 1 in this chapter has an alternate atlas definition that constructs the manifold from the images of the charts in \mathbb{R}^n and the transition functions.

The second definition is a compromise between the atlas version and a purely sheaf theoretic approach. It is, however, essentially the definition of a submanifold of Euclidean space where local parametrizations are given as local graphs.

DEFINITION 1.1.2. A *smooth structure* is a collection \mathcal{D} consisting of continuous functions, $f : O \rightarrow \mathbb{R}$, where the domains $O \subset M$ are open, such that for each $p \in M$, there is an open neighborhood $U \ni p$ and functions $x^i \in \mathcal{D}$, $i = 1, \dots, n$ such that:

- (1) The domains of x^i contain U .
- (2) The map $x = (x^1, \dots, x^n) : U \rightarrow \mathbb{R}^n$ is a homeomorphism onto its image $V \subset \mathbb{R}^n$, which is assumed to be open.
- (3) For each $f : O \rightarrow \mathbb{R}$ in \mathcal{D} there is a smooth function $F : x(U \cap O) \rightarrow \mathbb{R}$ such that $f = F \circ x$ on $U \cap O$.

The map in (2) is again a *chart* or *coordinate system* on U . The inverse $x^{-1} : V \rightarrow U$ is also called a *parametrization*. Many of our coordinate systems will in fact be given as parametrizations. Observe that in condition (3), $F = f \circ x^{-1}$, but, it is usually possible to find F without having to invert x . F is called the *coordinate representation* of f and is normally also denoted by f . The smooth structure is maximal provided we cannot add a function to it and still have a smooth structure.

Note that it is very easy to see that these two definitions are equivalent. Given an atlas the differentiable structure simply consists of all of the individual coordinate functions x_α^i . Conversely, a differentiable system creates an atlas as there are coordinates $x_p = (x_p^1, \dots, x_p^n)$ on a neighborhood around every $p \in M$. The transition functions $x_p \circ x_q^{-1}$ are smooth as the individual functions $x_p^i \circ x_q^{-1}$ are assumed to be smooth.

Both definitions have advantages. The first in certain proofs. The latter is generally easier to work with when showing that a concrete space is a manifold and is also useful when it comes to defining foundational concepts.

DEFINITION 1.1.3. A continuous function $f : O \rightarrow \mathbb{R}$ is said to be smooth with respect to \mathcal{D} if $\mathcal{D} \cup \{f\}$ is also a smooth structure. In other words we only need to check that condition (3) still holds when we add f to our collection \mathcal{D} . We can more generally define what it means for f to be C^k for any k with smooth being C^∞ and continuous C^0 . We shall generally only use smooth or continuous functions.

The space of all smooth functions is a *maximal smooth structure*. We use the notation $C^k(M)$ for the space of C^k functions defined on all of M and $C^k(O)$ for the space of C^k functions defined on open sets in M , $f : O \rightarrow \mathbb{R}$ with $O \subset M$ being open and f is C^k . The collection $C^k(M)$ is an example of a (pre)sheaf.

It is often the case that all the functions in a \mathcal{D} have domain M . In fact, we will show that it is always possible to use a finite collection \mathcal{D} (see theorem 3.4.4).

The next proposition shows that the dimension of the manifold is unique.

PROPOSITION 1.1.4. *If $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ are open sets that are diffeomorphic, then $m = n$.*

PROOF. The differential of the diffeomorphism is forced to be a linear isomorphism. This shows that $m = n$. \square

A map $F : M \rightarrow N$ between spaces has a natural dual or *pull-back* that takes functions defined on subsets of N to functions defined on subsets of M . Specifically, if $f : A \subset N \rightarrow \mathbb{R}$, then $F^*(f) = f \circ F : F^{-1}(A) \subset M \rightarrow \mathbb{R}$. Here it could happen that $F^{-1}(A) = \emptyset$. Note that if F is continuous, then its pull-back will map continuous functions on open subsets of N to continuous functions on open subsets of M . Conversely, if N is normal, and the pull-back takes continuous functions to continuous functions, then it will be continuous. To see this, fix $O \subset N$ that is open and select a continuous function $\lambda : N \rightarrow [0, \infty)$ such that $\lambda^{-1}(0, \infty) = O$. Thus, $(\lambda \circ F)^{-1}(0, \infty) = F^{-1}(O)$ and is, in particular, open as we assumed that $\lambda \circ F$ was continuous.

DEFINITION 1.1.5. A map $F : M \rightarrow N$ is said to be *smooth* if F^* takes smooth functions to smooth functions, i.e., $F^*(C^\infty(N)) \subset C^\infty(M)$. A bijective map $F : M \rightarrow N$ such that both F and F^{-1} are smooth is called a *diffeomorphism*.

PROPOSITION 1.1.6. *Let $F : M \rightarrow N$ be continuous. The following conditions are equivalent:*

- (1) F is smooth.
- (2) If \mathcal{D} is a differentiable structure on N , then $F^*(\mathcal{D}) \subset C^\infty(M)$.
- (3) If $x_\alpha : U_\alpha \rightarrow \mathbb{R}^m$ is an atlas for M and $y_\beta : V_\beta \rightarrow \mathbb{R}^n$ an atlas for N , then the coordinate representations $y_\beta \circ F \circ x_\alpha^{-1}$ are smooth when- and where-ever they are defined.

PROOF. The implications $1 \Rightarrow 2 \Rightarrow 3$ are obvious. For $3 \Rightarrow 1$ note that differentiability can be checked locally. Given $f : O \subset N \rightarrow \mathbb{R}$ we can locally rewrite

$$f \circ F = \left(f \circ y_\alpha^{-1} \right) \circ \left(y_\alpha \circ F \circ x_\beta^{-1} \right) \circ x_\beta.$$

Here $(f \circ y_\alpha)$ and $(y_\alpha \circ F \circ x_\beta^{-1})$ are smooth so $f \circ F \circ x_\beta^{-1}$ is smooth, which shows that $f \circ F$ is smooth. \square

1.2. Examples

If we start with $M \subset \mathbb{R}^k$ as a subset of Euclidean space, then the obvious choices are to use the induced topology and the ambient coordinate functions $x^i|_M : M \rightarrow \mathbb{R}$ as the potential differentiable structure \mathcal{D} . Depending on what subset we start with this might or might not work. Even when it doesn't, there might be other obvious ways that could make it work. For example, we could start with a subset which has corners, such as a triangle. While the obvious choice of a differentiable structure will not work we note that the subset is homeomorphic to a circle, which does have a valid differentiable structure. This structure will be carried over to the triangle via the homeomorphism. This is a rather subtle point and begs the very difficult question: Does every topological manifold carry a smooth structure? The answer is yes in dimensions 1, 2, and 3, but no in dimension 4 and higher. There are also subsets where the induced topology won't make the space even locally homeomorphic to Euclidean space. A figure eight 8 is a good example. However, there is an interesting bijective continuous map $\mathbb{R} \rightarrow 8$. It "starts" at the crossing moving NW, wraps around in the figure 8 and then ends at the crossing on the opposite side still pointing NW. As the interval was open every point on 8 only gets covered once in this process. This map is clearly also continuous. However, it is not a homeomorphism onto its image. Thus we see again that an even more subtle game can be played where we can refine the topology of a given subset to make it a manifold.

We start with a general example, that given the right set-up, can be used to show that a space is a manifold. It'll become even easier to use after it has been generalized to maps between manifolds (see section 1.4).

EXAMPLE 1.2.1. The graph of a smooth map $F : O \subset \mathbb{R}^n \rightarrow \mathbb{R}^k$, where O is open is given by

$$\text{Graph}F = \{(x, F(x)) \mid x \in O\}.$$

This is clearly an n -manifold with one chart that is diffeomorphic to O . More generally, any subset $M \subset \mathbb{R}^{n+k}$, which is locally given as a smooth graph $F : O \subset \mathbb{R}^n \rightarrow \mathbb{R}^k$ is an n -dimensional manifold. The implicit function theorem gives us a criterion for when such a situation occurs. Consider a smooth map $F : O \subset \mathbb{R}^{n+k} \rightarrow \mathbb{R}^k$, where O is open and a point $y_0 \in \mathbb{R}^k$. If the rank of $DF|_x : \mathbb{R}^{n+k} \rightarrow \mathbb{R}^k$ is k (surjective) for all $x \in F^{-1}(y_0)$, then $M = F^{-1}(y_0)$ is an n -manifold. The implicit function theorem shows that $M \subset \mathbb{R}^{n+k}$ is locally a graph over n of the $n+k$ coordinates.

However, as we shall soon see, there are many examples where spaces are not naturally presented as equations of smooth functions. Nevertheless, they can always locally be described as smooth graphs inside a suitable vector space.

There are constructions of manifolds where it is not immediately clear that they lie in a vector space. For instance, a covering map $M \rightarrow N$, where N is a manifold. Even when N lies in a vector space it is not clear that M does! The same with the reverse construction where M is assumed to be a manifold. More abstractly, one might consider the space of equivalence classes of an equivalence relation on a manifold. This will generally not lead to a quotient that is a manifold. So the question is: when does it become a manifold? These examples will be examined in detail in chapter 3.

1.2.1. Spheres. The n -sphere is defined as

$$S^n = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}.$$

Thus we have $n + 1$ natural functions. If we use the function

$$\begin{aligned} F : \mathbb{R}^{n+1} &\rightarrow \mathbb{R} \\ x &\mapsto |x|^2 \end{aligned}$$

then $DF|_{x_0}(h) = 2x_0 \cdot h$ which has rank 1 as long as $x_0 \neq 0$ so the implicit function theorem shows that it is a manifold.

More concretely, we see that any open hemisphere $O_i^\pm = \{x \in S^n \mid \pm x^i > 0\}$ is a graph over the coordinate system that comes from using the n functions x^j where $j \neq i$. The remaining coordinate function is obtained as a smooth expression:

$$\pm x^i = \sqrt{1 - \sum_{j \neq i} (x^j)^2}$$

on O_i^\pm .

A somewhat different atlas of charts can be constructed via *stereographic projection* from points $e \in S^n$. The map is geometrically given by drawing a line through the points $z \in e^\perp = \{z \in \mathbb{R}^{n+1} \mid z \perp e\}$ and e followed by checking where it intersects the sphere. The equator where $x \cdot e = 0$ stays fixed, while the hemisphere closest to e is mapped outside this equatorial band, and the hemisphere farthest from e is mapped inside the band. The map is defined on $S^n - \{-e\}$. The maps can most easily be found by first constructing them for the circle in the plane. This situation can be transferred to higher dimensions by observing that $\text{span}\{e, x\} = \text{span}\{e, z\}$ are 2-dimensional when they are defined. In general, we know that

$$z = s(x - e) + e$$

and

$$x = t(z - e) + e.$$

This implies that $st = 1$. The proviso $z \cdot e = 0$ shows that $s = \frac{1}{1 - x \cdot e}$ which leads to the formulas:

$$z = \frac{1}{1 - x \cdot e} (x - e) + e$$

and

$$x = \frac{2}{1 + |z|^2} (z - e) + e.$$

Any two of these maps suffice to create an atlas. But, we must check that the transition functions are also smooth. To be specific, we consider the ones coming from antipodal

points, say e and $-e$. In this case the transition is an inversion in the equatorial band and is given by (again do it first for the circle with $\pm e$ being on the x -axis)

$$z \mapsto \frac{z}{|z|^2}.$$

In particular, we see that the sphere is naturally identified with the one point compactification of Euclidean space of the same dimension.

1.2.2. Basic Matrix Groups. First some notation. Matrices with m rows and n columns and entries in a ring R are denoted by $\text{Mat}_{m \times n}(R)$. In case the ring is a field, \mathbb{F} , or more generally a division algebra the invertible $n \times n$ matrices are denoted by

$$Gl_n(\mathbb{F}) = Gl(n, \mathbb{F}) \subset \text{Mat}_{n \times n}(\mathbb{F})$$

and is called the *general linear group*. When $\mathbb{F} = \mathbb{R}, \mathbb{C}$ the invertible elements form an open subset of all the matrices since invertibility is equivalent to having nonvanishing determinant. The only division algebra we need to consider is \mathbb{H} , the space of *quaternions* (The letter H stands for Hamilton who invented these numbers). They are defined in section 4.2.5 and in this case $Gl_n(\mathbb{H}) \subset \text{Mat}_{n \times n}(\mathbb{H})$ is also open. This, however, has to be justified in a different and more general way. In a neighborhood of the identity the inverse of a matrix is given by a power series

$$(I - X)^{-1} = I + X + X^2 + \dots = \sum_{k=0}^{\infty} X^k$$

that converges as long as the norm of X is less than 1. Here “norm” refers to any reasonable norm that ensures convergence. We will mostly use the Euclidean norm of the matrix, thinking of it as lying in a real vector space, however, the operator norm works equally well. In a neighborhood of an invertible matrix A we can similarly invert nearby elements via

$$(A - AX)^{-1} = A^{-1}(I - X)^{-1} = A^{-1}(I + X + X^2 + \dots) = A^{-1} \sum_{k=0}^{\infty} X^k.$$

As the general linear groups are open sets in a vector space they become manifolds. Moreover, the group operations of multiplication and inverse are smooth. A manifold that is also a group with smooth group operations is called a *Lie group*.

A slight warning is necessary about quaternions as multiplication is not commutative. In particular, left scalar multiplication on \mathbb{H}^n makes the matrices in $\text{Mat}_{n \times n}(\mathbb{H})$ real linear if they also act on the left, but \mathbb{H} linear if the matrices act on the right!

Using that the operation of taking adjoints $A \mapsto A^*$ is smooth (in fact real linear) we obtain a smooth map $F : \text{Mat}_{n \times n}(\mathbb{F}) \rightarrow \text{Sym}_n(\mathbb{F})$ defined by $A \mapsto AA^*$, where $\text{Sym}_n(\mathbb{F})$ denotes the real vector space of self-adjoint operators (symmetric or Hermitian depending on the field.) Note that the image of this map consists of the self-adjoint matrices that are nonnegative definite, i.e., have nonnegative eigenvalues. The differential of this map can be calculated using the standard coordinates of the vector spaces. At the identity we have the first order Taylor expansion

$$F(I + tX) = (I + tX)(I + tX^*) = I + t(X + X^*) + o(t).$$

Consequently, the differential is

$$DF|_I(X) = X + X^*.$$

This has maximal rank since it is simply multiplication by 2 when restricted to $\text{Sym}_n(\mathbb{F})$. More generally the differential at an invertible $A \in \text{Gl}$ is given by

$$DF|_A(X) = XA^* + AX^*$$

which also has maximal rank as it is a bijection when restricted to the real subspace $\{X(A^{-1})^* \mid X \in \text{Sym}_n(\mathbb{F})\}$. Thus we obtain a smooth map:

$$F : \text{Gl}(n, \mathbb{F}) \rightarrow \text{Sym}_n(\mathbb{F})$$

that has maximal rank at all points. The preimage of the identity is identified with the *orthogonal group*:

$$O(n) = \{O \in \text{Gl}(n, \mathbb{R}) \mid OO^* = I\}$$

or the *unitary group*

$$U(n) = \{U \in \text{Gl}(n, \mathbb{C}) \mid UU^* = I\}.$$

These are manifolds of dimension $\binom{n}{2}$ and n^2 respectively. The group operations are also smooth so we have further examples of Lie groups.

1.2.3. Basic Geometry of Projective Spaces. Given a vector space V we define $\mathbb{P}(V)$ as the set of 1-dimensional subspaces or lines through the origin. It is called the *projective space* of V . In the concrete case where $V = \mathbb{F}^{n+1}$ we use the notation $\mathbb{P}(\mathbb{F}^{n+1}) = \mathbb{F}\mathbb{P}^n = \mathbb{P}^n$.

One can similarly develop a theory of the space of subspaces of any given dimension. The space of k -dimensional subspaces is denoted $\text{Gr}_k(V)$ and is called the Grassmannian of k -planes.

The space of operators or endomorphisms on V is denoted $\text{End}(V)$ and the invertible operators or automorphisms by $\text{Aut}(V)$. When $V = \mathbb{F}^n$ these are represented by matrices $\text{End}(\mathbb{F}^n) = \text{Mat}_{n \times n}(\mathbb{F})$ and $\text{Aut}(\mathbb{F}^n) = \text{Gl}_n(\mathbb{F})$. Since invertible operators map lines to lines we see that $\text{Aut}(V)$ acts in a natural way on $\mathbb{P}(V)$. In fact this action is transitive, i.e., if we have $p, q \in \mathbb{P}(V)$, then there is an operator $A \in \text{Aut}(V)$ such that $A(p) = q$. In fact, as any two bases in V can be mapped to each other by invertible operators it follows that any collection of k independent lines p_1, \dots, p_k , i.e., $p_1 + \dots + p_k = p_1 \oplus \dots \oplus p_k$ can be mapped to any other collection of k independent lines q_1, \dots, q_k . This means that the action of $\text{Aut}(V)$ on $\mathbb{P}(V)$ is k -point homogeneous for all $k \leq \dim(V) + 1$. Note that this action is not *effective* as all homotheties $A = \lambda 1_V$ act trivially on $\mathbb{P}(V)$.

Since an endomorphism might have a kernel it is not true that it maps lines to lines, however, if we have $A \in \text{End}(V)$, then we do get a map $A : \mathbb{P}(V) - \mathbb{P}(\ker A) \rightarrow \mathbb{P}(V)$ defined on lines that are not in the kernel of A .

Let us now assume that V is an inner product space with an inner product $\langle v, w \rangle$ that can be real (Euclidean) or complex (Hermitian). The key observation in relation to subspaces is that they are completely characterized by the orthogonal projections onto the subspaces. Thus the space of k -dimensional subspaces is the same as the space of orthogonal projections of rank k . It is convenient to know that an endomorphism $E \in \text{End}(V)$ is an orthogonal projection iff it is a projection, $E^2 = E$ that is self-adjoint, $E^* = E$. In the case of a one-dimensional subspace $p \in \mathbb{P}(V)$ spanned by a unit vector $v \in V$, the orthogonal projection is given by

$$\text{proj}_p(x) = \langle x, v \rangle v.$$

Clearly we get the same formula for all unit vectors in p . Note that the formula is quadratic in v . This yields an inclusion $\mathbb{P}(V) \rightarrow \text{End}(V)$ and endows $\mathbb{P}(V)$ with a natural topology. One can also easily see that $\mathbb{P}(V)$ is compact.

The angle between lines in V gives a natural metric on $\mathbb{P}(V)$. Automorphisms clearly do not preserve angles between lines and so are not necessarily isometries. However, if we

restrict attention to unitary or orthogonal transformations $L \subset \text{Aut}(V)$, then we know that they preserve inner products of vectors. Therefore, they must also preserve angles between lines. Thus L acts by isometries on $\mathbb{P}(V)$. This action is again homogeneous so $\mathbb{P}(V)$ looks geometrically the same everywhere.

One way of finding coordinates around $p \in \mathbb{P}(V)$ is to consider the set of 1-dimensional subspaces, $\mathbb{P}(V) - \mathbb{P}(p^\perp)$, that are not perpendicular to p . This is clearly an open set in $\mathbb{P}(V)$ and we claim that there is a coordinate map $G_p : \text{Hom}(p, p^\perp) \rightarrow \mathbb{P}(V) - \mathbb{P}(p^\perp)$. To construct this map decompose $V \simeq p \oplus p^\perp$ and note that any 1-dimensional subspace not in p^\perp is a graph over p given by a unique homomorphism in $\text{Hom}(p, p^\perp)$. The next thing to check is that G_p is a homeomorphism onto its image and is differentiable as a map into $\text{End}(V)$. Neither fact is hard to verify. Finally observe that $\text{Hom}(p, p^\perp)$ is a vector space of dimension $\dim V - 1$. In this way $\mathbb{P}(V)$ becomes a manifold of dimension $\dim V - 1$.

Note that while we showed that $\mathbb{P}(V)$ is locally a graph in the vector space $\text{End}(V)$ we did not use that the projective space was given by an equation so that we could appeal to the implicit function theorem.

1.2.4. Projective Coordinates. We saw that the n -dimensional (real) projective space $\mathbb{R}\mathbb{P}^n$ can be identified with the space of orthogonal projections of rank 1. More concretely, if

$$x = \begin{bmatrix} x^0 \\ x^1 \\ \vdots \\ x^n \end{bmatrix} \in \mathbb{R}^{n+1} - \{0\},$$

then the matrix that describes the orthogonal projection onto $\text{span}\{x\}$ is given by

$$\begin{aligned} E_x &= \frac{1}{|x|^2} \begin{bmatrix} x^0 x^0 & x^0 x^1 & \cdots & x^0 x^n \\ x^1 x^0 & x^1 x^1 & \cdots & x^1 x^n \\ \vdots & \vdots & \ddots & \vdots \\ x^n x^0 & x^n x^1 & \cdots & x^n x^n \end{bmatrix} \\ &= \frac{1}{|x|^2} x x^*. \end{aligned}$$

Clearly $E_x^* = E_x$ and as $x^* x = |x|^2$ we have $E_x^2 = E_x$ and $E_x x = x$. Thus E_x is the orthogonal projection onto $\text{span}\{x\}$. Finally note that $E_x = E_y$ if and only if $x = \lambda y$, $\lambda \neq 0$. With that in mind we obtain a natural differentiable system by using the coordinate functions

$$f^{ij}(E_x) = \frac{x^i x^j}{|x|^2}.$$

If we fix j and consider the $n+1$ functions f^{ij} , $i = 1, \dots, n+1$, then we have the relationship

$$f^{jj} = (f^{jj})^2 + \sum_{i \neq j} (f^{ij})^2.$$

This describes a sphere of radius $\frac{1}{2}$ centered at the point where $f^{ij} = 0$ for $i \neq j$ and $f^{jj} = \frac{1}{2}$. The origin on this sphere corresponds to all points where $x^j = 0$. But, any other point on the sphere corresponds to a unique element of $O_j = \{E_x \mid x_j \neq 0\}$. This means that around any given point in O_j we can use n of the functions f^{ij} as a coordinate chart. The remaining function is then expressed smoothly in terms of the other coordinate functions. This still

leaves us with the other functions f^{kl} , but, they satisfy

$$f^{kl} = \frac{f^{kj} f^{lj}}{f^{jj}}$$

and so on the given neighborhood in O_j they are also smoothly expressed in terms of our chosen coordinate functions. The more efficient collection of functions f^{ij} , $i \leq j$ yields the *Veronese map*

$$\mathbb{RP}^n \rightarrow \mathbb{R}^{\frac{(n+2)(n+1)}{2}}.$$

A more convenient differentiable system can be constructed using *homogeneous coordinates* on \mathbb{RP}^n . These are written $[x^0 : x^1 : \cdots : x^n]$ and represent the equivalence class of non-zero vectors that are multiples of x . The notation is suggestive of the fact that all elements in the equivalence class have the same ratios $x^i : x^j = \frac{x^i}{x^j}$ on O_j . We can now define a differentiable system by using the functions

$$f_j^i \left([x^0 : x^1 : \cdots : x^n] \right) = \frac{x^i}{x^j} = \frac{f^{ij}}{f^{jj}}.$$

These have domain O_j and are smoothly expressed in terms of the coordinate functions we already considered. Conversely note that on $O_i \cap O_j$ the old coordinates are also expressed smoothly in terms of the new functions:

$$f^{ij} = \left(\sum_k f_i^k f_j^k \right)^{-1}.$$

On O_j we can use f_j^i , $i \neq j$ as a coordinate chart. The other coordinate functions f_l^k can easily be expressed as smooth combinations by noting that on $O_l \cap O_j$ we have

$$f_l^k = \frac{f_j^k}{f_j^l}.$$

Thus using the obvious coordinate functions works, but, it is often desirable to use a different collection of functions for a differentiable system.

Homogeneous coordinates also work over \mathbb{C} . We offer a few extra formulas of these coordinates and how they tie in with the geometry of projective space.

For $z = (z^0, \dots, z^n) \in \mathbb{F}^{n+1} - \{0\}$ denote the 1-dimensional subspace generated by z as $[z^0 : \cdots : z^n]$. Thus $[z^0 : \cdots : z^n] = [w^0 : \cdots : w^n]$ iff and only if z and w are proportional. If we let $p = [1 : 0 : \cdots : 0]$, then the coordinate map is simply $G_p(z^1, \dots, z^n) = [1 : z^1 : \cdots : z^n]$.

Keeping in mind that p is the only line perpendicular to all lines in p^\perp we see that $\mathbb{P}^n - p$ can be represented by

$$\mathbb{P}^n - p = \left\{ [z : z^1 : \cdots : z^n] \mid (z^1, \dots, z^n) \in \mathbb{F}^n - \{0\} \text{ and } z \in \mathbb{F} \right\}.$$

Here the subset

$$\mathbb{P}(p^\perp) = \left\{ [0 : z^1 : \cdots : z^n] \mid (z^1, \dots, z^n) \in \mathbb{F}^n - \{0\} \right\}$$

can be identified with \mathbb{P}^{n-1} . Using the projection

$$R_0 = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix},$$

$$\ker(R_0) = p$$

we obtain a retraction $R_0 : \mathbb{P}^n - p \rightarrow \mathbb{P}^{n-1}$, with preimages

$$R_0^{-1}([0 : z^1 : \cdots : z^n]) = \{[z : z^1 : \cdots : z^n] \mid z \in \mathbb{F}\} \simeq \mathbb{F}.$$

Using the family of transformations

$$R_t = \begin{bmatrix} t & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

we see that R_0 is in fact a deformation retraction, i.e., a retraction that is homotopic to the identity, where the homotopy stays fixed on the corresponding retract.

Finally we check the projective spaces in the lowest dimensions. When $\dim V = 1$, $\mathbb{P}(V)$ is just a point and that point is in fact V itself. Thus $\mathbb{P}(V) = \{V\}$.

When $\dim V = 2$, we note that for each $p \in \mathbb{P}(V)$ the orthogonal complement p^\perp is again a one-dimensional subspace and therefore an element of $\mathbb{P}(V)$. This gives us an involution $p \rightarrow p^\perp$ on $\mathbb{P}(V)$ just like the antipodal map on the sphere. In fact

$$\begin{aligned} \mathbb{P}(V) &= (\mathbb{P}(V) - \{p\}) \cup (\mathbb{P}(V) - \{p^\perp\}), \\ \mathbb{P}(V) - \{p\} &\simeq \mathbb{F} \simeq \mathbb{P}(V) - \{p^\perp\}, \\ \mathbb{F} - \{0\} &\simeq (\mathbb{P}(V) - \{p\}) \cap (\mathbb{P}(V) - \{p^\perp\}). \end{aligned}$$

Thus $\mathbb{P}(V)$ is simply a one point compactification of \mathbb{F} . In particular, we have that $\mathbb{R}\mathbb{P}^1 \simeq S^1$ and $\mathbb{C}\mathbb{P}^1 \simeq S^2$, (you need to convince yourself that these maps are diffeomorphisms). Since the geometry doesn't allow for distances larger than $\frac{\pi}{2}$ it is natural to identify these projective "lines" with spheres of radius $\frac{1}{2}$ in \mathbb{F}^2 .

1.2.5. Matrices of Constant Rank. Define $\text{Mat}_{n \times m}^k \subset \text{Mat}_{n \times m}$ as the matrices with n rows, m columns, and rank k . We will focus on real matrices but everything carries over to the complex case with the modification that all dimensions become complex dimensions. Note that $\text{Mat}_{n \times m}^k$ is not a closed set, however, it is closed inside the open set $\text{Mat}_{n \times m}^{\geq k}$ of matrices with rank $\geq k$.

The special case where $k = n = m$ is the general linear group GL_n .

In the general case $\text{Mat}_{n \times m}^k$ is still a subspace of a Euclidean space so it is natural to suspect that the entries will suffice as a differentiable system. Like projective spaces there is no equation to help us as we only have the the rank, which is not even continuous, as a constraint. The trick is to discover how many coordinates are needed to create a coordinate system. To that end, assume that we look at the matrices of rank k where the first k rows and the first k columns are linearly independent. Such a matrix can be written in block form

$$\begin{bmatrix} A & C \\ B & D \end{bmatrix}.$$

Here the rank k condition dictate that the last $n - k$ rows are linear combinations of the first k , in particular, $B = YA$, $Y \in \text{Mat}_{(n-k) \times k}$. Similarly the last $m - k$ columns are linear combinations of the first k , in particular, $C = AX$, $X \in \text{Mat}_{k \times (m-k)}$. Finally, the last $(n - k) \times (m - k)$ block is given by $D = YAX$. Thus the matrix is uniquely represented by the invertible matrix A and the two matrices X, Y . Next observe that $Y = BA^{-1}$ and $X = A^{-1}C$. Thus we can use the $nm - (n - k)(m - k)$ entries that correspond to A, B, C as a coordinate chart on this set of matrices. The remaining entries corresponding to D are then smooth functions of these coordinates as $D = BA^{-1}C$.

More generally we define the sets $O_{i_1, \dots, i_k, j_1, \dots, j_k} \subset \text{Mat}_{n \times m}^k$ as the rank k matrices where the rows indexed by i_1, \dots, i_k and columns by j_1, \dots, j_k are linearly independent. On these sets all entries that lie in the corresponding rows and columns are used as coordinates and the remaining entries are smoothly expressed in terms of these using the above expression with the necessary index modifications.

When $m = n$ we can add other conditions such as having constant determinant, being skew- or self-adjoint, orthogonal, unitary and much more. These will be discussed further in section 4.2 and example 1.2.2.

1.2.6. Grassmannians. A particularly intricate situation is the *Grassmannian* of k -planes in \mathbb{R}^n (or \mathbb{C}^n). These are, as indicated, the k -dimensional subspaces of an n -dimensional vector space. We offer three approaches. The first two similar to what we saw for projective spaces, the third seems a little more straightforward but requires that we identify subspaces as fixed point sets rather than as projections.

We start with the abstract description. Let V be an n -dimensional inner product space. The k -dimensional subspaces are the elements in $\text{End}(V)$ that are orthogonal projections of rank k . This means that $\text{Gr}_k(V) \subset \text{End}(V)$ as a closed subset. Now consider a k -dimensional subspace $U \subset V$. All of the k -dimensional subspaces that have trivial intersection with U^\perp can be described as graphs over U and are uniquely represented by an element in $\text{Hom}(U, U^\perp)$. This defines a homeomorphism from $\text{Hom}(U, U^\perp)$ onto an open set in $\text{Gr}_k(V)$ that contains U . Moreover, this map is also smooth as a map into $\text{End}(V)$. We see that $\dim \text{Gr}_k(V) = \dim U \cdot \dim U^\perp = k(n - k)$.

The second description is more concrete and considers the subspaces more explicitly as orthogonal projections:

$$\text{Gr}_k = \text{Gr}_k(\mathbb{F}^n) = \{E \in \text{Mat}_{n \times n}^k \mid E^2 = E \text{ and } E^* = E\}.$$

Any k -dimensional subspace is generated by a basis, i.e., an element $X \in \text{Mat}_{k \times n}^k$, where we think of the rows as the k vectors that span the subspace. Given such an X the corresponding orthogonal projection can be expressed as

$$E_X = X^*(XX^*)^{-1}X \in \text{Gr}_k.$$

Moreover, $E_X = E_Y$ if and only if $X = AY$ where $A \in \text{Gl}_k$. Instead of analyzing the entries of E_X as our differentiable system, we will imitate the construction of homogeneous coordinates to create an efficient way of parametrizing suitable open sets in Gr_k . Let $O_{i_1, \dots, i_k} \subset \text{Gr}_k$ be the open set with the property that the rows of E corresponding to the indices i_1, \dots, i_k are linearly independent. As E is self-adjoint the corresponding columns are also linearly independent. If $E = E_X$, then O_{i_1, \dots, i_k} corresponds to the $X \in \text{Mat}_{k \times n}^k$ where the columns indexed by i_1, \dots, i_k are linearly independent. We can then consider the matrix $A_X \in \text{Gl}_k$ which consists of those columns from X . The remaining columns in $A_X^{-1}X$ parametrize $E_X = E_{A_X^{-1}X}$. To see this more explicitly assume that the first k columns

are linearly independent. In this case we can use

$$X = \begin{bmatrix} I_k & Z \end{bmatrix}, Z \in \text{Mat}_{k \times (n-k)}$$

and

$$E_X = \begin{bmatrix} A & C \\ B & D \end{bmatrix} = \begin{bmatrix} (I_k + ZZ^*)^{-1} & (I_k + ZZ^*)^{-1} Z \\ Z^* (I_k + ZZ^*)^{-1} & Z^* (I_k + ZZ^*)^{-1} Z \end{bmatrix}.$$

Here $Z = BA^{-1}$ depends smoothly on the entries in E regardless of how $E \in O_{1, \dots, k}$ is expressed as a matrix. In this way we have created smooth bijections

$$\text{Mat}_{k \times (n-k)} \rightarrow O_{i_1, \dots, i_k} \subset \text{Gr}_k \subset \text{Mat}_{n \times n}^k.$$

This shows that $\dim \text{Gr}_k = k(n-k)$. The inverse maps will now yield the differentiable system or equivalently atlas for Gr_k . The formula $Z = BA^{-1}$ makes it clear that these coordinates are smooth on an overlap $O_{i_1, \dots, i_k} \cap O_{j_1, \dots, j_k}$.

Finally, we use the abstract set-up from the first representation but instead think of subspaces as fixed point sets in V . A k -dimensional subspace $U \subset V$ can be identified with the reflection in U , i.e., the map, $I_U \oplus -I_{U^\perp}$, that has U as the 1-eigenspace and U^\perp as the -1 -eigenspace. This automorphism is both self-adjoint and orthogonal (or unitary). Differently said, it is an orthogonal (or unitary) transformation with real eigenvalues and trace $2k - n$. If we select an orthonormal basis where the first k vectors span U , then a nearby subspace is the graph of a linear map $B : U \rightarrow U^\perp$. The corresponding matrix in block form will look like

$$R = \begin{bmatrix} A & B^* \\ B & D \end{bmatrix}$$

where $A^* = A$, $D^* = D$, and $R^*R = I$. These conditions imply that

$$RR^* = \begin{bmatrix} A^2 + B^*B & AB^* + B^*D \\ BA + DB & D^2 + BB^* \end{bmatrix} = \begin{bmatrix} I_k & 0 \\ 0 & I_{n-k} \end{bmatrix}.$$

Here $B = 0$ corresponds to $A = I_k$ and $D = -I_{n-k}$. The nearby subspaces should be determined by $B \approx 0$ with $A \approx I_k$ and $D \approx -I_{n-k}$. Thus we claim that the entries of B determine the coordinates. In other words, for suitably small B the other entries in A and D are smooth functions of B .

The equation $A^2 + B^*B = I_k$ tells us that A is the square root of $I_k - B^*B$ that is close to I_k :

$$A = I_k - \frac{1}{2}B^*B + \dots = \sum_{i=0}^{\infty} \binom{\frac{1}{2}}{i} (B^*B)^i.$$

This power series converges as long as the eigenvalues for B^*B are < 1 . Similarly we obtain

$$D = -I_{n-k} + \frac{1}{2}BB^* - \dots = -\sum_{i=0}^{\infty} \binom{\frac{1}{2}}{i} (BB^*)^i.$$

This gives us smooth formulas for A and D when we are in the open set where $\|B\| < 1$. Moreover, these formulas show that $BA + DB = 0$ is also satisfied:

$$BA = \sum_{i=0}^{\infty} \binom{\frac{1}{2}}{i} B (B^*B)^i = \sum_{i=0}^{\infty} \binom{\frac{1}{2}}{i} (BB^*)^i B = -DB.$$

In conclusion, if $O \subset \text{Mat}_{(n-k) \times k}$ is the open unit ball around the origin, then we have created an injection $O \rightarrow \text{Gr}_k$ that is smooth as a map into the real vector space, $\text{Sym}(V)$, of self-adjoint maps on V . Moreover, the image in Gr_k is an open set around a fixed k -dimensional subspace.

1.2.7. Tangent Spaces to Spheres. The last example for now is somewhat different in nature and can easily be generalized to manifolds that come from subsets of Euclidean space where standard coordinate functions give a differentiable system.

We consider the set of vectors tangent to a sphere. By tangent to the sphere we mean that they are velocity vectors for curves in the sphere. If $c : I \rightarrow S^n$, then $|c|^2 = 1$ and consequently $\dot{c} \cdot c = \langle \dot{c}, c \rangle = 0$. Thus the velocity is always perpendicular to the base vector. This means that we are considering the set

$$TS^n \simeq \{(x, v) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid |x| = 1 \text{ and } \langle x, v \rangle = 0\}$$

Conversely we see that for $(x, v) \in TS^n$ the curve

$$c(t) = x \cos t + v \sin t$$

is a curve on the sphere that has velocity v at the base point x . Now suppose that we are considering the points $x \in O_j^\pm$ with $\pm x^j > 0$. We know that on this set we can use x^i , $i \neq j$ as coordinates. It seems plausible that we could similarly use v^i , $i \neq j$ for the vector component. We already know that x^j is a smooth function of x^i , $i \neq j$. So we now have to write v^j as a smooth function of v^i and x^i . The equation $\langle x, v \rangle = 0$ tells us that

$$v^j = -\frac{\sum_{i \neq j} x^i v^i}{x^j}$$

so this is certainly possible.

This also helps us in the general case where we might be considering tangent vectors to a general M . For simplicity assume that $x^{n+1} = F(x^1, \dots, x^n)$. If c is a curve, then we also have $c^{n+1}(t) = F(c^1(t), \dots, c^n(t))$. Thus

$$\dot{c}^{n+1}(t) = \frac{\partial F}{\partial x^i} \dot{c}^i(t).$$

This means that for the tangent vectors

$$v^{n+1} = \frac{\partial F}{\partial x^i} v^i.$$

Thus we have again written v^{n+1} as a smooth function of our chosen coordinates given that x^{n+1} is already written as a smooth function of x^1, \dots, x^n .

This argument is general enough that we can use it to create a differentiable structure for similarly defined tangent spaces TM for $M^m \subset \mathbb{R}^n$ where we used the n -coordinate functions from \mathbb{R}^n to generate the differentiable structure on M . The only difference is that we now need $n - m$ functions to describe $n - m$ of the coordinates on any given set where we've used a specific set of m coordinates as a chart. For instance

$$x^j = F^j(x^1, \dots, x^m), \quad j > m$$

yields

$$v^j = \sum_{i=1}^m \frac{\partial F^j}{\partial x^i} v^i, \quad j > m.$$

1.3. Topological Properties of Manifolds

The goal is to show that there exists partitions of unity on smooth manifolds and in particular that manifolds are paracompact. The simplest topological assumptions for this to work is that the manifold is second countable (there is a countable basis for the topology) and Hausdorff (points can be separated by disjoint open sets). For a manifold, as defined

above, this means that the topology will henceforth be assumed to be second countable and Hausdorff. The Hausdorff condition is essential for many obvious properties, but, it will also seem as if it is rarely used explicitly. Two essential consequences come from the Hausdorff axiom. First, that limits of sequences are uniquely defined. Second, compact subsets satisfy all of the usual equivalent conditions and are closed sets. In particular, manifolds are locally compact.

Checking that the topology is second countable generally follows by checking that the space can be covered by countably many coordinate charts. Clearly open subsets of \mathbb{R}^n are second countable. So this means that the space is a countable union of open sets that are all second countable and thus itself second countable. We will on two occasions have to prove that a manifold is second countable.

Checking that it is Hausdorff is generally also easy. Either two points will lie the same chart in which case they can easily be separated. Otherwise they'll never lie in the same chart and one must then check that there are charts around the points whose domains don't intersect.

1.3.1. Bump Functions. The first important result is that manifolds can be exhausted by compact sets: A *compact exhaustion* is an increasing countable collection of compact sets $K_1 \subset K_2 \subset \dots$ such that $M = \cup K_i$ and $K_i \subset \text{int}K_{i+1}$ for all i . The crucial ingredients for finding such an exhaustion is second countability and local compactness. A space that admits a compact exhaustion is said to be σ -compact.

A cover of sets is *locally finite* if each $p \in M$ has a neighborhood U_p such that only finitely many sets in the cover have nonempty intersections with U_p . A space is paracompact if every open cover has a locally finite refinement.

THEOREM 1.3.1. *A smooth manifold has a compact exhaustion and is also paracompact.*

PROOF. Around each $p \in M$ select an open neighborhood U_p such that the closure is compact. Since M is second countable (or just Lindelöf) we can select a countable collection U_{p_i} that covers M . Define $K_1 = \overline{U_{p_1}}$ and given K_i let $K_{i+1} = \overline{U_{p_1} \cup \dots \cup U_{p_k}}$, where p_1, \dots, p_k are chosen so that $k \geq i$ and $K_i \subset U_{p_1} \cup \dots \cup U_{p_k}$.

To show that the space is paracompact consider the compact "annuli" $C_i = K_i - \text{int}K_{i-1}$ and note that $C_i \cap C_j = \emptyset$ when $|i - j| > 1$. Extend this to a cover of open sets $U_i = \text{int}K_{i+1} - K_{i-1} \supset C_i$ and note that $U_i \cap U_j = \emptyset$ when $|i - j| > 4$. In other words these are locally finite covers. Given an open cover B_α we can consider the doubly indexed refinement $B_\alpha \cap U_i$ of B_α . For fixed i we can then extract a finite collection of $B_{i_j} \cap U_i$, $j = 1, \dots, n_i$, that cover the compact set C_i . This leads to a locally finite refinement of the original cover B_α . Clearly, any $p \in \text{int}(C_i \cup C_{i+1})$, for some i . Only finitely many U_k intersect the neighborhood $\text{int}(C_i \cup C_{i+1})$ and so only finitely many $B_{k_j} \cup U_k$ can intersect $\text{int}(C_i \cup C_{i+1})$. \square

The above topological properties of manifolds lead to a general principle that offers an abstract general condition for when a statement holds for manifolds.

Consider a class \mathcal{M}^n manifolds with the following properties:

- (1) Every $M \in \mathcal{M}$ has dimension n .
- (2) $\mathbb{R}^n \in \mathcal{M}^n$.
- (3) If $M \in \mathcal{M}^n$ and $U \subset M$ is open, then $U \in \mathcal{M}^n$.
- (4) If $M \in \mathcal{M}^n$ and M is diffeomorphic to N , then $N \in \mathcal{M}^n$.

This can for example be the class of all n -manifolds or all oriented n -manifolds or simply all open subsets of a given manifold.

The goal is to consider the validity of a statement $P(M)$ for all $M \in \mathcal{M}^n$. We will assume that the statement only depends on the diffeomorphism type of the manifold.

THEOREM 1.3.2. *The statement $P(M)$ is true for all manifolds in \mathcal{M}^n provided the following conditions hold:*

- (1) $P(\mathbb{R}^n)$ is true.
- (2) If $A, B \subset M \in \mathcal{M}^n$ are open and $P(A), P(B), P(A \cap B)$ are true, then $P(A \cup B)$ is true.
- (3) If $A_i \subset M \in \mathcal{M}^n$ form a countable collection of pairwise disjoint open sets such that $P(A_i)$ are true, then $P(\coprod A_i)$ is true.

PROOF. We start by showing that $P(U)$ is true for all open sets $U \subset \mathbb{R}^n$. Observe first that any open box $(a_1, b_1) \times \cdots \times (a_n, b_n)$ is diffeomorphic to \mathbb{R}^n and that the intersection of two boxes is either empty or a box. Consider next an open subset of \mathbb{R}^n that is a finite union of open boxes. The claim follows for such sets by induction on the number of boxes. To see this, assume it holds for any union of k or fewer open boxes and consider $k+1$ open boxes B_i . Then the statement holds for $B_1 \cup \cdots \cup B_k, B_{k+1}$, and the intersection as it is a union of k or fewer boxes:

$$(B_1 \cup \cdots \cup B_k) \cap B_{k+1} = (B_1 \cap B_{k+1}) \cup \cdots \cup (B_k \cap B_{k+1}).$$

This in turn shows that we can prove the theorem for all open sets in \mathbb{R}^n . Fix an open set $U \subset \mathbb{R}^n$ and a compact exhaustion K_i of U . For each compact set $C_i = K_i - \text{int}K_{i-1}$ select an open set $U_i \supset C_i$ that is a finite union of open boxes such that $U_i \cap U_j = \emptyset$ when $|i-j| \geq 2$. Thus the theorem holds for $\bigcup U_{2i}, \bigcup U_{2i+1}$. It also holds for the intersection $(\bigcup U_{2i}) \cap (\bigcup U_{2i+1}) = \bigcup (U_j \cap U_{j+1})$ as $U_i \cap U_{i+1} \cap U_j \cap U_{j+1} = \emptyset$ when $i \neq j$. Consequently, the statement holds for the entire union.

Having come this far we use the exact same strategy to prove the statement for an $M \in \mathcal{M}^n$ by considering the class of all open subsets $U \subset M$ and replacing the first statement with:

- (1) $P(U)$ is true for all open $U \subset M$ that are diffeomorphic to an open subset of \mathbb{R}^n , i.e., all charts $U \subset M$.

Using induction this shows that the statement is true for any open subset of M that is a finite union of charts. Next write $M = \bigcup U_i$ where each U_i is a finite union of charts and $U_i \cap U_j = \emptyset$ when $|i-j| \geq 2$. This means the theorem holds for $\coprod U_{2i}, \coprod U_{2i+1}$, and $(\coprod U_{2i}) \cap (\coprod U_{2i+1})$ and consequently for the entire union. \square

We can modify the theorem to hold for pairs (M, U) , where $U \subset M$ is open. We must then assume that $P(M, U)$ is a statement for a pair and that it is diffeomorphism invariant for pairs, i.e., if there is a diffeomorphism $F : M \rightarrow N$ and $P(M, U)$ is true, then $P(N, F(U))$ is also true.

THEOREM 1.3.3. *The statement $P(M, U)$ is true for all $M \in \mathcal{M}^n$ provided the following conditions hold:*

- (1) $P(\mathbb{R}^n, B)$ is true whenever $B = (a_1, b_1) \times \cdots \times (a_n, b_n)$.
- (2) If $A, B \subset M \in \mathcal{M}^n$ are open and $P(M, A), P(M, B), P(M, A \cap B)$ are true, then $P(M, A \cup B)$ is true.
- (3) If $A_i \subset M \in \mathcal{M}^n$ form a countable collection of pairwise disjoint open sets such that $P(M, A_i)$ are true, then $P(M, \coprod A_i)$ is true.

As a corollary we obtain a smooth version of Urysohn's lemma, a result first established by Whitney.

COROLLARY 1.3.4. (Smooth Urysohn Lemma) *If M is a smooth manifold and $C_0, C_1 \subset M$ are disjoint closed sets, then there exists a smooth function $f : M \rightarrow [0, 1]$ such that $C_0 = f^{-1}(0)$ and $C_1 = f^{-1}(1)$.*

PROOF. We first claim that for every open subset $U \subset M$, there exists a smooth bump function $f : M \rightarrow [0, \infty)$ such that $f^{-1}(0, \infty) = U$. Clearly (1) holds. For (2) one can simply add the bump functions for (M, A) and (M, B) . Similarly for (3) as the bump functions are positive on disjoint domains.

Finally, the Urysohn function is constructed by selecting $f_i : M \rightarrow [0, \infty)$ such that $f_i^{-1}(0) = C_i$ and defining

$$f(x) = \frac{f_0(x)}{f_0(x) + f_1(x)}.$$

This function is well-defined as $C_0 \cap C_1 = \emptyset$ and is the desired Urysohn function. \square

Based on the proof of theorem 1.3.2 we also obtain the following less abstract version. We consider a statement P about all open subsets of a fixed manifold. Thus we don't necessarily assume that the statement is invariant under diffeomorphisms.

COROLLARY 1.3.5. *The statement $P(M)$ is true for a manifold M provided*

- (1) *M has a cover of open sets O_α that are diffeomorphic to \mathbb{R}^n such that for all α the statement $P(B)$ is true for any box $B \subset O_\alpha$.*
- (2) *If $A, B \subset M \in \mathcal{M}^n$ are open and $P(A), P(B), P(A \cap B)$ are true, then $P(A \cup B)$ is true.*
- (3) *If $A_i \subset M \in \mathcal{M}^n$ form a countable collection of pairwise disjoint open sets such that $P(A_i)$ are true, then $P(\coprod A_i)$ is true.*

Using bump functions we can now easily construct the partitions of unity we need for numerous constructions.

LEMMA 1.3.6. *Let M be a smooth manifold. Any covering U_α of open sets admits partition of unity subordinate to this covering, i.e., there are smooth functions $\phi_\alpha : M \rightarrow [0, 1]$ such that the preimages satisfy: $\phi_\alpha^{-1}(0, \infty) \subset U_\alpha$; form a locally finite covering; and $1 = \sum_\alpha \phi_\alpha$.*

PROOF. First select a locally finite covering V_i subordinate to U_α . The previous result gives us functions $\lambda_i : M \rightarrow [0, 1]$ such that $\lambda_i^{-1}(0) = M - V_i$. As the cover is locally finite the sum $\sum_i \lambda_i$ is well-defined and positive. So after possibly dividing by $\sum_i \lambda_i$ we can assume λ_i form a *partition of unity*: $\sum_i \lambda_i = 1$. Next select for each i an index α_i so that $V_i \subset U_{\alpha_i}$ and define

$$\mu_\alpha = \sum_{\{i|\alpha_i=\alpha\}} \lambda_i.$$

Note that μ_α vanishes on M when α is not one of the chosen indices α_i and that μ_α always vanishes on $M - U_\alpha$. Moreover, $\mu_\alpha^{-1}(0, \infty)$ form a locally finite cover as V_i is locally finite. Finally, the index i is used when and only when $\alpha = \alpha_i$, in particular,

$$1 = \sum_i \lambda_i = \sum_\alpha \mu_\alpha.$$

\square

REMARK 1.3.7. We will often use a covering $U_p, p \in M$, where U_p is an open neighborhood of p .

PROPOSITION 1.3.8. *If $U \subset M$ is an open set in a smooth manifold and $f : U \rightarrow \mathbb{R}^n$ is smooth, then λf defines a smooth function on M provided $\lambda : M \rightarrow \mathbb{R}$ is smooth and has support in U , i.e.,*

$$\text{supp}\lambda = \overline{\{x \in M \mid \lambda(x) \neq 0\}} \subset U.$$

PROOF. Clearly λf is smooth on U and vanishes on the open set $M - \text{supp}\lambda$. Thus it is smooth on $M = U \cup (M - \text{supp}\lambda)$. \square

REMARK 1.3.9. Note that for any $p \in U$ the function λ can be chosen to be 1 on a neighborhood of p . In particular, any smooth function can locally be extended to a smooth function on all of M .

We can now also easily construct proper functions.

PROPOSITION 1.3.10. *A smooth manifold admits a proper smooth function.*

PROOF. Select a compact exhaustion $K_1 \subset K_2 \subset \dots$, where each K_i is compact, $K_i \subset \text{int}K_{i+1}$, and $M = \bigcup K_i$. Choose Urysohn functions $\phi_i : M \rightarrow [0, 1]$ such that $\phi_i(K_{i-1}) = 0$ and $\phi_i(M - \text{int}K_i) = 1$. Then use $\rho = \sum \phi_i$. \square

1.3.2. Metrizable. We finally mention several interesting results that help us understand the topological properties that are crucial for manifolds. Whitney's embedding theorem 3.4.4 also shows that manifolds are metrizable.

The Urysohn metrization theorem asserts that a second countable normal Hausdorff space is metrizable. In particular, manifolds are always metrizable. The proof of this result is remarkably simple.

THEOREM 1.3.11. *A second countable normal Hausdorff space is metrizable. Moreover, if the space admits a compact exhaustion, then it is metrizable with a complete metric.*

PROOF. We shall only use that the space is completely regular. In fact Tychonoff's Lemma shows that a regular Lindelöf space is normal. So it suffices to assume that the space is second countable and regular. There are second countable Hausdorff spaces that are not regular (79 in [Steen & Seebach]). Note that such spaces can't be locally compact.

The key is to use that the Hilbert cube: $\times_{i=1}^{\infty} I_i$ where $I_i = [0, 1]$ is a metric space with distance

$$d((x_i), (y_i)) = \sum_i 2^{-i} |x_i - y_i|.$$

The goal is then to show that our space is homeomorphic to a subset in the Hilbert cube.

Choose a countable collection of closed sets C such that their complements generate the topology of M . Enumerate the all pairs $(C_i, F_i) \in C \times C$ with $C_i \subset \text{int}F_i$, and for each such pair select a function $\phi_i : M \rightarrow [0, 1]$ such that $\phi_i(C_i) = 0$ and $\phi_i(M - \text{int}F_i) = 1$. This results in a map $\Phi : M \rightarrow \times_{i=1}^{\infty} I_i$ by defining $\Phi(x) = \times_{i=1}^{\infty} \phi_i(x)$.

This map is injective since distinct points can be separated by open sets whose complements are in C . Next we show that for each $C \in C$ the image $\Phi(C)$ is closed. Consider a sequence $c_n \in C$ such that $\Phi(c_n) \rightarrow \Phi(x)$. Note that for any fixed index i we have $\phi_i(c_n) \rightarrow \phi_i(x)$. If $x \notin C$, then we can find a pair (C_i, F_i) where $x \in M - \text{int}F_i$. Therefore, $\phi_i(c_n) = 0$ and $\phi_i(x) = 1$, which is impossible. Thus $x \in C$ and $\Phi(x) \in \Phi(C)$. This shows that the map is a homeomorphism onto its image.

An explicit metric on M can given by

$$d(x, y) = \sum_i 2^{-i} |\phi_i(x) - \phi_i(y)|.$$

In case the space also has a compact exhaustion we can find a proper function $\rho : M \rightarrow [0, \infty)$ and use the map: $(\rho, \Phi) : M \rightarrow [0, \infty) \times_{i=1}^{\infty} I_i$ which is also proper. Consider a Cauchy sequence x_n . Clearly $\rho(x_n)$ is also Cauchy and consequently has compact closure as a set in $[0, \infty)$. Thus x_n also lies in a compact set. In particular the sequence must have accumulation points and is consequently convergent. \square

For comparison it should be mentioned that if we use the topology on \mathbb{R} generated by the half open intervals $[a, b)$ then we obtain a paracompact space that is separable but not second countable and not locally compact (51 in [Steen & Seebach]).

THEOREM 1.3.12. *A connected locally compact metric space has a compact exhaustion.*

PROOF. Assume (M, d) is the metric space. For each $x \in M$ let

$$r(x) = \sup \left\{ r \mid \overline{B(x, r)} \text{ is compact} \right\}.$$

If $r(x) = \infty$ for some x we are finished. Otherwise $r(x)$ is a continuous function, in fact

$$|r(x) - r(y)| \leq d(x, y)$$

since

$$r(y) \leq d(x, y) + r(x)$$

and

$$r(x) \leq d(x, y) + r(y)$$

We now claim that for a fixed compact set C the set $C^\# = \{x \in M \mid \exists z \in C : d(x, z) \leq \frac{1}{2}r(z)\}$ is also compact and contains C in its interior. The latter statement is obvious since $B\left(x, \frac{1}{2}r(x)\right) \subset C^\#$ for all $x \in C$. Next select a sequence $x_i \in C^\#$ and select $z_i \in C$ such that $d(x_i, z_i) \leq \frac{1}{2}r(z_i)$. Since C is compact we can after passing to a subsequence assume that $z_i \rightarrow z \in C$ and that $d(z, z_i) < \frac{1}{4}r(z)$ for all i . Then $d(z, x_i) \leq d(z, z_i) + d(z_i, x_i) < \frac{1}{4}r(z) + \frac{1}{2}r(z_i)$. Continuity of $r(z_i)$ then shows that $x_i \in B\left(z, \frac{3}{4}r(z)\right)$ for large i . As $\overline{B\left(z, \frac{3}{4}r(z)\right)}$ is compact we can then extract a convergent subsequence of x_i .

Finally consider the compact sets $K_{i+1} = K_i^\#$ where K_1 is any non-empty compact set. We claim that $\cup_i K_i$ is both open and closed. The set is open since $B\left(x, \frac{1}{2}r(x)\right) \subset K_i^\# = K_{i+1}$ for any $x \in K_i$. To see that the set is closed select a convergent sequence $x_n \in \cup_i K_i$ and let x be the limit point. We have $r(x_n) \rightarrow r(x)$ and $d(x_i, x) \rightarrow 0$. So it follows that for large n we have $x \in B\left(x_n, \frac{1}{2}r(x_n)\right)$ showing that $x \in K_i^\#$ if $x_n \in K_i$. So the fact that M is connected shows that it has a compact exhaustion. \square

COROLLARY 1.3.13. *A second countable locally compact metric space has a compact exhaustion and is paracompact.*

PROOF. There are at most countably many connected components and each of these has a compact exhaustion. We can then proceed as in theorem 1.3.1. \square

THEOREM 1.3.14 (Baire Category Theorem). *A Hausdorff space that is locally compact satisfies: A countable union of closed sets without interiors has no interior.*

PROOF. Let $C_i \subset M$ be a countable collection of closed sets with no interior points. Select an open set $V_0 \subset X$. Then $V_0 - C_1$ is a nonempty open set as C_1 has no interior points. As M is locally compact we can find an open set V_1 such that $\overline{V_1} \subset V_0 - C_1$ is compact. Similarly we can find open sets V_i such that $\overline{V_i} \subset V_{i-1} - C_i \subset V_{i-1}$ is compact. By

compactness $\bigcap_{i=1}^{\infty} \bar{V}_i$ is nonempty and we also have $\bigcap_{i=1}^{\infty} \bar{V}_i \subset V_0 - \bigcup_{i=1}^{\infty} C_i$. In particular, $V_0 - \bigcup_{i=1}^{\infty} C_i$ is nonempty for any open set V_0 . This shows that $\bigcup_{i=1}^{\infty} C_i$ has no interior points. \square

EXAMPLE 1.3.15. The set of rationals $\mathbb{Q} \subset \mathbb{R}$ is a metric space that does not admit a complete metric nor is it locally compact.

1.4. Local Theory of Smooth Maps

We define the rank of smooth maps and use it to define and study immersions, embeddings, submersions, and maps of constant rank.

1.4.1. The Rank of a Map.

DEFINITION 1.4.1. The *rank* of a smooth map at $p \in M$ is denoted $\text{rank}_p F$ and is defined as the rank of the differential, $D(y \circ F \circ x^{-1})$, in local coordinates at $x(p)$. This definition is independent of the coordinate systems we choose due to the chain rule and the fact that the transition functions have nonsingular differentials at all points.

PROPOSITION 1.4.2. If $F : M \rightarrow N$ and $G : N \rightarrow O$ are smooth maps, then

$$\text{rank}_p (G \circ F) \leq \min \{ \text{rank}_p F, \text{rank}_{F(p)} G \}.$$

PROOF. Using coordinates x around $p \in M$, y around $F(p) \in N$, and z around $G(F(p)) \in O$ we can consider the composition

$$z \circ G \circ F \circ x^{-1} = (z \circ G \circ y^{-1}) \circ (y \circ F \circ x^{-1})$$

The chain rule then implies

$$D(z \circ G \circ F \circ x^{-1})|_p = D(z \circ G \circ y^{-1})|_{y \circ F(p)} \circ D(y \circ F \circ x^{-1})|_{x(p)}$$

This reduces the claim to the corresponding result for linear maps. \square

REMARK 1.4.3. Note that $\text{rank} A \geq k$ is an open condition on $\text{Mat}_{n \times m}$. Thus if $\text{rank}_p F = k$, then $\text{rank}_x F \geq k$ for all x in a neighborhood of p . Similarly, when $V \subset \mathbb{R}^n$ is a subspace, then the condition that $\text{im} A + V = \mathbb{R}^n$ is also an open condition for $A \in \text{Mat}_{n \times m}$.

1.4.2. Coordinates.

PROPOSITION 1.4.4. Let $y : U \rightarrow \mathbb{R}^k$ be smooth where $U \subset M$ is an open subset. If $\text{rank}_p y = k = \dim M = m$, then y is a chart on a neighborhood of p . Moreover, if $\text{rank}_p y = k < \dim M$, then it is possible to select functions y^{k+1}, \dots, y^m such that y^1, \dots, y^m form coordinates around p .

PROOF. The first claim follows from the inverse function theorem. Select a chart $x : V \rightarrow \mathbb{R}^m$ on a neighborhood of p and consider the smooth map $y \circ x^{-1} : x(U \cap V) \rightarrow \mathbb{R}^m$. By the definition of rank, the map has nonsingular differential at $x(p)$ and is consequently a diffeomorphism from a neighborhood around $x(p)$ to its image. This shows that y is a diffeomorphism on some neighborhood of p onto its image.

For the second claim select an arbitrary coordinate system z^1, \dots, z^m around p . The map $(y \circ z^{-1}, z^1, \dots, z^m)$ has a differential at $z(p)$ that looks like

$$\begin{bmatrix} D(y \circ z^{-1}) \\ I_m \end{bmatrix}$$

where I_m is the identity matrix and $D(y \circ z^{-1})$ has linearly independent rows. We can then use the replacement procedure to eliminate k of the bottom m rows so as to get a nonsingular $m \times m$ matrix. Assuming after possibly rearranging indices that the remaining rows are the last $m - k$ rows we see that $(y \circ z^{-1}, z^{k+1}, \dots, z^m)$ has rank m at p and thus forms a coordinate system around p . \square

1.4.3. Immersions.

DEFINITION 1.4.5. We say that $F : M \rightarrow N$ is an *immersion* if $\text{rank}_p F = \dim M$ for every $p \in M$.

PROPOSITION 1.4.6. For a smooth map $F : M \rightarrow N$ the following conditions are equivalent:

- (1) F is an immersion.
- (2) For each $p \in M$ there are charts $x : U \rightarrow \mathbb{R}^m$ and $y : V \rightarrow \mathbb{R}^n$ with $p \in U$ and $F(p) \in V$ such that

$$y \circ F \circ x^{-1}(x^1, \dots, x^m) = (x^1, \dots, x^m, 0, \dots, 0).$$

- (3) If \mathcal{D} is a differentiable structure on N , then $F^*(\mathcal{D})$ is a differentiable structure on M .

PROOF. It is obvious that 2 implies 1. For 1 implies 2. Select coordinates $u : U \rightarrow \mathbb{R}^m$ around p and $v : V \rightarrow \mathbb{R}^n$ around $F(p) \in N$. The composition $v \circ F \circ u^{-1}$ has rank m at $u(p)$. After possibly reordering the indices for the v -coordinates we can assume that $(v^1, \dots, v^m) \circ F \circ u^{-1}$ also has rank m at $u(p)$. But, this means that it is a diffeomorphism on some neighborhood around $u(p)$. Consequently, $x = (v^1, \dots, v^m) \circ F$ is a chart around p . Consider the functions

$$\begin{aligned} y^i(q) &= v^i(q), i = 1, \dots, m, \\ y^i(q) &= v^i(q) - v^i \circ F \circ x^{-1}(v^1(q), \dots, v^m(q)), i > m. \end{aligned}$$

These are defined on a neighborhood of $F(p)$ and when $i > m$ we have

$$y^i \circ F = v^i \circ F - v^i \circ F \circ x^{-1}(v^1 \circ F, \dots, v^m \circ F) = 0.$$

So it remains to check that $y = (y^1, \dots, y^n)$ are coordinates at $F(p)$. The composition $y \circ v^{-1}$ has a differential that is in lower triangular block form

$$\begin{bmatrix} I_m & 0 \\ * & I_{n-m} \end{bmatrix}$$

where the diagonal entries are the identity matrices on first m and last $n - m$ coordinate subspaces. This shows that they will form coordinates on some neighborhood of $F(p)$.

As 1 and 2 are equivalent we can now use the proof that 1 implies 2 to show that if 1 or 2 hold, then 3 also holds.

Conversely, assume that 3 holds. Select a coordinate chart $z^i = y^i \circ F$ around $p \in M$, where $y^i \in \mathcal{D}$, $i = 1, \dots, m$. The chart z has rank m at p , so it follows that the corresponding smooth map y must have rank at least m at $F(p)$. However, the rank can't be greater than m as it maps into \mathbb{R}^m . We can now add $n - m$ coordinate functions z^i from some other coordinate system around $F(p)$ so as to obtain a map $(y^1, \dots, y^m, z^{m+1}, \dots, z^n)$ that has rank n at $F(p)$. These coordinate choices show that 1 holds. \square

COROLLARY 1.4.7. *A smooth map $F : M \rightarrow N$ is an immersion iff for any smooth map $G : L \rightarrow M$ and $o \in L$ we have*

$$\text{rank}_o(F \circ G) = \text{rank}_o G.$$

DEFINITION 1.4.8. We say that F is an *embedding* if it is an immersion, injective, and $F : M \rightarrow F(M)$ is a homeomorphism, where the image is endowed with the induced topology.

PROPOSITION 1.4.9. *For a smooth map $F : M \rightarrow N$ the following conditions are equivalent:*

- (1) F is an embedding.
- (2) $F^*(C^\infty(N)) = C^\infty(M)$, i.e., F^* is surjective on smooth functions.

PROOF. Start by assuming that 2 holds. Given $p, q \in M$ select $f \in C^\infty(M)$ such that $f(p) \neq f(q)$. Next find $g \in C^\infty(N)$ such that $f = g \circ F$. Thus $g(F(p)) \neq g(F(q))$ showing that F is injective. To see that the topology of M agrees with the induced topology on $F(M)$ select an open set $O \subset M$ and $\lambda : M \rightarrow [0, \infty)$ such that $\lambda^{-1}(0, \infty) = O$. Select $\mu : U \subset N \rightarrow \mathbb{R}$ such that $\lambda = \mu \circ F$. Note that $F(M) \subset U$ as λ is defined on all of M . Thus

$$\mu^{-1}(0, \infty) \cap F(M) = F(\lambda^{-1}(0, \infty)) = F(O)$$

and $F(O)$ is open in $F(M)$. Finally select coordinates x around $p \in M$ and write $x^i = y^i \circ F$ for smooth functions on some neighborhood of $F(p)$. The composition $y \circ F \circ x^{-1}$ has rank m at $x(p)$. So the map $F \circ x^{-1}$ must have rank at least m at $x(p)$. However, the rank can't exceed m so this shows that $\text{rank}_p F = m$ and in turn that F is an immersion.

Conversely assume that F is an embedding and $f : O \subset M \rightarrow \mathbb{R}$ a smooth function. Using that F is an immersion we can for each $p \in M$ select charts $x_p : O_p \rightarrow \mathbb{R}^m$ around p and $y_p : U_p \rightarrow \mathbb{R}^n$ around $F(p)$ such that $y_p^j|_{F(O_p) \cap U_p} = 0$ for $j > m$. Since F is an embedding $U_p \cap F(O_p) \subset F(M)$ is open. This means that we can assume that U_p is chosen so that $F(O_p) = U_p \cap F(M)$. On each U_p define g_p such that $g_p \circ y_p^{-1}(a^1, \dots, a^n) = f \circ x_p^{-1}(a^1, \dots, a^m)$. We can then define $g = \sum_p \mu_p g_p$, where μ_p is a partition of unity for U_p . This gives us a function on the open set $\cup U_p$. Since F is injective it follows that $g \circ F = f$. \square

COROLLARY 1.4.10. *If $F : M \rightarrow N$ is an embedding such that $F(M) \subset N$ is closed, then $F^*(C^\infty(N)) = C^\infty(M)$.*

PROOF. The only additional item to worry about is whether the function g just constructed can be extended to N and remain fixed on $F(M)$. When the image is a closed subset this is easily done by finding a smooth Urysohn function μ that is 1 on $F(M)$ and vanishes on $N - U$. The function μg is then a smooth function on N that can be used instead of g . \square

REMARK 1.4.11. We note that when an injective immersion is also a proper map, then it becomes an embedding whose image is a closed subset. Such maps are called *proper embeddings* or *proper submanifolds*. Calling them ‘‘closed submanifolds’’ might cause confusion as closed manifolds are generally compact manifolds without boundary.

DEFINITION 1.4.12. A subset $S \subset M$ is a *submanifold* if it admits a topology such that the restriction of the differentiable structure on M to S is a differentiable structure. The dimension of the structure on S will generally be less than that of M unless S is an open subset with the induced topology. Note that the topology on S can be different than the induced

topology. However, it has to be finer as we require all smooth functions on M to be smooth on S . In this way we see that a submanifold is in fact the image of an injective immersion. A submanifold is embedded when the topology on S is the induced topology. This is equivalent to saying that any point $p \in S$ has a *slice neighborhood*, i.e., there exist coordinates x^1, \dots, x^n on a neighborhood $U \ni p$ such that $S \cap U = \{x \in U \mid x^{k+1} = \dots = x^n = 0\}$.

An embedded submanifold can always be realized as a properly embedded submanifold inside a suitable neighborhood.

PROPOSITION 1.4.13. *If $S \subset M$ is an embedded submanifold, then there is an open set $S \subset O \subset M$ such that $S \subset O$ is a properly embedded submanifold.*

PROOF. For any point $p \in S$, there exists a neighborhood $p \in V_p \subset M$ such that \bar{V}_p and $S \cap \bar{V}_p$ are both compact. We can then simply define $O = \bigcup_{p \in S} V_p$. \square

Suppose $F : M \rightarrow N$ is a smooth map whose image lies in a submanifold $S \subset N$. When is $F : M \rightarrow S$ smooth? This is definitely the case when S is embedded and also in case S is immersed provided $F : M \rightarrow S$ is continuous as we shall see next.

PROPOSITION 1.4.14. *Let $F : M \rightarrow N$ be a smooth map whose image lies in a submanifold $S \subset N$. If $F : M \rightarrow S$ is continuous, then $F : M \rightarrow S$ is smooth.*

PROOF. We fix $p \in M$ and with it $q = F(p) \in S$. There are coordinates y^1, \dots, y^n on a neighborhood of $q \in N$, such that y^1, \dots, y^k restrict to coordinates on a neighborhood $q \in V \subset S$. Since F is continuous, the preimage $U = F^{-1}(V)$ is open. Smoothness of $F|_U : U \rightarrow N$, then shows that $y^i \circ F|_U$ is smooth for $i = 1, \dots, k$. This shows that also $F|_U : U \rightarrow S$ is smooth. \square

We finish with a useful lemma about when a map that is an embedding when restricted to a submanifold can be extended to an embedding on a neighborhood of the submanifold. Note that the crucial assumption that $S \subset M$ is proper can be eliminated as proposition 1.4.13 shows that the submanifold is properly embedded in a suitable neighborhood.

LEMMA 1.4.15. *If $F : M \rightarrow N$ is a proper immersion that is an embedding when restricted to a properly embedded submanifold $S \subset M$, then F is an embedding on an open set containing S .*

PROOF. The fact that $S \subset M$ is properly embedded implies that neighborhoods of S have the property that their boundaries are disjoint from $\bar{S} = S$, while for a general embedded submanifold some neighborhoods have boundaries that intersect the closure of the submanifold. As F is proper on the closure of neighborhoods it suffices to show that F is injective on a neighborhood of S (see part (1) of proposition 1.4.20). We say that a sequence x_i converges to S provided that for any open set $U \supset S$ there exists $N > 0$ such that $\{x_i \mid i > N\} \subset U$. If F is not injective on any neighborhood, then we can find sequences with $x_i \neq y_i$ that converge to S and such that $F(x_i) = F(y_i)$. If the closure of either of the sets $\{x_i\}$ or $\{y_i\}$ does not intersect S , then the complement will be an open set that contains S , which is a contradiction. Thus both sets have accumulation points that lie in S . By passing to suitable subsequences we obtain convergent sequences $x_{i_j} \rightarrow x \in S$ and $y_{i_j} \rightarrow y \in S$ such that $F(x_{i_j}) = F(y_{i_j})$. In particular, $F(x) = F(y)$ so $x = y$ and $x_{i_j} = y_{i_j}$ for large j as they lie in a neighborhood of $x = y$ where F is injective. \square

1.4.4. Submersions.

DEFINITION 1.4.16. We say that $F : M \rightarrow N$ is a *submersion* if $\text{rank}_p F = \dim N$ for all $p \in M$.

PROPOSITION 1.4.17. For a smooth map $F : M \rightarrow N$ the following conditions are equivalent:

- (1) F is a submersion.
- (2) For each $p \in M$ there are charts $x : U \rightarrow \mathbb{R}^m$ and $y : V \rightarrow \mathbb{R}^n$ with $p \in U$ and $F(p) \in V$ such that

$$y \circ F \circ x^{-1} \left(x^1, \dots, x^m \right) = \left(x^1, \dots, x^n \right).$$

- (3) For each $f \in C^\infty(N)$ and $p \in M$ we have that $\text{rank}_p (f \circ F) = \text{rank}_{F(p)} (f)$.

PROOF. Assume that 1 holds and select a chart y around $F(p)$. In particular, $y \circ F$ has rank n at p . We can now supplement with $m - n$ coordinate functions x^i from any coordinate system around p such that $x^1 = y^1 \circ F, \dots, x^n = y^n \circ F, x^{n+1}, \dots, x^m$ are coordinates around p . This yields the desired coordinates.

Clearly 2 implies 3. If we assume that 3 holds and that we have a chart y around $F(p)$. Then we can consider smooth functions $f = \sum \alpha_i y^i$, where $\alpha_i \in \mathbb{R}$. These have rank 1 at $F(p)$ unless $\alpha^1 = \dots = \alpha^n = 0$. If we choose coordinates x around p , then

$$D \left(f \circ F \circ x^{-1} \right) |_{x^{-1}(p)} = \sum \alpha_i D \left(y^i \circ F \circ x^{-1} \right) |_{x^{-1}(p)}.$$

It follows that $D \left(y^i \circ F \circ x^{-1} \right) |_{x^{-1}(p)}$ are linearly independent, which in turn implies that $y \circ F \circ x^{-1}$ has rank n at $x^{-1}(p)$. \square

COROLLARY 1.4.18. Let $F : M \rightarrow N$ be a submersion. If $f : O \subset F(M) \rightarrow \mathbb{R}$ is a function on an open set such that $f \circ F$ is smooth, then f is smooth.

PROOF. Smoothness is clearly a local property so we can confine ourselves to functions that are defined on the coordinate systems guaranteed from 2 in proposition 1.4.17. But, then the claim is obvious. \square

COROLLARY 1.4.19. A smooth map $F : M \rightarrow N$ is a submersion iff for any smooth map $G : N \rightarrow O$ and $p \in M$ we have

$$\text{rank}_p (G \circ F) = \text{rank}_{F(p)} G$$

Finally we mention a few useful properties.

PROPOSITION 1.4.20. Let $F : M^m \rightarrow N^n$ be a smooth map.

- (1) If F is proper, then it is closed.
- (2) If F is proper, $K \subset N$ is compact, and $O \supset F^{-1}(K)$ is open, then there exists an open set $V \supset K$ such that $F^{-1}(V) \subset O$.
- (3) If F is a submersion, then it is open.
- (4) If F is a proper submersion and N is connected then it is surjective.

PROOF. 1. Let $C \subset M$ be a closed set and assume $F(x_i) \rightarrow y$, where $x_i \in C$. The set $\{y, F(x_i)\}$ is compact. Thus the preimage is also compact. This implies that $\{x_i\}$ has an accumulation point. If we assume that $x_{i_j} \rightarrow x \in C$, then continuity shows that $F(x_{i_j}) \rightarrow F(x)$. Thus $y = F(x) \in F(C)$.

2. The set $M - O$ is closed, so by 1 we obtain an open neighborhood

$$V = N - F(M - O)$$

around K . If $F(x) \in V$, then $x \notin M - O$ and consequently $F^{-1}(V) \subset O$.

3. Consequence of local coordinate representation of F .

4. Follows directly from properties 1 and 3. \square

1.4.5. Constant Rank. The canonical forms for immersions and submersions can be combined into a more general result for maps that have constant rank on all of the manifold. A map of constant rank is also called a *subimmersion* in some texts. The reason being that it is locally the composition of a submersion followed by an immersion as the next theorem shows.

THEOREM 1.4.21 (Rank Theorem). *Let $F : M^m \rightarrow N^n$ be a map of constant rank k on all of M . For each $p \in M$ there are charts $x : U \rightarrow \mathbb{R}^m$ and $y : V \rightarrow \mathbb{R}^n$ with $p \in U$ and $F(p) \in V$ such that $x(p) = 0$, $y(F(p)) = 0$, and*

$$y \circ F \circ x^{-1}(x^1, \dots, x^m) = (x^1, \dots, x^k, 0, \dots, 0).$$

In particular, if $F^{-1}(q)$ is nonempty, then it is an $(m - k)$ -dimensional submanifold.

PROOF. We start with general charts around p and $F(p)$ such that $u(p) = 0$, $v(F(p)) = 0$, and

$$v \circ F \circ u^{-1}(u^1, \dots, u^m) = (A(u), B(u))$$

where A takes up the first k coordinates and B the remaining $n - k$. After possibly reordering these two coordinate systems we can assume that $\frac{\partial A}{\partial u^1}, \dots, \frac{\partial A}{\partial u^k}$ are linearly independent on a neighborhood of 0. Now consider the map $u \mapsto x(u) = (A(u), u^{k+1}, \dots, u^m)$. This map has rank m at 0 and is consequently a local diffeomorphism. This gives us a new chart x around p where

$$v \circ F \circ x^{-1}(x^1, \dots, x^m) = (x^1, \dots, x^k, B(x)).$$

Since this map has constant rank k it must follow that

$$\frac{\partial B}{\partial x^i} = 0, i = k + 1, \dots, m.$$

After possibly shrinking the domain of the chart we have that

$$v \circ F \circ x^{-1}(x^1, \dots, x^m) = (x^1, \dots, x^k, B(x^1, \dots, x^k)).$$

We can now define $y = (v^1, \dots, v^k, v^{k+1} - B^{k+1}, \dots, v^n - B^n)$. This map is nonsingular at 0 and

$$y \circ F \circ x^{-1}(x^1, \dots, x^m) = (x^1, \dots, x^k, 0, \dots, 0).$$

\square

REMARK 1.4.22. Note that as all immersions have constant rank we can't expect the image to be a submanifold.

1.4.6. Regular and Critical Points. We say that $F : M \rightarrow N$ is *non-singular* on M if it is both a submersion and an immersion. This is evidently equivalent to saying that it is locally a diffeomorphism.

A point $p \in M$ is called a *regular point* if $\text{rank}_p F = \dim N$, otherwise it is a *critical point*. A point $q \in N$ is called a *regular value* if $F^{-1}(q)$ is empty or only contains regular points, otherwise it is a *critical value*.

REMARK 1.4.23. It follows from remark 1.4.3 that if $p \in M$ is a regular point for $F : M \rightarrow N$, then there is a neighborhood $p \in U \subset M$ such that all $x \in U$ are regular points. Thus the set of regular points is open. This however does not tell us that the set of regular values is open. In case F is proper we can use proposition 1.4.20 to conclude that the set of regular values is open.

THEOREM 1.4.24 (*The Preimage Theorem*). *If $q \in N$ is a regular value for a smooth function $F : M^m \rightarrow N^n$, then $F^{-1}(q)$ is empty or a properly embedded submanifold of M of dimension $m - n$.*

PROOF. We use the induced topology on the preimage. As the preimage is closed its intersections with a compact set is compact. To check that the preimage is embedded we show that $C^\infty(M)$ restricts to a differential system on the preimage.

If we select coordinates $y^i, i = 1, \dots, n$ around $q \in N$, then the functions $y^i \circ F$ are part of a coordinate system x^i around any point $p \in F^{-1}(q)$. This means that we can find a neighborhood $p \in U$ such that $U \cap F^{-1}(q) = \{x \in U \mid y^i(F(x)) = y^i(F(q))\}$, i.e., $x^i = y^i \circ F$ are constant on the preimage. Given $f \in C^\infty(M)$ defined around p we have that $f = F(x^1, \dots, x^m)$. Now on $U \cap F^{-1}(q)$ the first n coordinates are constant so it follows that $f|_{U \cap F^{-1}(q)} = F(x^1(p), \dots, x^n(p), x^{n+1}, \dots, x^m)$. Thus the restriction can be written as a smooth function of the last $m - n$ coordinates. Finally we note that these last $m - n$ coordinates also define the desired chart on $U \cap F^{-1}(q)$ as they are injective and yield a homeomorphism onto the image. \square

REMARK 1.4.25. The constant rank theorem implies that the preimage theorem remains true as long as the map has constant rank k on M . In this case the preimages have dimension $m - k$ provided they are nonempty.

To complement the preimage theorem we next prove.

THEOREM 1.4.26 (Brown, 1935, A.P. Morse, 1939 and Sard, 1942). *The set of regular values for a smooth function $F : M^m \rightarrow N^n$ is a countable intersection of open dense sets and in particular dense. Moreover, the set of critical values has measure 0.*

PROOF. We prove Brown's original statement: the set of critical values has no interior points. The proof we give is fairly standard and is very close to Brown's original proof. The same proof is easily adapted to prove Sard's measure zero version, but, this particular statement is in fact rarely used. A.P. Morse proved the measure theoretic result when the target space is \mathbb{R} .

Note that the set of critical points is closed but its image need not be closed. However, the set of critical points is a countable union of compact sets and thus the image is also a countable union of compact sets. This means that we rely on the Baire category theorem: a set that is the countable union of closed sets with empty interiors also has empty interior. Thus we only need to show that there are no interior points in the set of critical values that come from critical points in a compact set. Further note that it suffices to prove the theorem for the restriction of F to any open covering of M .

To clarify the meaning of measure 0 and prove Sard's theorem in the case where it is most used, we make some simple observations.

Consider a map $F : O \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$. When F is locally Lipschitz, then it maps sets of measure zero to sets of measure zero. Moreover, any differentiable map that has bounded derivative on compact sets is locally Lipschitz. Thus C^1 diffeomorphisms preserve sets of measure zero. This shows that the notion of sets of measure zero is well-defined in a smooth manifold. Now consider $F : M^m \rightarrow N^n$, where $m < n$ and construct $\bar{F} : M \times \mathbb{R}^{n-m} \rightarrow N$, by $\bar{F}(x, z) = F(x)$. Then $F(M) = \bar{F}(M \times \{0\})$ has measure zero as $M \times \{0\} \subset M \times \mathbb{R}^{n-m}$ has measure zero.

In the general case the proof uses induction on m . For $m = 0$ the claim is trivial as M is forced to be a countable set with the discrete topology. As mentioned above, it suffices to prove it for maps $F : U \subset \mathbb{R}^m \rightarrow \mathbb{R}^n$, where U is open. For such a map let C_0 be the set of critical points and define $C_k \subset C_0$ as the set of critical points where all derivatives of order $\leq k$ vanish. Note that all of these sets are closed.

First we show that $F(C_k)$ has no interior points when $k \geq m/n$: Fix a compact set K . Taylor's theorem shows that we can select $r > 0$ and $C > 0$ such that for any $x \in B(p, r)$ with $p \in C_k \cap K$ we have

$$|F(p) - F(x)| \leq C |p - x|^{k+1}.$$

Now cover $C_k \cap K$ by finitely many cubes I_i^ϵ of side length $\epsilon < r$, then $F(I_i^\epsilon)$ lies in a cube J_i^ϵ of side length $\leq C(m, n)\epsilon^{k+1}$ for a constant $C(m, n)$ that depends on C, m , and n . Thus

$$\begin{aligned} |J_i^\epsilon| &\leq (C(m, n))^n \epsilon^{n(k+1)} \\ &= (C(m, n))^n \epsilon^{n(k+1)-m} |I_i^\epsilon|. \end{aligned}$$

Since $C_k \cap K$ is compact we can assume that $\sum |I_i^\epsilon|$ remains bounded as $\epsilon \rightarrow 0$. Thus $\sum |J_i^\epsilon|$ will converge to 0 since $n(k+1) > m$. This shows that $F(C_k \cap K)$ does not contain any interior points as it could otherwise not be covered by cubes whose total volume is arbitrarily small.

Next we show that $F(C_k - C_{k+1})$ has no interior points for $k > 0$: Denote by ∂^k some specific partial derivative of order k . Thus $(\partial^k F)(p) = 0$ for $p \in C_k - C_{k+1}$ but some partial derivative $\frac{\partial \partial^k F}{\partial x^j}(p) \neq 0$. Without loss of generality we can assume that $\frac{\partial \partial^k F}{\partial x^1}(p) \neq 0$. This means that near p the set where $\partial^k F = 0$ will be a submanifold of dimension $m - 1$. Since p is critical for F it'll also be a critical point for the restriction of F to any submanifold. By induction hypothesis the image of such a set has no interior points. Thus for any fixed compact set K the set $K \cap (C_k - C_{k+1})$ can be divided into a finite collection of sets whose images have no interior points.

Finally we show that $F(C_0 - C_1)$ has no interior points: Note that when $n = 1$ it follows that $C_0 = C_1$ so there is nothing to prove in this case. Assume that $p \in C_0 - C_1$ is a point where $\frac{\partial F^1}{\partial x^1} \neq 0$. After rearranging the coordinates in \mathbb{R}^m and \mathbb{R}^n we can assume that $\frac{\partial F^1}{\partial x^1} \neq 0$. In particular, the set $L = \{x \mid F^1(x) = F^1(p)\}$ is a submanifold of dimension $m - 1$ in a neighborhood of p . Let $G = (F^2, \dots, F^n) : L \rightarrow \mathbb{R}^{n-1}$. Now observe that if $F(p)$ is an interior point in $F(C_0 - C_1)$, then $G(p)$ is an interior point for $G(L \cap (C_0 - C_1))$. This, however, contradicts our induction hypothesis since all the points in $L \cap (C_0 - C_1)$ are critical for G . (For the measure zero statement, this last part requires a precursor to the Tonelli/Fubini theorem or Cavalieri's principle: A set has measure zero if its intersection with all parallel hyperplanes has measure zero in the hyperplanes.)

Putting these three statements together implies that the set of critical values has no interior points. \square

1.5. Exercises

- (1) Gluing definition of a manifold: Consider a collection of open sets $U_i \subset \mathbb{R}^n$ and possibly empty open subsets $U_{ji} \subset U_i$ together with smooth maps $\phi_{ji} : U_{ji} \rightarrow U_{ij}$ that satisfy: $\phi_{ii} = id_{U_i}$ and $\phi_{kj} \circ \phi_{ji} = \phi_{ki}$.
- (a) Show that each $\phi_{ji} : U_{ji} \rightarrow U_{ij}$ is a diffeomorphism.
- (b) We say that $x_i \sim x_j$ provided $x_j = \phi_{ji}(x_i)$. Show that this defines an equivalence relation and that the quotient space $M = (\coprod_i U_i) / \sim$ with the quotient topology is a manifold. Hint: Show that the natural map $U_i \rightarrow M$ is a homeomorphism onto the image.
- (c) Show that M is second countable provided there are countably many U_i .
- (d) Show that M is Hausdorff provided that for each pair i, j the set

$$\{(x_i, x_j) \in U_i \times U_j \mid x_j = \phi_{ji}(x_i)\}$$

is closed.

- (e) Show that if $U_1 = U_2 = \mathbb{R} - \{0\}$ and $\phi_{21} = id_{U_1}$, then the corresponding manifold is not Hausdorff.
- (2) This exercise relies on the definition of a manifold from the previous exercise.
- (a) Show that if $U_1 = U_2 = \mathbb{R} - \{0\}$ and $\phi_{21}(x) = \frac{1}{x}$, then the corresponding manifold is S^1 via stereographic projection and $\mathbb{R}P^1$ via homogeneous coordinates.
- (b) Show that if $U_1 = U_2 = \mathbb{C} - \{0\}$ and $\phi_{21}(z) = \frac{1}{z}$, then the corresponding manifold is $\mathbb{C}P^1$.
- (c) Show that if $U_1 = U_2 = \mathbb{C} - \{0\}$ and $\phi_{21}(z) = \frac{1}{\bar{z}}$, then the corresponding manifold is S^2 .
- (d) Show that $\mathbb{C}P^1$ and S^2 are diffeomorphic.
- (3) Show that the space of symmetric matrices of rank k , $k \leq n$, in $\text{Mat}_{n \times n}(\mathbb{R})$ is a manifold of dimension $\binom{k+1}{2} + k(n-k)$.
- (4) Let M^m be a path connected manifold. Show that if $m > 1$, then $M - \{p\}$ is path connected.
- (5) Let $F : \mathbb{R}^m \rightarrow \mathbb{R}^n$ be a smooth map such that $F(\lambda x) = \lambda F(x)$ for all $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^m$. Show that F is linear.
- (6) Let $F : \mathbb{R}^m \rightarrow \mathbb{R}^n$ be a smooth map such that $F(0) = 0$ and define

$$H(t, x) = \begin{cases} t^{-1}F(tx) & t \neq 0 \\ DF|_0(x) & t = 0 \end{cases}$$

Show that $H(t, x) : \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is smooth.

- (7) Let $p : \mathbb{C} \rightarrow \mathbb{C}$ be a nontrivial polynomial and define $P : \mathbb{C}P^1 \rightarrow \mathbb{C}P^1$ by

$$P([z : 1]) = [p(z) : 1] \text{ and } P([1 : 0]) = [1 : 0].$$

- (a) Show that P is smooth.
- (b) Show that a similar definition works for any smooth proper map $f : \mathbb{C} \rightarrow \mathbb{C}$ and will always define a continuous extension $F : \mathbb{C}P^1 \rightarrow \mathbb{C}P^1$. Will it always be smooth?
- (8) Let $\frac{p}{q}$ be a rational function, where p and q are complex polynomials without common roots. Show that

$$R([z : 1]) = [p(z) : q(z)]$$

can be extended to a smooth map $R : \mathbb{C}\mathbb{P}^1 \rightarrow \mathbb{C}\mathbb{P}^1$ (the extension depends on the degrees of the polynomials).

- (9) Show that $\text{Aut}(V)$ acts by diffeomorphisms on $\mathbb{P}(V)$.
- (10) Let $F : M \rightarrow N$ be a smooth map. We say that F has constant rank at $p \in M$ provided there is a neighborhood $U \ni p$ such that $F|_U$ has constant rank.
- (a) Show that the set of points in M where F has constant rank is open and dense.
- (b) Show that if F is (locally) injective, then $\text{rank}_p F = \dim M$ for an open dense set of points in M .
- (c) Show that if F is open, then $\text{rank}_p F = \dim N$ for an open dense set of points in M .
- (11) Show that
- (a) $\mathbb{F}\mathbb{P}^{n-1} \subset \mathbb{F}\mathbb{P}^n$ is an embedded submanifold
- (b) and

$$\{[z_0 : \cdots : z_n] \in \mathbb{F}\mathbb{P}^n \mid \lambda_0 z_0 + \cdots + \lambda_n z_n = 0\}$$

defines an embedded submanifold diffeomorphic to $\mathbb{F}\mathbb{P}^{n-1}$ as long as $(\lambda_0, \dots, \lambda_n) \neq 0$.

- (12) Consider the immersion $\mathbb{R} \rightarrow \mathbb{R}^2$ whose image looks like a figure 8 with loops that are in the first and third quadrants and is invariant under the involution $A(x, y) = (y, x)$. Show that the restriction of A to this immersed submanifold is not continuous.
- (13) Show that for any manifold M^n there exists an open set $O \subset \mathbb{R}^n$ and a surjective local diffeomorphism $F : O \rightarrow M$. Conversely, give a definition of a smooth manifold from a surjective local homeomorphism $F : O \rightarrow M$, where $O \subset \mathbb{R}^n$ is open. Moreover, show that M is both second countable and Hausdorff.
- (14) Let $F : M^m \rightarrow N^n$ be a map of constant rank k . Show that there is an immersion $\bar{F} : \bar{M}^k \rightarrow N$ such that $F(M) = \bar{F}(\bar{M})$.
- (15) Let

$$V_k(\mathbb{R}^n) = \{(v_1, \dots, v_k) \in \mathbb{R}^n \times \cdots \times \mathbb{R}^n \mid v_i \cdot v_j = \delta_{ij}\} \subset \text{Mat}_{n \times k}^k(\mathbb{R})$$

be the Stiefel manifold of k ordered orthonormal vectors. Show that this is a manifold of dimension $kn - \binom{k+1}{2}$.

- (16) Let M, N be manifolds. If $S \subset M$ is a closed subset and $q \in N$, then there is a smooth map $F : M \rightarrow N$ such that $S = F^{-1}(q)$.
- (17) Show that if $F : M \rightarrow N$ admits a section $s : N \rightarrow M$, i.e., $F \circ s = \text{id}_N$, then s is an embedding. Show that there exists a neighborhood U of $s(N) \subset M$ such that $F|_U : U \rightarrow N$ is a submersion.
- (18) Let $F : M \rightarrow N$ be a submersion. Show that if $S \subset N$ is an embedded submanifold, then $F^{-1}(S) \subset M$ is an embedded submanifold that, when nonempty, satisfies

$$\dim N - \dim S = \dim M - \dim F^{-1}(S).$$

Hint: Start with the case where $S = G^{-1}(z)$ and z is a regular value for $G : N \rightarrow \mathbb{R}^k$, $k = \dim N - \dim S$. Then localize to prove the result.

- (19) Note that the torus $S^1 \times S^1$ can be embedded in \mathbb{R}^3 .
- (a) Show that $S^p \times S^q$ can be embedded in \mathbb{R}^{p+q+1} .
- (b) Show that the n torus $S^1 \times \cdots \times S^1$ can be embedded in \mathbb{R}^{n+1} .
- (c) Show that $S^{p_1} \times \cdots \times S^{p_k}$ can be embedded in $\mathbb{R}^{p_1 + \cdots + p_k + 1}$.
- (20) Let $F : S^1 \rightarrow \mathbb{R}$.

- (a) Show that if $y \in \mathbb{R}$ is a regular value, then it has an even number of preimages.
 (b) Show that there are at least as many critical points as there are preimages of a regular value.
- (21) Show that $\mathbb{R}P^n \subset \mathbb{R}P^{n+1}$ is not the preimage of a regular value of a function $\mathbb{R}P^{n+1} \rightarrow \mathbb{R}$.
- (22) Without quoting theorem 1.4.26 show that if $F : M^m \rightarrow N^n$ has constant rank $k < n$, then the image $F(M) \subset N$ has measure zero.
- (23) Consider the map $\pi : Gl_n \rightarrow Gr_k$ that maps $A \in Gl_n$ to the span of the first k rows.
- (a) Show that this map is a submersion.
 (b) Is the map

$$\begin{aligned} Gr_k &\rightarrow Gl_n \\ E^k &\mapsto I_E \oplus (-I_{E^\perp}) \end{aligned}$$

a section of π ? Recall that the basis of the vector space is fixed so the linear map $I_E \oplus (-I_{E^\perp})$ has to be represented in that basis.

- (24) Let Gr_k be the Grassmannian of k -dimensional subspaces in \mathbb{R}^n or \mathbb{C}^n .
- (a) Show that the map

$$\begin{aligned} Mat_{k \times n}^k &\rightarrow Gr_k \\ X &\mapsto E_X = X^* (XX^*)^{-1} X \end{aligned}$$

is a submersion whose preimages are diffeomorphic to Gl_k .

- (b) Show that there is a submersion $V_k(\mathbb{R}^n) \rightarrow Gr_k(\mathbb{R}^n)$ whose preimages can be identified with $O(k)$.
- (25) Show that the map

$$\begin{aligned} \mathbb{F}P^m \times \mathbb{F}P^n &\rightarrow \mathbb{F}P^{mn+m+n} \\ ([\cdots : x_i : \cdots], [\cdots : y_j : \cdots]) &\mapsto [\cdots : x_i y_j : \cdots] \end{aligned}$$

gotten by multiplying all of the homogeneous coordinates is well-defined and an embedding.

- (26) Show that

$$\{[z_1 : z_2 : z_3] \in \mathbb{C}P^2 \mid z_1^n + z_2^n + z_3^n = 0\}$$

is a compact submanifold.

- (27) More generally, show that

$$\{[z_1 : z_2 : z_3] \in \mathbb{C}P^2 \mid p(z_1, z_2, z_3) = 0\}$$

is a compact submanifold when p is homogeneous and irreducible. What happens in the real case?

- (28) Show that

$$\{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} \mid z_0^2 + \cdots + z_n^2 = 1\}$$

defines a submanifold and that it is diffeomorphic to TS^n .

- (29) Show that the Brieskorn variety $W^{2n-1}(d) \subset \mathbb{C}^{n+1}$ defined by the equations

$$\begin{aligned} z_0^2 + \cdots + z_n^2 &= 0 \\ z_0 \bar{z}_0 + \cdots + z_n \bar{z}_n &= 2 \end{aligned}$$

is a manifold of dimension $2n - 1$.

(30) Show that the Milnor manifold with $m \leq n$ given by

$$H(m, n) = \{([z_0 : \cdots : z_m], [w_0 : \cdots : w_n]) \in \mathbb{F}\mathbb{P}^m \times \mathbb{F}\mathbb{P}^n \mid z_0 w_0 + \cdots + z_m w_n = 0\}$$

is a manifold of dimension $\dim_{\mathbb{R}} \mathbb{F} \cdot (m + n - 1)$.

(31) A classical way of embedding $\mathbb{R}\mathbb{P}^n$ into S^{n+k} uses a symmetric bilinear map $b : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+k+1}$ with the property that if $b(x, y) = 0$, then $x = y = 0$.

Define $F : S^n \rightarrow S^{n+k}$, by $F(x) = \frac{b(x, x)}{|b(x, x)|}$.

(a) Show that $F(x) = F(y)$ if and only if $x = \pm y$. Hint: Consider $b(x + \lambda y, x - \lambda y)$ when $b(x, x) = \lambda^2 b(y, y)$.

(b) Show that F induces an embedding $\mathbb{R}\mathbb{P}^n \rightarrow S^{n+k}$.

(c) Use $b(x, y) = (z^0, \dots, z^{2n})$, where $x = (x^0, \dots, x^n)$, $y = (y^0, \dots, y^n)$ and $z^k = \sum_{i+j=k} x^i y^j$, to obtain an embedding $\mathbb{R}\mathbb{P}^n \rightarrow S^{2n}$.

(d) Use the multiplicative structure on $\mathbb{R}^2 \simeq \mathbb{C}$ to obtain a diffeomorphism $\mathbb{R}\mathbb{P}^1 \rightarrow S^1$.

(32) A regular closed curve $c : S^1 \rightarrow \mathbb{R}^2$ is said to have a crossing at $q \in \mathbb{R}^2$ provided $F^{-1}(q) = \{p_1, p_2\}$ consists of exactly two points and the derivatives of c at these two points are linearly independent. For each positive integer k there exists a curve with k crossings (draw it). It can even be realized as the level set $F^{-1}(0)$ of a polynomial $F(x, y) : \mathbb{R}^2 \rightarrow \mathbb{R}$.

(a) Check this in these three cases: $F(x, y) = x^2 + y^2 - 1$ for $p = 0$, $F(x, y) = (x^2 + y^2 - 4)^2$ for $p = 1$, $F(x, y) = (4x^2(1 - x^2) - y^2)^2$ for $p = 2$.

(b) Show that if $F^{-1}(0)$ is the image of a curve with p crossings, then the zero level set of $F(x, y) + z^2 - \epsilon$ is a compact surface of genus $p + 1$ ($p + 1$ holes) for sufficiently small ϵ . Essentially the curve has been fattened so that each of $p + 1$ enclosed regions of the curve correspond to a hole in the surface and the crossings become necks.

Tangent Spaces and Differentials of Maps

In this chapter we introduce the tangent space and the differential of smooth maps. Vector bundles and constructions with such bundles as well as their sections are also discussed. In the last section we prove the local and global versions of the Frobenius theorem. This will be used in chapter 4 to establish the correspondence between subgroups of Lie groups and subalgebras of their Lie algebras.

2.1. The Tangent Bundle

2.1.1. Motivation. To motivate the more abstract definitions let us start by selecting a differentiable system $\{f^i\}$ of functions $f^i : M \rightarrow \mathbb{R}$.

Tangent vectors can be viewed as velocities to curves on the manifold. These vectors have, as such, no place to live unless we know that the manifold is inside a vector space. In the general case we can use the collection f^i of smooth functions coming from a differential structure to measure the coordinates of the velocities by calculating the derivatives

$$\frac{d(f^i \circ c)}{dt}$$

for a smooth curve $c : I \rightarrow M$. Thus a tangent vector $v \in TM$ looks like a collection v^i of its coordinates. However, around any given point we know that there will be n coordinate functions, say f^1, \dots, f^n , that yield a chart and smooth functions F^j , $j > n$ such that $f^j = F^j(f^1, \dots, f^n)$. Thus we also have the relations

$$v^j = \sum_{i=1}^n \frac{\partial F^j}{\partial x^i} v^i, \quad j > n.$$

In other words the n coordinates v^1, \dots, v^n determine the rest of the coordinate components of v . Note that at a fixed point p , the tangent vectors $v \in T_p M$ form an n -dimensional vector space, which is an n -dimensional subspace of a fixed infinite dimensional vector space. Moreover, this tangent space is well-defined as the set of vectors tangent to curves going through p and is thus not dependent on the chosen coordinates. However, the coordinates help us select a basis for this vector space and thus to create suitable coordinates that yield a differentiable structure on TM .

As it stands, the definition does depend on our initial choice of a differentiable system. To get around this we could simply use the entire space of smooth functions $C^\infty(M)$ to get around this. This is essentially what we shall do below.

2.1.2. Einstein Summation Convention. We shall often use the index and summation convention introduced by Einstein. Given a vector space V we use subscripts for vectors in V . Thus a basis of V is denoted by e_1, \dots, e_n . Given a vector $v \in V$ we can then write it as

a linear combination of these basis vectors as follows

$$v = \sum_i v^i e_i = v^i e_i = \begin{bmatrix} e_1 & \cdots & e_n \end{bmatrix} \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix}.$$

Here we use superscripts on the coefficients and then automatically sum over indices that are repeated as both subscripts and superscripts. If we define a dual basis e^i for the dual space $V^* = \text{Hom}(V, \mathbb{R})$ as follows: $e^i(e_j) = \delta_j^i$, then the coefficients can be computed as $v^i = e^i(v)$. Thus we decide to use superscripts for dual bases in V^* . The matrix representation $[L_i^j]$ of a linear map $L : V \rightarrow V$ is found by solving

$$\begin{aligned} L(e_i) &= L_i^j e_j, \\ \begin{bmatrix} L(e_1) & \cdots & L(e_n) \end{bmatrix} &= \begin{bmatrix} e_1 & \cdots & e_n \end{bmatrix} \begin{bmatrix} L_1^1 & \cdots & L_1^n \\ \vdots & \ddots & \vdots \\ L_n^1 & \cdots & L_n^n \end{bmatrix} \end{aligned}$$

In other words

$$L_i^j = e^j(L(e_i)).$$

As already indicated, subscripts refer to the column number and superscripts to the row number.

When the objects under consideration are defined on manifolds, the conventions carry over as follows: Cartesian coordinates on \mathbb{R}^n and coordinates on a manifold have superscripts (x^i) as they are coordinate coefficients; coordinate vector fields then look like

$$\partial_i = \frac{\partial}{\partial x^i},$$

and consequently have subscripts (This odd way of thinking about vector fields will be discussed at length below). This is natural, as they form a basis for the tangent space. The dual 1-forms dx^i satisfy $dx^j(\partial_i) = \delta_i^j$ and consequently form the natural dual basis for the cotangent space.

Einstein notation is not only useful when one doesn't want to write summation symbols, it also shows when certain coordinate- (or basis-) dependent definitions are invariant under change of coordinates. Let us just consider a very simple situation, namely, the velocity field of a curve $c : I \rightarrow \mathbb{R}^n$. In coordinates, the curve is written

$$\begin{aligned} c(t) &= (x^i(t)) \\ &= x^i(t) e_i, \end{aligned}$$

if e_i is the standard basis for \mathbb{R}^n . The velocity field is defined as the vector $\dot{c}(t) = (\dot{x}^i(t))$. Using the coordinate vector fields this can also be written as

$$\dot{c}(t) = \frac{dx^i}{dt} \frac{\partial}{\partial x^i} = \dot{x}^i(t) \partial_i.$$

In a coordinate system on a general manifold we could then try to use this as our definition for the velocity field of a curve. In this case we must show that it gives the same answer in different coordinates. This is simply because the chain rule tells us that

$$\dot{x}^i(t) = dx^i(\dot{c}(t)),$$

and then observing that we have used the above definition for finding the components of a vector in a given basis.

When offering coordinate dependent definitions we shall be careful that they are given in a form where they obviously conform to this philosophy and are consequently easily seen to be invariantly defined.

2.1.3. Abstract Derivations. The space of all smooth functions $C^\infty(M)$ is not a vector space as we can't add functions that have different domains, especially if these domains do not even intersect. If we fix $p \in M$, then we consider the subset $C_p(M) \subset C^\infty(M)$ of smooth functions whose domain contains p . Thus any two functions in $C_p(M)$ can now be added in a meaningful way by adding them on the intersection of their domains and then noting that this is again an open set containing p . Thus we get a nice and very large collection of smooth functions defined on neighborhoods of p . To get a logically meaningful theory this space is often modified by considering instead equivalence classes of function in $C_p(M)$, the relation being that two functions are equivalent if they are equal on some neighborhood of p . This quotient space is denoted $\mathcal{F}_p(M)$ and the elements are called *germs* of functions at p . Note that $\mathcal{F}_p(M)$ really is a vector space.

Now consider a curve $c : I \rightarrow M$ with $c(t_0) = p$. The goal is to make sense of the velocity of c at t_0 . If $f \in C_p(M)$, then $f \circ c$ measures how c changes with respect to f . If f had been a coordinate function this would be the corresponding coordinate component of c in a chart. Similarly the derivative $\frac{d}{dt}(f \circ c)$ measures the change in velocity with respect to f , i.e., what should be the f -component of the velocity.

DEFINITION 2.1.1. The velocity $\dot{c}(t_0)$ of c at t_0 is the map

$$\begin{aligned} C_p(M) &\rightarrow \mathbb{R} \\ f &\mapsto \frac{d}{dt}(f \circ c)(t_0). \end{aligned}$$

Thus $\dot{c}(t_0)$ is implicitly defined by specifying how it creates directional derivatives

$$D_{\dot{c}(t_0)}f = \frac{d}{dt}(f \circ c)(t_0)$$

for all smooth functions defined on a neighborhood of $p = c(t_0)$.

DEFINITION 2.1.2. A derivation at p or on $C_p(M)$ is a linear map $D : C_p(M) \rightarrow \mathbb{R}$ that also satisfies the product rule for differentiation at p :

$$D(fg) = D(f)g(p) + f(p)D(g).$$

There is an alternate way of defining derivations as linear functions on $C_p(M)$. Let $C_p^0(M) \subset C_p(M)$ be the maximal ideal of functions that vanish at p and $(C_p^0(M))^2 \subset C_p^0(M)$ the ideal generated by products of elements in $C_p^0(M)$.

LEMMA 2.1.3. *The derivations at p are isomorphic to the subspace of linear maps on $C_p^0(M)$ that vanish on $(C_p^0(M))^2$.*

PROOF. If D is a derivation, then the derivation property shows that it vanishes on $(C_p^0(M))^2$. Furthermore, it also vanishes on constant functions as linearity and the derivation property implies

$$D(a) = aD(1) = aD(1 \cdot 1) = a(D(1) + D(1)).$$

Conversely, any linear map D on $C_p^0(M)$ that vanishes on $(C_p^0(M))^2$ defines a unique linear map on $C_p(M)$ by also defining it to vanish on constant functions. If $f, g \in C_p(M)$, then we have

$$\begin{aligned} 0 &= D((f - f(p))(g - f(p))) \\ &= D(fg) - f(p)Dg - g(p)Df + D(f(p)g(p)) \\ &= D(fg) - f(p)Dg - g(p)Df \end{aligned}$$

showing that it is a derivation. \square

Next we show that derivations exist.

PROPOSITION 2.1.4. *The map $f \mapsto \frac{d}{dt}(f \circ c)(t_0)$ is a derivation on $C_p(M)$.*

PROOF. That it is linear in f is obvious from the fact that differentiation is linear. The derivation property follows from the product rule for differentiation:

$$\frac{d}{dt}((fg) \circ c)(t_0) = \left(\frac{d}{dt}(f \circ c)(t_0) \right) (g \circ c)(t_0) + (f \circ c)(t_0) \frac{d}{dt}(g \circ c)(t_0).$$

\square

DEFINITION 2.1.5. The tangent space T_pM for M at p is the vector space of derivations on $C_p(M)$.

PROPOSITION 2.1.6. *If $p \in U \subset M$, where U is open, then $T_pU = T_pM$.*

PROOF. We already saw that derivations must vanish on constant functions. Next consider a function f that vanishes on a neighborhood of p . We can then find $\lambda : M \rightarrow \mathbb{R}$ that is 1 on a neighborhood of p and $\lambda = 0$ on the complement of the region where f vanishes. Thus $\lambda f = 0$ on M and

$$0 = D(\lambda f) = D(\lambda)f(p) + \lambda(p)D(f) = D(f).$$

This in turn shows that if two functions f, g agree on a neighborhood of p , then $D(f) = D(g)$. This means that a derivation D on $C_p(M)$ restricts to a derivation on $C_p(U)$ and conversely that any derivation on $C_p(U)$ also defines a derivation on $C_p(M)$. This proves the claim. \square

We are now ready to prove that there are no more derivations than one would expect.

LEMMA 2.1.7. *The natural map $\mathbb{R}^n \rightarrow T_0\mathbb{R}^n$ that maps v to $D_v f = \left(\frac{df}{dt} \right)(tv) |_{t=0}$ is an isomorphism.*

PROOF. The map is clearly linear and as

$$D_v x^i = v^i$$

it follows that its kernel is trivial. Thus we need to show that it is surjective. This claim depends crucially on the fact that derivations are defined on C^∞ functions. The key observation is that we have a Taylor formula

$$f(x) = f(0) + x^i f_i(x)$$

where f_i are also smooth and $f_i(0) = \frac{\partial f}{\partial x^i}(0)$. The functions are defined by

$$f_i(x) = \int_0^1 \frac{\partial f}{\partial x^i}(tx) dt$$

and the formula follows from the fundamental theorem of calculus applied to the identity:

$$\frac{d}{dt}(f(tx)) = x^i \frac{\partial f}{\partial x^i}(tx).$$

Now select an abstract derivation $D \in T_0\mathbb{R}^n$ and observe that

$$D(f) = D(f(0)) + D(x^i) f_i(0) + 0D(f_i) = \frac{\partial f}{\partial x^i}(0) D(x^i)$$

So if we define a vector $v = (D(x^1), \dots, D(x^n))$, then in fact

$$D(f) = D_v(f).$$

□

REMARK 2.1.8. The space of linear maps on $C^k(\mathbb{R}^n)$, $1 \leq k < \infty$, that satisfy the product rule

$$D(fg) = D(f)g(0) + f(0)D(g)$$

is infinite dimensional! It clearly suffices to show this for $n = 1$. Observe that if $Z \subset C^k(\mathbb{R})$ is the subset of functions that vanish at 0, then we merely need to show that Z/Z^2 is infinite dimensional. To see this, first note that if f is C^0 and $g \in Z$, then fg is differentiable with derivative $f(0)g'(0)$ at 0. This in turn implies that functions in Z^2 are not only C^k but also have derivatives of order $k+1$ at 0. However, there is a vast class of functions in Z that do not have derivatives of order $k+1$ at 0.

2.1.4. Concrete Derivations. To avoid the issue of crucially using C^∞ functions we can finesse the definition.

DEFINITION 2.1.9. T_pM is the space of derivations that are constructed from the derivations coming from curves that pass through p .

Without the above result it is not obvious that this is a vector space so a little more work is needed.

PROPOSITION 2.1.10. *If x^1, \dots, x^n are coordinates on a neighborhood of p , then two curves c_i passing through p at $t = 0$ define the same derivations if and only if for all $i = 1, \dots, n$*

$$\frac{d(x^i \circ c_1)}{dt}(0) = \frac{d(x^i \circ c_2)}{dt}(0).$$

PROOF. The necessity is obvious. Conversely note that any $f \in C_p(M)$ can be expressed smoothly as $f = F(x^1, \dots, x^n)$ on some neighborhood of p . Thus

$$\begin{aligned} \frac{d(f \circ c_1)}{dt}(0) &= \frac{d(F(x^1 \circ c_1, \dots, x^n \circ c_1))}{dt}(0) \\ &= \frac{\partial F}{\partial x^i} \frac{d(x^i \circ c_1)}{dt}(0) \\ &= \frac{\partial F}{\partial x^i} \frac{d(x^i \circ c_2)}{dt}(0) \\ &= \frac{d(f \circ c_2)}{dt}(0). \end{aligned}$$

□

PROPOSITION 2.1.11. *The subset of derivations on $C_p(M)$ that come from curves through p form a subspace.*

PROOF. First note that for a curve c through p we have

$$\alpha \frac{d(f \circ c)}{dt}(0) = \frac{d(f \circ c)(\alpha t)}{dt}(0).$$

So scalar multiplication preserves this subset.

Next assume that we have two curves c_i and select a coordinate system x^i around p . Define

$$c = x^{-1} \left(x^1 \circ c_1 + x^1 \circ c_2, \dots, x^n \circ c_1 + x^n \circ c_2 \right)$$

where x^{-1} is the inverse of the chart map $x : U \rightarrow V \subset \mathbb{R}^n$. Then

$$x^i \circ c = x^i \circ c_1 + x^i \circ c_2$$

and

$$\frac{d(f \circ c)}{dt}(0) = \frac{d(f \circ c_1)}{dt}(0) + \frac{d(f \circ c_2)}{dt}(0).$$

This shows that addition of such derivations also remain in this subset. \square

DEFINITION 2.1.12. The velocity of a curve $c : I \rightarrow M$ at t_0 is denoted by $\dot{c}(t_0) \in T_{c(t_0)}M$ and is the derivation corresponding to the map:

$$f \mapsto \frac{d(f \circ c)}{dt}(t_0).$$

As any vector $v \in T_pM$ can be written as $v = \dot{c}(t_0)$ we can also define the directional derivative of f by

$$D_v f = \frac{d(f \circ c)}{dt}(t_0).$$

Most books also use the notation

$$v(f) = D_v f.$$

EXAMPLE 2.1.13. We can now definitively identify

$$T_p S^n = \{(p, v) \mid v \in \mathbb{R}^{n+1}, v \perp p\}$$

since curves through p satisfy $|c|^2 = 1$ and consequently have velocity that is perpendicular to p .

EXAMPLE 2.1.14. The tangent spaces to Grassmannians and, in particular, projective spaces also have a nice representation. If $U \in \text{Gr}_k(V)$, then $T_U \text{Gr}_k = \text{Hom}(U, U^\perp)$. To see this note that any curve through U is a curve of subspaces that near U can be represented as graphs over U , i.e., as a curve in $\text{Hom}(U, U^\perp)$. The velocity is then also defined as a vector in $\text{Hom}(U, U^\perp)$. Note that, like in the case of the sphere, this identification is canonical, i.e., does not depend on any particular choices of bases or coordinates.

2.1.5. Local Coordinate Formulas, Differentials, and the Tangent Bundle. Finally let us use coordinates to specify a basis for the tangent space. Fix $p \in M$ and a coordinate system x^i around p . For any $f \in C_p(M)$ write $f = F(x^1, \dots, x^n)$ and define

$$\frac{\partial f}{\partial x^i} = \frac{\partial F}{\partial x^i}.$$

The map $f \mapsto \frac{\partial f}{\partial x^i}(p)$ is a derivation on $C_p(M)$. We denote it by $\frac{\partial}{\partial x^i}|_p$. These tangent vectors in fact form a basis as we saw that

$$D(f) = D(x^i) \frac{\partial f}{\partial x^i}|_p$$

i.e.,

$$D = v^i \frac{\partial}{\partial x^i} \Big|_p,$$

where the components v^i are uniquely determined. Moreover, as

$$\frac{d(f \circ c)}{dt}(0) = \frac{\partial f}{\partial x^i} \Big|_p \frac{d(x^i \circ c)}{dt}(0)$$

we also get this as a natural basis if we use curves to define the tangent space.

DEFINITION 2.1.15. The cotangent space T_p^*M to M at $p \in M$ is the vector space of linear functions on T_pM . Alternately this can also be defined as the quotient space $C_p^0(M) / (C_p^0(M))^2$ without even referring to tangent vectors.

Using coordinates we obtain a natural dual basis dx^i satisfying

$$dx^i \left(\frac{\partial}{\partial x^j} \right) = \frac{\partial x^i}{\partial x^j} = \delta_j^i.$$

In particular,

$$dx^i(v) = dx^i \left(v^j \frac{\partial}{\partial x^j} \right) = v^i$$

calculates the i^{th} coordinate of a vector.

We also obtain a natural set of transformation laws when we have another coordinate system y^i around p :

$$dy^i = \frac{\partial y^i}{\partial x^j} dx^j$$

and

$$\frac{\partial}{\partial y^i} = \frac{\partial x^j}{\partial y^i} \frac{\partial}{\partial x^j}.$$

Here the matrices $\left[\frac{\partial y^i}{\partial x^j} \right]$ and $\left[\frac{\partial x^j}{\partial y^i} \right]$ have entries that are functions on the common domain of the charts and are inverses of each other. These are also the natural transformation laws for a change of basis as well as the change of the dual basis.

The differential d also has a coordinate free definition. Let $f \in C_p(M)$, then we can define $df \in T_p^*M$ by

$$df(v) = D_v f = \frac{d(f \circ c)}{dt}(0)$$

if c is a curve with $\dot{c}(0) = v$. In coordinates we already know that

$$df(v) = \frac{\partial f}{\partial x^i} v^i$$

so in fact

$$df = \frac{\partial f}{\partial x^i} dx^i.$$

This shows that our definition of dx^i is consistent with the more abstract definition and that the transformation law for switching coordinates is simply just the law of how to write a vector or co-vector out in components with respect to a basis.

It now becomes very simple to define a differentiable structure on the tangent bundle TM . This space is the disjoint union of the tangent spaces T_pM where $p \in M$. There is also a natural base point projection $p : TM \rightarrow M$ that takes a vector in T_pM to its base point p . Starting with a differentiable system $\{f^i\}$ for M , we obtain a differentiable system $\{f^i \circ p, df^i\}$ for TM . Moreover, when f^1, \dots, f^n form a chart on $U \subset M$, then

$f^1 \circ p, \dots, f^n \circ p, df^1, \dots, df^n$ form a chart on TU . This takes us full circle back to our preliminary definition of tangent vectors.

IMPORTANT: The isomorphism between T_pM and \mathbb{R}^n depends on a choice of coordinates and is not canonically defined. We just saw that in a coordinate system we have a natural identification

$$TU \rightarrow U \times \mathbb{R}^n$$

which for fixed $p \in U$ yields a linear isomorphism

$$T_pU \rightarrow \{p\} \times \mathbb{R}^n \simeq \mathbb{R}^n.$$

However, this does not mean that TM has a natural map to $M \times \mathbb{R}^n$, that is a linear isomorphism when restricted to tangent spaces. Manifolds that admit such maps are called *parallelizable*. Euclidean space is parallelizable as are all matrix groups. But, as we shall see, S^2 is not parallelizable.

2.2. Derivatives of Maps and Vector Fields

2.2.1. Derivatives of Maps. Given a smooth function $F : M \rightarrow N$ we obtain a *derivative* or *differential* $DF|_p : T_pM \rightarrow T_{F(p)}N$. If we let $D = v = \dot{c}(0) \in T_pM$ represent a tangent vector, then there are three ways of writing this more explicitly:

$$DF|_p(D) = D \circ F^*,$$

$$D_{DF|_p(v)}f = D_v(f \circ F),$$

$$DF|_p(v) = \left. \frac{d(F(c(t)))}{dt} \right|_{t=0}.$$

When using coordinates around $p \in M$ we can also create the partial derivatives

$$\frac{\partial F}{\partial x^i} \in TN$$

as the velocities of the x^i -curves for $F \circ x^{-1}$, where the other coordinates are kept constant. In fact

$$\frac{\partial F}{\partial x^i}|_p = DF \left(\frac{\partial}{\partial x^i} \Big|_p \right).$$

Note that $\frac{\partial F}{\partial x^i}$ is a map from (a subset of) M to TN which at $p \in M$ is mapped to $T_{F(p)}N$. These partial derivatives represent the columns in a matrix representation for DF since

$$DF(v) = DF \left(\frac{\partial}{\partial x^i} v^i \right) = DF \left(\frac{\partial}{\partial x^i} \right) v^i = \frac{\partial F}{\partial x^i} v^i.$$

If we also have coordinates around $F(p)$ in N , then

$$DF(v) = \frac{\partial F}{\partial x^i} v^i = \frac{\partial (y^j \circ F)}{\partial x^i} v^i \frac{\partial}{\partial y^j}.$$

So the matrix representation for DF is precisely the matrix of partial derivatives

$$[DF] = \left[\frac{\partial (y^j \circ F)}{\partial x^i} \right] = \left[\frac{\partial (y^j \circ F \circ x^{-1})}{\partial x^i} \right].$$

We can now reformulate what it means for a smooth function to be an immersion or submersion.

DEFINITION 2.2.1. The smooth function $F : M \rightarrow N$ is an immersion if $DF|_p$ is injective for all $p \in M$. It is a submersion if $DF|_p$ is surjective for all $p \in M$.

REMARK 2.2.2. When we consider a map $F : M \rightarrow \mathbb{R}^k$, then we also have a differential

$$dF = \begin{bmatrix} dF^1 \\ \vdots \\ dF^k \end{bmatrix} : TM \rightarrow \mathbb{R}^k.$$

The identification $I : \mathbb{R}^k \times \mathbb{R}^k \rightarrow T\mathbb{R}^k$ defined by $I(p, v) = \frac{d}{dt}(p + tv)|_{t=0}$ shows that $DF = I(F, dF)$.

EXAMPLE 2.2.3. Based on example 1.2.2 the tangent spaces to the orthogonal and unitary groups are given as the kernel of the differential of the map $A \rightarrow AA^*$ and are thus the skew-adjoint matrices

$$T_I O(n) = \{X \in \text{Mat}_{n \times n}(\mathbb{R}) \mid X^* = -X\} = \mathfrak{so}(n),$$

$$T_I U(n) = \{X \in \text{Mat}_{n \times n}(\mathbb{C}) \mid X^* = -X\} = \mathfrak{u}(n).$$

At a general point A in $O(n)$ or $U(n)$ we can write any curve as $c(t) = A\hat{c}(t)$, where $\hat{c}(0) = I$. In particular $\dot{c}(0) = AX$, where $X^* = -X$. Thus

$$T_A O(n) = \{AX \in \text{Mat}_{n \times n}(\mathbb{R}) \mid X^* = -X\},$$

$$T_A U(n) = \{AX \in \text{Mat}_{n \times n}(\mathbb{C}) \mid X^* = -X\}.$$

We can use differentials together with some elementary linear algebra to prove an interesting result for retracts. A *retraction* $F : M \rightarrow M$ is an idempotent map, $F \circ F = F$. Linear projections are examples of retractions. The image of a retraction is called a *retract* and we say that M retracts on to $F(M)$.

THEOREM 2.2.4. *Let M be a connected manifold. If $F : M \rightarrow M$ satisfies $F \circ F = F$, then the image $F(M) \subset M$ is a submanifold.*

PROOF. First note that the image is a connected closed subset. We start by showing that the rank is constant on the image. Any $p \in F(M)$ is clearly a fixed point of F and $DF|_p : T_p M \rightarrow T_p M$ satisfies $(DF|_p)^2 = DF|_p$. In particular, $DF|_p$ can only have eigenvalues 0 or 1 and $\text{tr} DF|_p = \text{rank} DF|_p$. As the image is connected this shows that the rank is a constant k .

For a general $x \in M$ with $F(x) = p$ we have

$$DF|_p \circ DF|_x = DF|_x$$

implying that $\text{rank} DF \leq k$ on M . On the other $\text{rank} DF \geq k$ is an open condition so there must be an open set containing $F(M)$ where $\text{rank} DF = k$.

We use theorem 1.4.21 to show that $F(M)$ is an embedded submanifold. This is not an immediate consequence of the theorem as any immersion has constant rank. We select coordinates on a common neighborhood U around $p \in F(M)$ such that

$$y \circ F \circ x^{-1}(x^1, \dots, x^n) = (x^1, \dots, x^k, 0, \dots, 0).$$

The crucial point is that F is the identity map on its own image. First note that in U the image lies in the slice

$$y^i = 0, \text{ for } i = k+1, \dots, n.$$

Conversely, the points $z \in U$ where $y^i(z) = 0$, $i = k+1, \dots, n$ lie in the image as

$$F \circ x^{-1}(x^1, \dots, x^n) = y^{-1}(x^1, \dots, x^k, 0, \dots, 0).$$

Thus

$$U \cap F(M) = \{z \in U \mid y^i(z) = 0, i = k+1, \dots, n\}$$

□

2.2.2. Vector Fields. A *vector field* is a smooth map (called a *section*) $X : M \rightarrow TM$ such that $X|_p \in T_pM$. We use $X|_p$ instead of $X(p)$ as X is a map that can also be evaluated on functions. In fact, we obtain a derivation

$$D_X : C^\infty(M) \rightarrow C^\infty(M)$$

by defining

$$(D_X f)(p) = D_{X|_p} f.$$

Most books use the notation

$$X : C^\infty(M) \rightarrow C^\infty(M)$$

and

$$X(f)(p) = X|_p f = D_{X|_p} f.$$

Conversely, any such derivation corresponds to a vector field in the same way that tangent vectors correspond to derivations at a point.

In local coordinates we obtain

$$X = D_X(x^i) \frac{\partial}{\partial x^i}.$$

Given two vector fields X and Y we can construct their *Lie bracket* $[X, Y]$ implicitly as a derivation

$$D_{[X, Y]} = D_X D_Y - D_Y D_X = [D_X, D_Y].$$

This clearly defines a linear map and is a derivation as

$$\begin{aligned} D_{[X, Y]}(fg) &= D_X(gD_Y f + fD_Y g) - D_Y(gD_X f + fD_X g) \\ &= D_X g D_Y f + D_X f D_Y g + g D_X D_Y f + f D_X D_Y g \\ &\quad - D_Y g D_X f - D_Y f D_X g - g D_Y D_X f - f D_Y D_X g \\ &= g [D_X, D_Y] f + f [D_X, D_Y] g. \end{aligned}$$

In local coordinates this is conveniently calculated by ignoring second order partial derivatives:

$$\begin{aligned} \left[X^i \frac{\partial}{\partial x^i}, Y^j \frac{\partial}{\partial x^j} \right] &= X^i \frac{\partial Y^j}{\partial x^i} \frac{\partial}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j} \frac{\partial}{\partial x^i} \\ &\quad + X^i Y^j \frac{\partial^2}{\partial x^i \partial x^j} - Y^j X^i \frac{\partial^2}{\partial x^j \partial x^i} \\ &= X^i \frac{\partial Y^j}{\partial x^i} \frac{\partial}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j} \frac{\partial}{\partial x^i} \\ &= \left(X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j} \right) \frac{\partial}{\partial x^i}. \end{aligned}$$

Since tangent vectors are also velocities to curves it would be convenient if vector fields had a similar interpretation. A curve $c(t)$ such that

$$\dot{c}(t) = X|_{c(t)}$$

is called an *integral curve* for X . Given an initial value $p \in M$, there is in fact a unique integral curve $c(t)$ such that $c(0) = p$ and it is defined on some maximal interval I that contains 0 as an interior point.

In local coordinates we can write $x^i \circ c(t) = x^i(t)$ and $X = X^i \frac{\partial}{\partial x^i}$. The condition that c is an integral curve then comes down to

$$\dot{c}(t) = \frac{dx^i}{dt} \frac{\partial}{\partial x^i} = X^i \frac{\partial}{\partial x^i}$$

or

$$\frac{dx^i}{dt}(t) = X^i(c(t)).$$

This is a first order ODE and as such will have a unique solution given an initial value.

To get a maximal interval for an integral curve we have to use the local uniqueness of solutions and patch them together through a covering of coordinate charts.

We state the main theorem on integral curves that will be used throughout the notes.

THEOREM 2.2.5. *Let X be a vector field on a manifold M . For each $p \in M$ there is a unique integral curve $c_p(t) : I_p \rightarrow M$ where $c_p(0) = p$, $\dot{c}_p(t) = X_{c_p(t)}$ for all $t \in I_p$, and I_p is the maximal open interval for any curve satisfying these two properties. Moreover, the map $(t, p) \mapsto c_p(t)$ is defined on an open subset of $\mathbb{R} \times M$ and is smooth. Finally, for given $p \in M$ the interval I_p either contains $[0, \infty)$ or $c_p(t)$ is not contained in a compact set as $t \rightarrow b$, for some $b < \infty$.*

PROOF. The first part is simply existence and uniqueness of solutions to ODEs. The second part is that such solutions depend smoothly on initial data. The last statement is a basic compactness argument. \square

We use the general notation that $\Phi^t(p) = \Phi_X^t(p) = c_p(t)$ is the flow corresponding to a vector field X , i.e.

$$\frac{d}{dt} \Phi_X^t = X|_{\Phi_X^t} = X \circ \Phi_X^t.$$

REMARK 2.2.6. If we have a smooth family of vector fields $X_\lambda : L \times M \rightarrow TM$, $\lambda \in L$, then the corresponding flows $\Phi_{X_\lambda}^t$ are also smooth with respect to λ .

Let $F : M^m \rightarrow N^n$ be a smooth map between manifolds. If X is a vector field on M and Y a vector field on N , then we say that X and Y are *F-related* provided $DF(X|_p) = Y|_{F(p)}$, or in other words $DF(X) = Y \circ F$. Given that tangent vectors are defined as derivations we note that it is equivalent to say that for all $f \in C^\infty(N)$ we have $(D_Y f) \circ F = D_X(f \circ F)$. In particular, when X_i are *F-related* to Y_i for $i = 1, 2$, it follows that $[X_1, X_2]$ is *F-related* to $[Y_1, Y_2]$.

We can also tie this concept to the integral curves for the vector fields.

PROPOSITION 2.2.7. *X and Y are F -related iff $F \circ \Phi_X^t = \Phi_Y^t \circ F$ whenever both sides are defined. In particular, then RHS is defined as long as the LHS is defined.*

PROOF. Assuming that $F \circ \Phi_X^t = \Phi_Y^t \circ F$ we have

$$\begin{aligned} DF(X) &= DF\left(\frac{d}{dt}\Big|_{t=0} \Phi_X^t\right) \\ &= \frac{d}{dt}\Big|_{t=0} (F \circ \Phi_X^t) \\ &= \frac{d}{dt}\Big|_{t=0} (\Phi_Y^t \circ F) \\ &= Y \circ \Phi_Y^0 \circ F \\ &= Y \circ F. \end{aligned}$$

Conversely $DF(X) = Y \circ F$ implies that

$$\begin{aligned} \frac{d}{dt}(F \circ \Phi_X^t) &= DF\left(\frac{d}{dt}\Phi_X^t\right) \\ &= DF\left(X|_{\Phi_X^t}\right) \\ &= Y|_{F \circ \Phi_X^t}. \end{aligned}$$

This shows that $t \mapsto F \circ \Phi_X^t$ is an integral curve for Y . At $t = 0$ it agrees with the integral curve $t \mapsto \Phi_Y^t \circ F$ so by uniqueness we obtain $F \circ \Phi_X^t = \Phi_Y^t \circ F$. \square

Let X be a vector field and $\Phi^t = \Phi_X^t$ the corresponding locally defined flow on a smooth manifold M . The derivative of a function $f : M \rightarrow \mathbb{R}$ in the direction of X is the first order term in a Taylor expansion:

$$f(\Phi^t(p)) = f(p) + t(D_X f)(p) + o(t)$$

or more succinctly

$$f \circ \Phi^t = f + tD_X f + o(t).$$

The *Lie derivative* of f in the direction of X measures how f changes along the flow. Consequently, $L_X f = D_X f$.

The Lie bracket $[X, Y]$ similarly turns out to be a first order term in a Taylor expansion. We wish to consider how Y changes along the flow of X , i.e., how $Y|_{\Phi^t}$, changes. However, this can't be directly compared to Y as the vectors live in different tangent spaces. Thus we look at the curve $t \mapsto D\Phi^{-t}(Y|_{\Phi^t(p)})$ that lies in $T_p M$. The derivative with respect to t is called the *Lie derivative*, $L_X Y$, and satisfies:

$$D\Phi^{-t}(Y|_{\Phi^t(p)}) = Y|_p + t(L_X Y)|_p + o(t).$$

This Lie derivative of a vector field is in fact the Lie bracket.

PROPOSITION 2.2.8. *For vector fields X, Y on M we have*

$$L_X Y = [X, Y].$$

PROOF. The Lie derivative satisfies

$$D\Phi^{-t}(Y|_{\Phi^t}) = Y + tL_X Y + o(t)$$

or equivalently

$$Y|_{\Phi^t} = D\Phi^t(Y) + tD\Phi^t(L_X Y) + o(t).$$

Consequently, it is natural to consider the directional derivative of a function f in the direction of $Y|_{\Phi^t} - D\Phi^t(Y)$.

$$\begin{aligned} D_{(Y|_{\Phi^t} - D\Phi^t(Y))} f &= D_{Y|_{\Phi^t}} f - D_{D\Phi^t(Y)} f \\ &= (D_Y f) \circ \Phi^t - D_Y(f \circ \Phi^t) \\ &= D_Y f + tD_X D_Y f + o(t) \\ &\quad - D_Y(f + tD_X f + o(t)) \\ &= t(D_X D_Y f - D_Y D_X f) + o(t) \\ &= tD_{[X, Y]} f + o(t). \end{aligned}$$

This shows that

$$\begin{aligned} L_X Y &= \lim_{t \rightarrow 0} \frac{Y|_{\Phi^t} - D\Phi^t(Y)}{t} \\ &= [X, Y]. \end{aligned}$$

□

Proposition 2.2.7 together with this new interpretation of Lie brackets leads to several equivalent conditions for when vector fields *commute*.

PROPOSITION 2.2.9. *Consider two vector fields X, Y on M . The following are equivalent:*

- (1) $\Phi_X^t \circ \Phi_Y^s = \Phi_Y^s \circ \Phi_X^t$,
- (2) $D\Phi_X^t(Y) = Y \circ \Phi_X^t$, i.e., Y is Φ_X^t -related to itself,
- (3) $[X, Y] = 0$ on M .

PROOF. The fact that (1) and (2) are equivalent follows from proposition 2.2.7. The fact that (2) implies (3) follows from

$$[X, Y] = \lim_{t \rightarrow 0} \frac{Y|_{\Phi_X^t} - D\Phi_X^t(Y)}{t}.$$

Conversely, consider the curve $c(t) = D\Phi_X^{-t}(Y|_{\Phi_X^t(p)}) \in T_p M$. Its velocity at t_0 is calculated by considering the difference:

$$\begin{aligned} D\Phi_X^{-t_0-h}(Y|_{\Phi_X^{t_0+h}(p)}) - D\Phi_X^{-t_0}(Y|_{\Phi_X^{t_0}(p)}) &= D\Phi_X^{-t_0}(D\Phi_X^{-h}(Y|_{\Phi_X^h(\Phi_X^{t_0}(p))})) - D\Phi_X^{-t_0}(Y|_{\Phi_X^{t_0}(p)}) \\ &= D\Phi_X^{-t_0}(D\Phi_X^{-h}(Y|_{\Phi_X^h(\Phi_X^{t_0}(p))}) - Y|_{\Phi_X^{t_0}(p)}) \\ &= D\Phi_X^{-t_0}(h[X, Y]|_{\Phi_X^{t_0}(p)} + o(h)) \\ &= o(h). \end{aligned}$$

Showing that the curve is constant and consequently that (2) holds provided the Lie bracket vanishes. □

2.3. Vector Bundles

We collect the most important constructions and concepts about vector bundles.

2.3.1. Vector Space Constructions. Given two vector spaces E and F over the same field we have the product $E \times F$, direct sum $E \oplus F$, and the linear maps or homomorphisms $\text{Hom}(E, F)$ between the vector spaces. In the special case where $F = \mathbb{F}$ is the field we obtain the dual of E , $E^* = \text{Hom}(E, \mathbb{F})$. When $E = F$ we also denote $\text{Hom}(E, E) = \text{End}(E)$, the space of endomorphisms of E . In case $E \subset F$ is a subspace we also have the quotient space F/E that consists of the equivalence classes $x + E$ for $x \in F$.

Note that the difference between direct sums and direct products only becomes apparent when we have infinitely many vector spaces E_α . In this case $\oplus_\alpha E_\alpha \subset \times_\alpha E_\alpha$ consists of the tuples (e_α) where all but a finite number of entries vanish.

A particularly useful construction is that of the tensor product $E \otimes F$. It has the property that there is a bilinear map

$$\begin{aligned} E \times F &\rightarrow E \otimes F \\ (v, w) &\mapsto v \otimes w \end{aligned}$$

such that $E \otimes F = \text{span}\{v \otimes w \mid v \in E, w \in F\}$. In particular, if e_α is a basis for E and f_β a basis for F , then $e_\alpha \otimes f_\beta$ is a basis for $E \otimes F$. The rigorous construction is somewhat clumsy as it requires a very big vector space even when both vector spaces are finite dimensional. Let U be the vector space with basis (v, w) , $v \in E$, $w \in F$ and $W \subset U$ the subspace generated by

$$(\lambda_1 v_1 + \lambda_2 v_2, w) - \lambda_1 (v_1, w) - \lambda_2 (v_2, w), \lambda_1, \lambda_2 \in \mathbb{F}, v_1, v_2 \in E, w \in F$$

and

$$(v, \mu_1 w_1 + \mu_2 w_2) - \mu_1 (v, w_1) - \mu_2 (v, w_2), \mu_1, \mu_2 \in \mathbb{F}, w_1, w_2 \in F, v \in E.$$

With these constructions we can define the *tensor product* as the quotient space: $E \otimes F = U/W$. The inclusion map $E \times F \rightarrow U$ that takes each pair (v, w) to a basis vector becomes a bilinear map when composed with the quotient map $U \rightarrow U/W$. If we denote the image of the basis vector (v, w) by $v \otimes w \in E \otimes F$, then it is clear that the tensor product is spanned by such elements. This takes us back to the desired properties.

Tensor products can be iterated and it is easy to check that $(E \otimes F) \otimes G$ and $E \otimes (F \otimes G)$ are canonically isomorphic. With this in mind we can create the k -fold tensor product $E^{\otimes k}$ with k factors of E . Selecting a basis e_α for E allows us to write the elements of $T \in E^{\otimes k}$ as linear combinations

$$\sum T^{\alpha_1 \cdots \alpha_k} e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_k}.$$

We say that T is skew-symmetric if $T^{\cdots \alpha \cdots \alpha \cdots} = 0$ whenever two indices are equal. The space of skew-symmetric elements is denoted by $\bigwedge^k E \subset E^{\otimes k}$. The skew-symmetric elements are the image of the projection:

$$\begin{aligned} \text{Alt} : E^{\otimes k} &\rightarrow E^{\otimes k} \\ v_1 \otimes \cdots \otimes v_k &\mapsto \frac{1}{k!} \sum_{\sigma \in S_k} \text{sign}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}. \end{aligned}$$

The averaging factor $k!$ ensures that $\text{Alt} \circ \text{Alt} = \text{Alt}$. The skew-symmetrization operation leads to the wedge product

$$\begin{aligned} \bigwedge^k E \times \bigwedge^l E &\rightarrow \bigwedge^{k+l} E \\ (w_1, w_2) &\mapsto w_1 \wedge w_2 = \frac{(k+l)!}{k!l!} \text{Alt}(w_1 \otimes w_2). \end{aligned}$$

It is not difficult to check that this defines an (associative) ring structure on: $\bigwedge^* E = \bigoplus_{k=0}^{\infty} \bigwedge^k E$. Note that with this choice of factor we obtain

$$v_1 \wedge \cdots \wedge v_k = k! \text{Alt}(v_1 \otimes \cdots \otimes v_k) = \sum_{\sigma \in S_k} \text{sign}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}.$$

When the indices α for a basis of E are totally ordered (usually by integers) we obtain a basis $e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_k}$, $\alpha_1 < \cdots < \alpha_k$. Thus

$$\dim \bigwedge^k E = \binom{\dim E}{k}$$

and, in particular, $\wedge^n E$ is 1-dimensional when $n = \dim E$. In this case a choice of basis e_1, \dots, e_n creates a nonzero element in $\wedge^n E$ and if $v_i = \alpha_i^j e_j$, $i = 1, \dots, n$, then

$$\begin{aligned}
v_1 \wedge \cdots \wedge v_n &= \sum_{j_1, \dots, j_n=1}^n \alpha_1^{j_1} \cdots \alpha_n^{j_n} e_{j_1} \wedge \cdots \wedge e_{j_n} \\
&= \sum_{\{j_1, \dots, j_n\} = \{1, \dots, n\}} \alpha_1^{j_1} \cdots \alpha_n^{j_n} e_{j_1} \wedge \cdots \wedge e_{j_n} \\
&= \sum_{\sigma \in S_n} \alpha_1^{\sigma(1)} \cdots \alpha_n^{\sigma(n)} e_{\sigma(1)} \wedge \cdots \wedge e_{\sigma(n)} \\
&= \sum_{\sigma \in S_n} \text{sign}(\sigma) \alpha_1^{\sigma(1)} \cdots \alpha_n^{\sigma(n)} e_1 \wedge \cdots \wedge e_n \\
&= \det \left[\alpha_i^j \right] e_1 \wedge \cdots \wedge e_n.
\end{aligned}$$

In the case E is a real finite dimensional vector space we say that two bases define the same *orientation* provided their transition matrix has positive determinant. This can conveniently be formulated as saying that

$$f_1 \wedge \cdots \wedge f_n = \lambda e_1 \wedge \cdots \wedge e_n$$

for an $\lambda > 0$.

One can similarly construct subspaces of symmetric elements spanned by

$$v_1 \odot \cdots \odot v_k = \sum_{\sigma \in S_k} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}$$

leading to $E^{\odot k} \subset E^{\otimes k}$.

2.3.2. Vector Bundles.

DEFINITION 2.3.1. A rank k vector bundle over M , $\pi : E \rightarrow M$, is a submersion such that all the preimages $E_p = \pi^{-1}(p)$ are vector spaces isomorphic to a fixed vector space \mathbb{F}^k . In addition, the vector space structures are compatible in the sense that the bundle is locally *trivial*: For every $p \in M$ there is a neighborhood $U \ni p$ such that $\pi^{-1}(U)$ is diffeomorphic to $U \times \mathbb{F}^k$ and for each $x \in U$ the diffeomorphism restricts to a vector space isomorphism $E_x \rightarrow \{x\} \times \mathbb{F}^k \rightarrow \mathbb{F}^k$:

$$\begin{array}{ccc}
\pi^{-1}(U) & \longrightarrow & U \times \mathbb{F}^k \\
\searrow & & \swarrow \\
& U &
\end{array}$$

A good way to check that a vector bundle is trivial over $U \subset M$ is to find sections $s_1, \dots, s_k : U \rightarrow E$, i.e., $\pi \circ s_i = id_U$, such that for each $x \in U$ the vectors $s_1(x), \dots, s_k(x)$ form a basis for E_x . The trivial product bundle $E = M \times \mathbb{F}^k$ is often denoted by $\varepsilon^k = \varepsilon^k(M)$.

The vector space constructions from above lead to similar constructions for vector bundles over a fixed manifold. In particular, we say that a real rank k vector bundle is *orientable* provided each fiber can be oriented in such a way that locally there are trivializations where the sections are equivalent to the given orientations. This can more succinctly be stated as saying that the 1-dimensional bundle $\wedge^k E$ is trivial.

A subbundle $E \subset F$ of a vector bundle over M is a submanifold such that $E \cap F_p \subset F_p$ is a subspace of dimension l for all $p \in M$. It is not immediately clear that subbundles are locally trivial, especially as subbundles of trivial bundles need not be trivial. Consider for example $TS^n \subset S^n \times \mathbb{R}^{n+1}$. Here the tangent bundle is not trivial unless $n = 1, 3, 7$ and as

we shall see only admits nowhere vanishing sections when n is odd. The normal bundle $T^\perp S^n = \{(p, v) \in S^n \times \mathbb{R}^{n+1} \mid v \in \text{span}\{p\}\}$ is however trivial and

$$TS^n \oplus T^\perp S^n = S^n \times \mathbb{R}^{n+1}$$

or

$$TS^n \oplus \varepsilon^1(S^n) \simeq \varepsilon^{n+1}(S^n).$$

PROPOSITION 2.3.2. *The subbundle of a trivial bundle is locally trivial. Consequently, subbundles of vector bundles are also vector bundles.*

PROOF. We consider a subbundle $E \subset M \times \mathbb{R}^k$. As each $E_p \subset \{p\} \times \mathbb{R}^k \simeq \mathbb{R}^k$ there is a unique orthogonal projection $\text{proj}_p : \mathbb{R}^k \rightarrow E_p$. Since the subbundle is a smooth submanifold it follows that

$$\begin{aligned} P : M \times \mathbb{R}^k &\rightarrow E \\ (p, v) &\mapsto (p, \text{proj}_p(v)) \end{aligned}$$

is smooth. Let e_1, \dots, e_k be the canonical basis for \mathbb{R}^k and consider the sections $s_i(p) = \text{proj}_p(e_i)$. These sections always span E_p . For a fixed p we can select l of these sections $s_{i_1}(p), \dots, s_{i_l}(p)$ to form a basis for E_p . When x is sufficiently near p these sections must still be linearly independent and thus form a basis for E_x . This shows that E is locally trivial. \square

A bundle map between vector bundles E and F over M is simply a section of $\text{Hom}(E, F)$. If the restrictions to the fibers are always isomorphisms, then we say that the bundles are isomorphic. In case of two vector bundles over different manifolds $\pi_i : E_i \rightarrow M_i$, $i = 1, 2$ a bundle map consists of a map $f : M_1 \rightarrow M_2$ and a lift $\bar{f} : E_1 \rightarrow E_2$ that is linear when restricted to fibers

$$\begin{array}{ccc} E_1 & \xrightarrow{\bar{f}} & E_2 \\ \downarrow & & \downarrow \\ M_1 & \xrightarrow{f} & M_2 \end{array}$$

Given $f : M_1 \rightarrow M_2$ we can construct a pull-back bundle $f^*E_2 \rightarrow M_1$ by letting the fiber over p be the fiber over $f(p)$:

$$f^*E_2 = \{(p, v) \in M_1 \times E_2 \mid f(p) = \pi_2(v)\}.$$

With this construction the bundle map \bar{f} between E_1 and E_2 can be thought of as a section of $\text{Hom}(E_1, f^*E_2)$.

PROPOSITION 2.3.3. *Every vector bundle admits a smoothly varying inner product on its fibers. In particular, every subbundle is the image of a bundle map that is a projection on to it.*

PROOF. If $\pi : F \rightarrow M$ is a vector bundle and M is covered by open sets U_α such that $\pi^{-1}(U_\alpha) \approx U_\alpha \times \mathbb{R}^k$, then we can select an inner product g_α on \mathbb{R}^k for each α . Next use paracompactness to select a partition of unity $\lambda_\alpha : M \rightarrow [0, \infty)$ subordinate to the cover. The sum $g = \sum_\alpha \lambda_\alpha g_\alpha$ then defines an inner product on each fiber. Note that g is a (smooth) section of $(F^*)^{\otimes 2}$.

When we have a subbundle $E \subset F$, then we can use the inner product g to define the orthogonal projection fiberwise for each $p \in M$. This gives a smooth global projection onto E .

The orthogonal complement to $E \subset F$ is called the *normal bundle* to E and denoted E^\perp . It clearly depends on the metric but it is easy to verify that all normal bundles are isomorphic. \square

The tangent bundle TM of a manifold is a vector bundle, as is the dual T^*M and their corresponding wedge products and symmetric products $\wedge^k TM$, $\wedge^k T^*M$, $T^{\odot k}M$, and $(T^*M)^{\odot k}$. Tensor products of these bundles are called *tensor bundles* and their sections *tensors*.

A section of $(T^*M)^{\otimes k}$ is called a $(0, k)$ -tensor and is a k -linear map on each tangent space. Sections of

$$\text{Hom}(T^{\otimes k}M, T^{\otimes l}M) = (T^*M)^{\otimes k} \otimes T^{\otimes l}M$$

are called (l, k) -tensors. Thus vector fields are $(1, 0)$ -tensors and functions are $(0, 0)$ -tensors. In general the sections of a vector bundle E are denoted by $\Gamma(E)$ and in a natural way form a module over $C^\infty(M)$. The space of vector fields on M are often denoted by $X(M)$. Sections of $\wedge^k T^*M$ are called k -forms and the space of all such sections is denoted by $\Omega^k(M)$. The space of all forms on M^m is the direct sum

$$\Omega^*(M) = \bigoplus_{k=0}^m \Omega^k(M).$$

With the usual addition and the wedge product this forms a ring.

Occasionally tensors are defined globally on vector fields, in this case it is necessary to have a criterion that guarantees that it is a tensor.

LEMMA 2.3.4. *A k -linear map $T : (X(M))^{\times k} \rightarrow (X(M))^{\times l}$ that is also $C^\infty(M)$ -linear in each variable is an (l, k) -tensor.*

PROOF. The subtlety here lies in showing that it can be evaluated on vectors in a fixed tangent space. To simplify notation assume $k = 1$. We have to show that if $X|_p = Y|_p$, then $T(X)|_p = T(Y)|_p$. We start by localizing: If $X = Y$ on an open set $U \subset M$, then $T(X) = T(Y)$ on U . For each $p \in U$ select a bump function $\lambda : M \rightarrow [0, 1]$ that is 1 on p and has support in U . Thus $\lambda X = \lambda Y$ on all of M . In particular,

$$T(X)|_p = \lambda(p)T(X)|_p = T(\lambda X)|_p = T(\lambda Y)|_p = \lambda(p)T(Y)|_p = T(Y)|_p.$$

Now assume that U is the domain for a coordinate system x^1, \dots, x^m and write

$$X|_U = \sum X^i \partial_i.$$

If we multiply the right-hand side by λ we obtain a globally defined vector field. We can further assume that $\lambda = 1$ on some smaller neighborhood $V \ni p$. Thus

$$T(X)|_p = T\left(\sum \lambda X^i \partial_i\right)|_p = \sum X^i(p)T(\partial_i)|_p.$$

This shows that $T(X)|_p$ only depends on $X|_p$. We can also use this construction to find a vector field that equals any $v \in T_p M$ by using $X = \sum \lambda v^i \partial_i$ where $v = \sum v^i \partial_i|_p$. \square

The two propositions were only established for real vector bundles, but, can without effort be generalized to complex vector bundles with the caveat that one should use Hermitian inner products instead of Euclidean inner products.

2.3.3. Bundles over Projective Spaces. In this short section we discuss projective spaces and their associated bundles.

The *tautological* or *canonical line bundle* is defined as

$$\tau(\mathbb{P}^n) = \{(p, v) \in \mathbb{P}^n \times \mathbb{F}^{n+1} \mid v \in p\}.$$

This is a natural subbundle of the trivial vector bundle $\mathbb{P}^n \times \mathbb{F}^{n+1}$ and consequently has a natural orthogonal complement

$$\tau^\perp(\mathbb{P}^n) \simeq \{(p, v) \in \mathbb{P}^n \times \mathbb{F}^{n+1} \mid p \perp v\}$$

Note that in the complex case we are using Hermitian orthogonality. These are related to the tangent bundle in an interesting fashion. From our coordinatization around a point $p \in \mathbb{P}^n$ as in subsection 1.2.4 where we think of $p \subset \mathbb{F}^{n+1}$ as a 1-dimensional subspace we see that

$$T_p \mathbb{P}^n \simeq \text{Hom}(p, p^\perp)$$

and globally

$$T\mathbb{P}^n \simeq \text{Hom}(\tau(\mathbb{P}^n), \tau^\perp(\mathbb{P}^n)).$$

For each $p \in \mathbb{P}^n$ these bundles are trivial over the coordinate neighborhood $\mathbb{P}^n - \mathbb{P}(p^\perp)$.

The maps $\tau(\mathbb{P}^n) \rightarrow \mathbb{P}^n$ and $\mathbb{F}^{n+1} - \{0\} \rightarrow \mathbb{P}^n$ seem suspiciously similar. The fiber over the former is p while the latter has fiber $p - \{0\}$. Thus the latter map can be identified with the nonzero vectors in $\tau(\mathbb{P}^n)$. In other words the missing 0 in $\mathbb{F}^{n+1} - \{0\}$ is replaced by the zero section in $\tau(\mathbb{P}^n)$ in order to create a larger bundle. This process is called a *blow up* of the origin in \mathbb{F}^{n+1} . Essentially we have a map $\tau(\mathbb{P}^n) \rightarrow \mathbb{F}^{n+1}$ that maps the zero section to 0 and is otherwise a bijection. The map $\mathbb{F}^{n+1} - \{0\} \rightarrow \mathbb{P}^n$ when restricted to the unit sphere $S \subset \mathbb{F}^{n+1} - \{0\}$ is called a *Hopf fibration*.

The conjugate to the tautological bundle can also be seen internally in \mathbb{P}^{n+1} as the map

$$\mathbb{P}^{n+1} - \{p\} \rightarrow \mathbb{P}^n$$

where $\mathbb{P}^{n+1} - \{p\}$ is a tubular neighborhood of $\mathbb{P}^n \subset \mathbb{P}^{n+1}$. When $p = [1 : 0 : \dots : 0]$ this map is given by

$$[z : z^0 : \dots : z^n] \rightarrow [z^0 : \dots : z^n]$$

and looks like a vector bundle if we use fiberwise addition and scalar multiplication in the z -variable.

The equivalence is obtained by mapping

$$\mathbb{P}^{n+1} - \{[1 : 0 : \dots : 0]\} \rightarrow \tau(\mathbb{P}^n),$$

$$[z : z^0 : \dots : z^n] \rightarrow \left([z^0 : \dots : z^n], \bar{z} \frac{(z^0, \dots, z^n)}{|(z^0, \dots, z^n)|^2} \right)$$

It is necessary to conjugate z since $[z : z^0 : \dots : z^n]$ and $[\lambda z : \lambda z^0 : \dots : \lambda z^n]$ have to map to the same vector

$$\bar{z} \frac{(z^0, \dots, z^n)}{|(z^0, \dots, z^n)|^2} = \overline{\lambda z} \frac{(\lambda z^0, \dots, \lambda z^n)}{|(\lambda z^0, \dots, \lambda z^n)|^2}.$$

The conjugate to the tautological bundle can also be identified with the dual bundle $\text{Hom}(\tau(\mathbb{P}^n), \mathbb{C})$ via the natural inner product structure coming from $\tau(\mathbb{P}^n) \subset \mathbb{P}^n \times \mathbb{F}^{n+1}$. The relevant linear functional corresponding to $[z : z^0 : \dots : z^n]$ is given by

$$v \rightarrow \left\langle v, \bar{z} \frac{(z^0, \dots, z^n)}{|(z^0, \dots, z^n)|^2} \right\rangle.$$

This functional appears to be defined on all of \mathbb{F}^{n+1} , but, as it vanishes on the orthogonal complement to (z^0, \dots, z^n) we only need to consider the restriction to $\text{span}\{(z^0, \dots, z^n)\} = [z^0 : \dots : z^n]$.

2.4. Frobenius

The Frobenius theorem sets up a correspondence between suitable subbundles of the tangent bundle and decompositions of the manifold into equivalence classes of submanifolds of the same dimension. The original motivation comes from partial differential equations where certain overdetermined problems can be solved if the PDE satisfies suitable integrability conditions. The theory will be used later to set up a complete correspondence between Lie subgroups of a Lie group and subalgebras of the Lie algebra of the group.

2.4.1. The Local Theory. We begin with a motivational example. It's a classical problem going back to Clairaut to ask when a 1-form is an exact differential

$$Pdx + Qdy = du?$$

An obvious necessary condition is that $\partial_y P = \partial_x Q$, often called Clairaut's theorem. However, Clairaut more importantly showed that it was a sufficient condition, at least locally. The original question is an example of an overdetermined partial differential equation and the condition that guarantees solutions is called the *integrability condition*.

This problem can also be formulated in a slightly different manner. If we consider the graph of u as a submanifold in $\mathbb{R}^2 \times \mathbb{R}$, then the vector fields $\partial_x + P\partial_z$ and $\partial_y + Q\partial_z$ form a basis for the tangent space to this submanifold. Their Lie bracket $(\partial_x Q - \partial_y P)\partial_z$ must also be tangent to this submanifold. As the vector is vertical this is only possible when $\partial_x Q - \partial_y P = 0$. Note that any submanifold whose tangent space is spanned by $\partial_x + P\partial_z$ and $\partial_y + Q\partial_z$ is locally the graph of a function. In this way we have a more geometric interpretation of the original question.

The most general overdetermined system of PDEs that can easily be handled this way are of the following form. Given $P_k^i(x, u) = P_k^i(x^1, \dots, x^n, u^1, \dots, u^m)$, where $i = 1, \dots, m$ and $k = 1, \dots, n$ we consider the initial value problems for a system of first order PDEs $u(x)$:

$$\begin{aligned} \frac{\partial u^i}{\partial x^k} &= P_k^i(x, u(x)) \\ u(x_0) &= u_0 \end{aligned}$$

Since this implies that

$$\frac{\partial^2 u^i}{\partial x^k \partial x^l} = \frac{\partial P_l^i}{\partial x^k} + \frac{\partial P_l^i}{\partial u^j} P_k^j$$

an obvious necessary condition becomes

$$\frac{\partial P_l^i}{\partial x^k} + \frac{\partial P_l^i}{\partial u^j} P_k^j = \frac{\partial P_k^i}{\partial x^l} + \frac{\partial P_k^i}{\partial u^j} P_l^j.$$

Again this also becomes a sufficient condition for local existence. We can also reinterpret this as a graph/submanifold problem in $\mathbb{R}^n \times \mathbb{R}^m$. Here the tangent vector fields are $\frac{\partial}{\partial x^k} + \sum_{i=1}^m P_k^i \frac{\partial}{\partial u^i}$, $k = 1, \dots, n$ and their Lie brackets again become vertical vector fields

$$\sum_{i=1}^m \left(\frac{\partial P_l^i}{\partial x^k} + \sum_{j=1}^m \frac{\partial P_l^i}{\partial u^j} P_k^j - \frac{\partial P_k^i}{\partial x^l} - \sum_{j=1}^m \frac{\partial P_k^i}{\partial u^j} P_l^j \right) \frac{\partial}{\partial u^i}.$$

This leads to the following general question: When will a collection of linearly independent vector fields span the tangent space of a submanifold? With the now obvious answer being that their Lie brackets must be in the span of the vector fields.

While the question for functions can't always be solved globally it is possible to come up with a global solution for submanifolds. This requires a few more notions. A *distribution* D for a manifold M is a subbundle $D \subset TM$. We say that D is *involutive* if any two vector fields X, Y with values in D , $X, Y \in D$, have the property that $[X, Y] \in D$. The distribution is *integrable* if locally there is a connected submanifold $N \subset M$ through every point in M such that $TN = D|_N$. The submanifold N is called an *integral submanifold*. Clearly integrable distributions are involutive and we will show that the converse is true as well.

THEOREM 2.4.1 (Frobenius). *Let M be an m -manifold with a distribution D of rank k . If D is involutive, then it is integrable.*

PROOF. We show that in a neighborhood of a point $p \in M$ there exists a coordinate system (z^1, \dots, z^m) such that $D = \text{span} \left\{ \frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^k} \right\}$. The integral submanifolds will then be given as the slices:

$$(z^1, \dots, z^k) \mapsto (z^1, \dots, z^k, a^{k+1}, \dots, a^m)$$

where a^{k+1}, \dots, a^m are fixed. This implies the local existence and uniqueness of integral submanifolds.

To construct the coordinate system start by selecting coordinates (x^1, \dots, x^m) in a neighborhood of p such that

$$D_p = \text{span} \left\{ \frac{\partial}{\partial x^1} \Big|_p, \dots, \frac{\partial}{\partial x^k} \Big|_p \right\}$$

and $x^i(p) = 0, i = 1, \dots, m$. For the rest of the proof we can then work in suitable neighborhoods of $0 \in \mathbb{R}^m$.

Let π be the projection

$$(x^1, \dots, x^m) \mapsto (x^1, \dots, x^k).$$

This is a submersion and as $D\pi(D_p) = D_p$ we can shrink the neighborhood so that $D\pi|_{D_x}$ is an isomorphism for all x . We can then construct unique vector fields Z_1, \dots, Z_k with values in D that are π -related to $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^k}$. This implies that $[Z_i, Z_j]$ are π -related to $0 = [\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}]$ (see proposition 2.2.7). However, as the distribution is involutive we must have that $[Z_i, Z_j] \in D$ showing that $[Z_i, Z_j] = 0$. The corresponding flows $\Phi_i^{t_i}$ will then commute (see proposition 2.2.9). This shows that

$$\frac{\partial}{\partial t^i} \left(\Phi_1^{t^1} \circ \dots \circ \Phi_k^{t^k} (x) \right) = Z_i.$$

We can then define a map

$$(z^1, \dots, z^m) \mapsto \Phi_1^{z^1} \circ \dots \circ \Phi_k^{z^k} (0, \dots, 0, z^{k+1}, \dots, z^m).$$

This is a local diffeomorphism near the origin and defines coordinates such that

$$\frac{\partial}{\partial z^i} = Z_i \in D, i = 1, \dots, k.$$

□

There are two special cases worth noting: When D is 1-dimensional the integral submanifolds are unparametrized curves. This is a geometric version of finding solutions to first order differential equations. When D has codimension 1 the integral submanifolds locally become the level sets of a function by way of the coordinates constructed in theorem 2.4.1.

EXAMPLE 2.4.2. We will exhibit some 2-dimensional distributions that are far from involutive.

In \mathbb{R}^3 consider two vector fields $X = \partial_x + P\partial_z$ and $Y = \partial_y + Q\partial_z$ where P and Q are functions of (x, y) . These span a 2-dimensional distribution and their Lie bracket is given by

$$Z = [X, Y] = (\partial_x Q - \partial_y P) \partial_z.$$

So if $P = 1$ and $Q = x$, then $Z = \partial_z$ and X, Y, Z are everywhere linearly independent.

In higher dimensions we can similarly consider $X = \partial_1 + \sum_{i=3}^n P^i \partial_i$ and $Y = \partial_2 + \sum_{i=3}^n Q^i \partial_i$ with P^i and Q^i only being functions of (x^1, x^2) . We start with

$$Z_1 = [X, Y] = \sum_{i=3}^n (\partial_1 Q^i - \partial_2 P^i) \partial_i$$

and then iterate Lie brackets k more times to get:

$$Z_k = [X, \dots [X, [X, Y]]] = \sum_{i=3}^n \partial_1^k (\partial_1 Q^i - \partial_2 P^i) \partial_i.$$

If now $P^i = 1$ and $Q^i = (x^1)^{i-2}$, then

$$Z_k = \sum_{i=k+2}^n (i-2) \cdots (i-2-(k-1)) x^{i-2-k} \partial_i$$

creating n linearly independent fields from two simple vector fields.

2.4.2. The Global Theory. Our final goal is to obtain a global picture of the integral submanifolds of an involutive distribution. This will be used to understand subgroups of Lie groups in a later chapter.

A k -dimensional *foliation* of a manifold M is a smooth equivalence relation ($R \subset M \times M$ is a smooth submanifold with the projections $\pi_i : M \times M \rightarrow M$ restricting to submersions $\pi_i : R \rightarrow M$), where the equivalence classes consist of connected immersed submanifolds (also called *leaves*) all of dimension k . The equivalence class that contains $p \in M$ can be identified with the slice $(\{p\} \times M) \cap R$. The normal form for a submersion (see proposition 1.4.17) then guarantees that the tangent spaces to a foliation is an integrable distribution of rank k . Again the converse is true, but, there are some subtle points.

THEOREM 2.4.3. *A rank k involutive distribution consists of the tangent spaces to a foliation.*

PROOF. Let D be a rank k involutive distribution on an m -manifold M .

We start by observing that the local uniqueness of integral submanifolds shows that if two integral submanifolds intersect, then their union is also an integral submanifold. In this way we can create maximal integral submanifolds through a point $p \in M$ by taking the union of all integral submanifolds that contain p . However, it is not clear that these maximal integral submanifolds are second countable. The topology on the maximal integral submanifold is defined locally via the locally defined embedded integral submanifolds from theorem 2.4.1.

To check that maximal integral submanifolds are second countable we cover M by a countable collection of slice charts $(x_\alpha^1, \dots, x_\alpha^m) : U_\alpha \rightarrow \mathbb{R}^m$ such that the first k coordinate vector fields always span D . Thus the embeddings

$$(x_\alpha^1, \dots, x_\alpha^k) \mapsto (x_\alpha^1, \dots, x_\alpha^k, a^{k+1}, \dots, a^m)$$

form integral submanifolds and maximal integral submanifolds are unions of such embeddings. We will show that only countably many embeddings from a given chart can be used for any integral submanifold. This will show that integral submanifolds must be second countable as there are only countably many charts.

Two points $p, q \in M$ lie in the same integral submanifold L if and only if there is a finite collection of charts U_{α_i} and connected integral submanifolds $L_i = \{(x_{\alpha_i}^1, \dots, x_{\alpha_i}^k, a_i^{k+1}, \dots, a_i^m)\} \subset U_{\alpha_i}$, $i = 1, \dots, k$, such that $p \in L_1$, $q \in L_k$ and $L_i \cap L_{i+1} \neq \emptyset$, $i = 1, \dots, k-1$. Consider a maximal integral submanifold $L \subset M$ and two embeddings

$$(x_\alpha^1, \dots, x_\alpha^k) \mapsto (x_\alpha^1, \dots, x_\alpha^k, a_i^{k+1}, \dots, a_i^m), i = 0, 1$$

with images in L . As two such embeddings can be connected via a finite collection of charts the intersection $L \cap U_\alpha$ can only consist of a countable number of such embeddings. \square

We finish with a crucial technical proposition about smoothness of maps in to submanifolds. The exercises to chapter 1 has examples of how this can go wrong.

PROPOSITION 2.4.4. *If $F : N \rightarrow M$ is smooth and the image lies in a leaf $L \subset M$ of a foliation, then $F : N \rightarrow L$ is also smooth.*

PROOF. Fix $p \in N$ and a coordinate chart U around $F(p) \in M$ such that $L \cap U$ consists of a countable collection of embeddings with connected images as described in theorem 2.4.3. Let V be a small connected neighborhood around $p \in N$ such that $F(V) \subset U$ (continuity of F). Since the image lies in L and is also connected, it can only lie in the embedding whose image contains $F(p)$. This shows that $F : N \rightarrow L$ is continuous and smooth. \square

2.5. Exercises

- (1) Show that if $E \subset F$ is a subbundle and E' is complementary to E , i.e., $F = E \oplus E'$, then E' is isomorphic to F/E .
- (2) Let E and F be vector spaces.
 - (a) Construct a canonical bilinear map

$$E \times F \rightarrow \text{Hom}(E^*, F).$$

- (b) Show that the corresponding linear map

$$E \otimes F \rightarrow \text{Hom}(E^*, F)$$

is an injection and an isomorphism when E is finite dimensional.

- (3) Show that a vector bundle over a compact manifold is always a subbundle of a trivial bundle. Hint: Use a partition of unity to map the bundle into a trivial bundle constructed from a finite covering of trivial bundles.
- (4) Let $E_1 \oplus E_2 \rightarrow M$ be the direct sum of two vector bundles over M . Show that if any two of the three bundles are orientable, then so is the third.
- (5) Show that the pull-back of a trivial bundle is trivial.

- (6) Let $F : M \rightarrow N$ be a smooth map and $q \in N$ a regular value. Show that the normal bundle to $F^{-1}(q) \subset M$ is trivial. Hint: Show that it is isomorphic to the pull-back bundle of $T_q N \rightarrow \{q\}$ via $F|_{F^{-1}(q)} : F^{-1}(q) \rightarrow \{q\}$.
- (7) A manifold M^n is said to be *parallelizable* iff the tangent bundle is trivial, $TM \simeq M \times \mathbb{R}^n$. Show that M is parallelizable if and only if there are n vector fields that span the tangent space at every point.
- (8) Let $\pi_i : E_i \rightarrow M_i, i = 1, 2$ be two vector bundles and consider the product bundle

$$\pi_1 \times \pi_2 : E_1 \times E_2 \rightarrow M_1 \times M_2.$$

- (a) Show that if $M_1 = M_2$, then $E_1 \oplus E_2$ is isomorphic to the pull-back of $E_1 \times E_2$ via the diagonal map $x \mapsto (x, x)$.
- (b) Let $F_i : M_1 \times M_2 \rightarrow M_i$ denote the projections. Show that $E_1 \times E_2$ is isomorphic to $F_1^*(E_1) \oplus F_2^*(E_2)$.
- (9) Show that $\mathbb{R} \times S^n$ and $S^1 \times S^n$ are parallelizable.
- (10) Show that $S^p \times S^q$ is parallelizable if p or q is odd. Hint: $S^{2n-1} \subset \mathbb{C}^n$ admits a unit vector field.
- (11) Show that S^3 is parallelizable. Hint: Use quaternions as in section 4.2.5 to check that $T_p S^3 = \text{span}\{ip, jp, kp\}$.
- (12) Let E and F be vector bundles over M .
- (a) Show that a section of $\text{Hom}(E, F)$ defines a $C^\infty(M)$ linear map $\Gamma(E) \rightarrow \Gamma(F)$.
- (b) Show that every $C^\infty(M)$ -linear map $\Gamma(E) \rightarrow \Gamma(F)$ comes from a section of $\text{Hom}(E, F)$.
- (13) Show that the Lie bracket $\mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X}$ is not a tensor when the dimension is ≥ 2 .
- (14) Show that a real or complex Grassmannian $G_k(V)$ admits a natural embedding

$$\text{span}\{v_1, \dots, v_k\} \mapsto \text{span}\{v_1 \wedge \dots \wedge v_k\}$$

into the projective space $\mathbb{P}(\wedge^k V)$. This is called the Plücker embedding.

- (15) Let D be a rank k distribution on M^m .
- (a) Show that locally there is a trivialization of the tangent bundle X_1, \dots, X_m such that X_1, \dots, X_k span D .
- (b) Let $\omega^1, \dots, \omega^m$ be the 1-forms dual to X_1, \dots, X_m , i.e., $\omega^i(X_j) = \delta_j^i$. Show that ω^i vanishes on D only when $i = k + 1, \dots, m$ and conclude that $D = \bigcap_{i=k+1}^m \ker \omega^i$.
- (c) Let $\mathcal{A} = \{\omega \in \Omega(M) \mid \omega|_D = 0\}$. Show that \mathcal{A} is an ideal that is locally generated by $\omega^{k+1}, \dots, \omega^m$, i.e. every element is locally of the form $\sum_{i=k+1}^m \phi_i \omega^i, \phi_i \in \Omega^*(U)$.
- (16) Use the gluing definition from exercise 1 in chapter 1 to define the tangent bundle using domains $U_i \times \mathbb{R}^n$ and suitable transitions $U_{ji} \times \mathbb{R}^n \rightarrow U_{ij} \times \mathbb{R}^n$ to give a gluing definition of the tangent bundle.
- (17) For a local trivialization $U_i \times \mathbb{F}^k$ of a k -dimensional vector bundle $E \rightarrow M$ show that there are gluing transitions $\phi_{ij} : U_i \cap U_j \rightarrow Gl_k(\mathbb{F})$ that satisfy $\phi_{ii} = I_k$ on U_i and $\phi_{ij} \circ \phi_{jk} = \phi_{ik}$ on $U_i \cap U_j \cap U_k$. Conversely, given such gluing transitions construct a k -dimensional vector bundle on M .

Global Theory of Smooth Maps

In this chapter we present a more global perspective of smooth maps and how they interact with the topology of the manifold. In the first section the focus is on covering maps and the fundamental group. The second section is about orientability which is also a global notion and uses covering spaces. The third section generalizes covering maps to fibrations and general quotient maps between manifolds. In the last section we present the Whitney embedding theorem and use it to establish the existence of tubular neighborhoods and approximations of continuous maps by smooth maps. The themes of this chapter are used and further expanded upon in chapter 5.

3.1. Covering Maps

We start with a more general result about proper maps.

LEMMA 3.1.1. *Let $F : M^m \rightarrow N^m$ be a smooth proper map. If $y \in N$ is a regular value, then there exists a neighborhood V around y such that $F^{-1}(V) = \bigcup_{x \in F^{-1}(y)} U_x$ where U_x are mutually disjoint and $F : U_x \rightarrow V$ is a diffeomorphism for all $x \in F^{-1}(y)$.*

PROOF. Use that y is regular to find neighborhoods W_x around $x \in F^{-1}(y)$ such that $F : W_x \rightarrow F(W_x)$ is a diffeomorphism. This shows that the the points in $F^{-1}(y)$ are isolated. Since the preimage is also compact it follows that it is a finite set. We can then shrink the neighborhoods W_x , if necessary, so that they are mutually disjoint. Finally, use proposition 1.4.20 to find an open neighborhood V of y such that $F^{-1}(V) \subset \bigcup_{x \in F^{-1}(y)} W_x$. We can then use $U_x = W_x \cap F^{-1}(V)$. \square

COROLLARY 3.1.2 (Fundamental Theorem of Algebra). *A complex polynomial $p(z) : \mathbb{C} \rightarrow \mathbb{C}$ that is not surjective is a constant map.*

PROOF. There can only be finitely many critical values. In particular, the complement of regular values $R \subset \mathbb{C}$ is open and connected. Lemma 3.1.1 shows that the number of points in $p^{-1}(y)$, $y \in R$, is locally constant and hence constant as R is connected. So if the preimages are empty, then $p(z)$ is forced to be constant. \square

DEFINITION 3.1.3. A continuous map $\pi : \bar{M} \rightarrow M$ between topological spaces is called a *covering map* if each point in M has a neighborhood that is *evenly covered*, i.e., for every $y \in M$ there is a neighborhood V around y such that $\pi^{-1}(V) = \bigcup_{x \in \pi^{-1}(y)} U_x$ where $\pi : U_x \rightarrow V$ is a homeomorphism and the sets U_x are pairwise disjoint. In other words: $\pi^{-1}(V)$ is homeomorphic to $\pi^{-1}(y) \times V$:

$$\begin{array}{ccc} \pi^{-1}(V) & \longrightarrow & \pi^{-1}(y) \times V \\ & \searrow & \swarrow \\ & V & \end{array}$$

It is generally convenient to assume that V is connected so that each U_x is a connected component of the preimage $\pi^{-1}(V) = \bigcup U_x$.

A *bundle map* between covering spaces over M is a map that commutes with the projections onto M :

$$\begin{array}{ccc} \bar{M} & \longrightarrow & \hat{M} \\ & \searrow & \swarrow \\ & M & \end{array}$$

Two coverings are *equivalent* provided there is a bundle map between them that is also a homeomorphism. A covering map that is evenly covered over all of M is called a *trivial covering*. Thus coverings are locally trivial.

Finally, in case M is a manifold we can pull a differentiable structure from M back to \bar{M} . In this way π becomes a local diffeomorphism since the differentiable structure on V induces each U_x with a differentiable structure via the identification $\pi : U_x \rightarrow V$. As for the topological properties inherited by \bar{M} it is clear that \bar{M} is Hausdorff when M is, however, it is not necessarily true that \bar{M} is second countable unless \bar{M} is connected (see lemma 3.1.8).

EXAMPLE 3.1.4. Examples of covering maps are

$$\exp(it) : \mathbb{R} \rightarrow S^1$$

and for any integer $n \neq 0$

$$\begin{array}{ccc} S^1 & \mapsto & S^1 \\ z & \mapsto & z^n. \end{array}$$

Both have the property that for any $q \in S^1$, the set $S^1 - \{q\}$, is evenly covered.

Finally,

$$\exp(z) : \mathbb{C} \rightarrow \mathbb{C}^* = \mathbb{C} - \{0\}$$

is a covering map where the complement of a single line emanating from the origin is evenly covered.

COROLLARY 3.1.5. If $\pi : \bar{M} \rightarrow M$ is a proper non-singular map with M connected, then π is a covering map.

A key concept for understanding covering maps is known as the lifting property. A *lift* of a continuous map $F : M \rightarrow N$ into the base of a covering map $\pi : \bar{N} \rightarrow N$ is a continuous map $\bar{F} : M \rightarrow \bar{N}$ such that $\pi \circ \bar{F} = F$. If $\bar{F}(x_0) = \pi(y_0)$, then we say that the lift goes through y_0 .

$$\begin{array}{ccc} & & \bar{N} \\ & \nearrow & \downarrow \\ M & \longrightarrow & N \end{array}$$

When F is smooth, then the lift is also forced to be smooth. Moreover, when the covering is trivial:

$$\begin{array}{ccc} \bar{N} & \longrightarrow & \pi^{-1}(F(x_0)) \times N \\ & \searrow & \swarrow \\ & N & \end{array}$$

then F has a lift through any $y_0 \in \pi^{-1}(F(x_0))$.

PROPOSITION 3.1.6. If M is connected, $x_0 \in M$, and $y_0 \in \bar{N}$ such that $F(x_0) = \pi(y_0)$, then there is at most one lift \bar{F} such that $\bar{F}(x_0) = y_0$.

PROOF. Assume that we have two lifts F_1 and F_2 with this property and let $A = \{x \in M \mid F_1(x) = F_2(x)\}$. Clearly A is non-empty and closed. The covering map property shows that A is open. So when M is connected $A = M$. \square

PROPOSITION 3.1.7. *If $F : M \rightarrow N$ is a continuous map and $\pi : \bar{N} \rightarrow N$ is a covering map, then the pull-back*

$$F^*\bar{N} = \{(p, \bar{q}) \in M \times \bar{N} \mid F(p) = \pi(\bar{q})\} \subset M \times \pi^{-1}(F(M))$$

is a covering of M with the covering map, $F^\pi : F^*\bar{N} \rightarrow M$, being the restriction of the projection onto M . In particular, we obtain a commutative diagram of coverings:*

$$\begin{array}{ccc} F^*\bar{N} & \rightarrow & \bar{N} \\ \downarrow & & \downarrow \\ M & \rightarrow & N \end{array}$$

PROOF. Fix $p_0 \in M$ and select a neighborhood $V \ni q_0 = F(p_0)$ that is evenly covered. We claim that $U = F^{-1}(V)$ is evenly covered in $F^*\bar{N}$. The preimage is

$$(F^*\pi)^{-1}(U) = \{(p, \bar{q}) \in U \times \pi^{-1}(F(U)) \mid F(p) = \pi(\bar{q})\}$$

Since $\pi^{-1}(V) \simeq \pi^{-1}(q_0) \times V$ it follows that $\pi^{-1}(F(U)) \simeq \pi^{-1}(q_0) \times F(U)$ and

$$(F^*\pi)^{-1}(U) \simeq \pi^{-1}(q_0) \times U.$$

□

LEMMA 3.1.8. *Let $\pi : \bar{M} \rightarrow M$ be a covering. If M is second countable and \bar{M} is connected, then \bar{M} is also second countable.*

PROOF. Fix $q \in M$ and $p \in \bar{M}$ such that $\pi(p) = q$. We select a countable basis, \mathcal{V} , for the topology of M such each element in \mathcal{V} is evenly covered and a countable dense subset $A \subset M$. Consider the countable set \mathcal{S} of finite strings

$$(V_0, q_1, V_1, \dots, q_p, V_n)$$

where $V_i \in \mathcal{U}$, $q \in V_0$, and $q_i \in V_{i-1} \cap V_i \cap A$, $i = 1, \dots, n$. Here V_0 is evenly covered so there is a unique $U_0 \subset \bar{M}$ that is homeomorphic to V_0 and contains p . Given $V_0, q_1, \dots, q_k, V_k$ construct p_{k+1}, U_{k+1} such that $p_{k+1} \in U_k$ and $\pi(p_{k+1}) = q_{k+1}$ and then U_{k+1} as the unique set that is homeomorphic to V_{k+1} and contains p_{k+1} . This gives us a unique finite string $(U_0, p_1, \dots, p_n, U_n)$ for each element in \mathcal{S} . We claim that the collection, \mathcal{U} , of sets U_n constructed this way cover \bar{M} and hence create a countable basis for the topology. The union, O , of the sets in \mathcal{U} is clearly open. Let $\bar{x} \in \bar{O}$ be a point in the closure. Select an open set $U \ni \bar{x}$ that is diffeomorphic to a set $V \in \mathcal{V}$. The intersection $O \cap U$ is nonempty so there exists a sequence $(U_0, p_1, \dots, p_n, U_n)$ such that $U_n \cap U \neq \emptyset$. We can then choose $p_{n+1} \in U_n \cap U$ such that $\pi(p_{n+1}) \in V_n \cap V \cap A$ and obtain a string $(U_0, p_1, \dots, U_n, p_{n+1}, U)$ where $\bar{x} \in U$ and in particular $\bar{x} \in O$. □

We next address the properties that force all coverings of a space to be trivial.

DEFINITION 3.1.9. We say that two maps $F_0, F_1 : M \rightarrow N$ are homotopic if there is a continuous map $H : [0, 1] \times M \rightarrow N$ such that $F_0(x) = H(0, x)$ and $F_1(x) = H(1, x)$. In the case of smooth manifolds, smoothness of such a homotopy near the boundary points hasn't been defined yet. However, using a smooth function $\lambda : \mathbb{R} \rightarrow [0, 1]$ where $\lambda|_{(-\infty, 0]} \equiv 0$ and $\lambda|_{[1, \infty)} \equiv 1$ we can alter any homotopy to a new "homotopy" where $t \in \mathbb{R}$:

$$H(\lambda(t), x) : \mathbb{R} \times M \rightarrow M.$$

This is useful not only for defining smooth homotopies but also when M has boundary as the product $[0, 1] \times M$ is not a manifold with boundary in this case (see section 5.1 for the definition of manifolds with boundary). Moreover, composing with λ forces the homotopy to be stationary in the t -direction at $t = 0, 1$. Thus we can smoothly concatenate

homotopies provided $H_1(1, x) = H_2(0, x)$. In particular, maps being smoothly homotopic is an equivalence relation.

DEFINITION 3.1.10. Curves are very simple homotopies between maps from a one point space. Thus curves can easily be concatenated to smooth curves if we don't care about how they are parametrized. The equivalence classes of points created by curves are the *path connected components* of a space. We say that a space is path connected if any two points can be joined by a curve. A space is *simply connected* if it is path connected and any closed curve is homotopic to a constant map.

DEFINITION 3.1.11. We say that a connected manifold (or locally connected topological space), M , has the *unique lifting property* provided any covering $\pi : \bar{M} \rightarrow M$ is trivial. A *universal covering* $\tilde{M} \rightarrow M$, is a covering where \tilde{M} where either \tilde{M} is simply connected or has the unique lifting property.

This definition is in line with the treatment in [**Chevalley**] and avoids the discussion of the fundamental group as a set of homotopy classes of loops at a point. For a more traditional approach see, e.g., [**Hatcher1**]. For manifolds the two approaches are equivalent after one shows that universal coverings exist in both approaches.

PROPOSITION 3.1.12. *Let $\pi : \bar{N} \rightarrow N$ be a covering map. If M has the unique lifting property, then any continuous $F : M \rightarrow N$ has lift through any $\bar{y}_0 \in \pi^{-1}(F(x_0))$.*

PROOF. The pull-back covering $F^*\pi : F^*\bar{N} \rightarrow M$ is trivial. The lift is then simply the section $M \rightarrow F^*\bar{N} \simeq (F^*\pi)^{-1}(x_0) \times M$ whose image contains (x_0, \bar{y}_0) . \square

Next we show that simple connectivity implies that all covers are trivial.

THEOREM 3.1.13. *If M is simply connected, then it has the unique lifting property.*

PROOF. Assume that $\pi : \bar{M} \rightarrow M$ is a covering and fix $x_0 \in M$. We need to show that the identity map on M has a lift that maps x_0 to any given point in $\pi^{-1}(x_0)$.

Cover M by connected open sets V_α that are evenly covered: $\pi^{-1}(V_\alpha) \simeq \pi^{-1}(x_0) \times V_\alpha$.

Next suppose that M is covered by a string of connected sets U_i , $i = 0, 1, 2, \dots$ such that $U_i \subset V_{\alpha_i}$. We can then lift id_M on each of the sets U_i to go through a given point in $\pi^{-1}(U_i)$. If we further have the property that $U_k \cap \left(\bigcup_{i=0}^{k-1} U_i\right)$ is non-empty and connected, then we can use the uniqueness of liftings to successively define the lift on U_k given that it is defined on $\bigcup_{i=0}^{k-1} U_i$. Note that the sets U_i need not be open.

Unfortunately not a lot of manifolds admit such covers. Clearly \mathbb{R}^k does as it can be covered by coordinate cubes. Also any interval, disc, and square has this property. However, the circle S^1 cannot be covered by such a string of sets. On the other hand spheres S^n , $n > 1$ do have this property. We will only use the property for the interval and square.

Given two curves $c_i : [0, 1] \rightarrow M$ where $c_i(0) = x_0$ and $c_i(1) = x \in M$, where $i = 0, 1$, we invoke simple connectivity of M to find a homotopy $H : [0, 1]^2 \rightarrow M$ where $H(s, 0) = x_0$, $H(s, 1) = x$, and $H(i, t) = c_i(t)$. We can then find a lift of $F \circ H$ such that $\overline{F \circ H}(s, 0) = y_0$. Uniqueness of lifts then guarantees that $\overline{F \circ H}(s, 1)$ is constant, and, in particular, that the lift of id_M at $x \in M$ does not depend on the path connecting it to x_0 . This gives us a well-defined lift of id_M . \square

COROLLARY 3.1.14. *Let $\pi : \bar{M} \rightarrow M$ be a covering. If every closed curve $c : S^1 \rightarrow M$ has a lift that passes through each point in $\pi^{-1}(c(t_0))$ for a fixed $t_0 \in S^1$, then the covering is trivial.*

PROOF. This proof is almost identical to the above proof. The one difference is that the curves are no longer necessarily homotopic to each other. However, the fact that lifts of closed curves in M are assumed to become closed shows that the construction is independent of the paths we choose. \square

Since manifolds are locally simply connected we can establish the existence of a universal covering space.

THEOREM 3.1.15. *Every connected manifold admits a universal covering in the sense of satisfying the unique lifting property.*

PROOF. Note that if we have two coverings $\pi_i : M_i \rightarrow M$, $i = 0, 1$, then the pull-back

$$\begin{array}{ccc} \pi_0^* M_1 = \pi_1^* M_0 & \rightarrow & M_1 \\ \downarrow & & \downarrow \\ M_0 & \rightarrow & M \end{array}$$

is a covering of both M_0 and M_1 . Moreover, if M_0 and M_1 are both connected then the pull-back is also connected. This can be generalized to any selection of coverings $\pi_\alpha : M_\alpha \rightarrow M$, $\alpha \in A$

$$\bar{M} = \left\{ (x_\alpha) \in \prod_{\alpha \in A} M_\alpha \mid \pi_\alpha(x_\alpha) \text{ are equal for all } \alpha \in A \right\}$$

with

$$\pi : \bar{M} \rightarrow M$$

being defined by

$$\pi((x_\alpha)_{\alpha \in A}) = \pi_\alpha(x_\alpha).$$

In fact, the collection of all connected coverings of M form a category with the morphisms being bundle maps. The equivalence classes of objects in this category form a set and we define $\pi : \bar{M} \rightarrow M$ as the combined pull-back of all of these equivalence classes. To see that this is a covering space choose a countable basis, \mathcal{U} , for the topology of M such that every set in \mathcal{U} is simply connected. Thus any covering of M is evenly covered over the sets in \mathcal{U} . This implies that $\bar{M} \rightarrow M$ is also locally trivial. It is also an *initial object* in the category and cannot have any nontrivial connected coverings.

It follows from the definition of the unique lifting property that all universal coverings are equivalent. \square

COROLLARY 3.1.16. (Hadamard) *Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a proper non-singular map, then F is a diffeomorphism.*

We now turn our attention to the relationship between coverings and fundamental groups. The fundamental group, $\pi_1(M, p)$, is defined as the homotopy classes of loops that begin and end at $p \in M$. The homotopies are assumed to fix p for $t = 0, 1$. The group structure comes from concatenating loops at p . With this definition it is not hard to show that for any connected manifold M there is a covering $\pi : \bar{M} \rightarrow M$, where \bar{M} is simply connected. The space is defined so that $\pi^{-1}(q)$ consists of homotopy classes of curves that emanate from p and end at q , where the homotopies fix p for $t = 0$ and fix q for $t = 1$. The fact that M is locally connected and simply connected shows that \bar{M} is a covering space. See [Hatcher1] for a detailed treatment. We offer a different approach here that only uses the universal covering from theorem 3.1.15.

DEFINITION 3.1.17. If $\pi : \tilde{M} \rightarrow M$ is a covering map, then any lift of π to $\bar{\pi} : \tilde{M} \rightarrow \tilde{M}$ is called a *deck transformation* or *covering transformation*. It is evidently a bundle map on \tilde{M} .

DEFINITION 3.1.18. An action of a group G on a manifold M , or just a topological space, is said to be *properly discontinuous* provided that for each $p \in M$ there is a neighborhood U around p such that if $U \cap gU \neq \emptyset$, then $g = e = id_M$.

THEOREM 3.1.19. If $\pi : \tilde{M} \rightarrow M$ is the universal covering, then the set of deck transformations form a group, $\pi_1(M)$, that acts properly discontinuously. Moreover, if \tilde{M} is simply connected, then this group is isomorphic to the fundamental group $\pi_1(M, p)$.

PROOF. A composition of deck transformations is clearly also a deck transformation. To show that inverses exist we note that for $p \in M$ and $\tilde{p} \in \tilde{M}$ with $\pi(\tilde{p}) = p$ a deck transformation is uniquely determined by its value $\bar{\pi}(\tilde{p}) \in \pi^{-1}(p)$. Furthermore, for each $q \in \pi^{-1}(p)$ there is a deck transformation with $\bar{\pi}(\tilde{p}) = q$. For such a deck transformation we can construct a new deck transformation $\bar{\pi}^{-1}$ with the property that $\bar{\pi}^{-1}(q) = \tilde{p}$. The compositions $\bar{\pi} \circ \bar{\pi}^{-1}$ and $\bar{\pi}^{-1} \circ \bar{\pi}$ are both deck transformations that fix \tilde{p} and must therefore both be the identity map. This shows that $\pi_1(M) \simeq \pi^{-1}(p)$.

For $p \in M$ select a neighborhood U that is evenly covered

$$\pi^{-1}(U) \simeq \pi^{-1}(p) \times U.$$

The deck transformations simply permute these components which are all disjoint in \tilde{M} . So the only way a deck transformation can map a point in such a component to the same component is by fixing the point, thus showing that the deck transformation is the identity. This shows that the group of deck transformations acts properly discontinuously.

For the remainder of the proof we need to assume that the universal covering is simply connected. If we take a loop in M that starts and ends at p , then the lift to \tilde{M} that starts at \tilde{p} will end up at a $q \in \pi^{-1}(p)$. In this way we obtain a corresponding deck transformation. If two such paths are homotopic via a homotopy that fixes p for $t = 0, 1$, then we obtain the same deck transformation. This gives us a map from the fundamental group, $\pi_1(M, p)$, of M at p to the group $\pi_1(M)$ of deck transformations. This map is a bijection since there is also a bijection between $\pi_1(M, p)$ and $\pi^{-1}(p)$. To see this, note that if a loop at p lifts to a loop at some $q \in \pi^{-1}(p)$, then the loop is contractible in \tilde{M} and hence also in M . For surjectivity we note that for any $q \in \pi^{-1}(p)$ there exists a path from \tilde{p} to q . This is mapped to a loop at p and is clearly the lift of this loop. The correspondence is easily seen to be a homomorphism as it does not depend on the choice of $\tilde{p} \in \pi^{-1}(p)$. \square

EXAMPLE 3.1.20. The deck transformation groups of

$$\exp(it) : \mathbb{R} \rightarrow S^1$$

and

$$\exp(z) : \mathbb{C} \rightarrow \mathbb{C}^* = \mathbb{C} - \{0\}$$

are both isomorphic to \mathbb{Z} .

The deck transformation groups of

$$\begin{aligned} S^1 &\mapsto S^1 \\ z &\mapsto z^n, n > 0 \end{aligned}$$

and

$$\begin{aligned} S^1 &\mapsto S^1 \\ z &\mapsto z^{-n}, n > 0 \end{aligned}$$

are the cyclic groups of order n , $\mathbb{Z}/n\mathbb{Z}$.

More generally, the deck transformation group of

$$\mathbb{R}^n \rightarrow T^n = S^1 \times \cdots \times S^1$$

is \mathbb{Z}^n .

PROPOSITION 3.1.21. *If G acts properly discontinuously on M , then the quotient space $N = M/G$ is a smooth manifold and the quotient map $\pi : M \rightarrow M/G$ is a covering map.*

PROOF. The quotient topology is the topology that makes π continuous. It follows that π is also open since

$$\pi^{-1}(\pi(O)) = \bigcup_{g \in G} gO$$

is an open set. We next show that it is a covering map. From this it follows that the quotient becomes a smooth manifold with a quotient map that is a local diffeomorphism. Let $U \subset M$ be chosen so that $U \cap gU = \emptyset$ for all $g \neq e$. We claim that $\pi(U)$ is evenly covered. We have that

$$\pi^{-1}(\pi(U)) = \bigcup_{g \in G} gU,$$

so the claim follows provided the sets gU are pairwise disjoint. However, if $g_1U \cap g_2U \neq \emptyset$, then $g_2^{-1}g_1U \cap U \neq \emptyset$ so it follows that $g_1 = g_2$. \square

COROLLARY 3.1.22. *Let $\pi : \tilde{M} \rightarrow M$ the universal covering space. To every subgroup $H \subset \pi_1(M)$, there is a covering $\tilde{M} \rightarrow M_H$ whose deck transformation group is H . Moreover, M_H is also a covering of M . Conversely, any connected covering of M is constructed this way.*

PROOF. We have that $M = \tilde{M}/\pi_1(M)$. Since $H \subset \pi_1(M)$ there is a quotient map $\tilde{M}/H \rightarrow \tilde{M}/\pi_1(M)$ which is clearly also a covering of M (for each $p \in M$, select a neighborhood $p \in U$ such that $\tilde{M} \rightarrow M$ is trivial over U , then $\tilde{M}/H \rightarrow M$ will also be trivial over U). If $\tilde{M} \rightarrow M$ is a covering with \tilde{M} being connected, then the projection $\pi : \tilde{M} \rightarrow M$ can be lifted to a map $\tilde{M} \rightarrow \tilde{M}$ that is also a covering map. This shows that $\tilde{M} = \tilde{M}/H$, where $H \subset G$ is the deck transformation group for $\tilde{M} \rightarrow \tilde{M}$. \square

3.2. Orientability

Recall from section 2.3.1 that two ordered bases of a finite dimensional real vector space are said to represent the same orientation if the transition matrix from one to the other is of positive determinant. This evidently defines an equivalence relation with exactly two equivalence classes. A choice of such an equivalence class is called an orientation for the vector space.

Given a smooth manifold each tangent space has two choices for an orientation. Thus we obtain a two fold covering map $O_M \rightarrow M$, where the preimage of each $p \in M$ consists of the two orientations for $T_p M$. A manifold is said to be *orientable* if the orientation covering is trivial. A choice of sheet in the covering will correspond to a choice of an orientation for each tangent space. Using $\wedge^n TM$ we similarly see that M is orientable if and only if $\wedge^n TM$ is trivial.

To see that O_M really is a covering note that each chart $(x^1, x^2, \dots, x^n) : U \subset M \rightarrow \mathbb{R}^n$ with U is connected has two choices of orientations over U , namely, the class determined by the basis $(\partial_1, \partial_2, \dots, \partial_n)$ and by the basis $(-\partial_1, \partial_2, \dots, \partial_n)$. Thus U is covered by two disjoint sets each diffeomorphic to U and parametrized by these two different choices of

orientations. This tells us that \mathbb{R}^n is orientable and has a canonical orientation given by the standard Cartesian coordinate frame $(\partial_1, \partial_2, \dots, \partial_n)$.

Orientability of real vector bundles can be defined in a similar way and be related to $\wedge^k E$ and a corresponding orientation two-fold covering map $O_E \rightarrow M$.

Note that as simply connected manifolds only have trivial covering spaces they are all orientable. In particular, S^n , $n > 1$ is always orientable.

An other important observation is that the orientation covering O_M is an orientable manifold since it is locally the same as M and an orientation at each tangent space has been picked for us.

THEOREM 3.2.1. *The following conditions for a connected n -manifold M are equivalent.*

1. M is orientable.
2. Orientation is preserved moving along loops.
3. M admits an atlas where the Jacobians of all the transitions functions are positive.
4. M admits a nowhere vanishing n -form.

PROOF. $1 \Leftrightarrow 2$: The unique path lifting property for the covering $O_M \rightarrow M$ tells us that orientation is preserved along loops if and only if O_M is disconnected.

$1 \Rightarrow 3$: Pick an orientation. Take any atlas (U_α, x_α) of M where U_α is connected. As in our description of O_M from above we see that either each x_α corresponds to the chosen orientation, otherwise change the sign of the first component of x_α . In this way we get an atlas where each chart corresponds to the chosen orientation. Then it is easily checked that the transition functions $x_\alpha \circ x_\beta^{-1}$ have positive Jacobian as they preserve the canonical orientation of \mathbb{R}^n .

$3 \Rightarrow 4$: Choose a locally finite partition of unity (λ_α) subordinate to an atlas (U_α, x_α) where the transition functions have positive Jacobians. On each U_α we have the nowhere vanishing form $\omega_\alpha = dx_\alpha^1 \wedge \dots \wedge dx_\alpha^n$. Now note that if we are in an overlap $U_\alpha \cap U_\beta$, then

$$\begin{aligned} dx_\alpha^1 \wedge \dots \wedge dx_\alpha^n \left(\frac{\partial}{\partial x_\beta^1}, \dots, \frac{\partial}{\partial x_\beta^n} \right) &= \det \left(dx_\alpha^i \left(\frac{\partial}{\partial x_\beta^j} \right) \right) \\ &= \det \left(D \left(x_\alpha \circ x_\beta^{-1} \right) \right) \\ &> 0. \end{aligned}$$

Thus the globally defined form $\omega = \sum \lambda_\alpha \omega_\alpha$ is always nonnegative when evaluated on $\left(\frac{\partial}{\partial x_\beta^1}, \dots, \frac{\partial}{\partial x_\beta^n} \right)$. What is more, at least one term must be positive according to the definition of partition of unity.

$4 \Rightarrow 1$: Pick a nowhere vanishing n -form ω . Define the two sets O_\pm according to whether ω is positive or negative when evaluated on a basis. This yields two disjoint open sets in O_M which cover all of M . \square

The generalization for vector bundles only needs to be slightly reformulated.

THEOREM 3.2.2. *Let $E \rightarrow M$ be a real rank k vector bundle over M . The following conditions for a connected n -manifold M are equivalent.*

1. E is orientable.
2. Orientation is preserved moving along loops.
3. E admits local trivialisations that define the same orientations on intersections of their domains.
4. $\wedge^k E$ and $\wedge^k E^*$ admit nowhere vanishing sections, i.e., both bundles are trivial.

With this behind us we can try to determine which manifolds are orientable and which are not. Conditions 3 and 4 are often good ways of establishing orientability. To establish non-orientability is a little more tricky. However, if we suspect a manifold to be non-orientable then 1 tells us that there must be a non-trivial 2-fold covering map $\pi : \hat{M} \rightarrow M$, where \hat{M} is oriented and the two given orientations at points over $p \in M$ are mapped to different orientations in M via $D\pi$. A different way of recording this information is to note that for a two fold covering $\pi : \hat{M} \rightarrow M$ there is only one nontrivial deck transformation $A : \hat{M} \rightarrow \hat{M}$ with the properties: $A(x) \neq x$, $A \circ A = id_M$, and $\pi \circ A = \pi$. With this in mind we can show

PROPOSITION 3.2.3. *Let $\pi : \hat{M} \rightarrow M$ be a non-trivial 2-fold covering and \hat{M} an oriented manifold. In this case M is orientable if and only if A preserves the orientation on \hat{M} .*

PROOF. First suppose A preserves the orientation of \hat{M} . Then given a choice of orientation $e_1, \dots, e_n \in T_x \hat{M}$ we can declare $D\pi(e_1), \dots, D\pi(e_n) \in T_{\pi(x)} M$ to be an orientation at $\pi(x)$. This is consistent as $DA(e_1), \dots, DA(e_n) \in T_{I(x)} \hat{M}$ is mapped to $D\pi(e_1), \dots, D\pi(e_n)$ as well (using $\pi \circ A = \pi$) and also represents the given orientation on \hat{M} since A was assumed to preserve this orientation.

Suppose conversely that M is orientable and choose an orientation for M . Since we assume that both \hat{M} and M are connected the projection $\pi : \hat{M} \rightarrow M$, being nonsingular everywhere, must always preserve or reverse the orientation. We can without loss of generality assume that the orientation is preserved. Then we just use $\pi \circ A = \pi$ as in the first part of the proof to see that A must preserve the orientation on \hat{M} . \square

We can now use these results to check some concrete manifolds for orientability.

We already know that $S^n, n > 1$ are orientable, but, what about S^1 ? One way of checking that this space is orientable is to note that the tangent bundle is trivial and thus a uniform choice of orientation is possible. This clearly generalizes to Lie groups and other parallelizable manifolds. Using that $S^n \subset \mathbb{R}^{n+1}$ another method for checking orientability presents itself. At each $p \in S^{n+1}$ declare $v_1, \dots, v_n \in T_p M$ positively oriented if $p, v_1, \dots, v_n \in \mathbb{R}^{n+1}$ is positively oriented. More generally, if $M^n \subset \mathbb{R}^{n+1}$ has trivial normal bundle, then M is orientable.

Recall that $\mathbb{R}P^n$ has S^n as a natural double covering with the antipodal map as a natural deck transformation. This deck transformation preserves the radial field $X = x^i \partial_i$ and thus its restriction to S^n preserves or reverses orientation according to what it does on \mathbb{R}^{n+1} . On the ambient Euclidean space the map is linear and therefore preserves the orientation iff its determinant is positive. This happens iff $n+1$ is even. Thus we see that $\mathbb{R}P^n$ is orientable iff n is odd.

3.3. Submersions

In this section we present a number of results about the deeper structure of submersions.

3.3.1. Submersion-Fibrations. We study the relationship of the topologies of the manifolds related to a submersion.

In case F is a submersion it is possible to construct vector fields in M that are F -related to a given vector field in N .

PROPOSITION 3.3.1. *Assume that F is a submersion. Given a vector field Y in N , there are vector fields X in M that are F -related to Y .*

PROOF. First we do a local construction of X . Since F is a submersion proposition 1.4.17 shows that for each $p \in M$ there are charts $x : U \rightarrow \mathbb{R}^m$ and $y : V \rightarrow \mathbb{R}^n$ with $p \in U$ and $F(p) \in V$ such that

$$y \circ F \circ x^{-1}(x^1, \dots, x^m) = (x^1, \dots, x^n).$$

This relationship evidently implies that $\frac{\partial}{\partial y^i}$ and $\frac{\partial}{\partial x^i}$ are F -related for $i = 1, \dots, n$. Thus, if we write $Y = Y^i \frac{\partial}{\partial y^i}$, then we can simply define $X = \sum_{i=1}^n Y^i \circ F \frac{\partial}{\partial x^i}$. This gives the local construction.

For the global construction assume that we have a covering U_α , vector fields X_α on U_α that are F -related to Y , and a partition of unity λ_α subordinate to U_α and define $X = \sum \lambda_\alpha X_\alpha$. Linearity shows that this works:

$$\begin{aligned} DF(X) &= DF\left(\sum \lambda_\alpha X_\alpha\right) \\ &= \sum \lambda_\alpha DF(X_\alpha) \\ &= \sum \lambda_\alpha Y \circ F \\ &= Y \circ F. \end{aligned}$$

□

It is also possible to globally relate the flows of F -related vector fields given F is proper thus improving proposition 2.2.7.

PROPOSITION 3.3.2. *Assume that F is proper and that X and Y are F -related vector fields. If $F(p) = q$ and $\Phi_Y^t(q)$ is defined on $[0, b)$, then $\Phi_X^t(p)$ is also defined on $[0, b)$. In other words the relation $F \circ \Phi_X^t = \Phi_Y^t \circ F$ holds for as long as the RHS is defined.*

PROOF. Assume $\Phi_X^t(p)$ is defined on $[0, a)$. If $a < b$, then the set

$$\begin{aligned} K &= \{x \in M \mid F(x) = \Phi_Y^t(p) \text{ for some } t \in [0, a]\} \\ &= F^{-1}(\{\Phi_Y^t(p) \mid t \in [0, a]\}) \end{aligned}$$

is compact in M since F is proper. The integral curve $t \mapsto \Phi_X^t(p)$ lies in K since $F(\Phi_X^t(p)) = \Phi_Y^t(p)$. From theorem 2.2.5 we know that a maximally defined integral curve is either defined for all time or leaves every compact set. In particular, $[0, a)$ is not the maximal interval on which $t \mapsto \Phi_X^t(p)$ is defined. □

These relatively simple properties lead to some very general and tricky results.

DEFINITION 3.3.3. A *fibration* $F : M \rightarrow N$ is a smooth map that is *locally trivial* in the sense that for every $p \in N$ there is a neighborhood U of p such that $F^{-1}(U)$ is diffeomorphic to $U \times F^{-1}(p)$. This diffeomorphism must commute with the natural maps of these sets on to U . In other words $(x, y) \in U \times F^{-1}(p)$ must be mapped to a point in $F^{-1}(x)$. Note that it is easy to destroy the fibration property by simply deleting a point in M . Note also, that in this context fibrations are necessarily submersions.

Special cases of fibrations are covering maps and vector bundles. The Hopf fibration $S^3 \subset \mathbb{C}^2 - \{0\} \rightarrow S^2 = \mathbb{P}^1$ is a more non-trivial example of a fibration, which we shall study further below. Tubular neighborhoods are also examples of fibrations as we will see at the end of this chapter.

THEOREM 3.3.4 (Ehresmann). *If $F : M \rightarrow N$ is a proper submersion, then it is a fibration.*

PROOF. As far as N is concerned this is a local result. In N we simply select a set U that is diffeomorphic to \mathbb{R}^n and claim that $F^{-1}(U) \approx U \times F^{-1}(0)$. Thus we just need to prove the theorem in case $N = \mathbb{R}^n$, or more generally a coordinate box around the origin.

Next select vector fields X_1, \dots, X_n in M that are F -related to the coordinate vector fields $\partial_1, \dots, \partial_n$. Proposition 3.3.2 shows that we can define a smooth map $G : \mathbb{R}^n \times F^{-1}(0) \rightarrow M$ by $G(t^1, \dots, t^n, x) = \Phi_{X_1}^{t^1} \circ \dots \circ \Phi_{X_n}^{t^n}(x)$. The inverse to this map is $G^{-1}(z) = (F(z), \Phi_{X_n}^{-t^n} \circ \dots \circ \Phi_{X_1}^{-t^1}(z))$, where $F(z) = (t^1, \dots, t^n)$. \square

REMARK 3.3.5. Note that proposition 1.4.20 shows in analogy with lemma 3.1.1 that if $F : M \rightarrow N$ is a proper map and $y \in F(M)$ a regular value, then there is an open neighborhood $V \ni y$ such that $F^{-1}(V) \simeq V \times F^{-1}(y)$.

The theorem also unifies several different results.

COROLLARY 3.3.6 (Basic Lemma in Morse Theory). *Let $f : M \rightarrow \mathbb{R}$ be a proper map. If f is regular on $(a, b) \subset \mathbb{R}$, then $f^{-1}(a, b) \simeq f^{-1}(c) \times (a, b)$ where $c \in (a, b)$.*

COROLLARY 3.3.7 (Reeb). *Let M be a closed manifold that admits a map with two critical points, then M is homeomorphic to a sphere.*

PROOF. This is a bit easier to show if we also assume that the critical points are nondegenerate, see, e.g., theorem 4.1 in [Milnor] where the whole story can be found. Note that it is not necessary to use the Poincaré conjecture or the h-cobordism theorem to solve this problem! \square

THEOREM 3.3.8. *Let $\pi : S \rightarrow B$ be a fibration, where S is a sphere. If the fibration admits a section, then B is diffeomorphic to S . In particular, the fibrations $S \rightarrow \mathbb{P}^n$ are nontrivial.*

PROOF. The proof uses that the identity map on B is not homotopically trivial (see proposition 5.4.5).

Note that as S is compact and connected so is B . In general, a section $s : B \rightarrow S$ is a lift of the identity map on B and the image must be compact and connected.

When $\dim B = \dim S$ the fibration is a covering map so the image of the section must be a connected component of S and hence all of S . This implies that S and B are diffeomorphic.

In case $\dim B < \dim S$ Sard's theorem shows that the section can't be surjective. In particular, the section is homotopically trivial as S is a sphere. This in turn implies that the identity map on B is homotopically trivial leading to a contradiction. \square

Finally we can extend the fibration theorem to the case when M has boundary (see section 5.1 for a definition of manifolds with boundary).

THEOREM 3.3.9. *Assume that M is a manifold with boundary and that N is a manifold without boundary, if $F : M \rightarrow N$ is proper and a submersion on M as well as on ∂M , then it is a fibration.*

PROOF. The proof is identical and reduced to the case when $N = \mathbb{R}^n$. The assumptions allow us to construct the lifted vector fields so that they are tangent to ∂M . The flows will then stay in ∂M or $\text{int}M$ for all time if they start there. \square

REMARK 3.3.10. This theorem is sometimes useful when we have a submersion whose fibers are not compact. It is then occasionally possible to add a boundary to M so as to make the map proper. A good example is a tubular neighborhood around a closed submanifold

$S \subset U$. By possibly making U smaller we can assume that it is a compact manifold with boundary such that the fibers of $U \rightarrow S$ are closed discs rather than open discs.

EXAMPLE 3.3.11. Consider the the projection $\mathbb{R}^2 \rightarrow \mathbb{R}$ onto the first axis. This is clearly a submersion and a trivial bundle. The standard vector field ∂_x on \mathbb{R} can be lifted to the related field $\partial_x + y^2\partial_y$ on \mathbb{R}^2 . However, the integral curves for this lifted field are not complete as they are given by $\left(t + t_0, \frac{x_0}{1 - x_0(t + t_0)}\right)$ and diverge as t approaches $\frac{1}{x_0} - t_0$. In particular, neither the above proposition or theorem 3.3.4 can be made to work when the submersion isn't proper even though the submersion is a trivial fibration.

3.3.2. Quotient Manifolds. Suppose that \sim is an equivalence relation on a manifold M : when is M/\sim a manifold and $\pi : M \rightarrow M/\sim$ a submersion? Clearly the equivalence classes must form a foliation and the leaves/equivalence classes be closed subsets of M . Also their normal bundles have to be trivial as preimages of regular values have trivial normal bundle.

The most basic and still very nontrivial case is that of a Lie group G and a subgroup H . The equivalence classes are the cosets gH in G and the quotient space is G/H . When H is dense in G the quotient topology is not even Hausdorff. However one can prove that if H is closed in G , so that the equivalence classes are all closed embedded submanifolds, then the quotient is a manifold and the quotient map a submersion.

A nasty example is $\mathbb{R}^2 - \{0\}$ with the equivalence relation being that two points are equivalent if they have the same x -coordinate and lie in the same component of the corresponding vertical line. This means that the above general assumptions are not sufficient as all equivalence classes are closed embedded submanifolds with trivial normal bundles. The quotient space is the line with double origin and so is not Hausdorff!

REMARK 3.3.12. The key to getting a Hausdorff quotient is to assume that the graph of the equivalence relation

$$R = \{(x, y) \mid x \sim y\} \subset M \times M$$

is a proper submanifold. We can in fact find necessary and sufficient conditions that guarantee that the quotient space becomes a manifold and $\pi : M \rightarrow M/\sim$ a submersion. We let $\pi_{1,2} : M \times M \rightarrow M$ denote that projections onto the first and second factor. The equivalence class $\pi(p)$ that contains $p \in M$ is both a subset in M and a point in the quotient. Note that

$$\begin{aligned} R \cap (M \times \{p\}) &= \pi(p) \times \{p\}, \\ R \cap (\{p\} \times M) &= \{p\} \times \pi(p). \end{aligned}$$

PROPOSITION 3.3.13. *If M/\sim has a manifold structure such that $\pi : M \rightarrow M/\sim$ becomes a submersion, then $R \subset M \times M$ is a properly embedded submanifold and the restrictions of the projection maps $\pi_{1,2}|_R : R \subset M \times M \rightarrow M$ are submersions.*

PROOF. Note that a submanifold is properly embedded exactly when it is a closed subset of the ambient manifold.

Consider the graph

$$G(\pi) = \{(p, \pi(p)) \in M \times (M/\sim) \mid p \in M\}.$$

We have that $\text{id} \times \pi : M \times M \rightarrow M \times (M/\sim)$ is a submersion and $R = (\text{id} \times \pi)^{-1}(G(\pi))$. A simple generalization of the preimage theorem 1.4.24 (see also chapter 1 exercise 18) now shows that R is a properly embedded submanifold as $G(\pi)$ is a properly embedded submanifold. This also tells us that $(\text{id} \times \pi)|_R : R \rightarrow G(\pi)$ becomes a submersion. Composing

this map with the diffeomorphism $\pi_1|_{G(\pi)} : G(\pi) \rightarrow M$, then implies that $\pi_1|_R : R \rightarrow M$ becomes a submersion. Since R is invariant under the involution $(p, q) \mapsto (q, p)$ we have also shown that $\pi_2|_R : R \rightarrow M$ is a submersion. \square

EXAMPLE 3.3.14. Consider $M = Gl_n(\mathbb{F})$ with the relation that two matrices are equivalent if their first k rows have the same span. Thus $A \sim B$ if and only if

$$B = \begin{bmatrix} X & 0 \\ Y & Z \end{bmatrix} A, \text{ where } X \in Gl_k, Y \in \text{Mat}_{(n-k) \times k}, Z \in Gl_{n-k}.$$

In this way the map $Gl_n(\mathbb{F}) \rightarrow Gr_k(\mathbb{F}^n)$ that maps a matrix to the span of the first k rows becomes a natural projection onto the orbit space. The orbit space is the Grassmannian of k -planes in \mathbb{F}^n . The equivalence class that contains A is a closed set and an embedding of dimension

$$k^2 + (n-k)^2 + k(n-k) = n^2 - k(n-k)$$

that is parametrized by X, Y, Z

$$[A] = \left[\begin{array}{c|c} X & 0 \\ \hline Y & Z \end{array} \right] A, \text{ where } X \in Gl_k, Y \in \text{Mat}_{k \times (n-k)}, Z \in Gl_{n-k}.$$

This forces $Gl_n(\mathbb{F}) \rightarrow Gr_k(\mathbb{F}^n)$ to become a submersion for dimension reasons alone. This map is in fact a fibration. To see this recall that a neighborhood of a point in $Gr_k(\mathbb{F}^n)$ is parametrized by $k \times (n-k)$ matrices. Specifically if the point is the span of the first k basis vectors then a neighborhood is parametrized as that span of the $k \times n$ matrices

$$\left[\begin{array}{c|c} I_k & W \\ \hline & \end{array} \right], \text{ where } W \in \text{Mat}_{k \times (n-k)}.$$

This means that we have a section on this neighborhood were

$$\left[\begin{array}{c|c} I_k & W \\ \hline & \end{array} \right] \mapsto \left[\begin{array}{c|c} I_k & W \\ \hline 0 & I_{n-k} \end{array} \right].$$

The whole preimage of the neighborhood then becomes a product via the map

$$\left(\left[\begin{array}{c|c} I_k & W \\ \hline 0 & I_{n-k} \end{array} \right], \left[\begin{array}{c|c} X & 0 \\ \hline Y & Z \end{array} \right] \right) \mapsto \left[\begin{array}{c|c} X & 0 \\ \hline Y & Z \end{array} \right] \left[\begin{array}{c|c} I_k & W \\ \hline 0 & I_{n-k} \end{array} \right] = \left[\begin{array}{c|c} X & XW \\ \hline Y & YW + ZI_{n-k} \end{array} \right].$$

The relation is parametrized by

$$R = \left\{ \left(A, \left[\begin{array}{c|c} X & 0 \\ \hline Y & Z \end{array} \right] A \right) \mid A \in Gl_n, X \in Gl_k, Y \in \text{Mat}_{k \times (n-k)}, Z \in Gl_{n-k} \right\}.$$

In particular, we see that R is a closed set and an embedding of dimension $2n^2 - k(n-k)$ with the property that the projection onto the first factor is a submersion.

The converse of the above proposition is also true and offers a particularly nice characterization of quotient manifolds that rarely makes it into text books. Our proof is borrowed from [Serre] and appears to be one of the most efficient proofs.

THEOREM 3.3.15 (Godement). *If \sim is an equivalence relation on a smooth manifold M , then M/\sim has a manifold structure such that $\pi : M \rightarrow M/\sim$ becomes a submersion provided $R \subset M \times M$ is a properly embedded submanifold and the restriction $\pi_1|_R : R \rightarrow M$ is a submersion.*

PROOF. We first settle the topological aspects of the quotient by showing that $\pi : M \rightarrow M/\sim$ is open and that M/\sim is Hausdorff. Let $O \subset M$ be open and note that by definition of

the quotient topology that $\pi(O)$ is open precisely when $\pi^{-1}(\pi(O))$ is open. The latter set is open since,

$$\pi^{-1}(\pi(O)) = \pi_1((M \times O) \cap R) = \{q \in M \mid \exists p \in O : p \sim q\}$$

and $\pi_1|_R$ is a submersion and in particular an open map. For the Hausdorff property fix two equivalence classes $\pi(p), \pi(q)$. Select shrinking open neighborhoods $U_i \ni p$ and $V_i \ni q$ with $\cap_i U_i = \{p\}$ and $\cap_i V_i = \{q\}$. If $\pi(U_i) \cap \pi(V_i) \neq \emptyset$ for all i , then there exists $x_i \in U_i$ and $y_i \in V_i$ such that $\pi(x_i) = \pi(y_i) \in \pi(U_i) \cap \pi(V_i)$. But, then $x_i \rightarrow p$, $y_i \rightarrow q$, and $x_i \sim y_i$. Since R is a closed set this implies that $p \sim q$ and consequently $\pi(p) = \pi(q)$.

Given $p \in M$ we seek a neighborhood $U \ni p$ and a retraction $r : U \rightarrow U$ such that $x, y \in U$ are equivalent iff $r(x) = r(y)$. The image of r will be a submanifold slice that intersects each equivalence class that intersects U in a unique point.

Note that Δ and $M \times \{p\}$ are *transverse* at $(p, p) = \Delta \cap M \times \{p\}$, i.e.,

$$T_{(p,p)}(M \times M) = T_{(p,p)}\Delta + T_{(p,p)}(M \times \{p\}).$$

In particular, $R \supset \Delta$ and $M \times \{p\}$ are transverse at (p, p) . The intersection of the tangent spaces, $T_{(p,p)}R \cap T_{(p,p)}(M \times \{p\})$, has dimension k if $\dim R = n + k$. This intersection is naturally isomorphic to the k -dimensional space $E_p = \{v \in T_p M \mid (v, 0) \in T_{(p,p)}R\}$. In this way we obtain a subbundle of TM . Select a submanifold $W \subset M$ whose tangent space $T_p W$ is a complement to $E_p \subset T_p M$. Since also $\pi_2|_R$ is a submersion a simple generalization of the preimage theorem 1.4.24 (see also chapter 1 exercise 18) shows that

$$V = R \cap (M \times W) = (\pi_2|_R)^{-1}(W) = \{(x, y) \in R \mid y \in W\}$$

is a submanifold of dimension n as $\text{codim} W = k$ and $\dim R = n + k$. We claim that the restriction $\pi_1|_V : V \rightarrow M$ is nonsingular at (p, p) . Note that $(v, w) \in T_{(p,p)}V$ iff $(v, w) \in T_{(p,p)}R$ and $w \in T_p W$. So if $D\pi_1|_V(v, w) = v = 0$, then $w \in E_p \cap T_p W = \{0\}$. This shows that we can find neighborhoods $U_1, U_2 \ni p$ such that $\pi_1|_V : V \cap (U_1 \times U_1) \rightarrow U_2$ becomes a diffeomorphism with an inverse of the form $f(x) = (x, r(x))$. In particular, $U_2 \subset U_1$ and $r : U_2 \rightarrow W$. Define

$$U = \{x \in U_2 \mid r(x) \in U_2 \cap W\}.$$

For $x \in U$ both $(r(x), r(x))$ and $(r(x), r^2(x)) = f(r(x))$ are mapped to $r(x)$ via π_1 , hence $r(x) = r^2(x)$ and U is invariant under r with $r(U) = U \cap W$. Let

$$(x, y) \in R \cap (U \times (U \cap W)) \subset V.$$

As $\pi_1(x, y) = x$ and $f(x) = (x, r(x)) = (x, y)$ it follows that $r(x)$ is the only point in $U \cap W$ that is equivalent to x . In other words, if $x, y \in U$ are equivalent then $r(x) = r(y)$.

To finish, assume that W was chosen so as to have global coordinates $\phi = (x^1, \dots, x^{n-k})$. These descend to coordinates $\bar{\phi} = (\bar{x}^1, \dots, \bar{x}^{n-k})$ on the open set $\pi(U) = \pi(U \cap W)$. To obtain a differentiable structure we note that any function f whose domain intersects $\pi(U \cap W)$ has the property that $f \circ \bar{\phi}^{-1} = f \circ \pi \circ \phi^{-1}$. Thus $f \circ \bar{\phi}^{-1}$ is smooth when $f \circ \pi$ is smooth. This shows that we can define a differentiable structure by declaring that $f \in C^\infty(M/\sim)$ if and only if $f \circ \pi \in C^\infty(M)$. \square

3.4. Embeddings

3.4.1. Embeddings into Euclidean Space. The goal is to show that any manifold can be properly embedded into Euclidean space. This requires most importantly that we can find a way to reduce the dimension of the ambient Euclidean space into which the manifold can be embedded.

THEOREM 3.4.1 (Whitney Embedding, Dimension Reduction). *If $F : M^m \rightarrow \mathbb{R}^n$ is an injective immersion, then there is also an injective immersion $M^m \rightarrow \mathbb{R}^{2m+1}$. Moreover, if one of the coordinate functions of F is proper, then we can keep this property. In particular, when M is compact we obtain an embedding.*

PROOF. For each $v \in \mathbb{R}^n - \{0\}$ consider the orthogonal projection onto the orthogonal complement of v :

$$f_v(x) = x - \frac{\langle x, v \rangle v}{|v|^2}.$$

The image is an $(n-1)$ -dimensional subspace. So if we can show that $f_v \circ F$ is an injective immersion, then the ambient dimension has been reduced by 1.

Note that $f_v \circ F(x) = f_v \circ F(y)$ iff $F(x) - F(y)$ is proportional to v . Similarly $d(f_v \circ F)(w) = 0$ iff $dF(w)$ is proportional to v . With that in mind consider the images of the following two maps:

$$\begin{aligned} H : M \times M \times \mathbb{R} &\rightarrow \mathbb{R}^n \\ h(x, y, t) &= t(F(x) - F(y)) \\ G : TM &\rightarrow \mathbb{R}^n \\ G(w) &= dF(w) \end{aligned}$$

As long as $2m+1 < n$ Sard's theorem implies that the union of the two images has dense complement. Therefore, we can select $v \in \mathbb{R}^n - (H(M \times M \times \mathbb{R}) \cup G(TM))$. Clearly $v \neq 0$ as 0 lies in the image of both maps. So if $F(x) - F(y) = sv$, then either $s = 0$ and $x = y$ or $v = \frac{1}{s}(F(x) - F(y))$ which is impossible. Similarly, if $dF(w) = sv$, then either $s = 0$ showing that $w = 0$ or $v = \frac{1}{s}(dF(w))$ which is impossible.

Note that the v we selected could be taken from $O - (H(M \times M \times \mathbb{R}) \cup G(TM))$, where $O \subset \mathbb{R}^n$ is any open subset. This gives us a bit of extra information. While we can't get the ultimate map $M^m \rightarrow \mathbb{R}^{2m+1}$ to target a specific $(2m+1)$ -dimensional subspace of \mathbb{R}^n , we can map it into a subspace arbitrarily close to a fixed subspace of dimension $2m+1$. To be specific simply assume that $\mathbb{R}^{2m+1} \subset \mathbb{R}^n$ consists of the first $2m+1$ coordinates. By selecting $v \in (-\varepsilon, \varepsilon)^{2m+1} \times (1-\varepsilon, 1+\varepsilon)^{n-2m-1}$ we see that f_v changes the first coordinates with an error that is small.

This can be used to obtain proper maps $f_v \circ F$. When the first coordinate for F is proper, then $f_v \circ F$ is also proper provided v is not proportional to e_1 . This means that we merely have to select v such that $|v| > 1$, $v \cdot e_1 < \varepsilon$ to obtain a proper injective immersion. \square

REMARK 3.4.2. Note also that if F starts out only being an immersion, then we can find an immersion into \mathbb{R}^{2m} . This is because $G(TM) \subset \mathbb{R}^n$ has measure zero as long as $n > 2m$.

LEMMA 3.4.3. *If $A, B \subset M^m$ are open sets that both admit injective immersions into \mathbb{R}^{2m+1} , then the union $A \cup B$ admits a proper embedding into \mathbb{R}^{2m+1} .*

PROOF. Select a partition of unity $\lambda_A, \lambda_B : A \cup B \rightarrow [0, 1]$, i.e., $\text{supp } \lambda_A \subset A$, $\text{supp } \lambda_B \subset B$, and $\lambda_A + \lambda_B = 1$. Further, choose injective immersions $F_A : A \rightarrow \mathbb{R}^{2m+1}$ and $F_B : B \rightarrow \mathbb{R}^{2m+1}$. Multiplying these maps with our bump functions we obtain well-defined maps $\lambda_A F_A, \lambda_B F_B : A \cup B \rightarrow \mathbb{R}^{2m+1}$. This gives us a map

$$\begin{aligned} F : A \cup B &\rightarrow \mathbb{R}^{2m+1} \times \mathbb{R}^{2m+1} \times \mathbb{R} \times \mathbb{R}, \\ F(x) &= (\lambda_A(x) F_A(x), \lambda_B(x) F_B(x), \lambda_A(x), \lambda_B(x)), \end{aligned}$$

which we claim is an injective immersion.

If $F(x) = F(y)$, then $\lambda_{A,B}(x) = \lambda_{A,B}(y)$. If, e.g., $\lambda_B(x) > 0$, then $F_B(x) = F_B(y)$. This shows that $x = y$ as F_B is an injection.

If $dF(v) = 0$ for $v \in T_p M$, then $d\lambda_{A,B}(v) = 0$. So if, e.g., $\lambda_A(p) > 0$, then by the product rule:

$$d(\lambda_A F_A)|_p = (d\lambda_A)|_p F_A(p) + \lambda_A(p) dF_A|_p = \lambda_A(p) dF_A|_p$$

and consequently

$$dF_A|_p(v) = 0$$

showing that $v = 0$.

If, in addition, we select a proper function $\rho : A \cup B \rightarrow [0, \infty)$, then we obtain a proper injective immersion

$$(\rho, F) : A \cup B \rightarrow \mathbb{R} \times \mathbb{R}^{2m+1} \times \mathbb{R}^{2m+1} \times \mathbb{R} \times \mathbb{R}$$

and consequently an embedding. The dimension reduction result above then gives us a (proper) embedding into \mathbb{R}^{2m+1} . \square

THEOREM 3.4.4 (Whitney Embedding, Final Version). *An m -dimensional manifold M admits a proper embedding into \mathbb{R}^{2m+1} .*

PROOF. We only need to check the hypotheses in theorem 1.3.2. Clearly the statement is invariant under diffeomorphisms and holds for \mathbb{R}^m . Condition (2) was established in the previous lemma. Condition (3) is almost trivial. Given embeddings $F_i : A_i \rightarrow \mathbb{R}^{2m+1}$, where $A_i \subset M$ are open and pairwise disjoint we can construct new embeddings $G_i : A_i \rightarrow \left(i, i + \frac{1}{2}\right)^{2m+1} \subset \mathbb{R}^{2m+1}$ with disjoint images. This yields an embedding $G : \bigcup_i A_i \rightarrow \mathbb{R}^{2m+1}$.

This shows that any m -manifold has an embedding into \mathbb{R}^{2m+1} . To obtain a proper embedding we select a proper function $\rho : M \rightarrow [0, \infty)$ and use the dimension reduction result on the proper embedding $(\rho, F) : M \rightarrow \mathbb{R} \times \mathbb{R}^{2m+1}$. \square

3.4.2. Tubular Neighborhoods.

LEMMA 3.4.5. *If $M \subset \mathbb{R}^n$ is a properly embedded submanifold, then there exists a neighborhood of the normal bundle of M in \mathbb{R}^n that is diffeomorphic to a neighborhood of M in \mathbb{R}^n .*

PROOF. The normal bundle is defined as

$$T^\perp M = T^\perp(M \subset \mathbb{R}^n) = \{(v, p) \in T_p \mathbb{R}^n \times M \mid v \perp T_p M\}.$$

There is a natural map

$$\begin{aligned} T^\perp M &\rightarrow \mathbb{R}^n, \\ (v, p) &\mapsto v + p. \end{aligned}$$

This map is proper provided $M \subset \mathbb{R}^n$ is properly embedded. It is also clearly an embedding when restricted to the zero section. Note that the image of the differential at a point $(0, p)$ contains $T_p M$ and $\{v \in T_p \mathbb{R}^n \mid v \perp T_p M\}$. Consequently the differential is nonsingular. This shows that it is a local diffeomorphism on some neighborhood of the zero section M . Lemma 1.4.15 then shows that it is a diffeomorphism on a neighborhood of the zero section. \square

THEOREM 3.4.6. *If $M \subset N$ is a properly embedded submanifold, then there exists a neighborhood of the normal bundle of M in N that is diffeomorphic to a neighborhood of M in N .*

PROOF. Any subbundle of $TN|_M$ that is transverse to TM is a normal bundle. It is easy to see that all such bundles are isomorphic. One specific choice comes from a proper embedding $N \subset \mathbb{R}^n$ and then defining

$$T^\perp M = \{(v, p) \in T_p N \times M \mid v \perp T_p M\}.$$

We don't immediately obtain a map $T^\perp M \rightarrow N$. First we select a neighborhood $N \subset U \subset \mathbb{R}^n$ as in the previous lemma. The projection $\pi : U \rightarrow N$ that takes $w + q \in U$ to $q \in N$ is a fibration. This gives us a neighborhood $V = \{(v, p) \in T^\perp M \mid v + p \in U\}$ of M . This allows us to define a map

$$\begin{aligned} V &\rightarrow N, \\ (v, p) &\mapsto \pi(v + p). \end{aligned}$$

that is a local diffeomorphism near the zero section and a proper embedding on the zero section. \square

As an application of Whitney's theorem and the existence of tubular neighborhoods we can show some crucial results about smooth approximations of continuous maps.

THEOREM 3.4.7. *Let $F : M \rightarrow \mathbb{R}^n$ be a continuous function and $C \subset M$ a closed subset such that F is smooth on a neighborhood U_0 of C . If $\epsilon : M \rightarrow [0, 1]$ is a continuous function that is positive on $M - C$, then there exists a smooth function $G : M \rightarrow \mathbb{R}^n$ such that $|F(x) - G(x)| < \epsilon(x)$. In particular, $F(x) = G(x)$ for all $x \in C$ provided ϵ vanishes on C .*

PROOF. Consider a cover that consists of U_0 and U_p , $p \in M - C$, where $U_p \subset M - C$ is a neighborhood of p where $|F(x) - F(p)| < \epsilon(x)$ for all $x \in U_p$. We can then select a partition of unity $\lambda_0, \lambda_p, p \in M - C$ subordinate to this cover. Note that as all λ_p have support in $M - C$, it follows that $\lambda_0|_C \equiv 1$. Clearly

$$F(x) = \lambda_0(x)F(x) + \sum_{p \in M - C} \lambda_p(x)F(x)$$

and we can define a smooth function that agrees with F on C :

$$G(x) = \lambda_0(x)F(x) + \sum_{p \in M - C} \lambda_p(x)F(p).$$

It follows that

$$|F(x) - G(x)| \leq \sum_{p \in M - C} \lambda_p(x) |F(x) - F(p)| \leq \epsilon(x).$$

\square

We can next use the tubular neighborhood theorem to approximate maps into general manifolds.

THEOREM 3.4.8. *Let $F : M \rightarrow N$ be a continuous function and $C \subset M$ a closed subset such that F is smooth on a neighborhood U_0 of C . There exists a smooth $G : M \rightarrow N$ such that $F(x) = G(x)$, for all $x \in C$ and F and G are homotopic to each other.*

PROOF. Let $N \subset \mathbb{R}^n$ be a proper embedding and U a tubular neighborhood of N in \mathbb{R}^n with $\pi : U \rightarrow N$ a retraction onto N . Select $\epsilon : M \rightarrow [0, 1]$ such that $\epsilon|_C \equiv 1$ and $B(F(x), \epsilon(x)) \subset U$ for all $x \in M$. From the theorem above we have smooth $\tilde{G} : M \rightarrow \mathbb{R}^n$ such that $|F(x) - \tilde{G}(x)| < \epsilon(x)$. The choice of ϵ guarantees that for all $x \in M$ the homotopy

$$H(t, x) = (1 - t)F(x) + t\tilde{G}(x) \in U, t \in [0, 1]$$

lies in U . This yields a homotopy $\pi \circ H : [0, 1] \times M \rightarrow N$ from F to a smooth map $G = \pi \circ \tilde{G}$ such that for all $x \in C$ we have

$$\pi \circ \tilde{G}(x) = \pi \circ F(x) = F(x).$$

□

REMARK 3.4.9. Note that if F is proper, then G will also be proper as the functions stay close to each other provided $\epsilon(x)$ is bounded.

COROLLARY 3.4.10. *If two smooth maps are homotopic via a continuous homotopy, then they are also smoothly homotopic.*

3.5. Exercises

- (1) Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a smooth proper map with finitely many critical values.
 - (a) Show that if $n \geq 2$, then F is surjective (Hint: The set of regular values is connected).
 - (b) Give a counter example when $n = 1$.
- (2) Show that $\mathbb{R}^n \rightarrow T^n = S^1 \times \cdots \times S^1$ is a covering map and use this to show that any map $M \rightarrow T^n$ is homotopic to a constant provided M is simply connected.
- (3) Let $A, B \subset M$ be open and $A \cap B$ connected.
 - (a) Show that if A, B are simply connected, then $A \cup B$ is simply connected.
 - (b) Let $\tilde{M} \rightarrow M$ be a covering. Show that if A, B are evenly covered, then $A \cup B$ is evenly covered.
- (4) Let M and N be manifolds.
 - (a) Show that if M and N are simply connected, then so is $M \times N$.
 - (b) Show that if M and N satisfy the unique lifting property, then so does $M \times N$.
- (5) Let $F : M \rightarrow N$ be a proper immersion. Show that if F is injective when restricted to a closed subset $C \subset M$, then F is also injective on an open neighborhood of C .
- (6) Show that the Klein bottle and Möbius band are nonorientable by finding suitable connected double covers that are oriented.
- (7) Give an example of an injective immersion $\mathbb{R} \rightarrow T^2$ whose image is dense, e.g., of the form $(e^{i2\pi t}, e^{i2\pi at})$. Extend this map to an immersion $\mathbb{R} \times (-\epsilon, \epsilon) \rightarrow T^2$ and show that it is not injective on any neighborhood of $\mathbb{R} \times \{0\}$.
- (8) Show that any rank k vector bundle $E^k \rightarrow M^n$ is a subbundle of a trivial bundle of dimension $k + n$. Hint: You need a rank reduction result that controls the rank of the trivial bundle in terms of the rank of E and the dimension of M .
- (9) Show that the submersions $V_k(\mathbb{R}^n) \rightarrow \text{Gr}_k(\mathbb{R}^n)$ and $O(n) \rightarrow \text{Gr}_k(\mathbb{R}^n)$ are fibrations.
- (10) Show that $V_k(\mathbb{R}^n)$ can be realized as the quotient $O(n)/O(n-k)$ and $\text{Gr}_k(\mathbb{R}^n)$ as the quotient $O(n)/(O(k) \times O(n-k))$.
- (11) Use theorem 3.3.15 on $V_k(\mathbb{R}^n)$ to show that the Grassmannian $\text{Gr}_k(\mathbb{R}^n)$ is a manifold.
- (12) Use theorem 3.3.15 on $O(n)$ to show that the Grassmannian $\text{Gr}_k(\mathbb{R}^n)$ is a manifold.
- (13) Show that the map $\text{Mat}_{k \times n}^k \rightarrow \text{Gr}_k$ that takes the matrix to the span of the rows is a fibration whose preimages are diffeomorphic to GL_k .

CHAPTER 4

Lie Groups

This chapter covers the basic aspects of Lie group theory: Lie groups, subgroups, Lie algebras, matrix groups, the exponential map, covering groups, and the correspondence between Lie (sub)groups and Lie (sub)algebras.

4.1. General Properties

A Lie group is a smooth manifold with a group structure that is also smooth, i.e., a manifold G with an associative multiplication

$$\begin{aligned} G \times G &\rightarrow G \\ (g, h) &\mapsto gh \end{aligned}$$

that is smooth and inverse operation

$$\begin{aligned} G &\rightarrow G \\ g &\mapsto g^{-1} \end{aligned}$$

that is smooth. The identity is generally denoted e . The most obvious example of a Lie group is simply a vector space with addition as the product structure. A more interesting example is the space of invertible matrices, $GL(n, \mathbb{F})$ with matrix multiplication as the product structure.

A Lie group homomorphism is a homomorphism between Lie groups that is also smooth. A Lie subgroup $H \subset G$ is a subgroup that is also an immersed submanifold such that the inherited group operations are smooth on the submanifold, i.e., it is the image of a Lie group under an injective immersion that is also a homomorphism.

A Lie group is homogeneous in a canonical way as left translation by group elements: $l_g(x) = g \cdot x$ maps the identity element e to g . Consequently, $l_{gh^{-1}}$ maps h to g . Since left translation is a diffeomorphism it can be used to calculate the differential of a Lie group homomorphism from the differential at the identity. For a smooth homomorphism $\phi : G_1 \rightarrow G_2$ between Lie groups the homomorphism property can be expressed as

$$\phi \circ l_g = l_{\phi(g)} \circ \phi$$

or

$$\phi = l_{\phi(g)} \circ \phi \circ l_{g^{-1}}$$

This shows that

$$D\phi|_g = Dl_{\phi(g)} \circ D\phi|_e \circ Dl_{g^{-1}}$$

In particular, ϕ has constant rank and its kernel $\ker \phi \subset G_1$ must be a properly embedded submanifold and a Lie subgroup.

We could equally well have used right translation $r_g(x) = xg$ for these observations.

A vector field is left invariant if it is l_g -related to itself for all g , i.e., $Dl_g(X|_h) = X|_{gh}$. This shows that $X|_e$ determines X . In particular, we have that

$$l_g \circ \Phi'_X = \Phi'_X \circ l_g$$

and

$$g\Phi'_X(e) = \Phi'_X(g).$$

Conversely, given $X|_e \in T_e G$ it is easy to see that $X|_g = Dl_g(X|_e)$ defines a smooth left invariant vector field. The space of left invariant vector fields is denoted by \mathfrak{g} and is identified with $T_e G$ as a vector space. However, there is an extra structure on \mathfrak{g} as the Lie bracket of left invariant fields is again left invariant (see section 2.2.2). This makes \mathfrak{g} into a Lie algebra, i.e., an algebra with a bracket operation $[X, Y]$ that is bilinear, skew-symmetric, and satisfies the Jacobi identity. Any associative algebra $(A, +, \cdot)$ has such a bracket structure defined by commutation $[a, b] = ab - ba$. The space of square matrices $\text{Mat}_{n \times n}(\mathbb{F})$ with this commutator bracket is denoted by $\mathfrak{gl}(n, \mathbb{F})$ when we think of it as a Lie algebra.

When $H \subset G$ is a Lie subgroup it follows that $\mathfrak{h} \subset \mathfrak{g}$ is a subalgebra as left multiplication l_g on G preserves H when $g \in H$. More generally, for a smooth homomorphism $\phi : H \rightarrow G$ we see that any $X \in \mathfrak{h}$ is ϕ -related to the left invariant field $Y \in \mathfrak{g}$ that is determined by $Y|_e = D\phi(X|_e)$ showing that we obtain a Lie algebra homomorphism $\phi_* : \mathfrak{h} \rightarrow \mathfrak{g}$, i.e., ϕ_* is linear and preserves Lie brackets, $\phi_*[X, Y] = [\phi_*(X), \phi_*(Y)]$.

4.2. Matrix Groups

We explain the various basic matrix groups that come from the general linear groups.

4.2.1. The General Linear Groups. The most obvious examples of Lie groups are matrix groups starting with the general linear groups

$$Gl(n, \mathbb{R}) \subset \text{Mat}_{n \times n}(\mathbb{R}) = \mathfrak{gl}(n, \mathbb{R}),$$

$$Gl(n, \mathbb{C}) \subset \text{Mat}_{n \times n}(\mathbb{C}) = \mathfrak{gl}(n, \mathbb{C}).$$

These are open subsets of the vector space of $n \times n$ matrices and the group operations are explicitly given in terms of the standard arithmetic operations of numbers. The identity is usually denoted $e = I$ for matrix groups. As such we have right and left translation on $\text{Mat}_{n \times n}(\mathbb{F})$ for any $A \in \text{Mat}_{n \times n}(\mathbb{F})$ defined by $l_A(X) = AX$ and $r_A(X) = XA$. These are linear maps but not invertible unless $A \in Gl(n, \mathbb{F})$. With that in mind we note that the equation for left invariant fields $X|_g = Dl_g(X|_I)$ becomes $X|_g = gX|_I = r_{X|_I}(g)$. This allows us to show that the Lie bracket of left invariant fields is the same as the Lie algebra $\mathfrak{gl}(n, \mathbb{F})$. Let $X = r_A$ and $Y = r_B$ be two left invariant fields and $f : \text{Mat}_{n \times n}(\mathbb{F}) \rightarrow \mathbb{R}$ a linear function. For any tangent vector $v \in \text{Mat}_{n \times n}(\mathbb{F})$ we have $D_v f = f(v)$. This shows that

$$(D_X f)|_g = D_{gA} f = f(gA) = f \circ r_A(g)$$

and as r_A is linear

$$(D_Y(D_X f))|_I = (D_Y(f \circ r_A))|_I = f \circ r_A \circ r_B(I) = f(BA).$$

Similarly,

$$(D_X(D_Y f))|_I = f(AB)$$

and we can conclude that

$$D_{[X, Y]} f|_I = f([A, B]) = f(AB - BA).$$

Thus $[X, Y]|_I = [A, B] = AB - BA$.

4.2.2. The Special Linear Groups. The determinant map $\det : \text{Mat}_{n \times n}(\mathbb{F}) \rightarrow \mathbb{F}$ is multiplicative and smooth with the general linear group being the open subset of matrices with non-zero determinant. Thus we obtain a surjective homomorphism $\det : \text{GL}(n, \mathbb{F}) \rightarrow \mathbb{F}^\times$, where $\mathbb{F}^\times = \mathbb{F} - \{0\}$ with multiplication as the group operation. As homomorphisms have constant rank this shows that it must be a homomorphism.

The derivative at I can also be calculated explicitly as follows:

$$\det(I + tX) = t^n \det\left(t^{-1}I + X\right) = 1 + t(\text{tr}X) + o(t)$$

and at $A \in \text{GL}$:

$$\det(A + tX) = \det A \left(1 + t \left(\text{tr}\left(A^{-1}X\right)\right) + o(t)\right).$$

This gives us the special linear groups $\text{SL}(n, \mathbb{F})$ of matrices with $\det = 1$. The tangent space $T_I \text{SL}$ is given as the kernel of the differential and is thus the space of traceless matrices:

$$T_I \text{SL} = \{X \in \text{Mat}_{n \times n} \mid \text{tr}X = 0\} = \mathfrak{sl}(n, \mathbb{F}).$$

4.2.3. The Polar Decomposition. Recall from example 1.2.2 the submersion to the space of positive definite self-adjoint matrices:

$$\begin{aligned} F : \text{GL}(n, \mathbb{F}) &\rightarrow \text{Sym}_n^+(\mathbb{F}). \\ A &\mapsto AA^*. \end{aligned}$$

Note that $\text{Sym}_n^+(\mathbb{F}) \subset \text{Sym}_n(\mathbb{F})$ is an open convex cone of a real vector space and diffeomorphic to a Euclidean space. Finally we observe that this submersion is also proper as $A_k A_k^* \rightarrow \infty$ when $A_k \rightarrow \infty$. In particular, we can use Ehresman's theorem 3.3.4 to conclude that $\text{GL}(n, \mathbb{F})$ is diffeomorphic to $\text{Sym}_n^+(\mathbb{F}) \times F^{-1}(I)$. The fiber over the identity is identified with the orthogonal group:

$$O(n) = \{O \in \text{GL}(n, \mathbb{R}) \mid OO^* = I\}$$

or the unitary group

$$U(n) = \{U \in \text{GL}(n, \mathbb{C}) \mid UU^* = I\}$$

and both are compact Lie groups. We note that left translates $l_A F^{-1}(I) = A \cdot F^{-1}(I)$ are diffeomorphic to each other and $A \cdot F^{-1}(I) \subset F^{-1}(AA^*)$. Thus fibers are precisely the left translates of the orthogonal or unitary groups. This is the content of the polar decomposition for invertible matrices.

These two families of groups can be intersected with the special linear groups to obtain the special orthogonal groups $\text{SO}(n) = O(n) \cap \text{SL}(n, \mathbb{R})$ and the special unitary groups $\text{SU}(n) = U(n) \cap \text{SL}(n, \mathbb{C})$.

It is not immediately clear that these new groups have well-defined smooth structures. However, it follows from the canonical forms of orthogonal matrices that $\text{SO}(n)$ is the connected component of $O(n)$ that contains I . The other component consists of the orthogonal matrices with $\det = -1$.

For the unitary group we obtain a Lie group homomorphism $\det : U(n) \rightarrow S^1 \subset \mathbb{C}$ where all values are regular values.

The tangent spaces are the traceless skew-adjoint matrices. In the real case skew-adjoint matrices are skew-symmetric and thus automatically traceless, this conforms with $\text{SO}(n) \subset O(n)$ being open. In the complex case, the skew-adjoint matrices have purely imaginary entries on the diagonal so the additional assumption that they be traceless reduces the real dimension by 1, this conforms with 1 being a regular value of $\det : U(n) \rightarrow S^1$.

4.2.4. The Matrix Exponential. The matrix exponential map $\exp : \text{Mat}_{n \times n}(\mathbb{F}) \rightarrow \text{Gl}(n, \mathbb{F})$ is defined using the usual power series expansion. The relationship

$$\det \exp(A) = \exp(\text{tr}A)$$

shows that its image is in the general linear group and in case $\mathbb{F} = \mathbb{R}$ that it maps into the matrices with positive determinant.

It also commutes with the operation of taking adjoints $\exp A^* = (\exp A)^*$. This shows that we obtain the following restrictions

$$\begin{aligned} \exp : T_I O(n) &\rightarrow SO(n), \\ \exp : T_I U(n) &\rightarrow U(n), \\ \exp : T_I SU(n) &\rightarrow SU(n), \end{aligned}$$

as well as

$$\exp : \text{Sym}_n(\mathbb{F}) = T_I \text{Sym}_n^+(\mathbb{F}) \rightarrow \text{Sym}_n^+(\mathbb{F}).$$

These maps are all surjective. In all cases this uses that a matrix in the target can be conjugated to a nice canonical form, OCO^* , where C is diagonal in the last three cases. In the first case it has a block diagonal form that consists of 2×2 rotations and diagonal entries that are ± 1 . In the unitary case the diagonal entries are of the form $e^{i\theta}$. Thus $C = \exp(iD)$, where D is a real diagonal matrix, and $OCO^* = O \exp(iD) O^*$. Similarly, in the last case C is a diagonal matrix with positive entries and $C = \exp(D)$ for a unique diagonal matrix D with real entries. The first case is the most intricate. First observe that rotations do come from skew-symmetric matrices:

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} = \exp \begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix}.$$

This also takes care of pairs of real eigenvalues of the same sign as they correspond to rotations where $\theta = 0$ or π . Since elements in $SO(n)$ have determinant 1 we can always ensure that the real eigenvalues get paired up except when n is odd, in which case the remaining eigenvalue is 1.

The polar decomposition diffeomorphism $\text{Gl}(n, \mathbb{C}) \cong \text{Sym}_n^+(\mathbb{R}) \times U(n)$ now tells us that $\text{Gl}(n, \mathbb{C})$ is connected. Similarly, $\text{Gl}^+(n, \mathbb{R}) \simeq \text{Sym}_n^+(\mathbb{R}) \times SO(n)$ is connected. As the elements of $O(n)$ with determinant -1 are diffeomorphic to $SO(n)$ via multiplication by any reflection in a coordinate hyperplane it follows that $\text{Gl}(n, \mathbb{R})$ has precisely two connected components.

4.2.5. Low Dimensional Groups and Spheres. There are several interesting connections between low dimensional Lie groups and low dimensional spheres.

First we note that rotations in the plane are also complex multiplication by numbers on the unit circle $S^1 \subset \mathbb{C}$ so:

$$SO(2) = U(1) = S^1.$$

The 3-sphere can be thought of as the unit sphere $S^3 \subset \mathbb{C}^2$ and thus

$$S^3 = \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}.$$

On the other hand:

$$SU(2) = U(2) \cap SI(2, \mathbb{C}) = \left\{ \begin{bmatrix} z & -\bar{w} \\ w & \bar{z} \end{bmatrix} \in U(2) \mid z\bar{z} + w\bar{w} = 1 \right\}$$

so we have:

$$SU(2) = S^3.$$

Next we note that

$$\begin{aligned} SO(3) &= \left\{ \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} \mid e_i \cdot e_j = \delta_{ij}, \det \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} = 1 \right\} \\ &= \left\{ \begin{bmatrix} e_1 & e_2 & e_1 \times e_2 \end{bmatrix} \mid e_1 \cdot e_2 = 0, |e_1| = |e_2| = 1 \right\} \\ &= T^1S^2 \end{aligned}$$

where $T^1S^2 = \{(p, v) \mid |p| = |v| = 1, p \cdot v = 0\}$ is the set of unit tangent vectors.

There is another important identification for this space

$$SO(3) = \mathbb{RP}^3.$$

This comes from exhibiting a homomorphism $SU(2) \rightarrow SO(3)$ whose kernel is $\{\pm I\}$. This shows that via the identification $SU(2) = S^3$ the preimages are precisely antipodal points. The specifics of the construction take a bit of work and will also lead us to quaternions. First make the identification

$$\mathbb{C}^2 = \left\{ \begin{bmatrix} z & -\bar{w} \\ w & \bar{z} \end{bmatrix} \mid (z, w) \in \mathbb{C}^2 \right\}.$$

On the right hand side we obtain a collection of matrices that is closed under addition and multiplication by real scalars. Since \mathbb{C} is a commutative algebra the right hand side is also closed under multiplication. Thus it forms an algebra over \mathbb{R} . It is also a division algebra as non-zero elements have $\det = |z|^2 + |w|^2 > 0$ and thus have inverses. This is the algebra of quaternions also denoted \mathbb{H} . Note that $X \in \mathbb{C}^2$ has Euclidean length

$$|X|^2 = |z|^2 + |w|^2 = \det X.$$

Any $A \in SU(2)$ acts by conjugation on this algebra as follows

$$A \cdot X = AXA^*.$$

The map $X \mapsto AXA^*$ is an orthogonal transformation as it doesn't alter the Euclidean length of X :

$$|AXA^*|^2 = \det(AXA^*) = \det A \det X \det A^* = \det X = |X|^2.$$

A natural orthonormal basis is given by

$$1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, i = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, j = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, k = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}.$$

Note that these matrices each have Euclidean norm $\sqrt{2}$. So the inner product is scaled to make them have norm 1. The last matrix is defined so that we obtain

$$\begin{aligned} ij &= k = -ji, \\ jk &= i = -kj, \\ ki &= j = -ik, \\ i^2 &= j^2 = k^2 = -1. \end{aligned}$$

Conjugation of a quaternion X is defined as

$$\overline{a+ib+jc+kd} = a-ib-jc-kd, a, b, c, d \in \mathbb{R}$$

or

$$\overline{z+jw} = \bar{z} - j\bar{w}, z, w \in \mathbb{C}.$$

In particular, $jX = \bar{X}j$ for any $X \in \mathbb{H}$. Warning: This conjugation is not the same as conjugation of complex 2×2 matrices! Below we switch to lower case letters when we think of quaternions this way.

Clearly conjugation by elements $A \in SU(2)$ fixes 1 so it also fixes the orthogonal complement spanned by i, j, k . Thus we obtain a homomorphism $\phi : SU(2) \rightarrow SO(3)$ by letting $A \in SU(2)$ act by conjugation on $\text{span}_{\mathbb{R}}\{i, j, k\}$. This map has constant rank and is in fact a submersion. Note that $T_I SU(2) = \text{span}_{\mathbb{R}}\{i, j, k\}$. The differential at $A = I$ is calculated by considering the path $a(t) = 1 + th$, $h \in \text{span}_{\mathbb{R}}\{i, j, k\}$:

$$(I + th)x(I - th) = I + t(hx - xh) + o(t).$$

Thus $D\phi(h)$ is the skew-symmetric map

$$x \mapsto hx - xh.$$

If this map vanishes then $jh = hj$ which shows that $h = \bar{h}$ and consequently $h \in \text{span}_{\mathbb{R}}(1)$ showing that $h = 0$. In particular, the image of ϕ is both open and closed and hence all of $SO(3)$.

Similarly, we note that the kernel of this map consists of quaternions $a \in \mathbb{H}$ that commute with all elements in \mathbb{H} since $ax = xa$. In particular, $aj = j\bar{a}$ which shows that $A \in \text{span}_{\mathbb{R}}(1)$. Consequently, the only possibilities are $\pm I = \pm 1$.

From all of this we obtain a special proof of the ‘‘Hairy Ball Theorem’’ for S^2 .

THEOREM 4.2.1. *Every vector field on S^2 vanishes somewhere.*

PROOF. The proof is by contradiction. If we have a non-zero vector field, then we also have a unit vector field $p \mapsto (p, v(p)) \in US^2$. This gives us a diffeomorphism

$$\begin{aligned} SO(3) &\rightarrow S^2 \times S^1 \subset S^2 \times \mathbb{R}^2 \\ [p, e, p \times e] &\mapsto (p, e \cdot v(p), (p \times e) \cdot v(p)). \end{aligned}$$

This contradicts that $SO(3) = \mathbb{RP}^3$, and hence has universal cover S^3 , as $S^2 \times S^1$ has a non-compact simply connected cover $S^2 \times \mathbb{R}$. \square

4.2.6. The Symplectic Group. There is another sequence of matrix groups, called the *symplectic groups*, with a definition that is similar to that of the orthogonal and unitary groups. They can be defined as

$$Sp(n) = \{X \in \text{Mat}_{n \times n}(\mathbb{H}) \mid XX^* = I\},$$

i.e., quaternionic matrices whose inverses are the adjoints. Here the adjoint is the conjugate transpose.

The natural inclusions $Gl(n, \mathbb{R}) \subset Gl(n, \mathbb{C}) \subset Gl(n, \mathbb{H})$ lead to the inclusions $O(n) \subset U(n) \subset Sp(n)$. Conversely, if we successively identify

$$\begin{aligned} \mathbb{C} &= \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \mid a, b \in \mathbb{R} \right\} \simeq \mathbb{R}^2, \\ \mathbb{H} &= \left\{ \begin{bmatrix} z & -\bar{w} \\ w & \bar{z} \end{bmatrix} \mid z, w \in \mathbb{C} \right\} \simeq \mathbb{C}^2, \end{aligned}$$

then we have natural inclusions $Sp(n) \subset U(2n) \subset O(4n)$.

Evidently the complex numbers correspond to the matrices in $\text{Mat}_{2 \times 2}(\mathbb{R})$ that commute with

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

While the quaternions are the matrices in $\text{Mat}_{2 \times 2}(\mathbb{C})$ that commute with

$$\begin{bmatrix} 0 & -j \\ j & 0 \end{bmatrix}$$

since $jz = \bar{z}j$ for $z \in \mathbb{C}$. This also establishes another low dimensional coincidence: $Sp(1) \simeq SU(2)$.

The preimage theorem or the implicit function theorem can be used to show that $Sp(n)$ is a manifold and a Lie group in the same way as for $O(n)$ and $U(n)$. The real dimension of the space of self-adjoint quaternionic matrices is

$$n + 4 \binom{n}{2} = n + 2n(n-1) = 2n^2 - n$$

so $Sp(n)$ has dimension

$$4n^2 - 2n^2 + n = 2n^2 + n = n(2n+1).$$

The orthogonal, unitary, and symplectic groups are known as the *classical compact groups*. The subgroups $SO(n)$, $n = 3, 5, 6, 7, \dots$ and $SU(n)$, $n \geq 2$ are simple as are $Sp(n)$, $n \geq 1$. There are five more simple compact Lie groups, known as the *exceptional groups*. They are more difficult to construct as they depend on understanding the octonions. These numbers denoted \mathbb{O} have the additive structure of \mathbb{R}^8 and a multiplication that is not associative but where every nonzero number has an inverse. In particular, they do not form a matrix algebra like \mathbb{C} and \mathbb{H} .

The name, “symplectic” comes from a related subgroup in $Gl(2n, \mathbb{C})$ that consists of the matrices that preserve a symplectic form. A symplectic form is a nondegenerate skew-symmetric form.

If as above we choose to identify

$$\mathbb{H}^n = \mathbb{H} \oplus \dots \oplus \mathbb{H} = (\mathbb{C} \oplus j\mathbb{C}) \oplus \dots \oplus (\mathbb{C} \oplus j\mathbb{C}) \simeq \mathbb{C}^{2n},$$

then the matrix of the symplectic form looks like

$$\begin{bmatrix} \Omega_1 & 0 & \dots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & \Omega_1 \end{bmatrix}$$

where the entries are 2×2 matrices and the diagonal entries are:

$$\Omega_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

A more convenient representation is to identify

$$\mathbb{H}^n = \mathbb{C}^n \oplus j\mathbb{C}^n \simeq \mathbb{C}^{2n}.$$

In this case the symplectic form takes the form

$$\Omega_n = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix},$$

where the entries are $n \times n$ matrices.

A matrix $X \in Gl(2n, \mathbb{C})$ is symplectic if it preserves the bilinear form represented by Ω_n :

$$X^t \Omega_n X = \Omega_n.$$

If we write $X = \begin{bmatrix} A & C \\ B & D \end{bmatrix}$, then the conditions become

$$\begin{aligned} B^t A &= A^t B, \\ D^t C &= C^t D, \\ B^t C - A^t D &= -I_n. \end{aligned}$$

The group represented by these matrices is denoted by $Sp(2n, \mathbb{C})$ and called the (complex) symplectic group. The symplectic group from above is in this representation described as:

$$Sp(n) = U(2n) \cap Sp(2n, \mathbb{C}).$$

To make this more explicit note that if $X \in U(2n)$, then preserving the symplectic form is equivalent to

$$\Omega_n X = \bar{X} \Omega_n.$$

This is analogous to how quaternions were constructed as 2×2 matrices with Ω_n behaving as multiplication by j . The matrix representation then looks like:

$$X = \begin{bmatrix} A & -\bar{B} \\ B & \bar{A} \end{bmatrix}$$

with the additional constraints that

$$AA^* + \bar{B}B^t = I_n, AB^* - \bar{B}A^t = 0$$

to ensure that the matrix is unitary.

4.3. The Exponential Map

It is easy to verify that the matrix exponential satisfies the law of exponents $\exp(A+B) = \exp A \exp B$ when A, B commute. In particular, the map $t \mapsto \exp(tA)$ is a homomorphism from the Abelian Lie group $(\mathbb{R}, +)$ to GL_n . This one-parameter group is also the integral curve through e for the left-invariant vector field defined by $X|_g = gA$ since

$$\left. \frac{d \exp(tA)}{dt} \right|_{t=t_0} = \exp(t_0 A) \left. \frac{d \exp(sA)}{ds} \right|_{s=0} = (\exp(t_0 A)) A.$$

With this in mind we define the exponential map on a general Lie group, $\exp: T_e G \rightarrow G$, by declaring $t \mapsto \exp(tX)$ to be the integral curve through e of the left-invariant field defined by $X|_g = D l_g(X)$, i.e., $\exp(tX) = \Phi_X^t(e)$. Smoothness of \exp then follows from remark 2.2.6. All of the integral curves for X can be recovered with

$$l_g \exp(tX) = g \exp(tX) = \Phi_X^t(g)$$

as X is l_g -related to itself.

If $\phi: G_1 \rightarrow G_2$ is a homomorphism and Y is the left invariant field defined by $D\phi(X)|_e$ then X and Y are ϕ -related and hence by proposition 2.2.7

$$\phi(\Phi_X^t(e)) = \Phi_Y^t(e).$$

In other words

$$\phi(\exp(tX)) = \exp(tD\phi|_e(X))$$

and the diagram

$$\begin{array}{ccc} T_e G & \xrightarrow{D\phi} & T_e H \\ \exp \downarrow & & \exp \downarrow \\ G & \xrightarrow{\phi} & H \end{array}$$

is commutative.

PROPOSITION 4.3.1. *The exponential map has the following properties.*

- (1) $D \exp : T_0 T_e G \rightarrow T_e G$ is an isomorphism. In particular, there is a neighborhood U around $0 \in T_e M$ such that $\exp : U \rightarrow \exp(U)$ is a diffeomorphism.
- (2) $\exp(sX + tY + O(s^2 + t^2)) = \exp(sX) \exp(tY)$. In particular, for integers m we have

$$\exp(X + Y) = \lim_{m \rightarrow \infty} \left(\exp\left(\frac{1}{m}X\right) \exp\left(\frac{1}{m}Y\right) \right)^m.$$

- (3) If $T_e G = V \oplus W$, then the map $(X, Y) \mapsto \exp(X) \exp(Y)$, $X \in V$, $Y \in W$ is a diffeomorphism near the origin onto a neighborhood of $e \in G$.

PROOF. Recall that we can identify $T_0 T_e G \simeq T_e G$ by sending $X \in T_e G$ to $\frac{d}{dt}(tX)|_0 \in T_0 T_e G$. As $\frac{d \exp(tX)}{dt}|_{t=0} = X$ we see that $D \exp|_0 = id_{T_e G}$. This proves (1).

For (2) let \log be the inverse of \exp on a neighborhood of $e \in G$ and consider the two maps

$$\begin{aligned} (s, t) &\mapsto \log(\exp(sX) \exp(tY)), \\ (s, t) &\mapsto sX + tY. \end{aligned}$$

From (1) it follows that the derivatives $\frac{\partial}{\partial s}|_{(0,0)} = X$ and $\frac{\partial}{\partial t}|_{(0,0)} = Y$ are the same for both maps. This proves the first part of the claim. For the second claim let $s = t = \frac{1}{m}$ and note that

$$\exp\left(\frac{1}{m}X\right) \exp\left(\frac{1}{m}Y\right) = \exp\frac{1}{m}\left(X + Y + O\left(\frac{1}{m}\right)\right).$$

Thus

$$\left(\exp\left(\frac{1}{m}X\right) \exp\left(\frac{1}{m}Y\right) \right)^m = \exp\left(X + Y + O\left(\frac{1}{m}\right)\right)$$

when m is an integer and the claim follows by letting $m \rightarrow \infty$.

For (3) we again use (1) and the identification $T_0 T_e G \simeq T_e G = V \oplus W$. As in (2) we note that

$$(s, t) \mapsto \log(\exp(sX) \exp(tY)), \quad X \in V, Y \in W$$

again has partial derivatives at $(0, 0)$ that respect the splitting $T_e G = V \oplus W$. This shows that

$$\begin{aligned} V \oplus W &\rightarrow T_e G \\ (X, Y) &\mapsto \log(\exp(X) \exp(Y)) \end{aligned}$$

is nonsingular at the origin which proves the claim. \square

THEOREM 4.3.2. *Let G and H be Lie groups with H connected. A homomorphism $\phi : H \rightarrow G$ is uniquely determined by its differential $D\phi|_e : T_e H \rightarrow T_e G$. In particular, a connected Lie subgroup $H \subset G$ is determined by $T_e H \subset T_e G$.*

PROOF. Part (1) of the previous proposition together with $\phi(\exp X) = \exp(D\phi|_e X)$ shows that the $D\phi|_e$ determines ϕ in a neighborhood of the identity. We also have $\phi(g \exp X) = \phi(g) \exp(D\phi|_e X)$ so in a neighborhood of any $g \in H$ the map ϕ is determined by $\phi(g)$ and $D\phi|_e$. Thus any two homomorphisms with the same differential at e must agree on a set that is clearly closed and by what we just saw also open. This establishes the claim. \square

With the use of the exponential map we can also offer a very simple topological criterion for when a subgroup is an embedded Lie group. However, most embedded subgroups are also preimages of submersions so we can generally apply the preimage theorem 1.4.24 or constant rank theorem as explained in remark 1.4.25.

THEOREM 4.3.3 (Cartan). *A closed subgroup $H \subset G$ of a Lie group, is an embedded submanifold and, in particular, also a Lie group.*

PROOF. Define the tangent space to H inside $T_e G$ as

$$V = \{X \in T_e G \mid \exp(tX) \in H \text{ for all } t \in \mathbb{R}\}$$

and let W be a complement such that $T_e G = \text{span}(V) \oplus W$.

Clearly $\alpha X \in V$ if $X \in V$ for any $\alpha \in \mathbb{R}$. If $X, Y \in V$, then the formula

$$\exp t(X+Y) = \lim_{m \rightarrow \infty} \left(\exp\left(\frac{1}{m}tX\right) \exp\left(\frac{1}{m}tY\right) \right)^m$$

shows that $X+Y \in H$ as the right-hand side is the limit of elements in H and H is closed. This shows that V is a vector space.

Consider the restriction $\exp : V \rightarrow H$. We claim that this is a bijection near the origin. If not, then we can find $h_m \in H$, with $h_m \rightarrow e$ such that $h_m = \exp X_m \exp Y_m$ where $X_m \in V$, $Y_m \in W - \{0\}$. Here $X_m, Y_m \rightarrow 0$ and we can assume that $\frac{Y_m}{|Y_m|} \rightarrow Y \in W$. Note that $\exp Y_m \in H$ as $h_m \cdot \exp X_m \in H$. For fixed $t \in \mathbb{R} - \{0\}$ let k_m be the integer closest to $\frac{t}{|Y_m|}$ so that $k_m Y_m \rightarrow tY$. Now note that

$$\exp k_m Y_m = (\exp Y_m)^{k_m} \in H$$

and has $\exp tY$ as a limit. Since H is closed we have shown that $Y \in V$ which is a contradiction.

The map $(X, Y) \mapsto \exp X \exp Y$, $(X, Y) \in V \oplus W$ is a chart around e and restricts to $\exp : V \rightarrow H$ when $Y = 0$. This gives a chart around $e \in H$. By left translation we obtain a chart around every point in H . This submanifold is properly embedded as it is a closed subset. \square

4.4. Coverings and Quotients of Lie Groups

THEOREM 4.4.1. *A surjective Lie group homomorphism $\phi : G_1 \rightarrow G_2$ with a differential that is bijective is a covering map. Moreover, when G_1 is connected the kernel is central and in particular Abelian.*

PROOF. Consider a surjective Lie group homomorphism $\phi : G_1 \rightarrow G_2$. The kernel, $\ker \phi$, is by definition the pre-image of the identity and the level of group theory there is an isomorphism $G_1/\ker \phi \rightarrow G_2$. We claim that the kernel acts properly discontinuously on G_1 provided the differential of ϕ is bijective. We can then invoke proposition 3.1.21 conclude that ϕ is a covering map.

By the regular value theorem $\ker \phi$ is a closed 0-dimensional submanifold of G . Thus we can select a neighborhood U around $e \in G_1$ that has compact closure, $\bar{U} \cap \ker \phi = \{e\}$, and is mapped diffeomorphically to $\phi(U)$. Using continuity of the group multiplication and that inversion is a diffeomorphism it follows that there is neighborhood around $e \in G_1$ such that $V^2 \subset U$ and $V^{-1} = V$ i.e., if $a, b \in V$ then $a \cdot b \in U$ and $a^{-1} \in V$. We claim that if $g, h \in \ker \phi$ and $g \cdot V \cap h \cdot V \neq \emptyset$, then $g = h$. In fact, if $g \cdot v_1 = h \cdot v_2$, then $g^{-1} \cdot h = v_2 \cdot v_1^{-1} \in U \cap \ker \phi$, which implies that $g^{-1} \cdot h = e$. Around any other point $g \in G_1$ we can use the neighborhood $V \cdot g$ to check that the kernel acts properly discontinuously.

Note that left translation by an element in kernel will yield a deck transformation and all deck transformations are obtained this way.

Finally assume that G_1 is connected. For a fixed $g \in G_1$ consider conjugation $x \mapsto gxg^{-1}$. We say that x is central if it commutes with all elements in G_1 . This comes down to checking that x is fixed by all conjugations. Now $\ker \phi \subset G_1$ is already a normal subgroup and thus preserved by all conjugations. Consider a path $g(t)$ from $e \in G_1$ to $g \in G_1$, then for fixed x we obtain a path $g(t) \cdot x \cdot (g(t))^{-1}$. When $x \in \ker \phi$ this path is necessarily in $\ker \phi$ and starts at x . However, $\ker \phi$ is discrete so the path must be constant. This shows that any $x \in \ker \phi$ commutes with all elements in G_1 . \square

There is also a converse to the above theorem.

THEOREM 4.4.2. *Let $f : \tilde{G} \rightarrow G$ be a covering map, with \tilde{G} connected. If G is a Lie group, then \tilde{G} has a Lie group structure that makes f a homomorphism. Moreover, the fundamental group of a connected Lie group is Abelian.*

PROOF. The most important and simplest case is when $\tilde{G} = \tilde{G}$ is simply connected. In that case we can simply use the unique lifting property to lift the composition map $\tilde{G} \times \tilde{G} \rightarrow G \times G \rightarrow G$ to a product structure on \tilde{G} . The inverse structure is created in a similar way. We then have to use the uniqueness of lifts to establish associativity as we would otherwise obtain two different lifts for multiplying three elements $\tilde{G} \times \tilde{G} \times \tilde{G} \rightarrow G$.

We can now invoke corollary 3.1.22 to see that $\tilde{G} = \tilde{G}/H$ for some group $H \subset \ker(\tilde{G} \rightarrow G)$. This also gives a group structure on \tilde{G} as H is central and in particular normal. This group structure is also a lift of $\tilde{G} \times \tilde{G} \rightarrow G \times G \rightarrow G$ and is consequently smooth. \square

We can now also address the question of when the coset space of a subgroup becomes a manifold.

THEOREM 4.4.3. *If $H \subset G$ is a closed subgroup of a Lie group, then the quotient space G/H is a manifold and $\pi : G \rightarrow G/H$ is a submersion.*

PROOF. By theorem 3.3.15 we have to check that the corresponding equivalence relation

$$R = \{(x, y) \in G \times G \mid xy^{-1} \in H\}$$

is a properly embedded submanifold such that the restriction $\pi_1|_R : R \rightarrow G$ is a submersion. Consider the smooth map $p : G \times G \rightarrow G$ defined by $p(x, y) = xy^{-1}$. Clearly $R = p^{-1}(H)$ and for a fixed $y \in G$ the map $x \mapsto p(x, y)$ is simply right translation by y^{-1} . This shows that the differential of p is always surjective. This shows that R is a properly embedded submanifold. Next observe that the map $G \times H \rightarrow G \times G$ defined by $(x, h) \mapsto (x, h^{-1}x)$ has image R . Moreover, the composition with π_1 is the projection $G \times H \rightarrow G$ which we know is a submersion. Thus also $\pi_1|_R$ becomes a submersion. \square

4.5. The Lie Group Lie Algebra Correspondence

We saw at the very end of section 4.1 that a connected Lie subgroup $H \subset G$ defines a subalgebra of left invariant fields $\mathfrak{h} \subset \mathfrak{g}$. The left translates of H form the coset space G/H . As subsets of G they are all submanifolds that create a foliation of G . The corresponding distribution consists of the left translates $Dl_g(T_e H)$. As we shall see this construction sets up a bijective correspondence between subalgebras and connected Lie subgroups.

THEOREM 4.5.1. *Let G be a Lie group. A subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is the Lie algebra for a unique connected Lie subgroup $H \subset G$.*

PROOF. The Lie algebra \mathfrak{h} consists of left invariant vector fields on G whose Lie brackets also lie in \mathfrak{h} . As such they define an involutive distribution. By Frobenius' theorem there is a unique maximal integral submanifold through $e \in G$. This submanifold $H \subset G$ is by definition connected and the left translates $l_g H$ are all maximal integral submanifolds for the distribution. Note that $g \in H \cap l_g H$ provided $g \in H$ and consequently $H = l_g H$. This shows that group multiplication on G defines a multiplication $H \times H \rightarrow H$. As $H \times H \rightarrow G$ is smooth proposition 2.4.4 guarantees that multiplication is also smooth in H .

It follows from theorem 4.3.2 that H is the only connected Lie subgroup with $T_e H \simeq \mathfrak{h}$. \square

This theorem can be used to construct homomorphisms from Lie algebra homomorphisms.

THEOREM 4.5.2. *Let G and H be connected Lie groups. Any Lie algebra homomorphism $L : \mathfrak{g} \rightarrow \mathfrak{h}$ corresponds to a unique homomorphism $\phi : \tilde{G} \rightarrow H$, where $\tilde{G} \rightarrow G$ is the universal cover of G .*

PROOF. It suffices to prove this in case G is itself simply connected as all covers of a Lie group have isomorphic Lie algebras. We start by observing that the graph of a smooth homomorphism $\phi : G \rightarrow H$

$$\text{Graph}(\phi) = \{(g, \phi(g)) \mid g \in G\} \subset G \times H$$

is a Lie subgroup that is isomorphic to G , via inclusion $g \mapsto (g, \phi(g))$ and projection $(g, h) \mapsto g$. Similarly the graph of a Lie algebra homomorphism $L : \mathfrak{g} \rightarrow \mathfrak{h}$

$$\text{Graph}(L) = \{(X, L(X)) \mid X \in \mathfrak{g}\} \subset \mathfrak{g} \times \mathfrak{h}$$

is a subalgebra isomorphic to \mathfrak{g} .

The graph $\text{Graph}(L)$ will by the previous theorem correspond to a unique maximal connected subalgebra $G' \subset G \times H$. The projection onto G restricts to a homomorphism $\pi_1|_{G'} = \pi : G' \rightarrow G$. By construction the tangent space $T_e G'$ is mapped isomorphically on to $T_e G$. Theorem 4.4.1 then tells us that π is a covering map. If we assume that G is simply connected, then π becomes a smooth isomorphism and the inverse followed by projection onto H defines the homomorphism whose differential corresponds to L . \square

COROLLARY 4.5.3. *For any Lie group homomorphism there is a commutative diagram of homomorphisms between Lie groups*

$$\begin{array}{ccc} G_1 & \xrightarrow{\phi} & G_2 \\ \downarrow & & \uparrow \\ G_1/\ker\phi & \rightarrow & \text{im}\phi \end{array}$$

These results lead to a Lie group-Lie algebra correspondence. One missing piece is Ado's theorem which we will not prove.

THEOREM 4.5.4 (Ado). *Each (complex) finite dimensional Lie algebra is a subalgebra of $\mathfrak{gl}(n, \mathbb{R})$ or $(\mathfrak{gl}(n, \mathbb{C}))$ for some n .*

Assuming Ado's theorem we obtain

THEOREM 4.5.5. *Each Lie group corresponds to a unique Lie algebra and each finite dimensional Lie algebra corresponds to a unique simply connected Lie group.*

There are however more direct proofs that construct a Lie group directly thus bypassing the issue of showing that all Lie algebras are matrix algebras. In outline one considers the image of

$$\begin{aligned} \text{ad} : \mathfrak{g} &\rightarrow \mathfrak{gl}(\mathfrak{g}) \\ X &\mapsto \text{ad}_X \end{aligned}$$

which creates a short exact sequence

$$0 \rightarrow \mathfrak{z} \rightarrow \mathfrak{g} \rightarrow \text{ad}(\mathfrak{g}) \rightarrow 0$$

where the kernel

$$\mathfrak{z} = \ker \text{ad} = \{X \in \mathfrak{g} \mid [X, Y] = \text{ad}_X Y = 0, \text{ for all } Y \in \mathfrak{g}\}$$

is the center of the Lie algebra. We can now put a Lie algebra structure on $\mathfrak{z} \times \text{ad}(\mathfrak{g})$ such it becomes isomorphic to \mathfrak{g} .

Here $\text{ad}(\mathfrak{g})$ is a matrix algebra with a corresponding Lie group $\text{Ad}(\mathfrak{g}) \subset \text{Gl}(\mathfrak{g})$. We will use the universal cover $\widetilde{\text{Ad}}(\mathfrak{g})$ as the Lie group that corresponds to $\text{ad}(\mathfrak{g})$. Similarly, for \mathfrak{z} we can use a vector space Z of the same dimension as a corresponding Lie group. It is then possible to create a short exact sequence

$$1 \rightarrow Z \rightarrow G \rightarrow \widetilde{\text{Ad}}(\mathfrak{g}) \rightarrow 1,$$

where G is a suitable semidirect product of Z and $\widetilde{\text{Ad}}(\mathfrak{g})$. The existence of this product depends crucially on using simply connected Lie groups, which in turn have the property that the second cohomology vanishes interpreted as Lie group and Lie algebra cohomology.

4.6. Actions and Exercises

Let G be a Lie group and M a connected manifold. An action of G on M is a generalization of left translation on a Lie group. It is a smooth map

$$\begin{aligned} G \times M &\rightarrow M \\ (g, x) &\mapsto gx \end{aligned}$$

such that $g_1(g_2x) = (g_1g_2)x$. It'll be convenient to introduce the action map

$$\begin{aligned} A : G \times M &\rightarrow M \times M \\ (g, x) &\mapsto (gx, x) \end{aligned}$$

Note that $\pi_2 \circ A(g, x) = x$ is a submersion.

The orbits are denoted by $G \cdot x = \{gx \mid g \in G\}$ and generate an equivalence relation. The corresponding relation $R \subset M \times M$ is in fact the image of the action map $R = A(G \times M)$. The quotient or orbit space is denoted by $G \backslash M$ as we are acting on the left. An action is *transitive* if $R = M \times M$ or equivalently $G \backslash M$ is one point. An action is *proper* if A is proper, in particular, actions by compact groups are always proper.

The *isotropy or stabilizer group* at $x \in M$ is $G_x = \{g \in G \mid gx = x\}$. As such, G_x is a closed subgroup and consequently also a Lie group by theorem 4.3.3. An action is *effective* if $\bigcap_{x \in M} G_x = \{e\}$, i.e., only the identity acts trivially on M . An action is *free* if $G_x = \{e\}$ for all $x \in M$.

- (1) Show that if a Lie (or just topological) group, G , is not connected, then the connected component, G_0 , containing e is an open and closed subgroup.

- (2) Let G be a connected Lie group. Show that G is generated by any neighborhood $U \ni e$. Hint A: Find a smaller neighborhood $e \in V \subset U$ such that $V^{-1} = V$ and consider $\bigcup_{m=1}^{\infty} V^m$, where $V^m = V^{m-1} \cdot V$. Hint B: Select a path $c : [0, 1] \rightarrow G$ with $c(0) = e$ and find a subdivision $0 = t_0 < \dots < t_k = 1$ such that $(c(t_{i-1}))^{-1} c(t_i) \in U$.
- (3) Show that a continuous homomorphism between Lie groups is necessarily smooth. (Hint: use the graph).
- (4) Show that the homomorphism $SU(2) \rightarrow SO(3)$ defined in section 4.2.5 is a submersion.
- (5) Show that $SU(2)/SO(2)$ is diffeomorphic to $S^2 = \mathbb{C}\mathbb{P}^1$.
- (6) Show that $Sp(1)$ is isomorphic to $SU(2)$.
- (7) Show that matrices in $Sp(2n, \mathbb{C})$ have $\det = 1$ and conclude that $Sp(n) \subset SU(2n)$.
- (8) Consider the two Lie groups $U(n)$ and $SU(n) \times S^1$.
- Show that they have isomorphic Lie algebras.
 - Show that they are not isomorphic by showing that their centers are not isomorphic.
 - Show that they are diffeomorphic by finding a section $s : S^1 \rightarrow U(n)$ for $\det : U(n) \rightarrow S^1$, i.e., $\det \circ s = id_{S^1}$.
 - Show that there is a homomorphism $SU(n) \times S^1 \rightarrow U(n)$ that is an n -fold covering map.
- (9) Let $\bar{G} \rightarrow G$ be a covering of Lie groups. Show that there is a natural isomorphism between the Lie algebras of these Lie groups.
- (10) Conjugation on a Lie group G defines what is called the *adjoint action*

$$\begin{aligned} \text{Ad} : G \times G &\rightarrow G \\ (g, x) &\mapsto \text{Ad}_g x = gxg^{-1} \end{aligned}$$

- Show that its differential with respect to x , $\text{Ad}_g = D\text{Ad}_g|_e$, defines an action on $T_e G = \mathfrak{g}$.
 - Show that taking the differential with respect to g , $\text{ad}_X = D\text{Ad}_g|_e$, defines a Lie algebra action on \mathfrak{g} , i.e., $\text{ad}_X \circ \text{ad}_Y - \text{ad}_Y \circ \text{ad}_X = \text{ad}_{[X, Y]}$.
 - Show that $\text{ad}_X(Y) = [X, Y]$.
- (11) The group GL_{n+1} acts on $\mathbb{F}\mathbb{P}^n$.
- Show that the action is not proper.
 - Show that the action is transitive.
 - Show that the isotropy group for $[1 : 0 : \dots : 0]$ is

$$\begin{bmatrix} \alpha & \alpha b^t \\ 0 & \alpha A \end{bmatrix}$$

where $\alpha \in GL_1 = \mathbb{F}^\times$, $b \in \mathbb{F}^n$, and $A \in GL_n$.

- Show that any element that acts trivially is a homothety λI , $\lambda \in \mathbb{F} - \{0\}$.
 - Show that the homotheties $C = \{\lambda I \mid \lambda \in \mathbb{F} - \{0\}\}$ are the center of GL_{n+1} and that $PL_{n+1} = GL_{n+1}/C$ is a Lie group that acts effectively on $\mathbb{F}\mathbb{P}^n$. PL stands for “projective linear”, the abbreviation PSL is also used as $SL \rightarrow PL$ is a proper submersion.
- (12) Let G be a Lie group. Show that there is a neighborhood around the identity such that the only subgroup in this neighborhood is the trivial subgroup. Hint: Use that the exponential map is a local diffeomorphism.
- (13) Show that if an embedded submanifold $H \subset G$ of a Lie group is a subgroup, then it is a closed subset of G . Recall that embedded submanifolds are in general

not closed subsets of the ambient space. Hint: Show that a slice neighborhood around e left translates to a slice neighborhood around any point in H .

- (14) Show that if a manifold has a group structure such that multiplication is smooth, then the inverse operation is also smooth. Hint: Consider the smooth map $(x, y) \mapsto (x, xy)$ and show that it is a bijection with non-singular differential at (e, e) .
- (15) Show that if a subgroup $H \subset G$ of a Lie group is an open subset, then it is also a closed subset.
- (16) Let $M = S^1$ and G the group with two elements that acts as a reflection in the x -axis. Is $R \subset S^1 \times S^1$ smoothly embedded?
- (17) Show that if $G \backslash M$ is a smooth manifold such that $M \rightarrow G \backslash M$ is a submersion, then $R \subset M \times M$ is properly embedded.
- (18) This is a generalization of theorem 4.4.1. Consider a smooth homomorphism $\phi : G \rightarrow H$, where H is connected. Show that this is a fibration when $D\phi|_e$ is surjective and in particular induces a smooth structure on the group $G/\ker \phi$ via H . Hint A: Lift left invariant fields to left invariant fields and argue as in theorem 3.3.4. Hint B: Use exponential maps to find $U \subset T_e G$ that $\exp \circ D\phi$ maps diffeomorphically on to a neighborhood of $e \in H$ and observe that the preimage is $\{g \exp U \mid g \in \ker \phi\} \simeq \exp(D\phi(U)) \times \ker \phi$.
- (19) Assume that the action of G on M is proper.
- Show that G_x is compact.
 - Show that there is a proper embedding $G/G_x \rightarrow M$ whose image is $G \cdot x$.
 - Show that $G \backslash M$ is Hausdorff and second countable.
- (20) Give an example of a free action of \mathbb{R} on $S^1 \times S^1$ that is not proper.
- (21) Assume that an action of G on M is proper and free.
- Show that $G \backslash M$ is a smooth manifold such that $M \rightarrow G \backslash M$ is a submersion.
 - Show that $M \rightarrow G \backslash M$ is a fibration. Hint: Theorem 3.3.4 cannot be applied directly, but, the proof can be adapted by showing that vector fields can be lifted to vector fields that are invariant under the action.

Transversality and Incidence Theory

The goal of this chapter is to introduce *transversality* and use it to define several important topological invariants such as degree, winding number, Lefschetz number, and Euler characteristic. In chapters 7 and 8 we will show that these invariants can also be calculated using de Rham cohomology. We prove several profound results that have been used widely in the literature: Brouwer's fixed point theorem, the Jordan-Brouwer separation theorem, the Borsuk-Ulam theorem, the Poincaré-Hopf and Lefschetz theorems, and finally the Hopf degree theorem. Each section contains one or more of these results.

5.1. Preimages

We say that a map $F : M \rightarrow N$ is transverse to a submanifold $S \subset N$ provided

$$T_{F(p)}S + DF(T_pM) = T_{F(p)}N$$

for all $p \in M$ with the property that $F(p) \in S$. When M is itself a submanifold of N , then F is the inclusion map. With this definition we obtain a very useful generalization of the preimage theorem.

THEOREM 5.1.1. *If $F : M \rightarrow N$ is transverse to a (properly) embedded submanifold $S \subset N$, then $F^{-1}(S) \subset M$ is a (properly) embedded submanifold. When $F^{-1}(S) \neq \emptyset$ its dimension satisfies:*

$$\text{codim}F^{-1}(S) = \dim M - \dim F^{-1}(S) = \dim N - \dim S = \text{codim}S.$$

PROOF. The preimage of S will be a closed subset of M provided S is a closed subset of N . To show the preimage is a submanifold fix $p \in F^{-1}(S)$ and let $q = F(p)$. Around q we can select coordinates $(y^1, \dots, y^n) : U \rightarrow \mathbb{R}^n$ such that $S \cap U = \{y^1 = 0, \dots, y^k = 0\}$, i.e., $0 \in \mathbb{R}^k$ is a regular value for $(y^1, \dots, y^k) : U \rightarrow \mathbb{R}^k$. Thus we have an open set $F^{-1}(U)$ around p such that $F^{-1}(U) \cap F^{-1}(S) = F^{-1}(U \cap S)$ is the preimage of $0 \in \mathbb{R}^k$ for the map $G = (y^1 \circ F, \dots, y^k \circ F)$. If $G(x) = 0$, then the kernel of $DG|_x$ consists of $(DF|_x)^{-1}(T_{F(x)}S)$. Let E_x be a complement to $(DF|_x)^{-1}(T_{F(x)}S) \subset T_xM$. The fact that F is transverse to S implies that $DF|_x(E_x) \oplus T_{F(x)}S = T_{F(x)}N$. The differential of $(y^1, \dots, y^k) : U \rightarrow \mathbb{R}^k$ is also surjective on $DF|_x(E_x)$ so it follows that $DG|_x : E_x \rightarrow T_0\mathbb{R}^k$ is surjective (in fact an isomorphism). This shows that $0 \in \mathbb{R}^k$ is a regular value for G and consequently that $F^{-1}(S)$ is a submanifold of codimension $k = \text{codim}S$. \square

COROLLARY 5.1.2. *Let $G : M \rightarrow N$ be transverse to an embedded submanifold $S \subset N$. A map $F : L \rightarrow M$ is transverse to $G^{-1}(S) \subset M$ if and only if $G \circ F$ is transverse to S .*

PROOF. This is essentially the second part of the above proof. Select $x \in L$ with $F(x) \in G^{-1}(S)$ and let $E_x \subset T_xL$ be a complement to

$$\begin{aligned} (DF|_x)^{-1}(T_{F(x)}G^{-1}(S)) &= (DF|_x)^{-1}\left((DG|_{F(x)})^{-1}T_{G \circ F(x)}S\right) \\ &= (D(G \circ F)|_x)^{-1}(T_{G \circ F(x)}S). \end{aligned}$$

As G is transverse to S it follows that $DF|_x(E_x)$ is a complement to $T_{F(x)}G^{-1}(S)$ if and only if $D(G \circ F)|_x(E_x)$ is a complement to $T_{G \circ F(x)}S$. \square

DEFINITION 5.1.3. *Manifolds with boundary* are defined like manifolds, but, modeled on open sets in $L^n = \{x \in \mathbb{R}^n \mid x^1 \leq 0\}$. The *boundary* ∂M is the set of points that correspond to elements in $\partial L^n = \{x \in \mathbb{R}^n \mid x^1 = 0\}$. The *interior* is $\text{int}M = M - \partial M$.

A map into L^n is smooth if it is smooth as a map into \mathbb{R}^n . A map on L^n is said to be smooth if in a neighborhood $p \in \partial L^n$ it can be extended to a smooth map on a neighborhood on $p \in \mathbb{R}^n$.

We need to show that the boundary is well-defined.

PROPOSITION 5.1.4. *Let $U, V \subset \mathbb{R}^n$ be open sets. If $F : U \rightarrow V$ is a diffeomorphism such that $F(U \cap L^n) = V \cap L^n$, then $F(U \cap \partial L^n) = V \cap \partial L^n$.*

PROOF. Since $F(\text{int}(U \cap L^n)) \subset \mathbb{R}^n$ is open it follows that $F(\text{int}(U \cap L^n)) \subset \text{int}L^n$, likewise $F^{-1}(\text{int}(V \cap L^n)) \subset \text{int}L^n$. This shows that $F(\text{int}(U \cap L^n) = \text{int}(V \cap L^n))$ and $F(U \cap \partial L^n) = V \cap \partial L^n$. \square

DEFINITION 5.1.5. It is not hard to prove that if $F : M \rightarrow \mathbb{R}$ has $a \in \mathbb{R}$ as a regular value then $F^{-1}(-\infty, a]$ is a manifold with boundary. If M is oriented, then the boundary is oriented in such a way that if we add the outward pointing normal to the boundary as the first basis vector then we get a positively oriented basis for M . Thus $\partial_2, \dots, \partial_n$ is the positive orientation for ∂L^n since ∂_1 points away from L^n and $\partial_1, \partial_2, \dots, \partial_n$ is the usual positive orientation for L^n .

When $F : M \rightarrow N$, then we denote the restriction to the boundary as $\partial F = F|_{\partial M}$.

THEOREM 5.1.6. *Let $F : M \rightarrow N$, where M has boundary. If $S \subset N$ has no boundary and both F and ∂F are transverse to S , then $F^{-1}(S)$ is a submanifold with $\partial(F^{-1}(S)) = F^{-1}(S) \cap \partial M$.*

PROOF. The transversality assumptions for F and ∂F at $x \in (\partial F)^{-1}(S)$ imply that we can find a subspace $E_x \subset T_x \partial M$ such that $E_x \oplus \ker D\partial F|_x = T_x \partial M$ and $E_x \oplus \ker DF|_x = T_x M$. In particular, $\ker DF|_x$ contains vectors that are not tangent to ∂M .

To see how this helps us we select coordinates around $q = F(p) \in S$, $p \in \partial M$, such that S is the preimage of $0 \in \mathbb{R}^k$. By also choosing coordinates around p we are reduced to a situation where $F : L^m \rightarrow \mathbb{R}^k$ and $0 \in \mathbb{R}^k$ is a regular value for both F and ∂F . By further restricting around $p \in \partial L$ we can assume that F extends to $\bar{F} : \mathbb{R}^m \rightarrow \mathbb{R}^k$ where $0 \in \mathbb{R}^k$ is a regular value. The preimage $\bar{F}^{-1}(0) \subset \mathbb{R}^m$ is a submanifold and

$$F^{-1}(0) = \{x \in \bar{F}^{-1}(0) \mid x^1(x) \leq 0\}.$$

Thus $F^{-1}(0)$ becomes a manifold with boundary $\partial F^{-1}(0) = \bar{F}^{-1}(0) \cap \partial L$ provided 0 is a regular value for $x^1|_{\bar{F}^{-1}(0)}$. This is equivalent to $\bar{F}^{-1}(0)$ being transverse to ∂L . If $x \in \bar{F}^{-1}(0) \cap \partial L$, then we saw at the beginning of the proof that

$$\ker DF|_p = T_x \bar{F}^{-1}(0) = T_x F^{-1}(0)$$

contains vectors that are not tangent to $T_x \partial M$. This shows that $\bar{F}^{-1}(0)$ is transverse to ∂L . \square

Before we can apply this to our first interesting result we need to classify one-dimensional manifolds.

THEOREM 5.1.7. *A connected one-dimensional manifold is diffeomorphic to either S^1 or \mathbb{R} when it has no boundary and either $[0, 1]$ or $[0, \infty)$ when it has boundary.*

PROOF. If M is orientable, then it has a vector field that never vanishes. It can be constructed locally to be nonzero and point in the positive direction and then using a partition of unity to create a global nowhere vanishing vector field. Any maximal integral curve will cover the manifold and thus parametrize it. To see this, assume that $c : (a, b) \rightarrow M$ is a maximal integral curve, where $0 \in (a, b)$. Any $x \in M$ is connected to $c(0)$ by a continuous path which is also a compact set $C \subset M$. The integral curve must either lie in C as $t \rightarrow b$ (or $t \rightarrow a$) or leave C . In the latter case it will hit x . In the former, the integral curve will have an accumulation point z as $t \rightarrow b$ (or $t \rightarrow a$). However, there is also an integral curve through z which must overlap with c and thus up to translation coincide with c . This shows that z lies in the image of c and thus that it was not a maximal integral curve.

If the manifold is not orientable, then the orientation covering has an involution that is orientation reversing. However, any diffeomorphism on $[0, \infty)$ clearly fixes the boundary. On $[0, 1]$ it either fixes the boundary points or reverses them, in the latter case the intermediate value theorem guarantees an interior fixed point as it must cross the diagonal $y = x$. When the manifold has no boundary we must in addition use that it is orientation reversing to get a fixed point. For \mathbb{R} the map is strictly decreasing and so is also forced to cross $y = x$. On S^1 we can lift the map to \mathbb{R} where it will have a fixed point. \square

COROLLARY 5.1.8. *A compact manifold with boundary admits no retractions onto the boundary.*

PROOF. Consider a map $F : M \rightarrow \partial M$, such that $F|_{\partial M} = id_{\partial M}$. If $p \in \partial M$ is a regular value, then $F^{-1}(p) \subset M$ is a one-dimensional manifold with $\partial(F^{-1}(p)) = F^{-1}(p) \cap \partial M = \{p\}$. Thus $F^{-1}(p)$ is noncompact and consequently M must also be noncompact. \square

COROLLARY 5.1.9 (Brouwer's Fixed Point Theorem). *Any continuous map on the closed unit ball in Euclidean space has a fixed point.*

PROOF. Consider a smooth map $F : \bar{B} \rightarrow \bar{B}$, where $\bar{B} \subset \mathbb{R}^n$ is the closed unit ball. If F has no fixed points, then there is a unique line through p and $F(p)$ for all $p \in \bar{B}$. Let $G(p) \in \partial \bar{B}$ be the point on this line closest to p . We offer an explicit formula by solving

$$|tp + (1-t)F(p)|^2 = 1.$$

This quadratic equation has no solutions on $(0, 1)$ as that corresponds to the point between p and $F(p)$ and there is a solution $t_0 \leq 1$ and another $t_1 \geq 1$. The latter corresponds to $G(p) = t_1p + (1-t_1)F(p)$. When $p \in \partial \bar{B}$ we have $G(p) = p$ so we can use the previous corollary to obtain a contradiction.

In case F is only continuous it can be approximated by a sequence smooth maps $G_i : \bar{B} \rightarrow \mathbb{R}^n$. Since they are close to F we can select $\epsilon > 0$ such that $F_i = (1-\epsilon)G_i$ maps into B for all large i . These maps have fixed points x_i . Selecting a convergent subsequence we obtain a limit that becomes a fixed point for F . \square

5.2. Thom's Transversality Theorem

Throughout the section we will consider maps from M^n (possibly with boundary) into N (without boundary). We are interested in finding maps that are transverse to a specific (properly) embedded submanifold $S \subset N$.

LEMMA 5.2.1. *Let L be a manifold without boundary and $F : M \times L \rightarrow N$. If F and ∂F are transverse to $S \subset N$, then $F_l : M \rightarrow N$ and $\partial F_l : \partial M \rightarrow N$ are transverse to S for almost all $l \in L$, where $F_l(x) = F(x, l)$.*

PROOF. Our assumptions allow us to conclude that $S^* = F^{-1}(S) \subset M \times L$ is a (properly) embedded submanifold with boundary $\partial S^* = S^* \cap \partial M \times L$. Consider the restriction of the projection onto L , $\pi : S^* \rightarrow L$. We claim that if $l \in L$ is a regular value for π and $\partial\pi$, then F_l and ∂F_l are transverse to S . For simplicity we focus on F_l as the argument is identical for ∂F_l .

For the given l consider $(x, l) \in S^*$ and $y = F(x, l) = F_l(x) \in S$. By assumption

$$DF|_{(x,l)}(T_x M \times T_l L) + T_y S = T_y N.$$

For any fixed $a \in T_y N$, there exists $(w, e) \in T_x M \times T_l L$ such that

$$DF|_{(x,l)}(w, e) - a \in T_y S.$$

Since l is a regular value for $\pi : S^* \rightarrow L$ we can also find $(u, e) \in T_{(x,l)} S^*$ such that $D\pi|_{(x,l)}(u, e) = e$. Now $DF|_{(x,l)}(u, e) \in T_y S$ as $S^* = F^{-1}(S)$. So if $v = w - u \in T_x M$, then

$$DF_l(v) - a = DF|_{(x,l)}(v, 0) = DF|_{(x,l)}(w, e) - a - DF|_{(x,l)}(u, e) \in T_y S.$$

This shows that F_l is transverse to $S \subset N$. \square

This lemma can be used to prove the Borsuk-Ulam theorem. A map $F : O \subset \mathbb{R}^m \rightarrow \mathbb{R}^n$ is said to be odd (or even), if O is invariant under $x \mapsto A(x) = -x$ and $F \circ A = A \circ F$ (or $F \circ A = -F$). Note that all matrices in $\text{Mat}_{n \times m}$ induce odd maps.

THEOREM 5.2.2. *The following statements are equivalent and true:*

- (1) *If $f : S^n \rightarrow \mathbb{R}^n$, then there exists $x \in S^n$ such that $f(x) = f(-x)$.*
- (2) *If $f : S^n \rightarrow \mathbb{R}^n$ is odd, then there exists $x \in S^n$ such that $f(x) = 0$.*
- (3) *There is no odd map $f : S^n \rightarrow S^{n-1}$.*

PROOF. We first establish equivalence and then prove (2).

Clearly (1) implies (2) and (2) implies (3). For (3) implies (1) simply note that if (1) fails for some f , then

$$g(x) = \frac{f(x) - f(-x)}{|f(x) - f(-x)|}$$

contradicts (3).

Let $L = \text{Mat}_{n \times (n+1)}^n$ be the open set of rank n matrices. If $B \in \text{Mat}_{n \times (n+1)}^n$, then it induces an odd map $B : S^n \rightarrow \mathbb{R}^n$ with exactly two zeros $\{\pm v_B\}$ both of which span $\ker(B : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n)$. Note that the rank n assumption also implies that 0 is a regular value for $B : S^n \rightarrow \mathbb{R}^n$.

Assume now that $f : S^n \rightarrow \mathbb{R}^n$ has no zeros and consider the linear homotopies

$$F(t, x, B) = H_B(t, x) = t f(x) + (1-t) B(x) : [0, 1] \times S^n \times \text{Mat}_{n \times (n+1)}^n \rightarrow \mathbb{R}^n.$$

We claim that F is transverse to $0 \in \mathbb{R}^n$. Suppose $F(t, x, B) = 0$. As f has no zeros we must have $t < 1$ and we can use that

$$DF|_{(t,x,B)}(0, 0, H) = (1-t) H(x).$$

Since $x \neq 0$ and $t < 1$ we can for any $u \in \mathbb{R}^n = T_0 \mathbb{R}^n$ find $H \in \text{Mat}_{n \times (n+1)}^n = T_B \text{Mat}_{n \times (n+1)}^n$ such that $u = (1-t) H(x)$. Thus F is actually a submersion when $t < 1$.

Lemma 5.2.1 then implies that there is a $B \in \text{Mat}_{n \times (n+1)}^n$ such that the homotopy H_B is transverse to $0 \in \mathbb{R}^n$. Let $N = H_B^{-1}(0)$. This a compact one-dimensional manifold with $\partial N = N \cap \{0, 1\} \times S^n$ showing that $\partial N = \{(0, \pm v_B)\}$. Thus N is a union of circles and one arc N_0 that joins the two zeros $\{\pm v_B\}$ on the boundary. Note that the homotopy is a homotopy of odd maps, $H_B(t, -x) = -H_B(t, x)$. Thus $A(N) = N$ and $A(N_0) = N_0$. Now parametrize the arc N_0 to be a unit speed curve $c : [0, b] \rightarrow [0, 1] \times S^n \subset \mathbb{R}^{n+2}$ with $c(0) = (0, v_B)$ and

$c(b) = (0, -v_B)$. Since A is an isometry that preserves N_0 we see that $A \circ c$ is also a unit speed curve with $A \circ c(0) = (0, -v_B)$ and $A \circ c(b) = (0, v_B)$. In other words $A \circ c$ is simply c parametrized backwards:

$$A(c(s)) = c(b - s).$$

This shows that $A\left(c\left(\frac{b}{2}\right)\right) = c\left(\frac{b}{2}\right) \in [0, 1] \times S^n$ which is impossible. \square

Since L is locally path connected lemma 5.2.1 shows that any F_l is homotopic to nearby maps that are transverse to $S \subset N$. This will allow us to show that any map is homotopic to a nearby map that is transverse. We will prove a slightly more complicated version that also works for maps that are sections, e.g., vector fields.

THEOREM 5.2.3 (Thom). *Any map $f : M \rightarrow N$ is homotopic to a nearby map that is transverse to $S \subset N$. Moreover, if f is a section for $\pi : N \rightarrow M$, i.e., $\pi \circ f = id_M$, then the homotopy $H : [0, 1] \times M \rightarrow N$ can be chosen so that all of the maps $H_t : M \rightarrow N$ are sections. Finally, if f is proper, then the homotopy is also proper.*

PROOF. In case were f is a section note that the property $\pi \circ f = id_M$ implies that $f : M \rightarrow N$ becomes an embedding. Moreover, $D\pi|_{f(x)} : T_{f(x)}N \rightarrow T_xM$ is a surjection and hence is a submersion on a neighborhood of $f(M)$. This also shows that it is transverse to each of the preimages of π . For the rest of the proof we will assume that N is a neighborhood of $f(M)$ on which π is a submersion.

Using a proper embedding $N \rightarrow \mathbb{R}^k$ we can orthogonally project the coordinate vector fields on to TN to obtain vector fields X_1, \dots, X_k on N that span the tangent space at every point. We can further orthogonally project on to the tangent spaces of the preimages of π to obtain vector fields Y_1, \dots, Y_k on N that span the tangent spaces to the preimages of π . Let $\Phi_1^{t_1}, \dots, \Phi_k^{t_k}$ be the flows of either X_1, \dots, X_k or Y_1, \dots, Y_k depending of which case we are considering. For each $y \in N$, there exists $0 < \epsilon(y) \ll 1$, such that

$$\begin{aligned} B(0, \epsilon(y)) &\rightarrow N, \\ (t_1, \dots, t_k) &\mapsto \Phi_1^{t_1} \circ \dots \circ \Phi_k^{t_k}(y) \end{aligned}$$

is a submersion to a neighborhood of y in N or in a preimage of π . The function $\epsilon(y)$ can be chosen to be smooth as the existence of flows is locally uniform. We can then scale the parameters $s_i = t_i / \epsilon(y)$ to obtain maps

$$\begin{aligned} B(0, 1) &\rightarrow N, \\ (s_1, \dots, s_k) &\mapsto \Phi_1^{\epsilon(y)s_1} \circ \dots \circ \Phi_k^{\epsilon(y)s_k}(y) \end{aligned}$$

that are submersions into N or preimages of π .

We claim that

$$\begin{aligned} F : M \times B(0, 1) &\rightarrow N, \\ (x, s_1, \dots, s_k) &\mapsto \Phi_1^{\epsilon(f(x))s_1} \circ \dots \circ \Phi_k^{\epsilon(f(x))s_k}(f(x)) \end{aligned}$$

is transverse to $S \subset N$. Note that $F_{(0, \dots, 0)}(x) = f(x)$. Moreover, the maps $F_{(s_1, \dots, s_k)} : M \rightarrow N$ are sections to π since the flows preserve preimages of π .

In case the vector fields span TN the map $(s_1, \dots, s_k) \mapsto F(x, s_1, \dots, s_k)$ is a submersion for each x and, in particular, transverse to S . In case the vector fields only span the tangent spaces to the preimages of π the whole map F becomes a submersion since each section $F_{(s_1, \dots, s_k)} : M \rightarrow N$ is transverse to the preimages of π .

The lemma 5.2.1 now guarantees a point (s_1, \dots, s_k) so that $F_{(s_1, \dots, s_k)} : M \rightarrow N$ is transverse to S . The homotopy is then defined by

$$H(t, x) = F(x, ts_1, \dots, ts_k).$$

Finally, assume f is proper. First observe that as $N \subset \mathbb{R}^k$ is properly embedded it follows that: $y_i \rightarrow \infty$ in N if and only if $y_i \rightarrow \infty$ in \mathbb{R}^k . Moreover, as the vector fields X_i or Y_i all have norm ≤ 1 in $TN \subset T\mathbb{R}^k$ we have that in the Euclidean distance:

$$\left| \Phi_1^{\epsilon(y)s_1} \circ \dots \circ \Phi_k^{\epsilon(y)s_k}(y) - y \right| \leq k\epsilon(y).$$

Thus $y_i \rightarrow \infty$ if and only if $\Phi_1^{\epsilon(y_i)s_1} \circ \dots \circ \Phi_k^{\epsilon(y_i)s_k}(y_i) \rightarrow \infty$. As f is proper we have that $y_i = f(x_i) \rightarrow \infty$ provided $x_i \rightarrow \infty$ in M . This shows that also $\Phi_1^{\epsilon(y_i)s_1} \circ \dots \circ \Phi_k^{\epsilon(y_i)s_k}(y_i) \rightarrow \infty$. In particular, each $F_{(s_1, \dots, s_k)} : M \rightarrow N$ is proper. This implies that the homotopy $H(t, x) = F(x, ts_1, \dots, ts_k)$ is proper when f is proper. \square

We obtain a series of useful consequences. The first is a relative version of the above theorem.

THEOREM 5.2.4. *If $f : M \rightarrow N$ is transverse to S for all $x \in C \cap f^{-1}(S)$, where $C \subset M$ is closed, then the homotopy can be chosen so that $H(t, x) = f(x)$ for all $x \in C$ and $x \mapsto H(1, x)$ is transverse to $S \subset N$.*

PROOF. We use the same notation as in theorem 5.2.3. Select a bump function $\lambda : M \rightarrow [0, 1]$ such that $\lambda^{-1}(0) = C$. Define

$$\begin{aligned} m : M \times B(0, 1) &\rightarrow M \times B(0, 1), \\ (x, s) &\mapsto (x, \lambda^2(x)s) \end{aligned}$$

and note that

$$Dm(v, e) = \left(v, \lambda^2(x)e + 2\lambda(x)d\lambda(v)s \right).$$

We can now define $G = F \circ m$ so that both $G(x, s) = f(x)$ and $DG|_{(x, s)}(v, e) = Df|_x(v)$ for $x \in C$. This shows that G is transverse to S for all $x \in C \cap f^{-1}(S)$. For $x \in M - C$ we have

$$DG|_{(x, s)}(0, e) = DF|_{(x, s)}(0, \lambda^2(x)e)$$

showing that $G_x : B(0, 1) \rightarrow N$ becomes a submersion. \square

COROLLARY 5.2.5. *Let $F : M \rightarrow N$. If $\partial F : \partial M \rightarrow N$ is transverse to $S \subset N$, then there is a homotopy $H : [0, 1] \times M \rightarrow N$, such that $H(t, x) = \partial F(x)$ for all $x \in \partial M$ and $x \mapsto H(1, x)$ is transverse to $S \subset M$.*

REMARK 5.2.6. In particular, if two maps are homotopic and transverse to S , then there exists a homotopy between the maps that is also transverse to S .

COROLLARY 5.2.7. *Any manifold admits a vector field that is transverse to the zero section $p \mapsto 0_p \in T_p M$.*

COROLLARY 5.2.8. *Any map $F : M \rightarrow M$ is homotopic to a map $G : M \rightarrow M$ such that*

$$\begin{aligned} (id_M, G) : M &\rightarrow M \times M \\ x &\mapsto (x, G(x)) \end{aligned}$$

is transverse to the diagonal $\Delta = \{(p, p) \mid p \in M\}$.

PROOF. Just use that (id_M, F) is a section of the projection $\pi_1 : M \times M \rightarrow M$ on to the first coordinate. \square

5.3. Mod 2 Intersection Theory

We start with some elementary observations about intersections of subspaces $V^k, W^l \in \mathbb{R}^n$. They will always intersect in the origin and when $k + l > n$ they actually intersect in a nontrivial subspace.

This leads to some observations about spheres and projective spaces. In S^2 any two great circles intersect in two points or coincide. However, we can always homotope one of these great circles away from the other. This means that the fact that great circles intersect is not a topological property. When we pass to the projective plane $\mathbb{R}P^2$, the great circles become projective lines $\mathbb{R}P^1 \subset \mathbb{R}P^2$ and they will intersect in one point or coincide (2-dimensional subspaces in \mathbb{R}^3 intersect in a line or coincide). In contrast to the sphere we will show that the fact that projective lines intersect is a topological property and cannot be changed via homotopies.

The general set-up in this section and the next is a map $F : M \rightarrow N$ where M is compact and N is connected. We wish to study how F intersects a closed submanifold $S \subset N$. When M has boundary we further assume that ∂F does not intersect S and, in particular, is transverse to S . If we assume that F is transverse to S and that

$$\dim M + \dim S = \dim N,$$

then $F^{-1}(S) \subset M$ is a finite collection of points none of which lie on the boundary. We define

$$I_2(F, S) = \#F^{-1}(S) \pmod{2} = \begin{cases} 0 & \text{if } \#F^{-1}(S) \text{ is even,} \\ 1 & \text{if } \#F^{-1}(S) \text{ is odd.} \end{cases}$$

We proceed to show that this intersection number is a homotopy invariant of F . Note by contrast that the number of preimages is not a homotopy invariant. In the next section we will define a more subtle integer valued intersection number.

THEOREM 5.3.1. *If $F_0, F_1 : M^m \rightarrow N^n$ are homotopic and transverse to $S^{n-m} \subset N$, then $I_2(F_0, S) = I_2(F_1, S)$. When $\partial M \neq \emptyset$, we assume that $\partial F_0 = \partial F_1$, does not intersect S , and that the homotopy is fixed on ∂M .*

PROOF. When $\partial M \neq \emptyset$ the space $[0, 1] \times M$ is not a manifold with boundary. However, we are assuming that any homotopy maps $[0, 1] \times \partial M$ to a set that is disjoint from S . Theorem 5.2.4 and its corollary can easily be reframed to work in this context. Thus we obtain a homotopy $H : [0, 1] \times M \rightarrow N$ that is transverse to S and such that $H([0, 1] \times \partial M) \cap S = \emptyset$. The preimage $H^{-1}(S) \subset [0, 1] \times \text{int}M$ is a compact one-manifold with boundary

$$\partial H^{-1}(S) = H^{-1}(S) \cap \{0, 1\} \times \text{int}M = \{0\} \times F_0^{-1}(S) \cup \{1\} \times F_1^{-1}(S).$$

As $\#\partial H^{-1}(S) = \#F_0^{-1}(S) + \#F_1^{-1}(S)$ is even it follows that the two terms on the right have the same parity. This proves the theorem. \square

This immediately explains why projective lines can't be homotoped away from each other as they intersect in one point. The theorem also allows us to define the intersection number of a map.

DEFINITION 5.3.2. *If $F : M^m \rightarrow N^n$ and $S^{n-m} \subset N$, then $I_2(F, S)$ is defined as the mod 2 intersection number of any map that is homotopic to F and transverse to S . In case M has boundary, ∂F does not intersect S and the homotopies are all equal to ∂F when restricted to ∂M .*

When $M^m \subset N^n$, then we define the intersection number as $I_2(M, S) = I_2(i, S)$, where $i : M \rightarrow N$ is the inclusion map.

We can now also place corollary 5.1.8 in a more general context. For this to be more clear note that $I_2(id_M, \{x\}) = 1$.

THEOREM 5.3.3. *Let B^{m+1} be a compact manifold with boundary $\partial B = M^m$ and $f : M^m \rightarrow N^n$ with $S^{n-m} \subset N^n$ a closed submanifold. If $f = \partial F$, where $F : B \rightarrow N$, then $I_2(f, S) = 0$.*

PROOF. We can use theorem 5.2.3 to find a map $G : B \rightarrow N$ that is homotopic to F and such that both G and ∂G are transverse to S . The preimage $G^{-1}(S)$ is a compact one-manifold with an even number of boundary components $\partial(G^{-1}(S)) = (\partial G)^{-1}(S)$. Thus $0 = I_2(\partial G, S) = I_2(f, S)$. \square

REMARK 5.3.4. All of the above results also work for proper maps in case M is not compact. Theorem 5.2.3 guarantees that proper maps are homotopic to proper maps that are transverse through homotopies that are proper. Likewise theorems 5.3.1 and its more general version 5.3.3 as long as we assume that the homotopy or extension map are proper.

DEFINITION 5.3.5. The mod 2 Euler characteristic of a manifold is defined as $\chi_2(M) = I_2(X, M_0)$, where $X : M \rightarrow TM$ is a vector field and $M_0 = \{(p, 0_p) \mid p \in M\}$ the zero section. Corollary 5.2.7 implies that this is well-defined as all vector fields are homotopy equivalent.

Similarly corollary 5.2.8 shows that the mod 2 Lefschetz number of a map $F : M \rightarrow M$, $L_2(F) = I_2((id_M, F), \Delta)$, is a well-defined homotopy invariant of F .

PROPOSITION 5.3.6. *We have $\chi_2(M) = L_2(id_M)$.*

PROOF. We identify M with the diagonal $\Delta \subset M \times M$ and TM with the normal bundle $T^\perp(\Delta) = \{(v, -v) \mid v \in TM\}$ to the diagonal in the product. For a vector field X on M we obtain a section $(X, -X)$ of $N(\Delta)$ that is homotopic to the zero section

$$\Delta_0 = \{(p, p), (0_p, -0_p) \mid p \in M\}.$$

This tells us that

$$L_2(id_M) = I_2((id_M, id_M), \Delta) = I_2(\Delta, \Delta) = I_2(\Delta_0, \Delta_0) = I_2((-X, X), \Delta_0) = I_2(X, M_0).$$

\square

The mod 2 Euler characteristic of a sphere is always 0. For odd dimensional spheres this is because they admit a nowhere vanishing vector field. For even dimensional spheres we can select such vector field on the equator and extend it to the entire space creating only two zeros. What is more interesting is that $\chi_2(\mathbb{R}P^{2n}) = 1$ as we can select a vector field on S^{2n} that is invariant under the antipodal map and has two zeros. In projective space this yields a vector field with one zero. By the same construction we also see that $\chi_2(\mathbb{R}P^{2n+1}) = 0$.

We can now prove another difficult result, the Jordan-Brouwer separation theorem.

THEOREM 5.3.7. *If $S \subset \mathbb{R}^{n+1}$ is a closed, connected, n -dimensional submanifold, then $\mathbb{R}^{n+1} - S$ has two connected components.*

Note that $\mathbb{R}P^n \subset \mathbb{R}P^{n+1}$ has a complement that is a disc and is thus connected. The theorem holds with virtually the same proof when the ambient space \mathbb{R}^{n+1} is replaced with a simply connected manifold. Note also that transversality can be used to show that the complement of a submanifold of codim ≥ 2 is always connected. Thus the complement of a finite set in a connected manifold is connected when the manifold has dimension ≥ 2 .

PROOF. We start with the observation that $I_2(f, S) = 0$ for any closed curve $f : S^1 \rightarrow \mathbb{R}^{n+1}$. This is where simple connectivity is used.

This observation implies that there exists a unit normal field $X : S \rightarrow \mathbb{R}^{n+1}$, i.e., $X(p) \perp T_p S$ for all $p \in S$. Note that at each point there are only two choices for this unit normal and the existence is equivalent to saying that S is orientable.

First note that the space of unit normal vectors

$$UN(S \subset \mathbb{R}^{n+1}) = \{v \in T_p \mathbb{R}^{n+1} \mid p \in S, v \perp T_p S, |v| = 1\}$$

is a two-fold covering space of S . We can now appeal to corollary 3.1.14 and obtain a section $S \rightarrow UN(S \subset \mathbb{R}^{n+1})$ provided the unique lift of a closed curve in S becomes a closed curve in $UN(S \subset \mathbb{R}^{n+1})$. Let $c : [0, 1] \rightarrow S$ be a curve, and $X : [0, 1] \rightarrow UN(S \subset \mathbb{R}^{n+1})$ the unique lift. In case $c(0) = c(1)$ we need to show that $X(0) = X(1)$. If not, then $X(0) = -X(1)$. Consider the curve $c_\epsilon(t) = c(t) + \epsilon X(t)$. Since X is nontrivial and transverse to S there must be a small ϵ such that $c_\epsilon(t)$ does not intersect S . Now join the end points $c(0) + \epsilon X(0)$ and $c(1) + \epsilon X(1) = c(0) - \epsilon X(0)$ by a straight line that intersects S orthogonally and only in $c(0) = c(1)$. This leads to a closed curve that intersects S only once, which is a contradiction.

We can now create a tubular neighborhood, more like a band neighborhood, by considering $H(s, p) = p + sX(p)$ on $(-\epsilon, \epsilon) \times S$. As S is compact and X transverse to S the differential of H is an isomorphism at all points $(0, p)$. Thus it is a diffeomorphism on a neighborhood of $\{0\} \times S$. By decreasing ϵ we obtain a diffeomorphism $H : (-\epsilon, \epsilon) \times S \rightarrow U$ onto a neighborhood of S .

This neighborhood allows us to deform curves $c : [0, 1] \rightarrow \mathbb{R}^{n+1}$ between points $p, q \in \mathbb{R}^{n+1} - S$ to curves with a minimal number of intersections with S . Note that $I_2(c, S)$ is well-defined for curves with $c(0) = p$ and $c(1) = q$ and is invariant under homotopies that fix p and q . We claim that given $p, q \in \mathbb{R}^{n+1} - S$ there is a curve that intersects S transversely and intersects S in $I_2(c, S)$ points, where c is any curve from p to q that is transverse to S . In the tubular neighborhood U we can write $c(t) = p(t) + s(t)X(p(t))$ and note that if $c(t_0) \in S$, i.e., $s(t_0) = 0$, then either it crosses from negative s to positive s , or the other way around. We say that it has a positive or negative crossing. Now assume that the first crossing t_0 is positive, the last crossing t_k can be negative or positive. If t_k is negative, then we can replace c on $[-\delta + t_0, t_k + \delta]$ by a curve in $H(\{-\delta\} \times S)$ (this is where connectivity of S is used) to obtain a new curve that does not intersect S . This gives a curve that does not intersect S provided $I_2(c, S) = 0$. While if t_k is positive we can replace c on $[-\delta + t_0, t_k - \delta]$ by a curve in $H(\{-\delta\} \times S)$ to obtain a new curve that intersects S in only one point (at t_k). This gives a curve that intersects S once when $I_2(c, S) = 1$.

We can now finish the proof. Fix $p_0 \in \mathbb{R}^{n+1} - S$ and define

$$O_{0 \text{ or } 1} = \{p \in \mathbb{R}^{n+1} - S \mid I_2(c, S) = 0 \text{ or } 1, c(0) = p_0, c(1) = p\}.$$

We claim that both sets are nonempty, open, and connected. Clearly $p_0 \in O_0$. For O_1 select a shortest line segment from p_0 to S . It'll intersect S orthogonally and its continuation will yield a slightly longer segment that intersects S orthogonally in exactly one point. Both sets are open as any point $p \in \mathbb{R}^{n+1} - S$ has a neighborhood $B(p, \delta) \subset \mathbb{R}^{n+1} - S$. Thus any point in $B(p, \delta)$ can be joined to p by a segment that doesn't intersect S , and hence to p_0 by a curve with the same intersection number as a curve from p_0 to p . Finally, both sets are connected since any two points $p, q \in O_{0 \text{ or } 1}$ are joined to p_0 by curves whose intersection number with S have the same parity. This leads to a concatenated curve from p to q with

an even number of intersections with S . By the above argument it can be replaced with a curve that doesn't intersect S . \square

5.4. Oriented Intersection Theory

We refine the mod 2 intersection numbers from the last section to integer valued intersection numbers provided all of the manifolds involved in our standard set-up

$$F : M \rightarrow N \supset S$$

are oriented. We shall further assume that M is closed.

5.4.1. The Oriented Intersection Number. Recall that an orientation for a vector space V is a choice of an equivalence class of ordered bases. It can be denoted $[V]$ or $[v_1, \dots, v_n]$ if it refers to a specific ordered basis.

Given a subspace $V_0 \subset V$ that also comes with an orientation we can select a unique orientation on a complement $V_0 \oplus V_1 = V$ so that a positively oriented basis on V_0 followed by a positively oriented on V_1 gives a positively oriented basis for V . We also write $[V_0] \oplus [V_1] = [V]$. Note, that we could also select an orientation $[V_1]'$ such that $[V_1]' \oplus [V_0] = [V]$. These orientations agree unless both subspaces are odd dimensional as it takes $\dim V_0 \cdot \dim V_1$ transpositions to switch the ordered bases from $V_0 \oplus V_1$ to $V_1 \oplus V_0$.

When M has boundary we orient the boundary by first declaring that outward pointing vectors are positively oriented: $[n_x] \oplus [T_x \partial M] = [T_x M]$. This is consistent with how $L^n = \{x \in \mathbb{R}^n \mid x^1 \leq 0\}$ and its boundary are oriented when we use $n = \partial_1$.

In case $M = [0, 1]$ we simply assign numbers ± 1 to the points on the boundary. Thus $\{1\}$ is assigned a $+1$ while $\{0\}$ gets a -1 . Note that these signs cancel. Thus any compact oriented one-manifold has the property that the sum of the signs assigned to the boundary points is 0. This will be fundamental for homotopy invariance of oriented intersection numbers.

Now suppose that $F : M \rightarrow N$ is transverse to S and that M, N, S are oriented manifolds. We wish to assign an orientation to $S^* = F^{-1}(S)$. Select $x \in S^*$ and a complement E_x to $T_x S^* \subset T_x M$, $E_x \oplus T_x S^* = T_x M$. Since F is transverse we note that $DF|_x(E_x) \oplus T_x S = T_x N$. Thus we select the orientation of this complement so that

$$[DF|_x(E_x)] \oplus [T_x S] = [T_x N].$$

Since E_x and $DF|_x(E_x)$ are isomorphic this also induces an orientation on E_x and thus we can orient S so that

$$[E_x] \oplus [T_x S] = [T_x M].$$

A slight consistency issue now develops when M has boundary and also ∂F is transverse. Here ∂S^* obtains two possible orientations, one as the boundary of S^* which is oriented by F and one simply via ∂F . We need to check what affects this possible difference in orientations. Fix $x \in \partial S^*$ and start by noting that an outward pointing $n_x \in T_x S^* \subset T_x M$ is also outward pointing for M . Next we need a complement E_x to $T_x S^* \subset T_x M$. We obtain such a complement by selecting a complement E_x for $T_x \partial S^* \subset T_x \partial M$, this will then also be a complement for $T_x S^* \subset T_x M$. We now have from S^* that ∂S^* gets oriented by via F by:

$$\begin{aligned} [E_x] \oplus [T_x S^*] &= [T_x M], \\ [n_x] \oplus [T_x \partial S^*] &= [T_x S^*]. \end{aligned}$$

In other words

$$[E_x] \oplus [n_x] \oplus [T_x \partial S^*] = [T_x M].$$

On the other hand via ∂F we obtain

$$[E_x] \oplus [T_x \partial S^*]' = [T_x \partial M],$$

where

$$[n_x] \oplus [T_x \partial M] = [T_x M],$$

i.e.,

$$[n_x] \oplus [E_x] \oplus [T_x \partial S^*]' = [T_x M].$$

Thus we conclude that

$$[n_x] \oplus [E_x] \oplus [T_x \partial S^*]' = [E_x] \oplus [n_x] \oplus [T_x \partial S^*].$$

Here the orientations $[n_x] \oplus [E_x]$ and $[E_x] \oplus [n_x]$ agree if $\dim E_x$ is even, and are opposite when $\dim E_x$ is odd. This shows that we have the predictable relationship

$$[T_x \partial S^*]' = (-1)^{\dim E_x} [T_x \partial S^*].$$

We can now define oriented intersection numbers. If $F : M \rightarrow N$ is transverse to S and M^m, N^n, S^{n-m} are oriented manifolds, then we assign a sign/orientation $[x] = \pm 1$ to each $x \in F^{-1}(S)$ with the understanding that it is $+1$ precisely when the orientation of $T_x M = E_x$ is mapped to the positive orientation for $DF(T_x M)$ that comes from

$$[DF(T_x M)] \oplus [T_{F(x)} S] = [T_{F(x)} N].$$

In other words:

$$[x] = \text{sign det } DF|_x,$$

where $\text{det } DF|_x$ is calculated with respect to positively oriented bases for $T_x M$ and $DF(T_x M)$.

THEOREM 5.4.1. *When $F = \partial G$, where $G : B \rightarrow N$ and B is compact and oriented, then*

$$\sum_{x \in F^{-1}(S)} [x] = 0.$$

PROOF. By theorem 5.2.4 we can assume that G is transverse to S . Here $G^{-1}(S)$ is a compact one-manifold with $\partial G^{-1}(S) = F^{-1}(S)$. Orientations assigned to points in $F^{-1}(S)$ differ by the same sign $(-1)^m$ depending on whether we use the definition from F or as the boundary of $G^{-1}(S)$. We conclude that they add up to 0 as they come in pairs of opposite signs corresponding to each arc in $G^{-1}(S)$. \square

REMARK 5.4.2. This shows that two homotopic and transverse maps on a closed manifold must have the same value for the sum $\sum [x]$. Also note that as in remark 5.3.4 we can generalize this theorem to the case where B is not compact provided G is proper.

DEFINITION 5.4.3. The oriented intersection number $I(F, S)$ is defined as

$$I(F_1, S) = \sum_{x \in F_1^{-1}(S)} [x]$$

for any map F_1 that is homotopic to F and transverse to S . This differs in absolute value from $\#F^{-1}(S)$ by cancelling pairs of opposite signs, in particular

$$I(F, S) = I_2(F, S) \pmod{2}.$$

When $M^m \subset N^n$ we obtain two possible intersection numbers $I(M, S)$ and $I(S, M)$. Since

$$[T_x M] \oplus [T_x S] = (-1)^{\dim M \dim S} [T_x S] \oplus [T_x M]$$

this intersection number vanishes when the submanifolds are odd dimensional.

Consider the intersection $\mathbb{F}\mathbb{P}^m, \mathbb{F}\mathbb{P}^{n-m} \subset \mathbb{F}\mathbb{P}^n$. When the subspaces are in generic position they will intersect in a point $\mathbb{F}\mathbb{P}^0$. When $\mathbb{F} = \mathbb{C}$ this is the oriented intersection number. When $\mathbb{F} = \mathbb{R}$ it is the mod 2 intersection number as at least one of the three spaces is even dimensional and so not orientable.

5.4.2. Degree and Winding Numbers. We can now also define the oriented degree of a map $F : M^n \rightarrow N^n$ between oriented manifolds where we assume that N is connected. Using remark 5.4.2 the degree for proper maps is also well-defined as long as we modify all extensions and homotopies to be proper.

We start by considering the intersection numbers $I(F, \{q\})$, $q \in N$. From lemma 3.1.1 it follows that if q is a regular value, i.e., F is transverse to $\{q\}$, then some connected neighborhood V is evenly covered: $F^{-1}(V) = \bigcup_{x \in F^{-1}(q)} U_x$, where $F : U_x \rightarrow V$ is a diffeomorphism. Thus DF preserves (or reverses) the orientation on U_x if it preserves (or reverses) orientations at just one point, i.e., $I(F, \{y\}) = I(F, \{q\})$ for all $y \in V$. Now for a given F there is always a map homotopic to F that is transverse to $\{q\}$. So we can again conclude that $I(F, \{y\}) = I(F, \{q\})$ for all y in a neighborhood of q . This means that $y \mapsto I(F, \{y\})$ is locally constant on N , and hence constant when N is connected.

DEFINITION 5.4.4. The oriented degree for $F : M^n \rightarrow N^n$ is well-defined as

$$\deg F = I(F, \{q\}), q \in N$$

when N is connected and F is proper.

We get several nice results using degree theory. The key observation is that the degree of a map is a homotopy invariant as it is simply an intersection number. However, as we can only compute degrees of proper maps it is important that the homotopies are through proper maps. When working on closed manifolds this is not an issue. However, if the manifold is Euclidean space, then all maps are homotopy equivalent, although not necessarily through proper maps.

PROPOSITION 5.4.5. *The identity map on a closed manifold is not homotopic to a constant map.*

PROOF. The constant map has degree 0 while the identity map has degree 1 on an oriented manifold. In case the manifold isn't oriented we can use the mod 2 degree. \square

THEOREM 5.4.6. *Even dimensional spheres do not admit non-vanishing vector fields.*

PROOF. A nowhere vanishing vector field X on S^n can be scaled so that it is a unit vector field. If we consider it as a function $X : S^n \rightarrow S^n \subset \mathbb{R}^{n+1}$, then it is always perpendicular to its foot point as $T_p S^n \perp p$ in \mathbb{R}^{n+1} . This yields a homotopy

$$H(p, t) = p \cos(\pi t) + X_p \sin(\pi t).$$

Since $p \perp X_p$ and both are unit vectors the Pythagorean theorem shows that $H(p, t) \in S^n$ as well. When $t = 0$ the homotopy is the identity, and when $t = 1$ it is the antipodal map. Since the antipodal map reverses orientations on even dimensional spheres it is not possible for the identity map to be homotopic to the antipodal map. \square

Next we offer two interesting results for proper maps. The first is related to corollary 3.1.16.

THEOREM 5.4.7. *If $F : M \rightarrow N$ is a proper nonsingular map of degree ± 1 between oriented connected manifolds, then F is a diffeomorphism.*

PROOF. Since F is non-singular everywhere it either reverses or preserves orientations at all points. Moreover by corollary 3.1.5 it is also a covering map. Thus $|\deg F| = \#F^{-1}(y)$ for all $y \in N$. This shows that it must be a diffeomorphism. \square

Next we offer an interesting and very broad extension of the Fundamental Theorem of Algebra.

THEOREM 5.4.8. *Let $F : M \rightarrow N$ be a proper map between oriented noncompact n -manifolds, where N is connected. If F is nonsingular and orientation preserving outside a compact set, then F is surjective.*

PROOF. We assume that all critical points lie in the compact set $C \subset M$ and consider a value $y \in F(M) - F(C)$. This is a regular value and by assumption $\deg F = \#F^{-1}(y) > 0$. In particular, F is surjective. \square

REMARK 5.4.9. Note that when $n = 1$ the function $f(x) = x^2$ is proper and nonsingular outside a compact set. When $n \geq 2$, it is often possible to ensure that $M - C$ is connected as long as M is itself connected. Thus it often suffices to assume that the map is proper and nonsingular outside a compact set.

The classical winding number for curves in the plane is the number of times a closed curve goes around a fixed point such as the origin. It can be calculated using degrees and as we shall see later also using integration.

DEFINITION 5.4.10. Let $F : M^n \rightarrow \mathbb{R}^{n+1}$, where M is closed and oriented. When $z \notin F(M)$ we define the winding number

$$W(F, z) = \deg \left(\frac{F(x) - z}{|F(x) - z|} : M \rightarrow S^n \right).$$

While it is simply a degree, and the degree is simply an intersection number, it is convenient to maintain these terminologies.

We note that $W(F, z) = W(F - z, 0)$ and that the winding number is a homotopy invariant under homotopies that map in to $\mathbb{R}^{n+1} - \{z\}$.

The winding number can also be calculated in a different way as an intersection number and in return intersection numbers can be calculated as degrees.

THEOREM 5.4.11. *Let $G : B^{n+1} \rightarrow \mathbb{R}^{n+1}$ have $0 \in \mathbb{R}^{n+1}$ as a regular value. If B is compact and oriented with boundary $M = \partial B$ and $F = \partial G$ does not contain 0 in its image, then*

$$W(F, 0) = I(G, \{0\}).$$

PROOF. We select pairwise disjoint coordinate balls $B_x \simeq B(0, \epsilon)$ around each $x \in G^{-1}(0)$ such that $G(h) = DG|_x h + o(h)$.

Let $N^{n+1} = B - \bigcup_{x \in G^{-1}(0)} B_x$. This is a new compact manifold with boundary $M \cup \bigcup_{x \in G^{-1}(0)} \partial B_x$. The boundaries ∂B_x come with two orientations. One from being the boundary of B_x and the opposite orientation from being part of the boundary of N . By theorem 5.4.1 we conclude that

$$0 = \deg \left(\frac{G}{|G|} : \partial N \rightarrow S^n \right).$$

Here the degree is the sum of the degrees from decomposing the boundary as $\partial N = M \cup \bigcup_{x \in G^{-1}(0)} \partial B_x$ but where the degrees from the restrictions to ∂B_x come with the opposite

sign. Thus

$$\deg\left(\frac{G}{|G|} : M \rightarrow S^n\right) = \sum_{x \in G^{-1}(0)} \deg\left(\frac{G}{|G|} : \partial B_x \rightarrow S^n\right),$$

when ∂B_x is oriented as the boundary of B_x .

We now have to calculate the terms on the right. Since $x \in G^{-1}(0)$ is regular the differential $DG|_x$ is nonsingular. In particular, we can assume that B_x is so small that $|o(h)| \ll |DG|_x h|$ for all $h \in \partial B_x$. Consequently, we obtain a homotopy from $DG|_x h$ to G defined by

$$H(t, h) = DG|_x h + t o(h) : [0, 1] \times \partial B_x \rightarrow \mathbb{R}^{n+1} - \{0\}.$$

This reduces the task to calculating the winding number of the differential. However, the space of nonsingular matrices $GL_n(\mathbb{R})$ has two components. The orientation preserving matrices contributing +1 and the orientation reversing matrices -1. This proves the theorem. \square

REMARK 5.4.12. In case 0 is not a regular value for G but still has a finite preimage that lies in the interior we instead obtain the formula

$$W(F, 0) = \deg\left(\frac{G}{|G|} : M \rightarrow S^n\right) = \sum_{x \in G^{-1}(0)} \deg\left(\frac{G}{|G|} : \partial B_x \rightarrow S^n\right) = \sum_{x \in G^{-1}(0)} W(G|_{\partial B_x}, 0).$$

as the condition that each $x \in G^{-1}(0)$ is regular is only used to calculate the local winding number and show that it agrees with the intersection number. Thus the total winding number can be split up into local winding numbers.

The above theorem also gives us a new proof of a stronger version of the Borsuk-Ulam theorem 5.2.2. Note that an even map on a sphere obviously has even degree.

THEOREM 5.4.13. *An odd map $F : S^n \rightarrow S^n$ has odd degree. In particular, there are no odd maps $S^n \rightarrow S^{n-1}$.*

PROOF. We use induction on n . For $n = 0$ we have $S^0 = \{\pm 1\}$. As the map is odd it is a bijection and so has degree ± 1 . When $n > 0$ select $S^{n-1} \subset S^n$ and $y \notin F(S^{n-1})$. As F is odd and S^{n-1} is invariant under the antipodal map also $-y \notin F(S^{n-1})$. We can additionally assume that $\{\pm y\}$ are regular values for F . Now project along great circles through $\pm y$ onto the orthogonal equator S_y^{n-1} to obtain a new odd map

$$G(x) = \frac{F(x) - \langle F(x), y \rangle y}{|F(x) - \langle F(x), y \rangle y|} = \frac{\pi \circ F}{|\pi \circ F|},$$

where π is the orthogonal projection along y in Euclidean space. Using counting and that F is odd we obtain

$$\begin{aligned} \deg F &= I(F, \{y\}) \\ &= \frac{1}{2} I(F, \{\pm y\}) \\ &= I(F|_{S_+^n}, \{\pm y\}), \text{ where } S_+^n \text{ is the hemisphere with pole } y \\ &= I(\pi \circ F, 0) \\ &= \deg G, \end{aligned}$$

where the previous theorem was used for the last equality. \square

5.4.3. Lefschetz numbers and the Euler Characteristic. To define Lefschetz numbers and the Euler characteristic we need to select orientations for $M \times M$ and TM . A closer look at how orientations are used tells us that the ambient space only needs to have orientations defined along the submanifolds that the maps are intersecting. For $\Delta \subset M \times M$ we note that $T_{(p,p)}M \times M = T_pM \times T_pM$ comes with a canonical orientation: any choice of an ordered basis e_1, \dots, e_m for T_pM gives the same choice of orientation

$$(e_1, 0), \dots, (e_m, 0), (0, e_1), \dots, (0, e_m).$$

Similarly for $M_0 \subset TM$, there is a natural identification $T_{0,p}TM = T_pM \times T_{0,p}T_pM = T_pM \times T_pM$ where the first factor corresponds to M .

DEFINITION 5.4.14. Let M be closed and oriented, $F : M \rightarrow M$, and X a vector field. The oriented Lefschetz number and Euler characteristic are defined by

$$L(F) = I((id_M, F), \Delta)$$

and

$$\chi(M) = I(X, M_0).$$

We can now reprove proposition 5.3.6.

PROPOSITION 5.4.15. For a closed and oriented manifold

$$L(id_M) = \chi(M).$$

PROOF. The proof is the same after we note that the identification of TM with $N(\Delta)$ respects the orientation choices we have made. Given a positively oriented basis e_1, \dots, e_m we assume that Δ is oriented by $(e_1, e_1), \dots, (e_m, e_m)$ and claim that $N(\Delta)$ is oriented by $(e_1, -e_1), \dots, (e_m, -e_m)$. We use column operations to verify this, noting that adding multiples of vectors to other vectors can't change orientations. Starting with

$$(e_1, -e_1), \dots, (e_m, -e_m), (e_1, e_1), \dots, (e_m, e_m)$$

we can add the last m vectors to the first m and obtain

$$(e_1, 0), \dots, (e_m, 0), (e_1, e_1), \dots, (e_1, e_m)$$

and then subtract the first m vectors from the last m vectors to get our standard basis

$$(e_1, 0), \dots, (e_m, 0), (0, e_1), \dots, (0, e_m).$$

□

COROLLARY 5.4.16. For an odd dimensional manifold $L(id_M) = \chi(M) = 0$.

In order to do calculations we need a way of checking orientations at intersection points.

LEMMA 5.4.17. Let M be closed and oriented and $F : M \rightarrow M$. The map $(id_M, F) : M \rightarrow M \times M$ is transverse to Δ at (p, p) if and only if $DF|_p : T_pM \rightarrow T_pM$ only has 0_p as a fixed point, i.e., $+1$ is not an eigenvalue for $DF|_p$. Moreover, in this case the intersection number is given by the sign of $\det(id_{T_pM} - DF|_p)$.

PROOF. Fix an oriented basis e_1, \dots, e_m for T_pM and for convenience denote $DF|_p = A$. The tangent space to the graph of F is spanned by

$$(e_1, A(e_1)), \dots, (e_m, A(e_m))$$

so transversality comes down to checking if

$$(e_1, A(e_1)), \dots, (e_m, A(e_m)), (e_1, e_1), \dots, (e_m, e_m)$$

is a basis and the intersection number is determined by whether or not this is a positively oriented basis. We subtract the first m vectors from the last m to obtain

$$(e_1, A(e_1)), \dots, (e_m, A(e_m)), (0, e_1 - A(e_1)), \dots, (0, e_m - A(e_m)).$$

This can only be a basis if the last m vectors are linearly independent, i.e., $\det(id_{T_p M} - DF|_p) \neq 0$. Moreover, when this happens then suitable linear combinations of the last m vectors can be used to obtain the basis

$$(e_1, 0), \dots, (e_m, 0), (0, e_1 - A(e_1)), \dots, (0, e_m - A(e_m))$$

which is positively oriented only if

$$e_1 - A(e_1), \dots, e_m - A(e_m)$$

is positively oriented. \square

REMARK 5.4.18. This lemma also shows that we don't have to know or use the orientation of $T_p M$ to calculate the intersection number as the sign of $\det(id_{T_p M} - DF|_p)$ does not depend on a choice of basis. This makes it particularly easy to calculate Lefschetz numbers.

Let us calculate the Lefschetz numbers for linear maps on projective spaces. The first general observation is that a map $A \in \text{Aut}(V)$ has a fixed point $p \in \mathbb{P}(V)$ iff p is an invariant one dimensional subspace for A . In other words fixed points for A on $\mathbb{P}(V)$ correspond to eigenvectors, but, without information about eigenvalues.

We start with the complex case as it is a bit simpler. The claim is that any $A \in \text{Aut}(V)$ with distinct eigenvalues is a Lefschetz map on $\mathbb{P}(V)$ with $L(A) = \dim V$. Since such maps are diagonalizable we can restrict attention to $V = \mathbb{C}^{n+1}$ and the diagonal matrix

$$A = \begin{bmatrix} \lambda_0 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

By symmetry we need only study the fixed point $p = [1 : 0 : \dots : 0]$. Note that the eigenvalues are assumed to be distinct and none of them vanish. To check the action of A on a neighborhood of p we use the coordinates $[1 : z^1 : \dots : z^n]$ and observe that

$$\begin{aligned} A[1 : z^1 : \dots : z^n] &= [\lambda_0 1 : \lambda_1 z^1 : \dots : \lambda_n z^n] \\ &= \left[1 : \frac{\lambda_1}{\lambda_0} z^1 : \dots : \frac{\lambda_n}{\lambda_0} z^n \right]. \end{aligned}$$

This is already (complex) linear in these coordinates so the differential at p must be represented by the complex $n \times n$ matrix

$$DA|_p = \begin{bmatrix} \frac{\lambda_1}{\lambda_0} & & 0 \\ & \ddots & \\ 0 & & \frac{\lambda_n}{\lambda_0} \end{bmatrix}.$$

As the eigenvalues are all distinct, 1 is not an eigenvalue of this matrix, showing that A really is a Lefschetz map. Next we need to check the sign of $\det(I - DA|_p)$, viewed as the determinant of a real transformation. Since $Gl_n(\mathbb{C})$ is connected it must lie in $Gl_{2n}^+(\mathbb{R})$

This matrix doesn't have 1 as an eigenvalue and

$$\begin{aligned} \det\left(I - \begin{bmatrix} -1 & 0 \\ 0 & R \end{bmatrix}\right) &= \det\begin{bmatrix} 2 & 0 \\ 0 & I - R \end{bmatrix} \\ &= \det\begin{bmatrix} 2 & & & & & \\ & 1 & 1 & & & \\ & & -1 & 1 & & \\ & & & & \ddots & \\ & & & & & 1 & 1 \\ & & & & & & -1 & 1 \end{bmatrix} \\ &= 2^{n+1}. \end{aligned}$$

So we see that the determinant is positive. For q we use the coordinates $[z^0 : 1 : z^2 : \cdots : z^n]$ and easily see that the differential is

$$\begin{bmatrix} -1 & 0 \\ 0 & -R \end{bmatrix}$$

which also doesn't have 1 as an eigenvalue and again gives us positive determinant for $I - DA_q$. This shows that $L(A) = 2$ if $A \in GL_{2n+2}^-(\mathbb{R})$.

In case $A \in GL_{2n+1}(\mathbb{R})$ it is only possible to compute the Lefschetz number mod 2 as $\mathbb{R}P^{2n}$ isn't orientable. We can select

$$A^\pm = \begin{bmatrix} \pm 1 & 0 \\ 0 & R \end{bmatrix} \in GL_{2n+1}^\pm(\mathbb{R})$$

with R as above. In either case we have only one fixed point and it is a Lefschetz fixed point since $DA_p^\pm = \pm R$. Thus $L(A^\pm) = 1$ and all $A \in G(2n+1, \mathbb{R})$ have $L(A) = 1$.

This last example can also be used to calculate the Euler characteristic of even dimensional spheres. In fact the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix} \in SO(2n+1)$$

is orthogonal and preserves the sphere with two fixed points $(\pm 1, 0, \dots, 0)$ and is homotopic to the identity map. The intersection numbers are both calculated as the sign of $\det(id_{T_p S^{2n}} - R) > 0$. So the Lefschetz number and the Euler characteristic are both 2.

5.4.4. Isotopies and Poincaré-Hopf-Lefschetz. We start with a useful localization procedure showing that any finite collection of points in a connected manifold lie in an open set diffeomorphic to \mathbb{R}^n .

DEFINITION 5.4.19. An isotopy is a homotopy of diffeomorphisms $H : [0, 1] \times M \rightarrow M$, i.e., for each t , the map $x \mapsto H(t, x)$ is a diffeomorphism. It is said to be compactly supported if there is a compact set $C \subset M$, such that $H(t, x) = x$ for all t and $x \in M - C$. Note that we can alter any such homotopy, using a function $\lambda : [0, 1] \rightarrow [0, 1]$, to a new homotopy $H(\lambda(t), x)$. If $\lambda = 0$ for $t < \epsilon$ and $\lambda = 1$ for $t > 1 - \epsilon$, then the new homotopy becomes stationary at the ends. This allows us to smoothly concatenate homotopies provided $H_1(1, x) = H_2(0, x)$.

PROPOSITION 5.4.20. If $p, q \in \mathbb{R}^n$, then there exists a compactly supported isotopy such that $H(0, x) = x$ for all $x \in \mathbb{R}^n$ and $H(1, p) = q$.

PROOF. Simply select a suitable compactly supported function $\phi : \mathbb{R}^n \rightarrow [0, 1]$ with $\phi(p) = 1$ and define

$$H_t(x) = H(t, x) = x + t\phi(x)(q - p).$$

This map is proper since it is the identity outside a compact set, it is also nonsingular provided $|d\phi| < \frac{1}{|q-p|}$. Thus corollary 3.1.16 shows that it is a diffeomorphism. \square

LEMMA 5.4.21. *Let M be connected with $\dim M \geq 2$. If $p_1, \dots, p_k \in M$ are distinct and $q_1, \dots, q_k \in M$ are distinct, then there exists a compactly supported isotopy such that $H(0, x) = x$ for all $x \in M$ and $H(1, p_i) = q_i$.*

PROOF. The proof is by induction on k .

For $k = 1$ we create a relation by saying that p, q are related provided the statement of the lemma holds. This is clearly an equivalence relation. The previous proposition shows that the equivalence classes are open. The fact that M is connected then finishes the proof.

Now assume the statement holds for $k - 1$ points. Since $\dim M \geq 2$ we know that $M - \{p_k, q_k\}$ and $M - \{p_1, \dots, p_{k-1}, q_1, \dots, q_{k-1}\}$ are connected. Therefore, there exist compactly supported isotopies H on $M - \{p_k, q_k\}$ and G on $M - \{p_1, \dots, p_{k-1}, q_1, \dots, q_{k-1}\}$ that are the identity when $t = 0$ and with $H(1, p_i) = q_i$, $i = 1, \dots, k - 1$ and $G(1, p_k) = q_k$. As they are compactly supported they extend to all of M . Now H fixes p_i, q_i , $i = 1, \dots, k - 1$ and G fixes p_k, q_k . We can then compose $H(t, G(t, x))$ to obtain the desired isotopy. \square

COROLLARY 5.4.22. *Any finite collection of points in a connected manifold lies in an open set diffeomorphic to \mathbb{R}^n .*

PROOF. The above lemma settles this when $n = \dim M \geq 2$. When $\dim M = 1$, it follows from our classification of one-manifolds. \square

Consider a vector field X on \mathbb{R}^n where p is an isolated zero. Trivializing the tangent bundle $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$ we can think of $X : \mathbb{R}^n \rightarrow \mathbb{R}^n$, where p is an isolated zero. We define the index of X :

$$\text{ind}_p X = W(X|_{\partial B(p, \epsilon)}, 0).$$

Similarly, if a map $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ has an isolated fixed point p , then consider $X(x) = x - F(x)$ and define the Lefschetz number:

$$L_p(F) = \text{ind}_p X.$$

These definitions make sense for all small ϵ and by remark 5.4.12 will give the same answer for all ϵ . In fact, instead of $B(p, \epsilon)$ we could have used any closed neighborhood, M , around p with smooth boundary and with the property that p is the only zero or fixed point in M . Both definitions also match the intersection numbers, as discussed in the previous subsection, when everything is transverse.

We can now define the index of an isolated zero of a vector field and the Lefschetz number of an isolated fixed point of a function on an oriented manifold. Simply select a positively oriented chart around the point and then use the definition from Euclidean space. It is easy to check that any two positively oriented charts give the same number. We just need to check that the definition in Euclidean space is independent of diffeomorphisms that fix, say, the origin. Such maps have an expansion $G(x) = DG|_0 x + o(x)$ and are thus isotopic to $DG|_0$ on a small neighborhood of the origin. As $DG|_0$ is an orientation preserving linear map it is in turn isotopic to the identity.

THEOREM 5.4.23 (Poincaré-Hopf). *If X is a vector field with finitely many zeros on an oriented compact oriented manifold, then*

$$\chi(M) = \sum_{p, X(p)=0} \text{ind}_p X.$$

THEOREM 5.4.24 (Lefschetz). *If $F : M \rightarrow M$ is a map with finitely many fixed points on a compact oriented manifold, then*

$$L(F) = \sum_{p, F(p)=p} L_p(F).$$

PROOF. The two proofs are virtually identical after restricting to an open set $U \subset M$ diffeomorphic to \mathbb{R}^n that contains all the zeros or fixed points. We focus on the second as it is more general. As such, we consider a map $F : M \rightarrow M$ that has a finite number of fixed points p_1, \dots, p_k in the interior of a closed ball $B = \bar{B}(0, R) \subset \mathbb{R}^n \simeq U \subset M$. To calculate the relevant winding numbers we consider the auxiliary vector field $X(x) = x - F(x)$ on B whose zeros are precisely the fixed points of F . From remark 5.4.12 we have for sufficiently small $\epsilon > 0$ that

$$\sum_i L_{p_i}(F) = \sum_i W(X|_{\partial B(p_i, \epsilon)}, 0) = W(X|_{\partial B}, 0).$$

We can now select a new function $G : M \rightarrow M$ that is homotopic to F , agrees with F on $M - \text{int}B$, and such that the graph of G is transverse to the diagonal. This implies that 0 is a regular value for $Y(x) = x - G(x)$. We can then invoke theorem 5.4.11 to conclude

$$\sum_i L_{p_i}(F) = W(X|_{\partial B}, 0) = W(Y|_{\partial B}, 0) = I(Y, \{0\}) = L(G) = L(F).$$

□

It is tempting to use the above constructions to define Lefschetz numbers for maps on noncompact manifolds. But, even on \mathbb{R}^n this runs in to some trouble. Clearly all maps are homotopic. However, there are maps without fixed points such as translations and maps with nontrivial Lefschetz numbers such as rotations in the plane. The same issue occurs for vector fields, as there exist vector fields that vanish only at the origin but with any integer as index. Similar issues occur for vector fields on compact manifolds with boundary such a closed ball in \mathbb{R}^n .

Finally we give an outline of how the Euler characteristic ties in with the traditional combinatorial definition. This works in all dimensions but is a little easier to define for surfaces.

DEFINITION 5.4.25. *A polygonal subdivision of a surface M is a decomposition $M = \cup P_\alpha$ such that each P_α is diffeomorphic to a polygon in the plane and such that $P_\alpha \cap P_\beta$ is a vertex or union of edges. The fact that M is a manifold without boundary means that each edge is the edge of exactly two polygons.*

With respect to such a decomposition it is easy to visualize a vector field that is tangent to the edges, has a sink at each vertex, a saddle at exactly one interior point of each edge, and a source at exactly one point in the interior of each polygon. As sinks and sources have index 1, while saddles have index -1 we end up with the formula

$$\chi(M) = V - E + F,$$

where V is the number of vertices, E the number of edges, and F the number of polygons, e.g., faces.

We shall in section 8.2 show a more general formula for the Euler characteristic and Lefschetz number which only depends on the cohomology of the space. This formula makes sense on a much broader class of compact spaces, but, it is less obvious why a map with nonzero Lefschetz number must have a fixed point. This topological Lefschetz number is invariant under homotopies. In particular, translations in a vector space have Lefschetz number 1 so compactness is a crucial assumption in order to guarantee fixed points.

5.4.5. Hopf's Degree Theorem. The Hopf degree theorem states that maps from a closed, connected, oriented n -manifold to the n -sphere are homotopic if and only if they have the same degree. The same statement holds for nonorientable manifolds if we use the mod 2 degree. Since the result is also important when $n = 1$ and has a much more direct proof we start with that case.

THEOREM 5.4.26. *A map $F : S^1 \rightarrow S^1$ is homotopic to $z \mapsto z^{\deg F}$.*

PROOF. The covering map $\pi : \mathbb{R} \rightarrow S^1$ given by the function $\theta \mapsto e^{2\pi i\theta}$ of period 1 allows us to lift F to a map $\bar{F} : \mathbb{R} \rightarrow \mathbb{R}$ such that $F(e^{2\pi i\theta}) = e^{2\pi i\bar{F}(\theta)}$. Clearly $\bar{F}(\theta + 1) - \bar{F}(\theta) \in \mathbb{Z}$ so it follows that it is a constant, say, k . We will show that F and $z \mapsto z^k$ are homotopic. In \mathbb{R} we have an obvious linear homotopy

$$\bar{H}(t, \theta) = (1-t)\bar{F}(\theta) + tk\theta.$$

Since

$$\bar{H}(t, \theta + 1) - \bar{H}(t, \theta) = k$$

it induces a homotopy

$$H(t, e^{2\pi i\theta}) = e^{2\pi i\bar{H}(t, \theta)}$$

between F and $z \mapsto z^k$. As the latter map has degree k the theorem follows. \square

Before moving on to the general case we start with two easy extension results.

PROPOSITION 5.4.27. *Let $B \subset \mathbb{R}^n$ be an open ball and N a manifold. If $F : \mathbb{R}^n - B \rightarrow N$ has the property that $F|_{\partial B} : \partial B \rightarrow N$ is homotopic to a constant, then there is an extension $\bar{F} : \mathbb{R}^n \rightarrow N$ that agrees with F on $\mathbb{R}^n - B$.*

PROOF. Let $H(t, x) : [0, 1] \times \partial B \rightarrow N$ be a smooth homotopy with $H(1, x) = F(x)$ and $H(0, x) = p$ for some $p \in N$. We can further assume that for $t < \epsilon$ we have $H(t, x) = p$ and for $t > 1 - \epsilon$ we have $H(t, x) = F(x)$. Parametrizing B by $[0, 1] \times \partial B \rightarrow B$ then shows that H induces a smooth map on B that is F near the boundary and thus smoothly extends F . \square

LEMMA 5.4.28. *Let B be a manifold with smooth boundary $\partial B = M$. Any map $F : M \rightarrow \mathbb{R}^n$ extends to a smooth map $G : B \rightarrow \mathbb{R}^n$ where $\partial G = M$.*

PROOF. We can assume that there is a proper embedding $B \subset \mathbb{R}^k$ and instead show that we can extend F to be defined on all of \mathbb{R}^k . The desired G is then gotten by restricting to B .

Select a retraction $\pi : U \rightarrow M$ on a tubular neighborhood $U \supset M$ and a bump function $\lambda : \mathbb{R}^k \rightarrow [0, 1]$ which is 1 on M (M is a closed subset as it is properly embedded) and 0 outside a neighborhood $V \supset M$ with $\bar{V} \subset U$. The extension is given by $G(x) = \lambda(x)F(\pi(x))$. This is certainly an extension to U and as it vanishes outside V it is well-defined on all of \mathbb{R}^k . \square

To prove the Hopf degree statement we start by considering maps of degree 0.

THEOREM 5.4.29. *Let M^n be a closed, connected, oriented n -manifold. If $F : M^n \rightarrow S^n$ has degree 0, then F is homotopic to a constant map.*

This has an immediate consequence

COROLLARY 5.4.30. *Let M^n be a closed, connected, oriented n -manifold. If $F : M^n \rightarrow \mathbb{R}^{n+1} - \{0\}$ has $W(F, 0) = 0$, then F is homotopic to a constant map in $\mathbb{R}^{n+1} - \{0\}$.*

PROOF. Theorem 5.4.29 shows that $\frac{F}{|F|} : M^n \rightarrow S^n$ is homotopic to a constant. \square

PROOF OF THEOREM 5.4.29. The proof is by induction on n with the case $n = 1$ being covered in full above.

Consider a map $F : M^n \rightarrow S^n$ of degree 0. If the map is not surjective, then it is clearly homotopic to a constant map. Otherwise, select a regular value $p \in S^n$ and an open set $U \simeq \mathbb{R}^n$ that contains $F^{-1}(p)$ and is disjoint from $F^{-1}(-p)$, i.e., $F|_U : U \rightarrow S^n - \{-p\}$. Select a diffeomorphism $A_1 : S^n - \{-p\} \rightarrow \mathbb{R}^n$ that maps p to the origin and a diffeomorphism $A_2 : \mathbb{R}^n \rightarrow U$ with the property that $F^{-1}(p) \subset A_2(B(0, 1))$. The composition $G = A_1 \circ F \circ A_2 : \mathbb{R}^n \rightarrow \mathbb{R}^n$ has 0 as a regular value so by assumption and theorem 5.4.11 it follows that

$$0 = \deg F = I(G, \{0\}) = W\left(G|_{S^{n-1}=\partial B(0,1)}, 0\right).$$

By the induction hypothesis $G|_{S^{n-1}} : S^{n-1} \rightarrow \mathbb{R}^n - \{0\}$ is homotopically trivial. Proposition 5.4.27 then gives us an extension of $G|_{\mathbb{R}^n - B(0,1)}$ to a map $\bar{G} : \mathbb{R}^n \rightarrow \mathbb{R}^n - \{0\}$. This gives us a map $A_1^{-1} \circ \bar{G} \circ A_2^{-1} : U \rightarrow S^n - \{p, -p\}$ that agrees with F outside a compact set in U and thus induces a map $\bar{F} : S^n \rightarrow S^n - \{p\}$.

Here $\bar{G}, G : \mathbb{R}^n \rightarrow \mathbb{R}^n$ are clearly homotopic via a linear homotopy that is independent of t on $\mathbb{R}^n - B(0, 1)$. Thus there is a similar homotopy of $A_1^{-1} \circ \bar{G} \circ A_2^{-1}$ and $F|_U$ that maps into $S^n - \{-p\}$ and is independent of t outside a compact set. This shows that F and \bar{F} are homotopic. As \bar{F} is not surjective it is homotopic to a constant. \square

This theorem implies an important extension that is a partial converse to theorem 5.4.1.

COROLLARY 5.4.31. *Let N^{n+1} be a compact, connected, oriented manifold with boundary. A map $F : \partial N \rightarrow S^n$ has an extension to $G : N \rightarrow S^n$ with $\partial G = F$, provided $\deg F = 0$.*

PROOF. By lemma 5.4.28 we can find an extension $\bar{F} : N \rightarrow \mathbb{R}^{n+1}$ with $\partial \bar{F} = F$. We can further assume that 0 is a regular value for \bar{F} and that $F^{-1}(0) \subset B$ where B is an open ball with smooth boundary ∂B . The map $\bar{F}|_{\partial B} : \partial B \rightarrow \mathbb{R}^{n+1} - \{0\}$ has winding number 0 by theorem 5.4.11 and is thus homotopic to a constant in $\mathbb{R}^{n+1} - \{0\}$. By proposition 5.4.27 we can then extend $\bar{F} : N - B \rightarrow \mathbb{R}^{n+1} - \{0\}$ to a smooth map $G : N \rightarrow \mathbb{R}^{n+1} - \{0\}$. This map agrees with F on ∂N and can thus be normalized to create the desired extension. \square

The full version of Hopf's theorem now follows.

THEOREM 5.4.32. *Let M^n be a connected, closed, and oriented manifold. Two maps $F_0, F_1 : M \rightarrow S^n$ are homotopic if they have the same degree.*

PROOF. Let $N = [0, 1] \times M$ with its natural orientation so that the boundaries have opposite orientations. Thus F_0, F_1 yield a map $\partial N \rightarrow S^n$ of degree 0. We can then apply the above corollary. \square

Our final result follows along similar lines:

THEOREM 5.4.33. *If M is a compact, connected, and oriented manifold, then $\chi(M) = 0$ if and only if M admits a nowhere vanishing vector field.*

PROOF. Clearly a nonzero vector field leads to vanishing Euler characteristic. Conversely select a vector field X that is transverse to the zero section and an open ball B with smooth boundary such that these zeros are contained in B . On a neighborhood of \bar{B} we can trivialize the tangent bundle and write $X(x) = (x, F(x))$. Now 0 is a regular value for $F : \bar{B} \rightarrow \mathbb{R}^n$ and

$$0 = \chi(X) = I(F^{-1}(0), \{0\}).$$

Thus $F|_{\partial B}$ has a smooth extension to a map $\bar{F} : \bar{B} \rightarrow \mathbb{R}^n - \{0\}$ that we can assume is smoothly joined to X outside B . This gives us a nowhere vanishing vector field. \square

5.5. Exercises

- (1) Let $S \subset \mathbb{R}^n - \{0\}$ be a submanifold. Show that almost all k -dimensional subspaces are transverse to S . Hint: consider the map

$$(\alpha^1, \dots, \alpha^k, v_1, \dots, v_k) \mapsto \sum \alpha^i v_i$$

where v_1, \dots, v_k are linearly independent.

- (2) Given maps between compact, connected, oriented n -manifolds:

$$L \xrightarrow{F} M \xrightarrow{G} N$$

show that

$$\deg(G \circ F) = \deg G \deg F.$$

- (3) Let $M^m, N^n \subset \mathbb{R}^{n+m+1}$ be two closed, oriented, disjoint submanifolds and define the *linking number*

$$l(M, N) = \deg(F : M \times N \rightarrow S^{n+m}), F(x, y) = \frac{x-y}{|x-y|}.$$

(a) Show that $l(M, N) = (-1)^{(m+1)(n+1)} l(N, M)$.

(b) Show that $l(M, N) = 0$ if $M = \partial B$, where B is compact, oriented, and disjoint from N .

- (4) Starting with S^1 show that there is a map of degree k on S^n for every integer k .
 (5) What is the degree of a rational map $\frac{p}{q}$ on $\mathbb{C}P^1$, where $p, q \in \mathbb{C}[X]$ have no roots in common?
 (6) Let M be a closed, connected, and oriented n -manifold. Show that there is a map $M \rightarrow S^n$ of degree k for every integer k .
 (7) If $M \rightarrow N$ is a k -fold covering of closed manifolds, then $\chi(M) = k\chi(N)$.
 (8) If M, N are manifolds, then $\chi(M \times N) = \chi(M)\chi(N)$.
 (9) Let M be connected and p_1, \dots, p_k and q_1, \dots, q_k two collections of distinct points as in lemma 5.4.21. Show that if $v_i \in T_{p_i}M - \{0\}$ and $w_i \in T_{q_i}M - \{0\}$, then there is a compactly supported isotopy H from id_M to a map $H_1(x) = H(1, x)$ that satisfies:

$$DH_1|_{p_i}(v_i) = w_i.$$

- (10) Calculate the intersection number of $\mathbb{C}P^k, \mathbb{C}P^{n-k} \subset \mathbb{C}P^n$.
 (11) Let X be a vector field on \mathbb{C} given by a complex polynomial that has no repeated roots. What is the index at each zero?
 (12) Calculate the indices at $0 \in \mathbb{C}$ of the vector fields given by $X(z) = z^m$ and $X(z) = \bar{z}^m$, $m = 1, 2, 3, \dots$
 (13) Calculate the Lefschetz numbers at $0 \in \mathbb{C}$ of the maps $F(z) = z - z^m$ and $F(z) = z - \bar{z}^m$, $m = 1, 2, 3, \dots$

- (14) Assume $L^{n-1} \subset M^n$ is properly embedded, both manifolds are connected, and M is simply connected. Show that $M - L$ has exactly two components.
- (15) Using theorem 3.4.8 and corollary 3.4.10 show that intersection numbers and Lefschetz numbers are well-defined for continuous maps. Conclude that the Borsuk-Ulam theorem and the Hopf degree theorem hold for continuous maps.
- (16) Show that there are antipodal points on Earth with the same the temperature and barometric pressure.
- (17) Let $U_i \subset \mathbb{R}^n$, $i = 1, \dots, n$ be open, bounded, and connected. Show that there exists a hyperplane H that bisects the n open sets, i.e., if $\mathbb{R}^n = A \cup B$, where $A \cap B = H$, then

$$\text{vol}(U_i \cap A) = \text{vol}(U_i \cap B), i = 1, \dots, n.$$

Hint: You can use that Borsuk-Ulam holds for continuous functions.

- (18) Consider the maps on $\mathbb{C}\mathbb{P}^2$:

$$F_k([z_0 : z_1 : z_2]) = [z_0^k : z_1^k : z_2^k],$$

$$\bar{F}_k([z_0 : z_1 : z_2]) = [\bar{z}_0^k : \bar{z}_1^k : \bar{z}_2^k],$$

where $k = 1, 2, 3, \dots$

- (a) Show that if U_k is the group of the k th roots of unity then U_k^3/Δ acts transitively on the preimages of these maps, here $\Delta = \{(\zeta, \zeta, \zeta) \mid \zeta \in U_k\}$.
- (b) Show that the degree is k^2 .
- (c) What happens with the analogous question on $\mathbb{R}\mathbb{P}^2$?
- (d) Show that F_k is transverse to

$$\mathbb{C}\mathbb{P}^1 = \{[w_0 : w_1 : w_2] \in \mathbb{C}\mathbb{P}^2 \mid w_0 + w_1 + w_2 = 0\}$$

and let M be the preimage so that we obtain a map $G_k : M \rightarrow \mathbb{C}\mathbb{P}^1$.

- (e) Show that U_k^3/Δ acts transitively on the preimages of G_k .
- (f) Show that except for three points where there are k preimages all other points have k^2 preimages.
- (g) Use this to show that $\chi(M) = k(3 - k)$ (hint: the image is the union of two triangles whose vertices are the special three points with k preimages).
- (h) Is F_k transverse to

$$\mathbb{C}\mathbb{P}^1 = \{[w_0 : w_1 : w_2] \in \mathbb{C}\mathbb{P}^2 \mid w_0 = 0\}?$$

Basic Tensor Analysis

In this chapter we expand the constructions that involve tensors to include the exterior derivative and integration of forms. This will culminate in the general Stokes' theorem and how it generalizes the three classical integral theorems.

Recall from section 2.3.2 that (k, l) -tensors are sections of tensor bundles $(TM)^{\otimes k} \otimes (T^*M)^{\otimes l}$. Here $(1, 0)$ -tensors are vector fields and $(0, 1)$ -tensors 1-forms. In section 2.2.2 the Lie derivative was introduced on functions and vector fields. Moreover, it was shown that on vector fields it was the same the Lie bracket.

6.1. The Lie Derivative

Let $F : M \rightarrow N$ be a smooth map, the pull-back operation

$$F^* : (T^*N)^{\otimes k} \rightarrow (T^*M)^{\otimes k}$$

is defined by

$$(F^*T)(v_1, \dots, v_k) = T(DF(v_1), \dots, DF(v_k))$$

if $T \in (T_{F(p)}^*N)^{\otimes k}$ is a k -linear map on $T_{F(p)}N$. This operation naturally extends to $(0, k)$ -tensors, i.e., sections of the respective bundles. There is also a corresponding push-forward

$$F_* : (TM)^{\otimes k} \rightarrow (TN)^{\otimes k}$$

defined by

$$F_*(v_1 \otimes \dots \otimes v_k) = DF(v_1) \otimes \dots \otimes DF(v_k).$$

This operation however does not necessarily extend to sections when F is not injective or surjective (try to push forward a vector field on M). In case F is a diffeomorphism the push-forward is well-defined for sections, e.g., for a vector field

$$F_*(X)|_q = DF(X|_{F^{-1}(q)}).$$

Let X be a vector field and $\Phi^t = \Phi_X^t$ the corresponding locally defined flow on a smooth manifold M . Thus $\Phi^t(p)$ is defined for small t and the curve $t \mapsto \Phi^t(p)$ is the integral curve for X that goes through p at $t = 0$. The Lie derivative of a tensor in the direction of X is defined as the first order term in a suitable Taylor expansion of the tensor when it is moved by the flow of X .

For a function $f : M \rightarrow \mathbb{R}$ we have

$$f(\Phi^t(p)) = f(p) + t(L_X f)(p) + o(t),$$

where the Lie derivative $L_X f$ is just the directional derivative $D_X f = df(X)$. We can also write this as

$$\begin{aligned} f \circ \Phi^t &= f + tL_X f + o(t), \\ L_X f &= D_X f = df(X). \end{aligned}$$

For a vector field Y

$$((\Phi^{-t})_* Y)|_p = D\Phi^{-t}(Y|_{\Phi^t(p)}) = Y|_p + t(L_X Y)|_p + o(t),$$

where it follows from proposition 2.2.8 that $L_X Y = [X, Y]$.

The Lie derivative of a $(0, k)$ -tensor T is constructed as for a function

$$(\Phi^t)^* T = T + t(L_X T) + o(t)$$

or more precisely

$$\begin{aligned} ((\Phi^t)^* T)(Y_1, \dots, Y_k) &= T(D\Phi^t(Y_1), \dots, D\Phi^t(Y_k)) \\ &= T(Y_1, \dots, Y_k) + t(L_X T)(Y_1, \dots, Y_k) + o(t). \end{aligned}$$

PROPOSITION 6.1.1. *If X is a vector field and T a $(0, k)$ -tensor on M , then*

$$(L_X T)(Y_1, \dots, Y_k) = D_X(T(Y_1, \dots, Y_k)) - \sum_{i=1}^k T(Y_1, \dots, L_X Y_i, \dots, Y_k)$$

PROOF. We restrict attention to the case where $k = 1$. The general case is similar but requires more notation. Using that

$$Y|_{\Phi^t} = D\Phi^t(Y) + tD\Phi^t(L_X Y) + o(t)$$

we get

$$\begin{aligned} ((\Phi^t)^* T)(Y) &= T(D\Phi^t(Y)) \\ &= T(Y|_{\Phi^t} - tD\Phi^t(L_X Y)) + o(t) \\ &= T(Y) \circ \Phi^t - tT(D\Phi^t(L_X Y)) + o(t) \\ &= T(Y) + tD_X(T(Y)) - tT(D\Phi^t(L_X Y)) + o(t). \end{aligned}$$

Thus

$$\begin{aligned} (L_X T)(Y) &= \lim_{t \rightarrow 0} \frac{((\Phi^t)^* T)(Y) - T(Y)}{t} \\ &= \lim_{t \rightarrow 0} (D_X(T(Y)) - T(D\Phi^t(L_X Y))) \\ &= D_X(T(Y)) - T(L_X Y). \end{aligned}$$

□

Finally, we have that Lie derivatives satisfy all possible product rules. From the above propositions this is already obvious when multiplying functions with vector fields or $(0, k)$ -tensors. However, it is less clear when multiplying tensors.

PROPOSITION 6.1.2. *Let T_1 and T_2 be $(0, k_i)$ -tensors, then*

$$L_X(T_1 \otimes T_2) = (L_X T_1) \otimes T_2 + T_1 \otimes (L_X T_2).$$

PROOF. Recall that for 1-forms and more general $(0, k)$ -tensors the tensor product is defined as

$$T_1 \otimes T_2(X_1, \dots, X_{k_1}, Y_1, \dots, Y_{k_2}) = T_1(X_1, \dots, X_{k_1}) \cdot T_2(Y_1, \dots, Y_{k_2}).$$

The proposition is then a simple consequence of the previous proposition and the product rule for derivatives of functions. □

PROPOSITION 6.1.3. *Let T be a $(0, k)$ -tensor and $f : M \rightarrow \mathbb{R}$ a function, then*

$$L_{fX} T(Y_1, \dots, Y_k) = fL_X T(Y_1, \dots, Y_k) + df(Y_i) \sum_{i=1}^k T(Y_1, \dots, X, \dots, Y_k).$$

PROOF. We have that

$$\begin{aligned}
L_{fX}T(Y_1, \dots, Y_k) &= D_{fX}(T(Y_1, \dots, Y_k)) - \sum_{i=1}^k T(Y_1, \dots, L_{fX}Y_i, \dots, Y_k) \\
&= fD_X(T(Y_1, \dots, Y_k)) - \sum_{i=1}^k T(Y_1, \dots, [fX, Y_i], \dots, Y_k) \\
&= fD_X(T(Y_1, \dots, Y_k)) - f \sum_{i=1}^k T(Y_1, \dots, [X, Y_i], \dots, Y_k) \\
&\quad + df(Y_i) \sum_{i=1}^k T(Y_1, \dots, X, \dots, Y_k)
\end{aligned}$$

□

The case where $X|_p = 0$ is of special interest when computing Lie derivatives. We note that $\Phi^t(p) = p$ for all t . Thus $D\Phi^t : T_pM \rightarrow T_pM$ and

$$\begin{aligned}
L_X Y|_p &= \lim_{t \rightarrow 0} \frac{D\Phi^{-t}(Y|_p) - Y|_p}{t} \\
&= \frac{d}{dt} (D\Phi^{-t})|_{t=0}(Y|_p).
\end{aligned}$$

This shows that $L_X = \frac{d}{dt} (D\Phi^{-t})|_{t=0}$ when $X|_p = 0$. From this we see that if θ is a 1-form, then $L_X \theta = -\theta \circ L_X$ at points p where $X|_p = 0$.

The *interior product* is simply evaluation of a vector field in the first argument of a tensor:

$$i_X T(X_1, \dots, X_k) = T(X, X_1, \dots, X_k)$$

We list 4 general properties of Lie derivatives on tensors that are easy to check:

$$\begin{aligned}
L_{[X, Y]} &= L_X L_Y - L_Y L_X, \\
L_X(fT) &= L_X(f)T + fL_X T, \\
L_X[Y, Z] &= [L_X Y, Z] + [Y, L_X Z], \\
L_X(i_Y T) &= i_{L_X Y} T + i_Y(L_X T).
\end{aligned}$$

6.2. The Exterior Derivative

6.2.1. General Properties. Forms are skew-symmetric $(0, p)$ -tensors. The wedge product was defined in section 2.3.1

$$\begin{aligned}
\Omega^p(M) \times \Omega^q(M) &\rightarrow \Omega^{p+q}(M), \\
(\omega, \psi) &\rightarrow \omega \wedge \psi.
\end{aligned}$$

This operation is bilinear and antisymmetric in the sense that:

$$\omega \wedge \psi = (-1)^{pq} \psi \wedge \omega.$$

The wedge product of a function and a form is simply standard multiplication.

The exterior derivative

$$d : \Omega^p(\mathbb{R}^n) \rightarrow \Omega^{p+1}(\mathbb{R}^n)$$

is defined as follows

$$\begin{aligned}
 d\omega &= d\left(\sum_{i_1 < \dots < i_p} \omega_{i_1 \dots i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p}\right) \\
 &= \sum_{i_1 < \dots < i_p} d\omega_{i_1 \dots i_p} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_p} \\
 &= \sum_{j, i_1 < \dots < i_p} \partial_j \omega_{i_1 \dots i_p} dx^j \wedge dx^{i_1} \wedge \dots \wedge dx^{i_p}.
 \end{aligned}$$

The same formula can be used in any coordinate system and thus also gives us an exterior derivative on all smooth manifolds. To check that the formula doesn't depend on the coordinate system can be done with a brute force calculation or by collecting the properties that characterize it.

PROPOSITION 6.2.1. *The exterior derivative is uniquely defined by the properties*

- (1) $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$.
- (2) $d(\omega_1 \wedge \omega_2) = (d\omega_1) \wedge \omega_2 + (-1)^p \omega_1 \wedge d\omega_2$, for $\omega_1 \in \Omega^p(\mathbb{R}^n)$.
- (3) $d(d\omega) = 0$ or $d^2 = 0$.
- (4) $df(v) = D_v f$, for $f \in C^\infty(\mathbb{R}^n) = \Omega^0(\mathbb{R}^n)$.

PROOF. (1) and (4) are obvious. This shows that for (2) we can assume that $\omega_1 = f_1 dx^I$ and $\omega_2 = f_2 dx^J$, where $dx^I = dx^{i_1} \wedge \dots \wedge dx^{i_p}$:

$$\begin{aligned}
 d(f_1 dx^I \wedge f_2 dx^J) &= d(f_1 f_2 dx^I \wedge dx^J) \\
 &= d(f_1 f_2) \wedge dx^I \wedge dx^J \\
 &= ((df_1) f_2 + f_1 df_2) \wedge dx^I \wedge dx^J \\
 &= df_1 \wedge dx^I \wedge f_2 dx^J + (-1)^p f_1 dx^I \wedge df_2 \wedge dx^J \\
 &= d(f_1 dx^I) \wedge f_2 dx^J + (-1)^p f_1 dx^I \wedge d(f_2 dx^J).
 \end{aligned}$$

For (3) we can similarly assume that $\omega = f dx^I$ so that:

$$d\omega = \sum_j \partial_j f dx^j \wedge dx^I.$$

This gives us

$$\begin{aligned}
 d^2\omega &= \sum_{i,j} \partial_{ij}^2 f dx^i \wedge dx^j \wedge dx^I \\
 &= \sum_{i \neq j} \partial_{ij}^2 f dx^i \wedge dx^j \wedge dx^I \\
 &= \sum_{i < j} (\partial_{ij}^2 f - \partial_{ji}^2 f) dx^i \wedge dx^j \wedge dx^I \\
 &= 0.
 \end{aligned}$$

Should there exist another exterior derivative \bar{d} with the same properties, then we need to check that

$$\bar{d}(dx^I) = 0.$$

By (4) we have $dx^I = \bar{d}x^I$. We can then use (3) to show that $\bar{d}(\bar{d}x^I) = 0$ by induction on the degree of the form as

$$\left(\bar{d}(\bar{d}x^{i_1} \wedge \cdots \wedge \bar{d}x^{i_p}) = \bar{d}^2 x^{i_1} \wedge \bar{d}x^{i_2} \wedge \cdots \wedge \bar{d}x^{i_p} - \bar{d}x^1 \wedge \bar{d}(\bar{d}x^{i_2} \wedge \cdots \wedge \bar{d}x^{i_p}) \right).$$

□

This proposition leads to a new invariant formula for the exterior derivative.

PROPOSITION 6.2.2. *If $X_0, \dots, X_p \in \mathcal{X}(M)$ and $\omega \in \Omega^p(M)$, then*

$$\begin{aligned} d\omega(X_0, \dots, X_k) &= \sum_{i=0}^k (-1)^i D_{X_i} \left(\omega(X_0, \dots, \widehat{X}_i, \dots, X_k) \right) \\ &\quad - \sum_{i < j} (-1)^i \omega(X_0, \dots, \widehat{X}_i, \dots, [X_i, X_j], \dots, X_k) \\ &= \sum_{i=0}^k (-1)^i D_{X_i} \left(\omega(X_0, \dots, \widehat{X}_i, \dots, X_k) \right) \\ &\quad + \sum_{i < j} (-1)^{i+j} \omega([X_i, X_j], X_0, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_k). \end{aligned}$$

For a 1-form this becomes

$$d\omega(X, Y) = D_X(\omega(Y)) - D_Y(\omega(X)) - \omega([X, Y]).$$

PROOF. First use lemma 2.3.4 to show that the right hand side defines a tensor. This shows that it suffices to check the formula for a p -form of the type $\omega = f dx^1 \wedge \cdots \wedge dx^p$. In this case

$$d\omega = \sum_{i=p+1}^n \partial_i f dx^i \wedge dx^1 \wedge \cdots \wedge dx^p.$$

We can then evaluate both sides on $p+1$ elements of a coordinate basis $\partial_1, \dots, \partial_n$. In this case the right hand side only depends on the first term and we see that both sides vanish unless the $p+1$ vectors are of the form $\partial_j, \partial_1, \dots, \partial_p$ for $j > p$. For this choice of vectors the left hand side becomes $\partial_j f$ and the right hand side

$$\partial_j(\omega(\partial_1, \dots, \partial_p)) = \partial_j f.$$

□

Lie derivatives, interior products, wedge products and exterior derivatives when evaluated on forms satisfy:

$$\begin{aligned} d(\omega \wedge \psi) &= (d\omega) \wedge \psi + (-1)^p \omega \wedge (d\psi), \\ i_X(\omega \wedge \psi) &= (i_X\omega) \wedge \psi + (-1)^p \omega \wedge (i_X\psi), \\ L_X(\omega \wedge \psi) &= (L_X\omega) \wedge \psi + \omega \wedge (L_X\psi), \end{aligned}$$

and

$$\begin{aligned}
d \circ d &= 0, \\
i_X \circ i_X &= 0, \\
L_X &= d \circ i_X + i_X \circ d, \\
F^* \circ d &= d \circ F^*, \\
L_X \circ d &= d \circ L_X.
\end{aligned}$$

The third property $L_X = d \circ i_X + i_X \circ d$ is also known as H. Cartan's formula (son of the geometer E. Cartan). Note it tells us that if we know how d is defined on p -forms, then we can define d on $(p+1)$ -forms by

$$i_{X_0} \circ d = L_{X_0} - d \circ i_{X_0}.$$

The formula is a consequence of the coordinate free definition from the above proposition. The other formulas are all linear and easy to verify directly in a coordinate system for forms of the type $f dx^I$.

6.2.2. Div, Grad, and Curl. We use the language of forms to explain some basic concepts from multivariable calculus in \mathbb{R}^3 .

The *gradient* of a function f is a vector field

$$\partial_x f \partial_x + \partial_y f \partial_y + \partial_z f \partial_z.$$

This formula depends on using Cartesian coordinates unlike the formula for the differential

$$df = \partial_x f dx + \partial_y f dy + \partial_z f dz.$$

The *volume form* is the 3-form $\text{vol} = dx \wedge dy \wedge dz$. We shall explain in section 6.3 why it is natural to integrate forms on manifolds. The *divergence* of a vector field $X = P\partial_x + Q\partial_y + R\partial_z$ is usually defined as

$$\text{div}(X) = \partial_x P + \partial_y Q + \partial_z R.$$

This also crucially depends on Cartesian coordinates. Again we have an alternate definition via the formula

$$L_X \text{vol} = \text{div}(X) \text{vol}.$$

To check the validity we calculate

$$\begin{aligned}
L_X \text{vol} &= L_X(dx \wedge dy \wedge dz) \\
&= (L_X dx) \wedge dy \wedge dz + dx \wedge L_X dy \wedge dz + dx \wedge dy \wedge L_X dz.
\end{aligned}$$

Here

$$L_X dx = L_P \partial_x dx + L_Q \partial_y dx + L_R \partial_z dx$$

and if $Y = \bar{P}\partial_x + \bar{Q}\partial_y + \bar{R}\partial_z$, then

$$\begin{aligned}
(L_X dx)(Y) &= D_X(dx(Y)) - dx[X, Y] \\
&= D_X \bar{P} - dx((D_X \bar{P} - D_Y P) \partial_x + \dots) \\
&= D_Y P.
\end{aligned}$$

Thus $L_X dx = dP$ and

$$(L_X dx) \wedge dy \wedge dz = (\partial_x P) dx \wedge dy \wedge dz.$$

Similar calculations for the other two terms then show that

$$L_X \text{vol} = \text{div}(X) \text{vol}.$$

While this formula still depends on our particular formula for the volume form it can be used for other coordinates as long as we change the volume form to those coordinates.

H. Cartan's formula for the Lie derivative of forms gives us a different way of finding the divergence

$$\begin{aligned}\operatorname{div}(X)\operatorname{vol} &= L_X\operatorname{vol} \\ &= di_X\operatorname{vol} + i_Xd\operatorname{vol} \\ &= di_X\operatorname{vol}.\end{aligned}$$

In particular, $\operatorname{div}(X)\operatorname{vol}$ is always exact.

This formula suggests that we should study the correspondence that takes a vector field X to the 2-form $i_X\operatorname{vol}$.

$$\begin{aligned}i_X\operatorname{vol} &= i_X\operatorname{vol} \\ &= Pi_{\partial_x}\operatorname{vol} + Qi_{\partial_y}\operatorname{vol} + Ri_{\partial_z}\operatorname{vol} \\ &= Pdy \wedge dz - Qdx \wedge dz + Rdx \wedge dy \\ &= Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy\end{aligned}$$

If we compose the grad and div operations we obtain the *Laplacian*:

$$\operatorname{div}(\operatorname{grad}f) = \Delta f$$

The curl operation is a little more involved

$$\operatorname{curl}X = (\partial_y R - \partial_z Q)\partial_x + (\partial_z P - \partial_x R)\partial_y + (\partial_x Q - \partial_y P)\partial_z..$$

There is a more invariant correspondence after checking that:

$$d(Pdx + Qdz + Rdz) = i_{\operatorname{curl}X}\operatorname{vol}.$$

If we define $\omega_X = Pdx + Qdz + Rdz$ or better yet by $\omega_X(v) = X \cdot v$, then all the formulas are:

$$\begin{aligned}\omega_{\operatorname{grad}f} &= df, \\ i_{\operatorname{curl}X}\operatorname{vol} &= d\omega_X, \\ \operatorname{div}(X)\operatorname{vol} &= di_X\operatorname{vol}.\end{aligned}$$

Using that $d \circ d = 0$ on all forms we obtain the classical vector analysis formulas

$$\begin{aligned}\operatorname{curl}(\operatorname{grad}f) &= 0, \\ \operatorname{div}(\operatorname{curl}X) &= 0,\end{aligned}$$

from

$$\begin{aligned}i_{\operatorname{curl}(\operatorname{grad}f)}\operatorname{vol} &= d\omega_{\operatorname{grad}f} =ddf, \\ \operatorname{div}(\operatorname{curl}X)\operatorname{vol} &= di_{\operatorname{curl}X}\operatorname{vol} = dd\omega_X.\end{aligned}$$

6.3. Integration of Forms

We shall assume that M is an oriented n -manifold. Thus, M comes with a covering of charts $x_\alpha = (x_\alpha^1, \dots, x_\alpha^n) : U_\alpha \longleftrightarrow B(0, 1) \subset \mathbb{R}^n$ such that the transition functions $x_\alpha \circ x_\beta^{-1}$ preserve the usual orientation on Euclidean space, i.e., $\det(D(x_\alpha \circ x_\beta^{-1})) > 0$. In addition, select a partition of unity, $\phi_\alpha : M \rightarrow [0, 1]$, subordinate to this covering.

Given an n -form ω on M we wish to define the integral:

$$\int_M \omega.$$

When M is not compact, it might be necessary to assume that the form has compact support. In each chart we can write

$$\omega = f_\alpha dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n.$$

Using the partition of unity, we then obtain

$$\begin{aligned} \omega &= \sum_\alpha \phi_\alpha \omega \\ &= \sum_\alpha \phi_\alpha f_\alpha dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n, \end{aligned}$$

where each of the forms $\phi_\alpha f_\alpha dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n$. Since U_α is identified with $V_\alpha = x_\alpha(U_\alpha) \subset \mathbb{R}^n$, we simply declare that

$$\int_{U_\alpha} \phi_\alpha f_\alpha dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n = \int_{V_\alpha} \phi_\alpha f_\alpha dx^1 \cdots dx^n.$$

Here the right-hand side is the integral of the function $\phi_\alpha f_\alpha$ viewed as a function on V_α . We define

$$\int_M \omega = \sum_\alpha \int_{U_\alpha} \phi_\alpha f_\alpha dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n$$

whenever this sum converges. Using the standard change of variables formula for integration on Euclidean space, we see that this definition is indeed independent of the choice of coordinates and partition of unity.

With these definitions behind us, we can now state and prove Stokes' theorem for manifolds with boundary.

THEOREM 6.3.1. *Let M be an oriented n -manifold. For any $\omega \in \Omega^{n-1}(M)$ with compact support we have*

$$\int_M d\omega = \int_{\partial M} \omega.$$

PROOF. If we use the trick

$$d\omega = \sum_\alpha d(\phi_\alpha \omega),$$

then we see that it suffices to prove the theorem in the case $M = L^n = (-\infty, 0] \times \mathbb{R}^{n-1}$ and ω has compact support. In that case we can write

$$\omega = \sum_{i=1}^n f_i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n,$$

The differential of ω is now easily computed:

$$\begin{aligned} d\omega &= \sum_{i=1}^n (df_i) \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \\ &= \sum_{i=1}^n (\partial_i f_i) dx^i \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \\ &= \sum_{i=1}^n (-1)^{i-1} \partial_i f_i dx^1 \wedge \cdots \wedge dx^i \wedge \cdots \wedge dx^n. \end{aligned}$$

Thus,

$$\begin{aligned} \int_{L^n} d\omega &= \int_{L^n} \sum_{i=1}^n (-1)^{i-1} \partial_i f_i dx^1 \wedge \cdots \wedge dx^n \\ &= \sum_{i=1}^n (-1)^{i-1} \int_{L^n} \partial_i f_i dx^1 \cdots dx^n \\ &= \sum_{i=1}^n (-1)^{i-1} \int \left(\int (\partial_i f_i) dx^i \right) dx^1 \cdots \widehat{dx^i} \cdots dx^n. \end{aligned}$$

Since each f_i has compact support the fundamental theorem of calculus tells us that

$$\begin{aligned} \int_{-\infty}^{\infty} (\partial_i f_i) dx^i &= 0, \text{ for } i > 1, \\ \int_{-\infty}^0 (\partial_1 f_1) dx^1 &= f_1(0, x^2, \dots, x^n). \end{aligned}$$

Thus

$$\int_{L^n} d\omega = \int_{\partial L^n} f_1(0, x^2, \dots, x^n) dx^2 \wedge \cdots \wedge dx^n.$$

Since $dx^1 = 0$ when restricted to ∂L^n it follows that

$$\omega|_{\partial L^n} = f_1 dx^2 \wedge \cdots \wedge dx^n.$$

This proves the theorem. \square

We get a very nice corollary out of Stokes' theorem.

THEOREM. *If M is a compact connected manifold with nonempty boundary, then there is no retraction $r : M \rightarrow \partial M$.*

PROOF. Note that if such a retraction exists then ∂M is connected.

If M is oriented and ω is a volume form on ∂M , then we have

$$\begin{aligned} 0 &< \int_{\partial M} \omega \\ &= \int_{\partial M} r^* \omega \\ &= \int_M d(r^* \omega) \\ &= \int_M r^* d\omega \\ &= 0. \end{aligned}$$

If M is not orientable, then we lift the situation to the orientation cover and obtain a contradiction there. \square

We shall briefly discuss how the classical integral theorems of Green, Gauss, and Stokes follow from the general version of Stokes' theorem presented above.

Green's theorem in the plane is the easiest.

THEOREM 6.3.2. (Green's Theorem) *Let $\Omega \subset \mathbb{R}^2$ be a domain with smooth boundary $\partial\Omega$. If $X = P\partial_x + Q\partial_y$ is a vector field defined on a region containing Ω then*

$$\int_{\Omega} (\partial_x Q - \partial_y P) dx dy = \int_{\partial\Omega} P dx + Q dy.$$

PROOF. Note that the integral on the right-hand side is a line integral which can also be interpreted as the integral of the 1-form $\omega = Pdx^1 + Qdx^2$ on the 1-manifold $\partial\Omega$. With this in mind we just need to observe that $d\omega = (\partial_1 Q - \partial_2 P) dx^1 \wedge dx^2$ in order to establish the theorem. \square

Gauss' Theorem is quite a bit more complicated, but, we did some of the ground work when we defined the divergence above.

THEOREM 6.3.3. (The divergence theorem or Gauss' theorem) *Let X be a vector field defined on a compact domain $M \subset \mathbb{R}^3$ with smooth boundary and N the outward pointing unit normal field to ∂M . If $\text{vol}_{\partial M} = i_N \text{vol}$ denotes the area form on the boundary, then*

$$\int_M (\text{div} X) \text{vol} = \int_{\partial M} X \cdot N \text{vol}_{\partial M}$$

PROOF. We know that

$$\text{div} X \text{vol} = di_X \text{vol}.$$

So by Stokes' theorem it suffices to show that

$$(i_X \text{vol})|_{\partial M} = X \cdot N \text{vol}_{\partial M}.$$

The orientation on $T_p \partial M$ is so that v_2, v_3 is a positively oriented basis for $T_p \partial M$ iff N, v_2, v_3 is a positively oriented basis for $T_p M$. With such a choice of basis we have

$$\begin{aligned} i_X \text{vol}(v_2, v_3) &= \text{vol}(X, v_2, v_3) \\ &= \text{vol}((X \cdot N)N, v_2, v_3) \\ &= X \cdot N \text{vol}(N, v_2, v_3) \\ &= (X \cdot N) i_N \text{vol}(v_2, v_3) \\ &= X \cdot N \text{vol}_{\partial M}(v_2, v_3) \end{aligned}$$

where we used that $X - (X \cdot N)N$, the component of X in $T_p \partial M$, is a linear combination of v_2, v_3 and therefore doesn't contribute to the form. \square

The divergence theorem can easily be generalized to domains in \mathbb{R}^n and even Riemannian manifolds with boundary. Stokes' Theorem is specific to 3 dimensions. Classically it holds for an oriented surface $S \subset \mathbb{R}^3$ with smooth boundary but can be formulated for oriented surfaces in oriented Riemannian 3-manifolds.

THEOREM 6.3.4. (Stokes' theorem) *Let $S \subset \mathbb{R}^3$ be an oriented compact surface with boundary ∂S . If X is a vector field defined on a region containing S and N is the unit normal field to S , then*

$$\int_S (\text{curl} X \cdot N) i_N \text{vol} = \int_{\partial S} \omega_X.$$

PROOF. Recall that ω_X is the 1-form defined by

$$\omega_X(v) = X \cdot v.$$

This form is related to $\text{curl} X$ by

$$d\omega_X = i_{\text{curl} X} \text{vol}.$$

So Stokes' Theorem tells us that

$$\int_{\partial S} \omega_X = \int_S i_{\text{curl} X} \text{vol}.$$

The integral on the right-hand side can now be understood in a manner completely analogous to our discussion of $i_X \text{vol}_{\partial M}$ in the proof of the divergence theorem. We note that N is

chosen perpendicular to $T_p S$ in such a way that $N, v_2, v_3 \in T_p M$ is positively oriented iff $v_2, v_3 \in T_p S$ is positively oriented. Thus we have again that

$$\text{vol}_S = i_N \text{vol}$$

and consequently

$$i_{\text{curl} X} \text{vol} = \text{curl} X \cdot N \text{vol}_{\partial M}.$$

□

6.4. Exercises

- (1) Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be smooth and define $\nabla f = \text{grad} f$ by $df(v) = v \cdot \nabla f$.
- Show that $v \cdot w = (\sum \delta_{ij} dx^i \otimes dx^j)(v, w)$.
 - Show that $\nabla f = \sum \delta^{ij} \partial_i f \partial_j$.
 - Show that

$$L_{\nabla f} \sum \delta_{ij} dx^i \otimes dx^j = \sum \partial_{kl}^2 f dx^k \otimes dx^l.$$

- (2) Show that if $\omega = f \cdot (-ydx + xdy - dz)$, where $f : \mathbb{R}^3 \rightarrow (0, \infty)$, then $d\omega \neq 0$ on all of \mathbb{R}^3 .
- (3) Show that the following relations hold when evaluated on forms:

$$\begin{aligned} i_X \circ i_X &= 0, \\ L_X &= d \circ i_X + i_X \circ d, \\ F^* \circ d &= d \circ F^*, \\ L_X \circ d &= d \circ L_X. \end{aligned}$$

- (4) Let

$$r^2 = \sum (x^i)^2.$$

- (a) Show that the gradient is the unit radial field:

$$\nabla r = \frac{\sum x^i \partial_i}{r}.$$

- (b) Show that on $\mathbb{R}^2 - \{0\}$

$$dx \wedge dy = r dr \wedge d\theta,$$

where θ is any unit speed parametrization of part of $S^1(1)$.

- (c) Show that on $\mathbb{R}^n - \{0\}$

$$\text{vol} := dx^1 \wedge \cdots \wedge dx^n = r dr \wedge \text{vol}_{S^{n-1}(1)},$$

where $\text{vol}_{S^{n-1}(1)} = (i_{\nabla r} \text{vol})|_{S^{n-1}(1)}$.

- (5) Let D be a rank k distribution on M^m .

- Show that locally there is a trivialization of the tangent bundle X_1, \dots, X_m such that X_1, \dots, X_k span D .
- Let $\omega^1, \dots, \omega^m$ be the 1-forms dual to X_1, \dots, X_m , i.e., $\omega^i(X_j) = \delta_j^i$. Show that ω^i vanishes on D only when $i = k+1, \dots, m$ and conclude that $D = \bigcap_{i=k+1}^m \ker \omega^i$.
- Let $\mathcal{A} = \{\omega \in \Omega(M) \mid \omega|_D = 0\}$. Show that \mathcal{A} is an ideal that is locally generated by $\omega^{k+1}, \dots, \omega^m$, i.e. every element is locally of the form $\sum_{i=k+1}^m \phi_i \omega^i$, $\phi_i \in \Omega^*(U)$.

- (d) Show that D is involutive if and only if there are 1-forms ϕ_i^j such that for $j = k + 1, \dots, n$ we have

$$d\omega^j = \sum_{i=k+1}^n \alpha_i^j \wedge \omega^i.$$

- (e) Show more abstractly that D is involutive if and only if the exterior derivative preserves \mathcal{A} , i.e., $d(\mathcal{A}) \subset \mathcal{A}$.
- (6) Let $\omega \in \Omega^1(M)$ with $d\omega = 0$. Show that on a neighborhood of a point where ω doesn't vanish we have $\omega = df$. (A similar statement is true for all forms but the proof a bit more involved.) Hint: Use the local version of the Frobenius theorem.
- (7) Let $\omega \in \Omega^1(M^3)$ be a nowhere vanishing 1-form and $D = \ker \omega$ the corresponding distribution of rank 2.
- (a) Show that D is involutive if and only if $\omega \wedge d\omega = 0$.
- (b) Show that D is involutive if and only if locally $\omega = gdf$ for functions f and $g > 0$.
- (8) Show that if

$$\omega = \frac{1}{n} \sum_{i=1}^n (-1)^{i-1} x^i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n,$$

then $d\omega = dx^1 \wedge \cdots \wedge dx^n$ and use this to show that

$$\frac{1}{n} \text{vol}_{n-1} S^{n-1}(1) = \text{vol} B(0, 1),$$

where $\bar{B}(0, 1)$ is the closed unit ball and $S^{n-1}(1) = \partial B(0, 1)$.

- (9) On $\mathbb{R}^n - \{0\}$ consider the $(n-1)$ -form

$$\omega = r^{-n} \sum_{i=1}^n (-1)^{i-1} x^i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n,$$

where $r^2 = \sum_{i=1}^n (x^i)^2$.

- (a) Show that $d\omega = 0$.
- (b) Show that

$$\int_{S^{n-1}(\epsilon)} \omega = n \text{vol} \bar{B}(0, 1),$$

where $\bar{B}(0, 1)$ is the closed unit ball and $S^{n-1}(\epsilon)$ the sphere of radius ϵ centered at the origin.

Basic Cohomology Theory

We define de Rham cohomology and establish the basic properties. Several examples are calculated using both Mayer-Vietoris and *invariant cohomology* when symmetries from actions are present. Next we streamline the axioms to line up with the universal principle from theorem 1.3.2. This is used to prove: uniqueness of cohomology, Poincaré duality, Künneth's theorem and the Leray-Hirsch theorem for fibrations. There is also a section on generalized cohomology that covers Alexander cohomology and relative cohomology in the context of de Rham cohomology.

7.1. De Rham Cohomology

Throughout we let M be an n -manifold. Using that $d \circ d = 0$, it follows that the *exact forms*

$$B^p(M) = d\left(\Omega^{p-1}(M)\right)$$

are a subset of the *closed forms*

$$Z^p(M) = \{\omega \in \Omega^p(M) \mid d\omega = 0\}.$$

The de Rham cohomology is defined as the quotient space:

$$H^p(M) = \frac{Z^p(M)}{B^p(M)}.$$

Given a closed form ψ , let $[\psi]$ denote the corresponding cohomology class.

The first simple property comes from the fact that any function with zero differential must be locally constant. As a consequence, on a connected manifold:

$$H^0(M) = \mathbb{R}.$$

Given a smooth map $F : M \rightarrow N$ the pull-back operation on forms induces a map in cohomology:

$$\begin{aligned} H^p(N) &\rightarrow H^p(M), \\ F^*([\psi]) &= [F^*\psi]. \end{aligned}$$

This definition is independent of the choice of ψ , since $F^* \circ d = d \circ F^*$.

The two key results that are needed for a deeper understanding of de Rham cohomology are the Mayer-Vietoris sequence and homotopy invariance of the pull-back map.

LEMMA 7.1.1. (The Mayer-Vietoris Sequence) *If $M = A \cup B$ for open sets $A, B \subset M$, then there is a long exact sequence*

$$\dots \rightarrow H^p(M) \rightarrow H^p(A) \oplus H^p(B) \rightarrow H^p(A \cap B) \rightarrow H^{p+1}(M) \rightarrow \dots.$$

PROOF. We start by defining a short exact sequence

$$0 \rightarrow \Omega^p(M) \rightarrow \Omega^p(A) \oplus \Omega^p(B) \rightarrow \Omega^p(A \cap B) \rightarrow 0.$$

The map $\Omega^p(M) \rightarrow \Omega^p(A) \oplus \Omega^p(B)$ is simply restriction $\omega \mapsto (\omega|_A, \omega|_B)$. The second is given by $(\omega, \psi) \mapsto (\omega|_{A \cap B} - \psi|_{A \cap B})$. With these definitions it is clear that $\Omega^p(M) \rightarrow \Omega^p(A) \oplus \Omega^p(B)$ is injective and that the sequence is exact at $\Omega^p(A) \oplus \Omega^p(B)$. It is a bit less obvious why $\Omega^p(A) \oplus \Omega^p(B) \rightarrow \Omega^p(A \cap B)$ is surjective. To see this select a partition of unity λ_A, λ_B with respect to the covering A, B . Given $\omega \in \Omega^p(A \cap B)$ we see that $\lambda_A \omega$ defines a form on B and $\lambda_B \omega$ defines a form on A . Consequently, $(\lambda_B \omega, -\lambda_A \omega) \mapsto \omega$.

These maps induce maps in cohomology

$$H^p(M) \rightarrow H^p(A) \oplus H^p(B) \rightarrow H^p(A \cap B)$$

such that this sequence is exact. The connecting homomorphisms

$$\delta : H^p(A \cap B) \rightarrow H^{p+1}(M)$$

are constructed using the diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & \Omega^{p+1}(M) & \rightarrow & \Omega^{p+1}(A) \oplus \Omega^{p+1}(B) & \rightarrow & \Omega^{p+1}(A \cap B) \rightarrow 0 \\ & & \uparrow d & & \uparrow d & & \uparrow d \\ 0 & \rightarrow & \Omega^p(M) & \rightarrow & \Omega^p(A) \oplus \Omega^p(B) & \rightarrow & \Omega^p(A \cap B) \rightarrow 0. \end{array}$$

If we take a form $\omega \in \Omega^p(A \cap B)$, then $(\lambda_B \omega, -\lambda_A \omega) \in \Omega^p(A) \oplus \Omega^p(B)$ is mapped onto ω . If $d\omega = 0$, then

$$\begin{aligned} d(\lambda_B \omega, -\lambda_A \omega) &= (d\lambda_B \wedge \omega, -d\lambda_A \wedge \omega) \\ &\in \Omega^{p+1}(A) \oplus \Omega^{p+1}(B) \end{aligned}$$

vanishes when mapped to $\Omega^{p+1}(A \cap B)$. So we obtain a well-defined form

$$\begin{aligned} \delta \omega &= \begin{cases} d\lambda_B \wedge \omega & \text{on } A \\ -d\lambda_A \wedge \omega & \text{on } B \end{cases} \\ &\in \Omega^{p+1}(M). \end{aligned}$$

It is easy to see that this defines a map in cohomology that makes the Mayer-Vietoris sequence exact.

The construction here is fairly concrete, but, it is a very general construction in homological algebra. \square

The first part of the Mayer-Vietoris sequence

$$0 \rightarrow H^0(M) \rightarrow H^0(A) \oplus H^0(B) \rightarrow H^0(A \cap B) \rightarrow H^1(M)$$

is particularly simple since we know what the zero dimensional cohomology is. In case $A \cap B$ is connected it reduces to a short exact sequence

$$0 \rightarrow H^0(M) \rightarrow H^0(A) \oplus H^0(B) \rightarrow H^0(A \cap B) \rightarrow 0$$

so the Mayer-Vietoris sequence for higher dimensional cohomology starts with

$$0 \rightarrow H^1(M) \rightarrow H^1(A) \oplus H^1(B) \rightarrow \dots$$

To study what happens when we have homotopic maps between manifolds we have to figure out how forms on the product $[0, 1] \times M$ relate to forms on M . Since M potentially has boundary we will instead investigate $\mathbb{R} \times M$.

On the product $\mathbb{R} \times M$ consider the vector field ∂_t tangent to the first factor and the corresponding one-form dt . In local coordinates on $\mathbb{R} \times M$ forms can be written

$$\omega = a_I dx^I + b_J dt \wedge dx^J$$

if we use summation convention and multi index notation

$$\begin{aligned} a_I &= a_{i_1 \dots i_k}, \\ dx^I &= dx^{i_1} \wedge \dots \wedge dx^{i_k} \end{aligned}$$

Given $a < b$ we can integrate out the dt factor as follows

$$\mathcal{I}_a^b(\omega) = \int_a^b \omega = \int_a^b b_J dt \wedge dx^J = \left(\int_a^b b_J dt \right) dx^J$$

Thus defining a map

$$\Omega^{k+1}(\mathbb{R} \times M) \rightarrow \Omega^k(M)$$

To see that this is well-defined note that it can be expressed as

$$\mathcal{I}_a^b(\omega) = \int_a^b dt \wedge i_{\partial_t} \omega$$

since

$$i_{\partial_t}(\omega) = b_J dx^J.$$

LEMMA 7.1.2. *If $j_t : M \rightarrow \mathbb{R} \times M$ is the inclusion map $j_t(x) = (t, x)$, then*

$$\mathcal{I}_a^b(d\omega) + d\mathcal{I}_a^b(\omega) = j_b^*(\omega) - j_a^*(\omega),$$

where $a < b$.

PROOF. We can assume without loss of generality that $a = 0$ and $b = 1$ and define $\mathcal{I} = \mathcal{I}_0^1$. The key is to prove that

$$\mathcal{I}(d\omega) + d\mathcal{I}(\omega) = \int_0^1 dt \wedge L_{\partial_t} \omega$$

Given this it follows that the right hand side is

$$\begin{aligned} \int_0^1 dt \wedge L_{\partial_t} \omega &= \int_0^1 dt \wedge L_{\partial_t} (a_I dx^I + b_J dt \wedge dx^J) \\ &= \int_0^1 dt \wedge (\partial_t a_I dx^I + \partial_t b_J dt \wedge dx^J) \\ &= \int_0^1 dt \wedge (\partial_t a_I) dx^I \\ &= \left(\int_0^1 dt \partial_t a_I \right) dx^I \\ &= (a_I(1, x) - a_I(0, x)) dx^I \\ &= j_1^*(\omega) - j_0^*(\omega) \end{aligned}$$

The first formula follows by noting that

$$\begin{aligned} \mathcal{I}(d\omega) + d\mathcal{I}(\omega) &= \int_0^1 dt \wedge i_{\partial_t} d\omega + d \left(\int_0^1 dt \wedge i_{\partial_t} \omega \right) \\ &= \int_0^1 dt \wedge i_{\partial_t} d\omega + \int_0^1 dt \wedge di_{\partial_t} \omega \\ &= \int_0^1 dt \wedge (i_{\partial_t} d\omega + di_{\partial_t} \omega) \\ &= \int_0^1 dt \wedge (L_{\partial_t} \omega) \end{aligned}$$

The one tricky move here is the identity

$$d\left(\int_0^1 dt \wedge i_{\partial_t} \omega\right) = \int_0^1 dt \wedge di_{\partial_t} \omega$$

On the left hand side it is clear what d does, but, on the right hand side we are computing d of a form on the product. However, as we are wedging with dt this does not become an issue. Specifically, if d is exterior differentiation on $\mathbb{R} \times M$ and d_x exterior differentiation on M , then

$$\begin{aligned} d_x\left(\int_0^1 dt \wedge i_{\partial_t} \omega\right) &= d_x\left(\int_0^1 b_J dt\right) \wedge dx^J \\ &= \sum_i \frac{\partial \int_0^1 b_J dt}{\partial x^i} \wedge dx^i \wedge dx^J \\ &= \sum_i \int_0^1 \frac{\partial b_J}{\partial x^i} dt \wedge dx^i \wedge dx^J \\ &= \left(\int_0^1 dt \wedge \left(\sum_i \frac{\partial b_J}{\partial x^i} dx^i\right)\right) \wedge dx^J \\ &= \left(\int_0^1 dt \wedge (d_x b_J)\right) \wedge dx^J \\ &= \left(\int_0^1 dt \wedge (db_J - \partial_t b_J dt)\right) \wedge dx^J \\ &= \left(\int_0^1 dt \wedge db_J\right) \wedge dx^J \\ &= \int_0^1 dt \wedge di_{\partial_t} \omega \end{aligned}$$

□

We can now establish homotopy invariance.

PROPOSITION 7.1.3. *If $F_0, F_1 : M \rightarrow N$ are smoothly homotopic, then they induce the same maps on de Rham cohomology.*

PROOF. The formula

$$\mathcal{I}(d\omega) + d\mathcal{I}(\omega) = j_1^*(\omega) - j_0^*(\omega)$$

shows that $j_1^*(\omega) - j_0^*(\omega)$ is exact provided $d\omega = 0$. In particular, j_0 and j_1 induce the same maps in cohomology:

$$j_0^* = j_1^* : H^*(\mathbb{R} \times M) \rightarrow H^*(M).$$

Assuming we have a homotopy $H : [0, 1] \times M \rightarrow N$, such that $F_0 = H \circ j_0$ and $F_1 = H \circ j_1$ it follows that

$$F_1^*(\omega) - F_0^*(\omega) = (H \circ j_1)^*(\omega) - (H \circ j_0)^*(\omega) = 0.$$

Note that by using a smooth function $\lambda : \mathbb{R} \rightarrow [0, 1]$ with $\lambda(t) = 0$ for $t \leq 0$ and $\lambda(t) = 1$ for $t \geq 1$, we can always obtain a smooth map $H(\lambda(t), x) : \mathbb{R} \times M \rightarrow N$ from a homotopy. □

COROLLARY 7.1.4. *If two manifolds, possibly of different dimension, are homotopy equivalent, then they have the same de Rham cohomology.*

PROOF. This follows from having maps $F : M \rightarrow N$ and $G : N \rightarrow M$ such that $F \circ G$ and $G \circ F$ are homotopic to the identity maps. \square

LEMMA 7.1.5. (The Poincaré Lemma) *The cohomology of a contractible manifold M is*

$$\begin{aligned} H^0(M) &= \mathbb{R}, \\ H^p(M) &= \{0\} \text{ for } p > 0. \end{aligned}$$

In particular, convex sets in \mathbb{R}^n have trivial de Rham cohomology.

PROOF. Being contractible is the same as being homotopy equivalent to a point. \square

While we can't definitively relate the cohomology of a covering space to its base there is a simple relationship.

LEMMA 7.1.6. *Let $F : M \rightarrow N$ be a finite covering map, then*

$$F^* : H^p(N) \rightarrow H^p(M)$$

is an injection.

PROOF. The trick lies in finding a so called *transgression map* $\tau : \Omega^p(M) \rightarrow \Omega^p(N)$ such that $d \circ \tau = \tau \circ d$ and $\tau \circ F^* = id_{\Omega^p(N)}$. This will induce a map $\tau^* : H^*(M) \rightarrow H^*(N)$ such that $\tau^* \circ F^* = id_{H^*(N)}$, which shows in particular that F^* is an injection.

While it'd be natural to try to average forms on M to make them descend to N , this won't work unless we have a finite group that acts transitively on the fibers. Instead we do the averaging in N . If $\omega \in \Omega^p(M)$ and $y \in N$ is covered by the points $x_i \in M$, $i = 1, \dots, k$, then we can push each of the linear forms $\omega|_{x_i}$ on $T_{x_i}M$ via $DF|_{x_i}$ to a linear p -form on T_yN and then define

$$\tau(\omega)|_y = \frac{1}{k} \sum \left((DF|_{x_i})^{-1} \right)^* \omega|_{x_i}.$$

This yields a smooth form as each point in N has a neighborhood that is evenly covered by k diffeomorphic sets. The composition property is immediate and the commutation with d follows from the fact that d commutes with pull backs of maps, in this case, the locally defined inverse of F . \square

7.2. Examples of Cohomology Groups

We calculate the cohomology of spheres and projective spaces in two ways. First the traditional way using Mayer-Vietoris and then by a completely different approach using the large group of symmetries on these spaces.

7.2.1. Spheres. For S^n we use that

$$\begin{aligned} S^n &= (S^n - \{p\}) \cup (S^n - \{-p\}), \\ S^n - \{\pm p\} &\simeq \mathbb{R}^n, \\ (S^n - \{p\}) \cap (S^n - \{-p\}) &\simeq \mathbb{R}^n - \{0\}. \end{aligned}$$

Since $\mathbb{R}^n - \{0\}$ deformation retracts onto S^{n-1} this allows us to compute the cohomology of S^n by induction using the Mayer-Vietoris sequence. We start with S^1 , which is a bit different as the intersection has two components. The Mayer-Vietoris sequence starting with $p = 0$ looks like

$$0 \rightarrow \mathbb{R} \rightarrow \mathbb{R} \oplus \mathbb{R} \rightarrow \mathbb{R} \oplus \mathbb{R} \rightarrow H^1(S^1) \rightarrow 0.$$

Showing that $H^1(S^1) \simeq \mathbb{R}$. For $n \geq 2$ the intersection is connected so the connecting homomorphism

$$H^{p-1}(S^{n-1}) \rightarrow H^p(S^n)$$

must be an isomorphism for $p \geq 1$. Thus

$$H^p(S^n) = \begin{cases} 0, & p \neq 0, n, \\ \mathbb{R}, & p = 0, n. \end{cases}$$

7.2.2. Projective Spaces. For \mathbb{P}^n we use the decomposition

$$\mathbb{P}^n = (\mathbb{P}^n - \mathbb{P}^{n-1}) \cup (\mathbb{P}^n - p),$$

where

$$\begin{aligned} p &= [1 : 0 : \cdots : 0], \\ \mathbb{P}^{n-1} &= \mathbb{P}(p^\perp) = \{[0 : z^1 : \cdots : z^n] \mid (z^1, \dots, z^n) \in \mathbb{F}^n - \{0\}\}, \end{aligned}$$

and consequently

$$\begin{aligned} \mathbb{P}^n - p &= \{[z : z^1 : \cdots : z^n] \mid (z^1, \dots, z^n) \in \mathbb{F}^n - \{0\} \text{ and } z \in \mathbb{F}\} \simeq \mathbb{P}^{n-1}, \\ \mathbb{P}^n - \mathbb{P}^{n-1} &= \{[1 : z^1 : \cdots : z^n] \mid (z^1, \dots, z^n) \in \mathbb{F}^n\} \simeq \mathbb{F}^n, \\ (\mathbb{P}^n - \mathbb{P}^{n-1}) \cap (\mathbb{P}^n - p) &= \{[1 : z^1 : \cdots : z^n] \mid (z^1, \dots, z^n) \in \mathbb{F}^n - \{0\}\} \simeq \mathbb{F}^n - \{0\}. \end{aligned}$$

We have already identified \mathbb{P}^1 so we don't need to worry about having a disconnected intersection when $\mathbb{F} = \mathbb{R}$ and $n = 1$. Using that $\mathbb{F}^n - \{0\}$ deformation retracts to the unit sphere S of dimension $\dim_{\mathbb{R}} \mathbb{F}^n - 1$ we see that the Mayer-Vietoris sequence reduces to

$$\begin{aligned} 0 &\rightarrow H^1(\mathbb{P}^n) \rightarrow H^1(\mathbb{P}^{n-1}) \rightarrow H^1(S) \rightarrow \cdots \\ \cdots &\rightarrow H^{p-1}(S) \rightarrow H^p(\mathbb{P}^n) \rightarrow H^p(\mathbb{P}^{n-1}) \rightarrow H^p(S) \rightarrow \cdots \end{aligned}$$

for $p \geq 2$. To get more information we need to specify the scalars and in the real case even distinguish between even and odd n . First assume that $\mathbb{F} = \mathbb{C}$. Then $S = S^{2n-1}$ and $\mathbb{C}\mathbb{P}^1 \simeq S^2$. A simple induction then shows that

$$H^p(\mathbb{C}\mathbb{P}^n) = \begin{cases} 0, & p = 1, 3, \dots, 2n-1, \\ \mathbb{R}, & p = 0, 2, 4, \dots, 2n. \end{cases}$$

When $\mathbb{F} = \mathbb{R}$, we have $S = S^{n-1}$ and $\mathbb{R}\mathbb{P}^1 \simeq S^1$. This shows that $H^p(\mathbb{R}\mathbb{P}^n) = 0$ when $p = 1, \dots, n-2$. The remaining cases have to be extracted from the last part of the sequence

$$0 \rightarrow H^{n-1}(\mathbb{R}\mathbb{P}^n) \rightarrow H^{n-1}(\mathbb{R}\mathbb{P}^{n-1}) \rightarrow H^{n-1}(S^{n-1}) \rightarrow H^n(\mathbb{R}\mathbb{P}^n) \rightarrow 0$$

where we know that

$$H^{n-1}(S^{n-1}) = \mathbb{R}.$$

This shows that $H^n(\mathbb{R}\mathbb{P}^n)$ is either 0 or \mathbb{R} . Next we observe that the natural map

$$H^k(\mathbb{R}\mathbb{P}^n) \rightarrow H^k(S^n)$$

is an injection by lemma 7.1.6. This means that we obtain the simpler exact sequence

$$0 \rightarrow H^{n-1}(\mathbb{R}\mathbb{P}^{n-1}) \rightarrow H^{n-1}(S^{n-1}) \rightarrow H^n(\mathbb{R}\mathbb{P}^n) \rightarrow 0$$

From this we conclude that $H^n(\mathbb{R}P^n) = 0$ iff $H^{n-1}(\mathbb{R}P^{n-1}) = \mathbb{R}$. Given that $H^1(\mathbb{R}P^1) = \mathbb{R}$ we then obtain the cohomology groups:

$$H^p(\mathbb{R}P^{2n}) = \begin{cases} 0, & p \geq 1, \\ \mathbb{R}, & p = 0, \end{cases}$$

$$H^p(\mathbb{R}P^{2n+1}) = \begin{cases} 0, & 2n \geq p \geq 1, \\ \mathbb{R}, & p = 0, 2n+1. \end{cases}$$

7.2.3. Invariant Cohomology. There is a very powerful general principle that allows us to calculate the cohomology of all of the above spaces, and more, using only homotopy invariance.

The general set-up is a manifold M with an action by a group G of diffeomorphisms. The action of each group element $g \in G$ will be denoted by

$$\begin{aligned} A_g : M &\rightarrow M \\ x &\mapsto A_g(x) = gx. \end{aligned}$$

The G -invariant p -forms are defined by

$$\Omega_G^p(M) = \{\omega \in \Omega^p(M) \mid A_g^* \omega = \omega \text{ for all } g \in G\}.$$

As $A_g^* \circ d = d \circ A_g^*$ we obtain a complex

$$\Omega_G^0(M) \xrightarrow{d} \Omega_G^1(M) \xrightarrow{d} \Omega_G^2(M) \xrightarrow{d} \dots$$

and a corresponding G -invariant cohomology

$$H_G^p(M) = \frac{\ker\left(\Omega_G^p(M) \xrightarrow{d} \Omega_G^{p+1}(M)\right)}{\text{im}\left(\Omega_G^{p-1}(M) \xrightarrow{d} \Omega_G^p(M)\right)}.$$

The inclusion

$$\Omega_G^p(M) \rightarrow \Omega^p(M)$$

induces a natural map

$$H_G^p(M) \rightarrow H^p(M)$$

that need not be an isomorphism or even an injection.

EXAMPLE 7.2.1. On \mathbb{R} consider the action that translates by integers $\mathbb{Z} \subset \mathbb{R}$. The invariant 1-forms are simply the forms $f(x)dx$ where f is a function with period 1. For such a form to be exact with respect to invariant forms requires that $f dx = dh$ for some function h with period 1. This, however, implies that

$$\int_0^1 f dx = h(1) - h(0) = 0.$$

So if $f \equiv 1$, then $[dx] \in H_{\mathbb{Z}}^1(\mathbb{R})$ creates a nontrivial cohomology class that is trivial in $H^1(\mathbb{R})$. In this case we have in fact that

$$H_{\mathbb{Z}}^*(\mathbb{R}) \simeq H^*(\mathbb{R}/\mathbb{Z}) = H^*(S^1).$$

THEOREM 7.2.2. *If G is a compact Lie group, in particular a finite group, then $H_G^p(M) \rightarrow H^p(M)$ is an injection. Moreover, if in addition $G \subset G^*$, where G^* is a connected Lie group that also acts on M , then $H_G^p(M) \rightarrow H^p(M)$ is an isomorphism.*

PROOF. Select a left invariant volume form vol_G on G . By compactness we can assume for simplicity that $\int_G \text{vol}_G = 1$. On a finite group integration is merely averaging over the elements in the group.

Integration of vector valued functions on G allows us to create a left inverse to the inclusion $\Omega_G^p(M) \rightarrow \Omega^p(M)$. For $\omega \in \Omega^p(M)$ fix a point $x \in M$ and average $(A_g^* \omega)|_x$ over $g \in G$:

$$\bar{\omega}|_x = \int_G (A_g^* \omega)|_x \text{vol}_G.$$

When $\omega \in \Omega_G^p(M)$ it follows that

$$\bar{\omega}|_x = \int_G (A_g^* \omega)|_x \text{vol}_G = \int_G \omega|_x \text{vol}_G = \omega|_x \int_G \text{vol}_G = \omega|_x.$$

Thus averaging really is a left inverse. To check that the averaged form is invariant we have to use that the volume form is left invariant:

$$\begin{aligned} A_h^* \bar{\omega} &= \int_G A_h^* A_g^* \omega \text{vol}_G \\ &= \int_G A_{hg}^* \omega l_h^* \text{vol}_G \\ &= \int_G A_{hg}^* \omega l_h^* \text{vol}_G \\ &= \int_G (A_g^* \omega) \text{vol}_G. \end{aligned}$$

Finally note that averaging also commutes with the exterior derivative on forms

$$d\bar{\omega} = \int_G dA_g^* \omega \text{vol}_G = \int_G A_g^* d\omega \text{vol}_G = \overline{d\omega}.$$

Thus averaging also induces a left inverse in cohomology. In particular, the induced map $H_G^p(M) \rightarrow H^p(M)$ is an injection.

When the elements $g \in G \subset G^*$ are part of a larger connected group, a path from g to e creates a homotopy from A_g to $A_e = id_M$. Thus the cohomology classes satisfy $[A_g^* \omega] = [\omega]$ for all $g \in G$. This shows that $[\bar{\omega}] = [\omega]$ and in particular that $H_G^p(M) \rightarrow H^p(M)$ is an isomorphism. \square

EXAMPLE 7.2.3. On S^{2n} the antipodal map A is orientation reversing. Since $A^2 = id_{S^{2n}}$ we can average over the group $\{id_{S^{2n}}, A\}$. For any volume form $\omega \in \Omega^{2n}(S^{2n})$ we have $\int_{S^{2n}} A^* \omega = -\int_{S^{2n}} \omega$. Thus averaging volume forms simply results in a form that integrates to zero. As we shall see in section 7.5.1 this implies that the cohomology class of an averaged volume form vanishes.

In order to calculate the cohomology of some basic examples it is convenient to reduce the task. We will consider manifolds M with a transitive action of a compact connected Lie group G , i.e., for each $x, y \in M$ there exists $g \in G$ such that $A_g x = gx = y$. The isotropy at a fixed point $x \in M$ is the closed subgroup $H = \{g \in G \mid A_g x = gx = x\}$. Since each A_h fixes x , when $h \in H$, the differential $DA_h|_x$ acts on $T_x M$. Thus H induces a linear action on $T_x M$. Since the action is transitive any G -invariant form ω is completely determined by its value at x . The linear form $\omega|_x$ on $T_x M$ is in addition invariant under the action of H on $T_x M$:

$$\omega|_x(v_1, \dots, v_p) = \omega|_x(DA_h|_x(v_1), \dots, DA_h|_x(v_p))$$

for all $v_1, \dots, v_p \in T_x M$ and $h \in H$. We claim that the converse is also true, i.e., any linear form ω_x on $T_x M$ that is invariant under the action of H extends to a G -invariant form on M . Note that if $y = gx = g'x$, then $g' = gh$, where $h = g^{-1}g' \in H$. We define

$$\omega|_y (DA_g|_x(v_1), \dots, DA_g|_x(v_p)) = \omega_x(v_1, \dots, v_p)$$

or more succinctly $\omega|_y \circ DA_g|_x = \omega_x$ or equivalently $\omega|_y = \omega_x \circ DA_{g^{-1}}|_y$. This is well-defined as

$$\begin{aligned} \omega|_y \circ DA_g|_x &= \omega_x \\ &= \omega_x \circ DA_h|_x \\ &= \omega|_y \circ DA_g|_x \circ DA_h|_x \\ &= \omega|_y \circ DA_{gh}|_x. \end{aligned}$$

It is also easy to see that it is a G -invariant form.

EXAMPLE 7.2.4. We can use the action of $SO(n+1)$ on S^n to calculate the cohomology of spheres. The action is by orthogonal transformations of unit vectors in \mathbb{R}^{n+1} that transform the standard basis e_0, \dots, e_n to the other positively oriented orthonormal bases of \mathbb{R}^{n+1} . In particular, the action is transitive on S^n . We fix $x = e_0$ as the first basis vector in the ambient Euclidean space. The elements of $SO(n+1)$ that fix x can be identified with $SO(n)$. So we consider the action of $SO(n)$ on $T_x S^n = x^\perp = \text{span}\{e_1, \dots, e_n\}$.

This reduces the problem to checking which constant coefficient p -forms

$$\omega = \sum_{i_1 < \dots < i_p} \omega_{i_1 \dots i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p}, \omega_{i_1 \dots i_p} \in \mathbb{R}$$

on \mathbb{R}^n are invariant under $SO(n)$. Clearly constant functions and the standard volume form have this property. So we have to consider the case where $0 < p < n$. Evaluating on e_{i_1}, \dots, e_{i_p} we can select $j \neq i_1, \dots, i_p$ and an element $g \in SO(n)$ such that $g(e_{i_1}) = -e_{i_1}$, $g(e_j) = -e_j$, $g(e_i) = e_i$, for $i \neq i_1, j$. This shows that

$$\omega_{i_1 \dots i_p} = \omega(e_{i_1}, \dots, e_{i_p}) = \omega(-e_{i_1}, \dots, e_{i_p}) = -\omega_{i_1 \dots i_p}.$$

Thus all linear p -forms that are $SO(n)$ invariant vanish.

We conclude that $\Omega_{SO(n+1)}^0(S^n) = \mathbb{R}$, $\Omega_{SO(n+1)}^p(S^n) = 0$, for $0 < p < n$, and $\Omega_{SO(n+1)}^n(S^n) = \mathbb{R}$ with a generator

$$\sum_i (-1)^i dx^0 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^n.$$

This generator restricts to the standard volume form $dx^1 \wedge \dots \wedge dx^n$ on $T_x S^n$. The invariant 0-forms are clearly all closed. This shows that the invariant n -forms are not exact when $n = 1$. For $n > 1$ the n -forms can't be exact as there are no nontrivial invariant $(n-1)$ -forms. This calculates the cohomology of S^n and agrees with the previous calculations:

$$H^*(S^n) \simeq H_{SO(n+1)}^*(S^n) = \Omega_{SO(n+1)}^*(S^n).$$

EXAMPLE 7.2.5. The previous example can be used to calculate the cohomology of \mathbb{RP}^n also using the action of $SO(n+1)$. We will think of points $x \in \mathbb{RP}^n$ as antipodal pairs $\{\pm y\} \in S^n$. In this way $T_x \mathbb{RP}^n$ also becomes equivalence classes in $T_{\pm y} S^n = \{\pm y\}^\perp$ where $(y, v) \in S^n \times \{y\}^\perp \subset S^n \times \mathbb{R}^{n+1}$ is identified with $(-y, -v) \in S^n \times \{-y\}^\perp$. The action of $g \in SO(n+1)$ on S^n becomes an action on TS^n :

$$g \cdot (y, v) = (A_g y, DA_g v) = (gy, gv).$$

As $-g \cdot (y, v) = g \cdot (-y, -v)$ this also tells us how g acts on $T\mathbb{R}P^n$:

$$g \cdot (\pm y, \pm v) = (\pm gy, \pm gv).$$

As for S^n it follows that $SO(n)$ fixes the point $x = \{\pm e_0\}$, however, the full isotropy consists of $S(O(1) \times O(n))$ which has two components. The other component consists of the orthogonal transformations that send e_0 to $-e_0$ and act on $\text{span}\{e_1, \dots, e_n\}$ with determinant -1 .

We thus immediately obtain: $\Omega_{SO(n+1)}^0(\mathbb{R}P^n) = \mathbb{R}$, $\Omega_{SO(n+1)}^p(\mathbb{R}P^n) = 0$, for $0 < p < n$. For $\Omega_{SO(n+1)}^n(\mathbb{R}P^n)$ the answer depends on the parity of n . In case $n = 2k + 1$ is odd, the antipodal map $-I \in SO(2k + 2)$ acts trivially on $\mathbb{R}P^{2k+1}$. The isotropy of x thus consists of elements $g \in SO(2k + 1)$ and $-Ig \in S(O(1) \times O(2k + 1))$ in the other component. As the actions of g and $-Ig$ are the same on $\mathbb{R}P^{2k+1}$ we only need to consider the linear action of $SO(n)$ on \mathbb{R}^n in order to understand the linear action of the isotropy on $T_x\mathbb{R}P^{2k+1}$. As for the sphere we conclude that $\Omega_{SO(2k+2)}^{2k+1}(\mathbb{R}P^{2k+1}) = \mathbb{R}$. When $n = 2k$ is even the antipodal map is not an element of $SO(2k + 1)$ and thus the action on $\mathbb{R}P^{2k}$ is effective, i.e., only the unit element fixes all points. This means that the isotropy $S(O(1) \times O(2k))$ is also effective and in particular contains an element that is orientation reversing on $T_x\mathbb{R}P^{2k}$, e.g., the element that maps e_0 to $-e_0$, e_1 to $-e_1$, and fixes all other basis vectors (on $\mathbb{R}P^{2k+1}$ this map preserves the orientation!). Since the linear volume form on \mathbb{R}^n is not preserved by orientation reversing orthogonal transformations we conclude that $\Omega_{SO(2k+1)}^{2k}(\mathbb{R}P^{2k}) = 0$. All in all we have shown that

$$H^*(\mathbb{R}P^n) \simeq H_{SO(n+1)}^*(\mathbb{R}P^n) = \Omega_{SO(n+1)}^*(\mathbb{R}P^n).$$

EXAMPLE 7.2.6. On complex projective space \mathbb{P}^n we can use the transitive action of $U(n + 1)$ that maps complex lines to complex lines in \mathbb{C}^{n+1} . We use the standard complex basis c_0, \dots, c_n which gives a real basis $e_0, f_0, \dots, e_n, f_n$, where $c_i = e_i + \sqrt{-1}f_i$. The isotropy at $x = \text{span}\{c_0\} = [1 : 0 : \dots : 0]$ is $U(1) \times U(n)$, where $U(1)$ acts trivially as it is simply multiplication by complex scalars on $\text{span}\{c_0\}$ and $U(n)$ acts on the tangent space in the way $U(n)$ acts on $\mathbb{C}^n = \text{span}\{c_1, \dots, c_n\}$.

We can then again simply check which constant coefficient forms on \mathbb{C}^n are invariant under $U(n)$. Consider the unitary transformations $g_i \in U(n)$ such that $g_i(c_i) = -c_i$ and $g_i(c_j) = c_j$ for $j \neq i$. Using these transformations it follows that a $(p + q)$ -form, $0 < p + q < 2n$, vanishes if it is evaluated on $e_{i_1}, \dots, e_{i_p}, f_{j_1}, \dots, f_{j_q}$ where one of e_{i_k} (resp. f_{j_l}) in the collection does not have its partner f_{i_k} (resp. e_{j_l}) in the collection. This means that we can restrict attention to the cases where $p = q$ and $i_1 = j_1, \dots, i_p = j_p$. However, in all of these cases the value $\omega(e_{i_1}, \dots, e_{i_p}, f_{i_1}, \dots, f_{i_p})$ must be the same as permutations of the complex basis vectors c_1, \dots, c_n are also unitary transformations.

This shows that $\Omega_{U(n+1)}^{2p+1}(\mathbb{P}^n) = 0$ and $\Omega_{U(n+1)}^{2p}(\mathbb{P}^n) = \mathbb{R}$ with a generator that when restricted to $T_x\mathbb{P}^n$ is given by

$$\left(\sum_{i=1}^n dx^i \wedge dy^i \right)^p = \binom{n}{p} \sum_{i_1 < \dots < i_p} dx^{i_1} \wedge dy^{i_1} \wedge \dots \wedge dx^{i_p} \wedge dy^{i_p}.$$

Since there are no invariant forms of odd degree all of the invariant forms of even degree are closed but not exact. This calculates the cohomology of complex projective space and in addition gives us generators $[\omega^p] \in H^{2p}$ that can be calculated from any generator $[\omega] \in H^2$:

$$H^*(\mathbb{P}^n) \simeq H_{U(n+1)}^*(\mathbb{P}^n) = \Omega_{U(n+1)}^*(\mathbb{P}^n).$$

EXAMPLE 7.2.7. The final example will be the torus $T^n = S^1 \times \cdots \times S^1 \subset \mathbb{C}^n$ which acts on itself via multiplication of unit complex numbers in each factor. This action is also transitive but has trivial isotropy. The 1-forms $d\theta^i, i = 1, \dots, n$ that are the standard volume forms on the factors are invariant under this action as are all of their wedge products. This shows that

$$\Omega_{T^n}^p(T^n) = \text{span} \{d\theta^{i_1} \wedge \cdots \wedge d\theta^{i_p} \mid 1 \leq i_1 < \cdots < i_p \leq n\}.$$

As these forms are all closed we have calculated the cohomology of T^n to be

$$H^*(T^n) \simeq H_{T^n}^*(T^n) = \Omega_{T^n}^*(T^n).$$

Based on this last example it is tempting to think that the cohomology of a connected compact Lie group is equally simple. It is true that we can select a basis of left invariant 1-forms and their wedge products to obtain similar spaces for $\Omega_G^p(G)$. However, these forms are not necessarily closed. For example, if we evaluate the differential of a left invariant 1-form on left invariant fields we obtain

$$d\omega(X, Y) = D_X\omega(Y) - D_Y\omega(X) - \omega([X, Y]) = -\omega([X, Y]).$$

Thus the differential is dictated by the Lie algebra, which in case of the torus was Abelian. Nevertheless with a good choice of basis for the Lie algebra it does become possible to calculate the cohomology.

7.3. Axiomatic Cohomology

In this section we specify the most basic properties of cohomology theories for manifolds.

In section 7.1 we introduced the functor

$$M \mapsto H^*(M) = H^0(M) \oplus H^1(M) \oplus \cdots \oplus H^n(M)$$

that maps an n -manifold to a graded vector space. The morphisms are the smooth maps between manifolds and the functor is contravariant as $F : M \rightarrow N$ induces a pull-back $F^* : H^*(N) \rightarrow H^*(M)$. Pull-back maps are natural in the sense that $(G \circ F)^* = F^* \circ G^*$. We established the basic, but not elementary, properties:

- (Point Axiom)

$$H^*(\{p\}) = H^0(\{p\}) = \mathbb{R}.$$

- (Mayer-Vietoris) If $A, B \subset M$ are open subsets, then we obtain a connecting homomorphism $\delta : H^*(A \cap B) \rightarrow H^{*+1}(A \cup B)$ that yields a long exact sequence $\cdots \rightarrow H^*(A \cup B) \rightarrow H^*(A) \oplus H^*(B) \rightarrow H^*(A \cap B) \rightarrow H^{*+1}(A \cup B) \rightarrow \cdots$.

- (Homotopy Invariance) The projection $\pi : \mathbb{R} \times M \rightarrow M$ induces an isomorphism

$$H^*(M) \rightarrow H^*(\mathbb{R} \times M),$$

where the inclusions $j_t : M \rightarrow \mathbb{R} \times M$ given by $j_t(x) = (t, x)$ all induce inverses to π in cohomology.

The last statement clearly follows from homotopy invariance of de Rham cohomology (proposition 7.1.3).

The above properties hold for all cohomology theories on topological spaces and essentially characterize them. On manifolds, or even just for all (orientable) n -manifolds, we can simplify these axioms to better align with theorem 1.3.2. This will also guide us in how to establish several isomorphism results.

First we narrow down the category. The objects can be all manifolds, all n -manifolds, or all oriented n -manifolds. The morphisms are the inclusion maps $A \subset M$ of open sets in manifolds. A *cohomology functor* on manifolds and inclusions,

$$M \mapsto \mathcal{H}^*(M) = \mathcal{H}^0(M) \oplus \mathcal{H}^1(M) \oplus \cdots \oplus \mathcal{H}^n(M),$$

is now only natural under inclusions:

$$j_{A \subset B}^* \circ j_{B \subset M}^* = (j_{B \subset M} \circ j_{A \subset B})^* = (j_{A \subset M})^*.$$

We impose the additional requirement that the cohomologies of diffeomorphic manifolds are isomorphic. The modified axioms for (oriented) n -manifolds now become:

- (1) (Poincaré Lemma)

$$\mathcal{H}^*(\mathbb{R}^n) = \mathcal{H}^0(\mathbb{R}^n) = \mathbb{R}.$$

- (2) (Mayer-Vietoris) If $A, B \subset M$ are open subsets, then we obtain a connecting homomorphism $\delta : \mathcal{H}^*(A \cap B) \rightarrow \mathcal{H}^{*+1}(A \cup B)$ that yields a long exact sequence $\cdots \rightarrow \mathcal{H}^*(A \cup B) \rightarrow \mathcal{H}^*(A) \oplus \mathcal{H}^*(B) \rightarrow \mathcal{H}^*(A \cap B) \rightarrow \mathcal{H}^{*+1}(A \cup B) \rightarrow \cdots$.

- (3) (Countable Disjointness) If $A_\alpha \subset M$ form a countable collection of pairwise disjoint open subsets, then

$$\mathcal{H}^*\left(\bigsqcup_{\alpha} A_{\alpha}\right) = \times_{\alpha} \mathcal{H}^*(A_{\alpha}).$$

We've already established all but the last property for de Rham cohomology. For last last property just note that any form on the union naturally restricts to forms on each of the open sets. Moreover, the form is exact iff it is exact on each set A_{α} .

The last axiom is really a countable version of Mayer-Vietoris for disjoint sets and is necessary for when we wish to prove results for general noncompact manifolds. Note that there is no homotopy invariance axiom, but, the three axioms together actually imply homotopy invariance. In fact theorem 1.3.2 immediately implies that these properties uniquely determines cohomology on n -manifolds. This is also known as the de Rham isomorphism theorem.

THEOREM 7.3.1. *Consider a cohomology theory \mathcal{H}^* on (oriented) n -manifolds that satisfies (1),(2),(3) and a natural map $\mathcal{H}^*(M) \rightarrow H^*(M)$ that respects inclusions:*

$$\begin{array}{ccc} \mathcal{H}^*(M) & \rightarrow & H^*(M) \\ \downarrow & & \downarrow \\ \mathcal{H}^*(A) & \rightarrow & H^*(A) \end{array}$$

The map $\mathcal{H}^(M) \rightarrow H^*(M)$ is an isomorphism for all (oriented) n -manifolds provided it is an isomorphism when $M = \mathbb{R}^n$.*

PROOF. Given that $P(\mathbb{R}^n)$ is true we still need to establish the other two conditions in theorem 1.3.2. Assume that $A, B \subset M$ are open and that $P(A)$, $P(B)$, and $P(A \cap B)$ are true. Using the Mayer-Vietoris property we obtain

$$\begin{array}{ccccccccc} \mathcal{H}^{*-1}(A) \oplus \mathcal{H}^{*-1}(B) & \rightarrow & \mathcal{H}^{*-1}(A \cap B) & \rightarrow & \mathcal{H}^*(A \cup B) & \rightarrow & \mathcal{H}^*(A) \oplus \mathcal{H}^*(B) & \rightarrow & \mathcal{H}^*(A \cap B) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^{*-1}(A) \oplus H^{*-1}(B) & \rightarrow & H^{*-1}(A \cap B) & \rightarrow & H^*(A \cup B) & \rightarrow & H^*(A) \oplus H^*(B) & \rightarrow & H^*(A \cap B) \end{array}$$

Each square in this diagram is commutative and all vertical arrows, except for the middle one, are assumed to be isomorphisms. It follows by a simple diagram chase that the middle

arrow is also an isomorphism. More precisely, the five lemma asserts that if we have a commutative diagram:

$$\begin{array}{ccccccccc} A_1 & \rightarrow & A_2 & \rightarrow & A_3 & \rightarrow & A_4 & \rightarrow & A_5 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ B_1 & \rightarrow & B_2 & \rightarrow & B_3 & \rightarrow & B_4 & \rightarrow & B_5 \end{array}$$

where the two horizontal rows are exact and $A_i \rightarrow B_i$ are isomorphisms for $i = 1, 2, 4, 5$, then $A_3 \rightarrow B_3$ is an isomorphism.

Finally assume that $A_\alpha \subset M$ are pairwise disjoint open sets and $P(A_\alpha)$ are true. It follows that

$$\mathcal{H}^*\left(\bigsqcup A_\alpha\right) = \times \mathcal{H}^*(A_\alpha) \simeq \times H^*(A_\alpha) = H^*\left(\bigsqcup A_\alpha\right)$$

and $P(\bigsqcup A_\alpha)$ is true. \square

REMARK 7.3.2. Note that condition that $\mathcal{H}^*(\mathbb{R}^n) \rightarrow H^*(\mathbb{R}^n)$ is an isomorphism is almost trivial as the two vector spaces are isomorphic to \mathbb{R} . Thus the natural map merely has to be nontrivial for it to become an isomorphism.

The squares in the five-lemma were all assumed to be commutative. This depends on how the horizontal maps in the Mayer-Vietoris sequence are defined. It can happen that a sign makes the squares anti-commute, but, this does not affect the validity of the statement.

EXAMPLE 7.3.3. We can check homotopy invariance by defining $\mathcal{H}^*(M) = H^*(\mathbb{R} \times M)$ and $\pi^* : H^*(M) \rightarrow \mathcal{H}^*(M)$, where $\pi : \mathbb{R} \times M \rightarrow M$ is the projection. All the conditions are easy to verify.

If instead we take products with a noncontractible space such as S^1 , then we can again check that most of the conditions still hold, only (1) fails. In this way we can construct generalized cohomology theories where only the point axiom fails.

7.4. Generalized Cohomology Theories

We introduce several cohomology theories that can be helpful in calculating the cohomology of spaces. Only compactly supported cohomology is needed for subsequent sections.

7.4.1. Compactly Supported Cohomology. Compactly supported cohomology is not a cohomology theory in the sense of theorem 7.3.1. In the next section we will see how it can be dualized to better fit in with cohomology. Here we establish the basic properties.

DEFINITION 7.4.1. *Compactly supported cohomology* is defined as follows: Let $\Omega_c^p(M)$ denote the compactly supported p -forms. With this we have the compactly supported exact and closed forms $B_c^p(M) \subset Z_c^p(M)$ (note that $d : \Omega_c^p(M) \rightarrow \Omega_c^{p+1}(M)$) and define

$$H_c^p(M) = \frac{Z_c^p(M)}{B_c^p(M)}.$$

Needless to say, for closed manifolds the two cohomology theories are identical. For connected open manifolds, on the other hand, we have that the closed 0-forms must be zero, as they also have to have compact support. Thus $H_c^0(M) = \{0\}$ if M has no compact connected components.

Note that only proper maps $F : M \rightarrow N$ have the property that they map $F^* : \Omega_c^p(N) \rightarrow \Omega_c^p(M)$. In particular, if $A \subset M$ is open, we do not have a restriction map. Instead, we observe that there is a natural inclusion $\Omega_c^p(A) \rightarrow \Omega_c^p(M)$, which induces

$$H_c^p(A) \rightarrow H_c^p(M).$$

Thus compactly supported cohomology looks more like a homology theory.

We start by establishing a version of the Poincaré lemma for this new cohomology theory.

LEMMA 7.4.2. *The compactly supported cohomology of Euclidean space is*

$$H_c^p(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \text{when } p = n, \\ 0 & \text{when } p \neq n. \end{cases}$$

PROOF. We focus on the case where $p = n$, the other cases will be handled in a similar way.

We will consider general oriented n -manifolds. The map

$$\begin{aligned} \Omega_c^n(M) &\rightarrow \mathbb{R}, \\ \omega &\mapsto \int_M \omega \end{aligned}$$

vanishes on closed forms by Stokes' theorem. Consequently, it induces a map

$$\begin{aligned} H_c^n(M) &\rightarrow \mathbb{R}, \\ [\omega] &\mapsto \int_M \omega. \end{aligned}$$

It is also surjective, since any form with the property that it is positive when evaluated on a positively oriented frame is integrated to a positive number.

Case 1: $M = S^n$. We know that $H^n(S^n) = \mathbb{R}$, so $\int : H^n(S^n) \rightarrow \mathbb{R}$ must be an isomorphism.

Case 2: $M = \mathbb{R}^n$. We can think of $M = S^n - \{x\}$. Any compactly supported form ω on M is thus also a form on S^n . Given that $\int_M \omega = 0$, we further note that $\int_{S^n} \omega = 0$. In particular, ω must be exact on S^n . Let $\psi \in \Omega^{n-1}(S^n)$ be chosen such that $d\psi = \omega$. Use again that ω is compactly supported to find an open disc U around x such that ω vanishes on U and $U \cup M = S^n$. Then ψ is clearly closed on U and must by the Poincaré lemma be exact. Thus, we can find $\theta \in \Omega^{n-2}(U)$ with $d\theta = \psi$ on U . This form doesn't necessarily extend to S^n , but, we can select a bump function $\lambda : S^n \rightarrow [0, 1]$ that vanishes on $S^n - U$ and is 1 on some smaller neighborhood $V \subset U$ around x . Now observe that $\psi - d(\lambda\theta)$ is actually defined on all of S^n . It vanishes on V and clearly

$$d(\psi - d(\lambda\theta)) = d\psi = \omega.$$

The case for p -forms proceeds in a similar way using that $H^p(S^n) = 0$ for $1 < p < n$. When $p = 1$, we obtain $\omega = d\psi$, where $\psi \in \Omega^0(S^n)$. Thus ψ is constant in a neighborhood of x and we can use $\psi - \psi(x)$ as a function with compact support in $S^n - \{x\}$ whose differential is ω .

Finally $H_c^0(M) = 0$ for all connected non-compact manifolds. \square

This result together with the fact that compactly supported cohomology respects inclusions of compact sets indicates that for an n -manifold we should consider

$$\mathcal{H}^p(M) = \text{Hom}(H_c^{n-p}(M), \mathbb{R}).$$

This in fact defines a cohomology functor. Clearly, \mathcal{H}^* is the same for diffeomorphic manifolds and is contravariant under inclusions:

$$\mathcal{H}^p(M) = \text{Hom}(H_c^{n-p}(M), \mathbb{R}) \rightarrow \text{Hom}(H_c^{n-p}(A), \mathbb{R}) = \mathcal{H}^p(A).$$

The above lemma tells us that the Poincaré lemma holds:

$$\mathcal{H}^*(\mathbb{R}^n) = \mathcal{H}^0(\mathbb{R}^n) = \text{Hom}(H_c^n(\mathbb{R}^n), \mathbb{R}) = \mathbb{R}.$$

Since $\Omega_c^*(\coprod_\alpha A_\alpha) = \oplus_\alpha \Omega_c^*(A_\alpha)$ for a pairwise disjoint union of open sets $A_\alpha \subset M$, we also obtain

$$\begin{aligned} \mathcal{H}^*\left(\coprod_\alpha A_\alpha\right) &= \text{Hom}\left(H_c^{n-*}\left(\coprod_\alpha A_\alpha\right), \mathbb{R}\right) \\ &= \text{Hom}\left(\bigoplus_\alpha H_c^{n-*}(A_\alpha), \mathbb{R}\right) \\ &= \bigotimes_\alpha \text{Hom}(H_c^{n-*}(A_\alpha), \mathbb{R}) \\ &= \bigotimes_\alpha \mathcal{H}^*(A_\alpha). \end{aligned}$$

Finally we need a Mayer-Vietoris sequence for open sets $A, B \subset M$ with $M = A \cup B$. This starts with the observation that we have exact sequences:

$$\begin{array}{ccccccc} 0 & \rightarrow & \Omega_c^{*+1}(A \cap B) & \rightarrow & \Omega_c^{*+1}(A) \oplus \Omega_c^{*+1}(B) & \rightarrow & \Omega_c^{*+1}(M) \rightarrow 0 \\ & & \uparrow d & & \uparrow d & & \uparrow d \\ 0 & \rightarrow & \Omega_c^*(A \cap B) & \rightarrow & \Omega_c^*(A) \oplus \Omega_c^*(B) & \rightarrow & \Omega_c^*(M) \rightarrow 0 \end{array}$$

where the horizontal arrows are defined by:

$$\begin{aligned} \Omega_c^*(A \cap B) &\rightarrow \Omega_c^*(A) \oplus \Omega_c^*(B), \\ [\omega] &\mapsto ([\omega], [\omega]), \end{aligned}$$

and

$$\begin{aligned} \Omega_c^*(A) \oplus \Omega_c^*(B) &\rightarrow \Omega_c^*(M), \\ ([\omega_A], [\omega_B]) &\mapsto [\omega_A - \omega_B]. \end{aligned}$$

This certainly leads to a long exact Mayer-Vietoris sequence:

$$\cdots \rightarrow H_c^*(A \cap B) \rightarrow H_c^*(A) \oplus H_c^*(B) \rightarrow H_c^*(M) \rightarrow H_c^{*+1}(A \cap B) \rightarrow \cdots.$$

However, we can also dualize to obtain a short exact sequence that algebraically looks similar (even with the sign choices) to the sequence used for Mayer-Vietoris:

$$0 \rightarrow \text{Hom}(\Omega_c^{n-p}(M), \mathbb{R}) \rightarrow \text{Hom}(\Omega_c^{n-p}(A), \mathbb{R}) \oplus \text{Hom}(\Omega_c^{n-p}(B), \mathbb{R}) \rightarrow \text{Hom}(\Omega_c^{n-p}(A \cap B), \mathbb{R}) \rightarrow 0$$

and differentials that map

$$d : \text{Hom}(\Omega_c^{n-p}(M), \mathbb{R}) \rightarrow \text{Hom}(\Omega_c^{n-p-1}(M), \mathbb{R}) = \text{Hom}(\Omega_c^{n-(p+1)}(M), \mathbb{R}).$$

This gives us a connecting homomorphism $\delta : \mathcal{H}^*(A \cap B) \rightarrow \mathcal{H}^{*+1}(M)$ and a long exact sequence

$$\cdots \rightarrow \mathcal{H}^*(M) \rightarrow \mathcal{H}^*(A) \oplus \mathcal{H}^*(B) \rightarrow \mathcal{H}^*(A \cap B) \rightarrow \mathcal{H}^{*+1}(M) \rightarrow \cdots.$$

Finally, we can also prove lemma 7.1.6 for compactly supported cohomology.

LEMMA 7.4.3. *If $F : M \rightarrow N$ is a finite covering map, then*

$$F^* : H_c^p(N) \rightarrow H_c^p(M)$$

is an injection.

PROOF. The proof uses the same transgression map after we note that it maps $\tau : \Omega_c^p(M) \rightarrow \Omega_c^p(N)$ since F takes compact sets to compact sets. \square

7.4.2. Relative Cohomology. Compactly supported cohomology can be used very effectively to define relative cohomology and also simplifies the calculation of some of the cohomology groups we have seen.

We start with the simplest and most important situation where $S \subset M$ is a closed submanifold of a closed manifold.

PROPOSITION 7.4.4. *If $S \subset M$ is a closed submanifold of a closed manifold, then*

- (1) *The restriction map $i^* : \Omega^p(M) \rightarrow \Omega^p(S)$ is surjective.*
- (2) *If $\theta \in \Omega^{p-1}(S)$ is closed, then there exists $\psi \in \Omega^{p-1}(M)$ such that $\theta = i^*\psi$ and $d\psi \in \Omega_c^p(M-S)$.*
- (3) *If $\omega \in \Omega^p(M)$ with $d\omega \in \Omega_c^{p+1}(M-S)$ and $i^*\omega \in \Omega^p(S)$ is exact, then there exists $\theta \in \Omega^{p-1}(M)$ such that $\omega - d\theta \in \Omega_c^p(M-S)$.*

PROOF. Select a neighborhood $S \subset U \subset M$ that deformation retracts $\pi : U \rightarrow S$. Then $i^* : H^p(U) \rightarrow H^p(S)$ is an isomorphism. We also need a function $\lambda : M \rightarrow [0, 1]$ that is compactly supported in U and is 1 on a neighborhood of S .

1. Given $\omega \in \Omega^p(S)$ let $\bar{\omega} = \lambda\pi^*(\omega)$.
2. This also shows that $d(\lambda\pi^*\theta) = d\lambda \wedge \pi^*\theta + \lambda d\pi^*\theta$ has compact support in $M-S$.
3. Conversely assume that $\omega \in \Omega^p(M)$ has $d\omega \in \Omega_c^{p+1}(M-S)$. By possibly shrinking U we can assume that it is disjoint from the support of $d\omega$. Thus, $d\omega|_U = 0$ since $i : S \rightarrow U$ is an isomorphism in cohomology and we assume that $i^*\omega$ is exact, it follows that $\omega|_U = d\psi$ for some $\psi \in \Omega^{p-1}(U)$. Define $\theta = \lambda\psi$ and then note that

$$\begin{aligned} \omega - d\theta &= \omega - \lambda d\psi - d\lambda \wedge \psi \\ &= \omega - \lambda\omega|_U - d\lambda \wedge \theta \\ &\in \Omega_c^p(M-S). \end{aligned}$$

\square

THEOREM 7.4.5. *Assume $S \subset M$ is a closed submanifold of a closed manifold, then there is a long exact sequence of cohomology groups:*

$$\rightarrow H_c^p(M-S) \rightarrow H^p(M) \rightarrow H^p(S) \rightarrow H_c^{p+1}(M-S) \rightarrow$$

PROOF. Part (1) of the above proposition shows that we have a short exact sequence

$$\begin{aligned} 0 &\rightarrow \Omega^p(M, S) \rightarrow \Omega^p(M) \rightarrow \Omega^p(S) \rightarrow 0, \\ \Omega^p(M, S) &= \ker(i^* : \Omega^p(M) \rightarrow \Omega^p(S)). \end{aligned}$$

We claim that (2) and (3) show that the natural inclusion

$$\Omega_c^p(M-S) \rightarrow \Omega^p(M, S)$$

induces an isomorphism $H_c^p(M-S) \rightarrow H^p(M, S)$.

To show that it is injective consider $\omega \in \Omega_c^p(M-S)$, such that $\omega = d\theta$, where $\theta \in \Omega^{p-1}(M, S)$. We can apply (3) to θ to find $\psi \in \Omega^{p-2}(M)$ such that $\theta - d\psi \in \Omega_c^{p-1}(M-S)$. This shows that $\omega = d(\theta - d\psi)$ for a form $\theta - d\psi \in \Omega_c^{p-1}(M-S)$.

To show that it is surjective consider $\omega \in \Omega^p(M, S)$ with $d\omega = 0$. By (3) we can find $\theta \in \Omega^{p-1}(M)$ such that $\omega - d\theta \in \Omega_c^p(M-S)$, but, we don't know that $\theta \in \Omega^{p-1}(M, S)$. To fix that problem use (2) to find $\psi \in \Omega^{p-1}(M)$ such that $i^*\theta = i^*\psi$ and $d\psi \in \Omega_c^p(M-S)$. Then $\omega - d(\theta - \psi) = (\omega - d\theta) - d\psi \in \Omega_c^p(M-S)$ and $\theta - \psi \in \Omega^{p-1}(M, S)$. \square

Good examples are $S^{n-1} \subset S^n$ with $S^n - S^{n-1}$ being two copies of \mathbb{R}^n and $\mathbb{P}^{n-1} \subset \mathbb{P}^n$ where $\mathbb{P}^n - \mathbb{P}^{n-1} \simeq \mathbb{F}^n$. This gives us a slightly different inductive method for computing the cohomology of these spaces. Conversely, given the cohomology groups of those spaces, it computes the compactly supported cohomology of \mathbb{R}^n .

It can also be used on manifolds with boundary:

$$\rightarrow H_c^p(\text{int}M) \rightarrow H^p(M) \rightarrow H^p(\partial M) \rightarrow H_c^{p+1}(\text{int}M) \rightarrow$$

where we can specialize to $M = D^n \subset \mathbb{R}^n$, the closed unit ball. The Poincaré lemma computes the cohomology of D^n so we get that

$$H_c^{p+1}(B^n) \simeq H^p(S^{n-1}).$$

For general connected compact manifolds with boundary we also obtain some interesting information.

THEOREM 7.4.6. *If M is a connected compact n -manifold with boundary, then*

$$H^n(M) = 0.$$

PROOF. If M is oriented, then we know that ∂M is also oriented and that

$$\begin{aligned} H^n(M, \partial M) &= H_c^n(\text{int}M) \simeq \mathbb{R} \\ H^n(\partial M) &= \{0\}, \\ H^{n-1}(\partial M) &\simeq \mathbb{R}^k, \end{aligned}$$

where k is the number of components of ∂M . The connecting homomorphism $H^{n-1}(\partial M) \rightarrow H_c^n(\text{int}M)$ can be analyzed from the diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & \Omega^n(M, \partial M) & \rightarrow & \Omega^n(M) & \rightarrow & \Omega^n(\partial M) \rightarrow 0 \\ & & \uparrow d & & \uparrow d & & \uparrow d \\ 0 & \rightarrow & \Omega^{n-1}(M, \partial M) & \rightarrow & \Omega^{n-1}(M) & \rightarrow & \Omega^{n-1}(\partial M) \rightarrow 0 \end{array}$$

Evidently any $\omega \in \Omega^{n-1}(\partial M)$ is the restriction of some $\bar{\omega} \in \Omega^{n-1}(M)$. Moreover, if $d\omega = 0$, then we can further assume that $d\bar{\omega} \in \Omega_c^n(\text{int}M)$. Stokes' theorem then tells us that

$$\int_M d\bar{\omega} = \int_{\partial M} \bar{\omega} = \int_{\partial M} \omega.$$

This shows that the map $H^{n-1}(\partial M) \rightarrow H_c^n(\text{int}M)$ is nontrivial and hence surjective, which in turn implies that $H^n(M) = \{0\}$.

If M is not orientable, then we can use lemma 7.1.6 on the orientation covering. \square

It is possible to extend the above long exact sequence to the case where M is non-compact by using compactly supported cohomology on M . This gives us the long exact sequence

$$\rightarrow H_c^p(M - S) \rightarrow H_c^p(M) \rightarrow H^p(S) \rightarrow H_c^{p+1}(M - S) \rightarrow$$

It is even possible to also have S be non-compact if we assume that the embedding is proper and then also use compactly supported cohomology on S

$$\rightarrow H_c^p(M - S) \rightarrow H_c^p(M) \rightarrow H_c^p(S) \rightarrow H_c^{p+1}(M - S) \rightarrow$$

We can generalize even further to a situation where S is simply a compact subset of M . In that case we define the deRham-Alexander cohomology groups $\check{H}^p(S)$ using

$$\check{\Omega}^p(S) = \frac{\{\omega \in \Omega^p(M)\}}{\omega_1 \sim \omega_2 \text{ iff } \omega_1 = \omega_2 \text{ on a ngbd of } S},$$

i.e., the elements of $\check{\Omega}^p(S)$ are germs of forms on M at S . We now obtain a short exact sequence

$$0 \rightarrow \Omega_c^p(M-S) \rightarrow \Omega_c^p(M) \rightarrow \check{\Omega}^p(S) \rightarrow 0.$$

This in turn gives us a long exact sequence

$$\rightarrow H_c^p(M-S) \rightarrow H_c^p(M) \rightarrow \check{H}^p(S) \rightarrow H_c^{p+1}(M-S) \rightarrow$$

Finally we can define a more general relative cohomology group. We take a differentiable map $F : S \rightarrow M$ between manifolds. It could, e.g., be an embedding of $S \subset M$, but, S need not be closed. Define

$$\Omega^p(F) = \Omega^p(M) \oplus \Omega^{p-1}(S)$$

and the differential

$$\begin{aligned} d : \Omega^p(F) &\rightarrow \Omega^{p+1}(F) \\ d(\omega, \psi) &= (d\omega, F^*\omega - d\psi) \end{aligned}$$

Note that $d^2 = 0$ so we obtain a complex and cohomology groups $H^p(F)$. These “forms” fit into a short exact sequence

$$0 \rightarrow \Omega^{p-1}(S) \rightarrow \Omega^p(F) \rightarrow \Omega^p(M) \rightarrow 0,$$

where the maps are just the natural inclusion and projection. When we include the differential we arrive at a large diagram where the left square is anti-commutative and the right one commutative

$$\begin{array}{ccccccc} 0 & \rightarrow & \Omega^p(S) & \rightarrow & \Omega^{p+1}(M) \oplus \Omega^p(S) & \rightarrow & \Omega^{p+1}(M) \rightarrow 0 \\ & & \uparrow d & & \uparrow (d, F^* - d) & & \uparrow d \\ 0 & \rightarrow & \Omega^{p-1}(S) & \rightarrow & \Omega^p(M) \oplus \Omega^{p-1}(S) & \rightarrow & \Omega^p(M) \rightarrow 0 \end{array}$$

This still leads us to a long exact sequence

$$\rightarrow H^{p-1}(S) \rightarrow H^p(F) \rightarrow H^p(M) \rightarrow H^p(S) \rightarrow$$

The connecting homomorphism $H^p(M) \rightarrow H^p(S)$ is in fact the pull-back map F^* as can be seen by a simple diagram chase.

In case $i : S \subset M$ is an embedding we also use the notation $H^p(M, S) = H^p(i)$. In this case it would seem that the connecting homomorphism is more naturally defined to be $H^{p-1}(S) \rightarrow H^p(M, S)$, but, we don't have a short exact sequence

$$0 \rightarrow \Omega^p(M) \oplus \Omega^{p-1}(S) \rightarrow \Omega^p(M) \rightarrow \Omega^p(S) \rightarrow 0$$

hence the tricky shift in the groups.

We can easily relate the new relative cohomology to the one defined above. This shows that the relative cohomology, while trickier to define, is ultimately more general and useful.

PROPOSITION 7.4.7. *If $i : S \subset M$ is a closed submanifold of a closed manifold, then the natural map*

$$\begin{aligned} \Omega_c^p(M-S) &\rightarrow \Omega^p(M) \oplus \Omega^{p-1}(S) \\ \omega &\rightarrow (\omega, 0) \end{aligned}$$

defines an isomorphism

$$H_c^p(M-S) \simeq H^p(i).$$

PROOF. Simply observe that we have two long exact sequences

$$\begin{aligned} &\rightarrow H^p(i) \rightarrow H^p(M) \rightarrow H^p(S) \rightarrow H^{p+1}(i) \rightarrow \\ &\rightarrow H_c^p(M-S) \rightarrow H^p(M) \rightarrow H^p(S) \rightarrow H_c^{p+1}(M-S) \rightarrow \end{aligned}$$

where two out of three terms are equal. \square

Now that we have a fairly general relative cohomology theory we can establish the well-known excision property.

THEOREM 7.4.8. *If $M = U \cup V$, where U and V are open, then the restriction map*

$$H^p(M, U) \rightarrow H^p(V, U \cap V)$$

is an isomorphism.

PROOF. First select a partition of unity λ_U, λ_V relative to U, V .

We start with injectivity. Take a class $[(\omega, \psi)] \in H^p(M, U)$, i.e.,

$$\begin{aligned} d\omega &= 0, \\ \omega|_U &= d\psi. \end{aligned}$$

If the restriction to $(V, U \cap V)$ is exact, then we can find $(\bar{\omega}, \bar{\psi}) \in \Omega^{p-1}(V) \oplus \Omega^{p-2}(U \cap V)$ such that

$$\begin{aligned} \omega|_V &= d\bar{\omega}, \\ \psi|_{U \cap V} &= \bar{\omega}|_{U \cap V} - d\bar{\psi}. \end{aligned}$$

Using that $\bar{\psi} = \lambda_U \bar{\psi} + \lambda_V \bar{\psi}$ we obtain

$$\begin{aligned} (\psi + d(\lambda_V \bar{\psi}))|_{U \cap V} &= (\bar{\omega} - d(\lambda_U \bar{\psi}))|_{U \cap V}, \\ \psi + d(\lambda_V \bar{\psi}) &\in \Omega^{p-1}(U), \\ \bar{\omega} - d(\lambda_U \bar{\psi}) &\in \Omega^{p-1}(V). \end{aligned}$$

Thus we have a form $\bar{\omega} \in \Omega^{p-1}(M)$ defined by $\psi + d(\lambda_V \bar{\psi})$ on U and $\bar{\omega} - d(\lambda_U \bar{\psi})$ on V . Clearly $d\bar{\omega} = \omega$ and $\psi = \bar{\omega}|_U - d(\lambda_V \bar{\psi})$ so we have shown that (ω, ψ) is exact.

For surjectivity select $(\bar{\omega}, \bar{\psi}) \in \Omega^p(V) \oplus \Omega^{p-1}(U \cap V)$ that is closed:

$$\begin{aligned} d\bar{\omega} &= 0, \\ \bar{\omega}|_{U \cap V} &= d\bar{\psi}. \end{aligned}$$

Using

$$\begin{aligned} \bar{\omega}|_{U \cap V} - d(\lambda_U \bar{\psi}) &= d(\lambda_V \bar{\psi}), \\ \bar{\omega} - d(\lambda_U \bar{\psi}) &\in \Omega^p(V), \\ d(\lambda_V \bar{\psi}) &\in \Omega^p(U) \end{aligned}$$

we can define ω as $\bar{\omega} - d(\lambda_U \bar{\psi})$ on V and $d(\lambda_V \bar{\psi})$ on U . Clearly ω is closed and $\omega|_U = d(\lambda_V \bar{\psi})$. Thus we define $\psi = \lambda_V \bar{\psi}$ in order to get a closed form $(\omega, \psi) \in \Omega^p(M) \oplus \Omega^{p-1}(U)$. Restricting this form to $\Omega^p(V) \oplus \Omega^{p-1}(U \cap V)$ yields $(\bar{\omega} - d(\lambda_U \bar{\psi}), \lambda_V \bar{\psi})$ which is not $(\bar{\omega}, \bar{\psi})$. However, the difference is exact:

$$\begin{aligned} (\bar{\omega}, \bar{\psi}) - (\bar{\omega} - d(\lambda_U \bar{\psi}), \lambda_V \bar{\psi}) &= (d(\lambda_U \bar{\psi}), \lambda_U \bar{\psi}) \\ &= d(\lambda_U \bar{\psi}, 0). \end{aligned}$$

Thus $[(\omega, \psi)] \in H^p(M, U)$ is mapped to $[(\bar{\omega}, \bar{\psi})] \in H^p(V, U \cap V)$. \square

7.5. Poincaré Duality and its Consequences

We explain several interesting results that follow from Poincaré Duality and begin to connect the numerical invariants defined geometrically using transversality to algebraic concepts in cohomology. We start by explaining Poincaré duality, then give some examples of its consequences and finish by showing how it can be used to calculate the degree of a map using integration.

7.5.1. Poincaré Duality. The last important piece of information is how the wedge product acts on cohomology. It is easy to see that we have a map

$$\begin{aligned} H^p(M) \times H^q(M) &\rightarrow H^{p+q}(M), \\ ([\omega], [\psi]) &\mapsto [\omega \wedge \psi]. \end{aligned}$$

This is well-defined as

$$(\omega + d\theta) \wedge (\psi + d\phi) = \omega \wedge \psi + d(\theta \wedge (\psi + d\phi)) \pm d(\omega \wedge \phi).$$

Thus the wedge product induces a ring structure on $H^*(M)$ that in a suitable sense will be shown to be dual to the intersection theory developed using transversality. To that end, we are particularly interested in understanding what happens in case $p + q = n$ as that will create a natural map from the cohomology functor $\mathcal{H}^*(M)$ to de Rham cohomology.

Note that this ring structure also gives us a well-defined map:

$$H^p(M) \times H_c^q(M) \rightarrow H_c^{p+q}(M).$$

When M is oriented and $p + q = n$ we can in addition integrate to obtain a pairing:

$$H^p(M) \times H_c^q(M) \rightarrow H_c^n(M) \xrightarrow{\int} \mathbb{R}.$$

THEOREM 7.5.1 (Poincaré Duality). *Let M be an oriented n -manifold. The pairing*

$$\begin{aligned} H^p(M) \times H_c^{n-p}(M) &\rightarrow \mathbb{R}, \\ ([\omega], [\psi]) &\mapsto \int_M \omega \wedge \psi \end{aligned}$$

is well-defined and non-degenerate. In particular, the two cohomology groups $H^p(M)$ and $H_c^{n-p}(M)$ are dual to each other and consequently have the same dimension when they are finite dimensional.

PROOF. The bilinear form defines a linear map on all oriented n -manifolds:

$$H^p(M) \rightarrow \text{Hom}(H_c^{n-p}(M), \mathbb{R}) = \mathcal{H}^p(M).$$

We claim that this map is an isomorphism for all orientable, but, not necessarily connected, manifolds. This will follow from theorem 7.3.1 provided we can show that it is an isomorphism when $M = \mathbb{R}^n$. This case follows from the proof of lemma 7.4.2. \square

There is also a map

$$H_c^{n-p}(M) \rightarrow \text{Hom}(H^p(M), \mathbb{R})$$

which is an isomorphism when $H_c^{n-p}(M)$ is finite dimensional, but, not necessarily otherwise. In fact the countable disjointness property generally fails in this case.

7.5.2. Consequences of Poincaré Duality.

COROLLARY 7.5.2. *If M^n is contractible, then*

$$H_c^p(M) = \begin{cases} \mathbb{R} & \text{when } p = n, \\ 0 & \text{when } p \neq n. \end{cases}$$

THEOREM 7.5.3. *On a closed oriented n -manifold M the cohomology groups $H^p(M)$ and $H^{n-p}(M)$ are isomorphic.*

PROOF. This requires that we know that $H^p(M)$ is finite dimensional for all p .

First note that if $O \subset \mathbb{R}^k$ is a finite union of open boxes, then the de Rham cohomology groups are finite dimensional. The proof of this uses Mayer-Vietoris and induction on the number of boxes. Specifically if $M = A \cup B$, where $H^*(A)$, $H^*(B)$, and $H^*(A \cap B)$ are finite dimensional, then also $H^*(M)$ is finite dimensional. To see this consider the part of the long exact sequence:

$$H^{*-1}(A \cap B) \rightarrow H^*(M) \rightarrow H^*(A) \oplus H^*(B).$$

Here the image of $H^{*-1}(A \cap B) \rightarrow H^*(M)$ is finite dimensional and as the sequence is exact any complement to the image is mapped injectively into $H^*(A) \oplus H^*(B)$ and is thus also finite dimensional. Next if $B_1, \dots, B_k \subset \mathbb{R}^k$ are boxes, then the intersection

$$B_k \cap (B_1 \cup \dots \cup B_{k-1}) = (B_k \cap B_1) \cup \dots \cup (B_k \cap B_{k-1})$$

consists of at most $k-1$ boxes. This allows us to complete the induction step.

This will give the result for $M \subset \mathbb{R}^k$ as we can find a tubular neighborhood $M \subset U \subset \mathbb{R}^k$ and a retraction $r : U \rightarrow M$, i.e., $r|_M = id_M$. Now cover M by open boxes that lie in U and use compactness of M to find $M \subset O \subset U$ with O being a union of finitely many open boxes. Since $r|_M = id_M$ the retraction $r^* : H^p(M) \rightarrow H^p(O)$ is an injection so it follows that $H^p(M)$ is finite dimensional. \square

Note that $\mathbb{R}P^2$ does not satisfy this duality between H^0 and H^2 . In fact we always have

THEOREM 7.5.4. *If M is a connected n -manifold that is not orientable, then*

$$H_c^n(M) = 0.$$

PROOF. We use the two-fold orientation cover $F : \hat{M} \rightarrow M$ and the involution $A : \hat{M} \rightarrow \hat{M}$ such that $F = F \circ A$. The fact that M is not orientable means that A is orientation reversing. This implies that pull-back by A changes integrals by a sign:

$$\int_{\hat{M}} \eta = - \int_{\hat{M}} A^* \eta, \quad \eta \in \Omega_c^n(\hat{M}).$$

To prove the theorem select $\omega \in \Omega_c^n(M)$ and consider the pull-back $F^* \omega \in \Omega_c^n(\hat{M})$. Since $F = F \circ A$ this form is invariant under pull-back by A

$$\int_{\hat{M}} F^* \omega = \int_{\hat{M}} A^* \circ F^* \omega.$$

On the other hand, as A reverses orientation we must also have

$$\int_{\hat{M}} F^* \omega = - \int_{\hat{M}} A^* \circ F^* \omega.$$

Thus

$$\int_{\hat{M}} F^* \omega = 0.$$

This shows that the pull-back is exact

$$F^* \omega = d\psi, \quad \psi \in \Omega_c^{n-1}(\hat{M}).$$

On the other hand, from lemma 7.4.3 we know that $F^* : H_c^n(M) \rightarrow H_c^n(\hat{M})$ is an injection. This shows the claim. \square

COROLLARY 7.5.5. *If M is an open connected n -manifold, then*

$$H^n(M) = 0.$$

PROOF. By lemma 7.4.3 it suffices to prove this for orientable manifolds. In this case it follows from Poincaré duality that

$$H^n(M) \simeq \text{Hom}(H_c^0(M), \mathbb{R}) \simeq 0.$$

\square

There are many more interesting results for compactly supported cohomology. In case of oriented manifolds Poincaré duality is a natural way of proving them, but, without that result one can often prove them using theorem 1.3.2. A good example is the compactly supported version of homotopy invariance:

$$H_c^*(M) \simeq H_c^{*+1}(\mathbb{R} \times M).$$

7.5.3. Degrees of Maps. Given the simple nature of the top cohomology class of a manifold, we see that maps between manifolds of the same dimension can act only by multiplication on the top cohomology class. This multiplicative factor is in fact an integer and coincides with the *degree* of the map.

To be precise, suppose we have two connected oriented n -manifolds M and N and also a proper map $F : M \rightarrow N$. Then we obtain a diagram

$$\begin{array}{ccc} H_c^n(N) & \xrightarrow{F^*} & H_c^n(M) \\ \downarrow \int & & \downarrow \int \\ \mathbb{R} & \xrightarrow{d} & \mathbb{R}. \end{array}$$

Since the vertical arrows are isomorphisms, the induced map F^* yields a unique map $d : \mathbb{R} \rightarrow \mathbb{R}$. This map must be multiplication by some number, which we call the degree of F , denoted by $\deg F$. Clearly, the degree is defined by the property

$$\int_M F^* \omega = \deg F \cdot \int_N \omega.$$

From the functorial properties of the induced maps on cohomology we see that

$$\deg(F \circ G) = \deg(F) \deg(G).$$

PROPOSITION 7.5.6. *If $F : M \rightarrow N$ is a diffeomorphism between oriented n -manifolds, then $\deg F = \pm 1$, depending on whether F preserves or reverses orientation.*

PROOF. Note that our definition of integration of forms is independent of coordinate changes. It relies only on a choice of orientation. If this choice is changed then the integral changes by a sign. This clearly establishes the lemma. \square

THEOREM 7.5.7. *If $F : M \rightarrow N$ is a proper map between oriented n -manifolds, then $\deg F$ is an integer and agrees with the oriented degree.*

PROOF. The proof will also give a recipe for computing the degree. By Sard's theorem there is a regular value $y \in N$. By lemma 3.1.1 there exists a neighborhood V around y such that $F^{-1}(V) = \bigcup_{k=1}^n U_k$, where U_k are mutually disjoint and $F : U_k \rightarrow V$ is a diffeomorphism. Now select $\omega \in \Omega_c^n(V)$ with $\int \omega = 1$ and note that:

$$F^* \omega = \sum_{i=1}^k F^* \omega|_{U_i},$$

where each $F^* \omega|_{U_i}$ has compact support in U_i . The above lemma now tells us that

$$\int_{U_i} F^* \omega|_{U_i} = \pm 1.$$

Hence,

$$\begin{aligned} \deg F &= \deg F \cdot \int_N \omega \\ &= \deg F \cdot \int_U \omega \\ &= \int_M F^* \omega \\ &= \sum_{i=1}^k \int_{U_i} F^* \omega|_{U_i} \end{aligned}$$

is an integer. Here $\int_{U_i} F^* \omega|_{U_i} = \pm 1$ depending simply on whether F preserves or reverses the orientations at x_i . Thus, the cohomologically defined degree also counts the number of preimages for regular values with sign just as the oriented degree from section 5.4.2. \square

On an oriented Riemannian manifold (M, g) we always have a canonical volume form denoted by $d\text{vol}_g$. Using this form, we see that the degree of a map between closed Riemannian manifolds $F : (M, g) \rightarrow (N, h)$ can be computed as

$$\deg F = \frac{\int_M F^*(d\text{vol}_h)}{\text{vol}(N)}.$$

In case F is locally a Riemannian isometry, we must have that:

$$F^*(d\text{vol}_h) = \pm d\text{vol}_g.$$

Hence,

$$\deg F = \pm \frac{\text{vol}M}{\text{vol}N}.$$

This gives the well-known formula for the relationship between the volumes of Riemannian manifolds that are related by a finite covering map.

On $\mathbb{R}^n - \{0\}$ there is an interesting closed $(n-1)$ -form

$$\omega = r^{-n} \sum_{i=1}^n (-1)^{i+1} x^i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n,$$

where $r^2 = \sum_{i=1}^n (x^i)^2$. If we restrict this to a sphere of radius ε around the origin, then

$$\begin{aligned}
 \int_{S^{n-1}(\varepsilon)} \omega &= \varepsilon^{-n} \int_{S^{n-1}(\varepsilon)} \sum_{i=1}^n (-1)^{i+1} x^i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \\
 &= \varepsilon^{-n} \int_{\bar{B}(0, \varepsilon)} d \left(\sum_{i=1}^n (-1)^{i+1} x^i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \right) \\
 &= \varepsilon^{-n} \int_{\bar{B}(0, \varepsilon)} n dx^1 \wedge \cdots \wedge dx^n \\
 &= n \varepsilon^{-n} \text{vol} \bar{B}(0, \varepsilon) \\
 &= n \text{vol} \bar{B}(0, 1) \\
 &= \text{vol}_{n-1} S^{n-1}(1).
 \end{aligned}$$

More generally, if $F : M^{n-1} \rightarrow \mathbb{R}^n - \{0\}$ is a smooth map, then it is clearly homotopic to the map $F_1 : M^{n-1} \rightarrow S^{n-1}(1)$ defined by $F_1 = F/|F|$ so we obtain an integral formula for the winding number

$$\begin{aligned}
 W(F, 0) &= \text{deg} F_1 \\
 &= \frac{1}{\text{vol}_{n-1} S^{n-1}(1)} \int_M F_1^* \omega \\
 &= \frac{1}{\text{vol}_{n-1} S^{n-1}(1)} \int_M F^* \omega.
 \end{aligned}$$

7.6. The Künneth-Leray-Hirsch Theorem

In this section we shall compute the cohomology of a fibration under certain simplifying assumptions. We start with the trivial fiber bundles $E = F \times B$. The standard projection for any fiber bundle is denoted $\pi : E \rightarrow B$ and when the bundle is trivial we also have a projection $\bar{\pi} : E \rightarrow F$ on to the fiber.

THEOREM 7.6.1 (Künneth). *If $H^*(F)$ is finite dimensional, then there is an isomorphism:*

$$\begin{aligned}
 \times : \bigoplus_{p+q=r} H^p(F) \otimes H^q(B) &\rightarrow H^r(E) \\
 \phi \otimes \beta &\mapsto \bar{\pi}^*(\psi) \wedge \pi^*(\omega).
 \end{aligned}$$

PROOF. We fix F and use theorem 1.3.2 with the statement $P(B)$ being that the theorem is true.

When $B = \mathbb{R}^n$, we have $H^*(B) = H^0(B) = \mathbb{R}$. Thus

$$\bigoplus_{p+q=r} H^p(F) \otimes H^q(B) = H^r(F)$$

and the statement follows from homotopy invariance of cohomology.

For condition (2) in theorem 1.3.2 assume that the result holds for open sets $A_1, A_2, A_1 \cap A_2 \subset B$, then we can use the same strategy as in the proof of theorem 7.3.1 to verify the statement for $A_1 \cup A_2$.

Finally for condition (3), assume the statement holds for pairwise disjoint open sets: $A_\alpha \subset B$. We have to show it also holds for the union. This depends crucially on $H^p(F)$

being finite dimensional as tensor products do not, in general, respect infinite products (see example below). Specifically, we use that

$$\mathrm{Hom}(\mathrm{Hom}(H^p(F), \mathbb{R}), V) \simeq \mathrm{Hom}(\mathrm{Hom}(H^p(F), \mathbb{R}), \mathbb{R}) \otimes V \simeq H^p(F) \otimes V.$$

In particular, if $V = \times_{\alpha} V_{\alpha}$, then

$$\begin{aligned} H^p(F) \otimes (\times_{\alpha} V_{\alpha}) &= \mathrm{Hom}(\mathrm{Hom}(H^p(F), \mathbb{R}), \times_{\alpha} V_{\alpha}) \\ &= \times_{\alpha} \mathrm{Hom}(\mathrm{Hom}(H^p(F), \mathbb{R}), V_{\alpha}) \\ &= \times_{\alpha} (H^p(F) \otimes V_{\alpha}). \end{aligned}$$

This leads us to the desired isomorphism:

$$\begin{aligned} \bigoplus_{p+q=r} H^p(F) \otimes H^q\left(\bigsqcup_{\alpha} A_{\alpha}\right) &= \bigoplus_{p+q=r} H^p(F) \otimes (\times_{\alpha} H^q(A_{\alpha})) \\ &= \bigoplus_{p+q=r} \times_{\alpha} (H^p(F) \otimes H^q(A_{\alpha})) \\ &= \times_{\alpha} \bigoplus_{p+q=r} (H^p(F) \otimes H^q(A_{\alpha})) \\ &= \times_{\alpha} H^k\left(\pi^{-1}(A_{\alpha})\right) \\ &= H^k\left(\pi^{-1}\left(\bigsqcup_{\alpha} A_{\alpha}\right)\right). \end{aligned}$$

□

EXAMPLE 7.6.2. In case both factors have infinite dimensional cohomology the result does not necessarily hold. Consider two 0-dimensional manifolds A, B , i.e., they are finite or countable sets. Here $H^0(A \times B)$ is isomorphic to the space of functions $A \times B \rightarrow \mathbb{R}$, while $H^0(A) \otimes H^0(B)$ consists of finite sums of elements of the form $f_A \otimes f_B$, where $f_C : C \rightarrow \mathbb{R}$. Thus the map $H^0(A) \otimes H^0(B) \rightarrow H^0(A \times B)$ is only an isomorphism when A or B is finite. To address the construction in the above proof note that

$$H^0(A) \otimes H^0(B) = H^0(A) \otimes \times_{b \in B} H^0(b)$$

while

$$\times_{b \in B} H^0(A) \otimes H^0(b) = \times_{b \in B} H^0(A) \otimes \mathbb{R} = \times_{b \in B} H^0(A) = \times_{a \in A, b \in B} \mathbb{R} = H^0(A \times B).$$

Künneth's theorem also has a direct counter part for compactly supported cohomology:

$$\bigoplus_{p+q=r} H_c^p(F) \otimes H_c^q(B) = H_c^r(F \times B)$$

as long as $H_c^*(F)$ is finite dimensional. The proof is similar with the caveat that homotopy invariance is replaced by

$$H_c^{*+n}(F \times \mathbb{R}^n) \simeq H_c^*(F).$$

We now assume that $\pi : E \rightarrow B$ is a submersion-fibration where the fibers are diffeomorphic to a manifold F . The key condition that is needed is that the restriction to any fiber $\pi^{-1}(p) \simeq F$ is a surjection in cohomology

$$H^*(E) \rightarrow H^*\left(\pi^{-1}(p)\right) \rightarrow 0, \text{ for all } p \in B.$$

In the case of a product this obviously holds since the projection $\bar{\pi} : F \times B \rightarrow F$ is a right inverse to the inclusions $F \rightarrow F \times \{s\} \subset F \times B$. The restriction assumption does not hold in general, e.g., the fibration $S^3 \rightarrow S^2$ is a good counter example.

It seems a daunting task to check the condition for all fibers in a general situation. Assuming we know it is true for a specific fiber $F = \pi^{-1}(p)$ we can select a neighborhood A around p such that $\pi^{-1}(A) = F \times A$. As long as A is contractible we see that $\pi^{-1}(A)$ and F are homotopy equivalent and so the restriction to any of the fibers over A will also give a surjection in cohomology. When B is connected a covering of such contractible sets shows that $H^*(E) \rightarrow N^*(\pi^{-1}(x))$ is a surjection for all $x \in B$. In fact, this construction gives us a bit more. We assume that for a specific fiber F there is a subspace $\mathcal{V}^* \in H^*(E)$ that is isomorphic to $H^*(F)$. The construction then shows that \mathcal{V}^* is isomorphic to $H^*(\pi^{-1}(x))$ for all $x \in B$ as long as B is connected.

THEOREM 7.6.3 (Leray-Hirsch). *Assume we have $\mathcal{V}^* \subset H^*(E)$ that is isomorphic to $H^*(\pi^{-1}(x))$ via restriction for all $x \in B$. If $H^*(F)$ is finite dimensional, then there is an isomorphism:*

$$\bigoplus_{p+q=r} \mathcal{V}^p \otimes H^q(B) \rightarrow H^r(E),$$

where the map $\mathcal{V}^p \otimes H^q(B) \rightarrow H^{p+q}(E)$ is defined by $\phi \otimes \beta \mapsto \phi \wedge \pi^*(\beta)$.

REMARK 7.6.4. Observe that for any map $E \rightarrow B$ the space $H^*(E)$ is naturally a $H^*(B)$ module:

$$H^*(B) \times H^*(E) \rightarrow H^*(E)$$

via pull-back $H^*(B) \rightarrow H^*(E)$ and wedge product in $H^*(E)$. The statement of the theorem can then be rephrased as offering a condition for when $H^*(E)$ is a free $H^*(B)$ -module.

PROOF. Note that for each open $A \subset B$ there is a natural restriction

$$\mathcal{V}^* \subset H^*(E) \rightarrow \mathcal{V}^*|_A \subset H^*(\pi^{-1}(A)).$$

This shows that the assumption of the theorem holds for all of the bundles $\pi^{-1}(A) \rightarrow A$, where $A \subset B$ is open.

With these constructions in mind we can employ the strategy from corollary 1.3.5. To that end, restrict attention to open subsets $A \subset M$ with the statement $P(A)$ being that for all r the map

$$\bigoplus_{p+q=r} \mathcal{V}^p|_A \otimes H^q(A) \rightarrow H^r(\pi^{-1}(A))$$

is an isomorphism.

To check condition (1) note that the statement holds for any $A \subset B$ that is diffeomorphic to $\mathbb{R}^{\dim B}$ and where the bundle is trivial $\pi^{-1}(A) \cong F \times A$. In particular, the statement also holds for any box in A .

Condition (2) in corollary 1.3.5 is established as in theorem 7.6.1 and the proof of theorem 7.3.1.

Finally for condition (3) we simply replace $H^p(F)$ with \mathcal{V}^p and proceed as in the proof of theorem 7.6.1. \square

In the general case of a fiber bundle the obvious generalization to a compactly supported result runs into some logistical problems. The best version uses forms on E that are compactly supported on fibers $\Omega_{cv}^*(E)$, thus $\Omega_c^*(E) \subset \Omega_{cv}^*(E) \subset \Omega^*(E)$. This leads to a cohomology theory $H_{cv}^*(E)$ that has the natural property that for $A \subset B$ there is a restriction map $H_{cv}^*(E) \rightarrow H_{cv}^*(\pi^{-1}(A))$. The proof from above can then be used again to show.

THEOREM 7.6.5. Assume we have $\mathcal{V}_{c_v}^* \subset H_{c_v}^*(E)$ that is isomorphic to $H_c^*(\pi^{-1}(x))$ via restriction for all $x \in B$. If $H_c^*(F)$ is finite dimensional, then there is an isomorphism:

$$\bigoplus_{p+q=r} \mathcal{V}_{c_v}^p \otimes H^q(B) \rightarrow H_{c_v}^r(E)$$

where the map $\mathcal{V}_{c_v}^p \otimes H^q(B) \rightarrow H_{c_v}^{p+q}(E)$ is defined by $\phi \otimes \beta \mapsto \phi \wedge \pi^*(\beta)$.

The important special case is when B is compact where the formulation becomes more natural.

COROLLARY 7.6.6. Assume that B is compact and $\mathcal{V}_c^* \subset H_c^*(E)$ is isomorphic to $H_c^*(\pi^{-1}(x))$ via restriction for all $x \in B$. If $H_c^*(F)$ is finite dimensional, then there is an isomorphism:

$$\bigoplus_{p+q=r} \mathcal{V}_c^p \otimes H^q(B) \rightarrow H_c^r(E)$$

where the map $\mathcal{V}_c^p \otimes H^q(B) \rightarrow H_c^{p+q}(E)$ is defined by $\phi \otimes \beta \mapsto \phi \wedge \pi^*(\beta)$.

This corollary follows directly from Poincaré duality when both E and B are oriented. The corresponding \mathcal{V}^* is defined via the Poincaré duality isomorphism $H^* \rightarrow \text{Hom}(H_c^{n-*}, \mathbb{R})$, i.e., $\mathcal{V}^p \simeq \text{Hom}(\mathcal{V}_c^{n-p}, \mathbb{R})$, where $n = \dim E$.

7.7. Exercises

- (1) Using theorem 3.4.8 and corollary 3.4.10 show that pull-backs $F^* : H^*(N) \rightarrow H^*(M)$ are well-defined for continuous maps $F : M \rightarrow N$.
- (2) Calculate the cohomology of the torus using Mayer-Vietoris and induction on dimension.
- (3) Calculate the cohomology of $S^p \times S^q$ using Mayer-Vietoris.
- (4) Let $\omega \in \Omega^1(M)$.
 - (a) Define $\int_c \omega$ for a piecewise smooth curve $c : [a, b] \rightarrow M$.
 - (b) If $d\omega = 0$, then $\int_{c_0} \omega = \int_{c_1} \omega$, where $c_{0,1} : [a, b] \rightarrow M$ agree at the end points and are homotopic via a homotopy that fixes the end points.
 - (c) Show that ω is exact provided $\int_c \omega$ only depends on $c(a)$ and $c(b)$.
 - (d) Show that a simply connected manifold has $H^1(M) = 0$.
- (5) Let M be a closed n -manifold. Calculate $H^*(M - \{p\})$ in terms of $H^*(M)$.
- (6) Show that if $F : M \rightarrow N$ is homotopic to a constant map, then $F^*(\omega)$ is exact for any closed form ω on N .
- (7) Show that if $F : M \rightarrow N$ admits a section $s : N \rightarrow M$, i.e., $F \circ s = id_N$, then $F^* : H^*(N) \rightarrow H^*(M)$ is an injection.
- (8) Show that there is a natural isomorphism

$$H_c^*(M) \simeq H_c^{*+1}(\mathbb{R} \times M).$$

- (9) Show that the cohomology of $S^p \times S^q$ is generated by a form $\omega_1 \in \Omega^p(S^p \times S^q)$, a form $\omega_2 \in \Omega^q(S^p \times S^q)$, and $\omega_1 \wedge \omega_2$. Hint: Use Künneth's theorem or that the action by $SO(p+1) \times SO(q+1)$ is transitive.
- (10) Let $M = T^n = S^1 \times \cdots \times S^1$ and let θ be a generator for $H^1(S^1)$. Define $\theta_i = \pi_i^*(\theta)$, where $\pi_i : T^n \rightarrow S^1$ is the projection onto the i^{th} factor. Use Künneth's theorem to show that $H^p(T^n)$ has a basis of the form $\theta_{i_1} \wedge \cdots \wedge \theta_{i_p}$, $i_1 < \cdots < i_p$. Conclude that $\dim H^p(T^n) = \binom{n}{p}$.

- (11) The Betti numbers and the Poincaré polynomial of a manifold are defined as

$$b_k(M) = \dim H^k(M),$$

$$p_M(t) = \sum_k b_k(M) t^k.$$

- (a) Show that $p_{S^n}(t) = 1 + t^n$.
 (b) Show that $p_{T^n}(t) = (1+t)^n$.
 (c) Show that $p_{\mathbb{C}\mathbb{P}^n}(t) = \frac{1-t^{2n+2}}{1-t^2}$.
 (d) Show that $p_{M \times N}(t) = p_M(t) p_N(t)$.
 (12) Show that if $\omega \in \Omega^2(\mathbb{C}\mathbb{P}^n)$ generates $H^2(\mathbb{C}\mathbb{P}^n)$, then ω^k generates $H^{2k}(\mathbb{C}\mathbb{P}^n)$.
 (13) Show that any map $\mathbb{C}\mathbb{P}^m \rightarrow \mathbb{C}\mathbb{P}^n$, $m > n$, induces a trivial map $H^k(\mathbb{C}\mathbb{P}^m) \rightarrow H^k(\mathbb{C}\mathbb{P}^n)$ for $k > 0$.
 (14) Show that any map $S^{p+q} \rightarrow S^p \times S^q$ has degree 0.
 (15) Let $p, q \in \mathbb{N}$. Show that any map $S^{2p} \times S^{2q} \rightarrow \mathbb{C}\mathbb{P}^{p+q}$ has degree 0 unless $p = q = 1$.
 (16) Let M, N be closed oriented n -manifolds with N connected. Show that if $F : M \rightarrow N$ has nonzero degree, then $F^* : N^*(N) \rightarrow H^*(M)$ is injective.
 (17) A *symplectic form* $\omega \in \Omega^2(M^{2n})$ is a closed form that is nondegenerate, i.e., for every $v \neq 0$ the linear function $w \mapsto \omega(w, v)$ is not trivial.
 (a) Show $\omega \in \Omega^2(M)$ is nondegenerate if and only if $\dim M$ is even and ω^n is a volume form where $2n = \dim M$. Hint: This is linear algebra. Find a normal form on a vector space for any skew-symmetric bilinear form.
 (b) Show that when M is closed, then a symplectic form generates a nontrivial element in cohomology.
 (18) Let M^{4n+2} be closed and oriented. Show that $\dim H^{2n+1}(M)$ is even.
 (19) For a smooth function $f : M^n \rightarrow \mathbb{R}$ define

$$\begin{aligned} d_f : \Omega^p(M) &\rightarrow \Omega^{p+1}(M) \\ d_f(\omega) &= d\omega + df \wedge \omega \end{aligned}$$

and

$$\begin{aligned} m_f : \Omega^p(M) &\rightarrow \Omega^p(M) \\ m_f(\omega) &= e^f \omega. \end{aligned}$$

- (a) Show that $d_f = m_{-f} \circ d \circ m_f$ and $d_f \circ d_f = 0$.
 (b) Show that the cohomology groups defined by d_f are isomorphic to de Rham cohomology.
 (20) For a 1-form $\theta \in \Omega^1(M)$ define

$$\begin{aligned} d_\theta : \Omega^p(M) &\rightarrow \Omega^{p+1}(M) \\ d_\theta(\omega) &= d\omega + \theta \wedge \omega \end{aligned}$$

- (a) Show that if $d\theta = 0$, then $d_\theta \circ d_\theta = 0$.
 (b) Show that if θ is closed but not exact, then the cohomology defined by d_θ is not necessarily isomorphic to de Rham cohomology. Hint: Show that the d_θ -cohomology of S^1 is trivial.
 (21) Let $S \subset \mathbb{R}^n$ be a closed or properly embedded oriented submanifold of codimension 1.

- (a) Use the long exact sequence for the pair (\mathbb{R}^n, S) to show that the number of components of $\mathbb{R}^n - S$ can be calculated with the formula:

$$\dim H^0(\mathbb{R}^n - S) = 1 + \dim H_c^{n-1}(S).$$

- (b) Generalize (a) to the case where \mathbb{R}^n is replaced by a connected oriented manifold M^n with $H^1(M) = 0$.
- (c) Give examples where (b) fails if one or both manifolds are not orientable.
- (22) Let G be a finite group that acts on M with trivial isotropy, i.e., if $gx = x$ for any $x \in M$ and $g \in G$, then $g = e$. Show that $M \rightarrow M/G$ defines a covering map and that $H^*(M/G) = H_G^*(M)$.
- (23) Let G/H be a homogeneous space where G is compact and simply connected and H is connected. Show that G/H is simply connected, e.g., $SU(n)/SO(n)$ is simply connected. Hint: Lift a loop based at the equivalence class H to a path in G that begins and ends in H .
- (24) Let G/H be an n -dimensional homogeneous space where G is compact and connected. Show that:

$$\dim H^p(G/H) \leq \binom{n}{p}.$$

- (25) Let G/H be an n -dimensional homogeneous space where G is compact and connected. Show that if the linear action of H on $T_H G/H$ contains an orientation reversing element, then $\Omega_G^n(G/H) = 0$ and G/H is not orientable.
- (26) Show that $SU(3)/SO(3)$ has Poincaré polynomial $1 + t^5$, i.e., has the same cohomology as S^3 . Comment: the long exact homotopy sequence shows that $SU(3)/SO(3)$ is simply connected, while $\pi_2(SU(3)/SO(3)) = \mathbb{Z}/2\mathbb{Z}$. Thus it is not a homotopy sphere and, in particular, not homeomorphic to a sphere.
- (27) For a product $E = F \times B$ consider the Künneth isomorphism:

$$\times : \bigoplus_{p+q=r} H^p(F) \otimes H^q(B) \rightarrow H^r(E).$$

Show that if $\phi_i \in H^{p_i}(F)$ and $\beta_i \in H^{q_i}(B)$, where $i = 1, 2$, then wedge product in $H^*(F) \otimes H^*(B)$ satisfies

$$\times(\phi_1 \otimes \beta_1) \wedge \times(\phi_2 \otimes \beta_2) = (-1)^{q_1 p_2} \times((\phi_1 \wedge \phi_2) \otimes (\beta_1 \wedge \beta_2)).$$

Intersection Theory Revisited

The goal of the chapter is to develop a cohomological approach to intersection numbers. This will allow us to calculate the Lefschetz and Euler numbers of maps on spheres, tori, and projective spaces.

8.1. Intersection Theory and the Poincaré Dual

Let $S^k \subset N^n$ be a closed oriented submanifold of an oriented manifold with finite dimensional de Rham cohomology. The codimension is denoted by $m = n - k$. By integrating k -forms on N over S we obtain a linear functional $H^k(N) \rightarrow \mathbb{R}$. Theorem 7.5.1 shows that $\text{Hom}(H^k(N), \mathbb{R}) \simeq H_c^m(N)$. The Poincaré dual to this functional is the cohomology class $[\eta_S^N] \in H_c^m(N)$ such that

$$\int_S \omega = \int_N \eta_S^N \wedge \omega$$

for all $\omega \in H^k(N)$. Any representative $\eta_S^N \in [\eta_S^N]$ is called a Poincaré dual to $S \subset N$. The obvious defect of this definition is that several natural submanifolds might not have nontrivial duals for the simple reason that $H_c^m(N)$ vanishes, e.g., $N = S^n$.

To find a nontrivial dual we observe that $\int_S \omega$ only depends on the values of ω in a neighborhood of S . Thus we can consider duals supported in any neighborhood U of S in N , i.e., $[\eta_S^U] \in H_c^m(U)$. We normally select a tubular neighborhood so that there is a deformation retraction $\pi : U \rightarrow S$, where the fibers $\pi^{-1}(p)$ are diffeomorphic to \mathbb{R}^m for all $p \in S$. In particular,

$$\pi^* : H^k(S) \rightarrow H^k(U)$$

is an isomorphism and $[\eta_S^U] \in H_c^m(U)$ is characterized as the dual to integration of k -forms on S , i.e., for all $\omega \in \Omega^k(S)$ we have

$$\int_S \omega = \int_U \eta_S^U \wedge \pi^*(\omega).$$

EXAMPLE 8.1.1. When $S = p$ is a point integration over S is simply evaluation of functions at p . The Poincaré dual is represented by any compactly supported n -form that integrates to 1.

EXAMPLE 8.1.2. When $S = S^1 \subset S^1 \times (-1, 1) \subset S^2$ we first note that $[\eta_{S^1}^{S^2}] = 0$ while

$$[\eta_{S^1}^{S^1 \times (-1, 1)}] \in H_c^1(S^1 \times (-1, 1)) \simeq H_c^0(S^1).$$

The Poincaré dual can be represented by $\pi_2^* \eta$, where $\pi_2 : S^1 \times (-1, 1) \rightarrow (-1, 1)$ is the projection and $\eta \in \Omega_c^1((-1, 1))$ any form with integral 1.

EXAMPLE 8.1.3. Consider the embedded submanifold $S_{p,q} \subset T^2 = S^1 \times S^1$ defined by $F(e^{i\theta}) = (e^{ip\theta}, e^{iq\theta})$, where $p, q \in \mathbb{Z}$ only have ± 1 as common divisors. Let $dt = \frac{d\theta}{2\pi} \in$

$\Omega^1(S^1)$ be the volume form with integral 1 and $\pi_{1,2} : T^2 \rightarrow S^1$ the projections onto the two factors. We obtain two forms $\eta_{1,2} = \pi_{1,2}^*(dt)$ that generate $H^1(T^2)$ and yield a volume form $\eta_1 \wedge \eta_2$ that integrates to 1. To find a representative $\eta_{S_{p,q}}^{T^2} = \alpha\eta_1 + \beta\eta_2$ we simply need to check that

$$\begin{aligned} \int_{T^2} (\alpha\eta_1 + \beta\eta_2) \wedge \eta_1 &= \int_{S_{p,q}} \eta_1, \\ \int_{T^2} (\alpha\eta_1 + \beta\eta_2) \wedge \eta_2 &= \int_{S_{p,q}} \eta_2. \end{aligned}$$

Here the left hand sides are $-\beta$ and α respectively, while the right hand sides are p and q respectively. Thus

$$\eta_{S_{p,q}}^{T^2} = q\eta_1 - p\eta_2.$$

The dual gives us an interesting isomorphism called the *Thom isomorphism*. A more general and abstract version was presented in corollary 7.6.6.

LEMMA 8.1.4 (Thom Isomorphism). Recall that $k + m = n$. If $\pi : U \rightarrow S$ is a tubular neighborhood, then the map

$$\begin{aligned} H_c^*(S) &\rightarrow H_c^{*+m}(U), \\ [\omega] &\mapsto [\eta_S^U \wedge \pi^*(\omega)] \end{aligned}$$

is an isomorphism.

PROOF. Using Poincaré duality twice we see that

$$\begin{aligned} H_c^{*+m}(U) &\simeq \text{Hom}(H^{n-*+m}(U), \mathbb{R}) \\ &\simeq \text{Hom}(H^{k-*}(S), \mathbb{R}) \\ &\simeq H_c^*(S). \end{aligned}$$

Thus it suffices to show that the map

$$\begin{aligned} H_c^*(S) &\rightarrow H_c^{*+m}(U) \\ [\omega] &\mapsto [\eta_S^N \wedge \pi^*(\omega)] \end{aligned}$$

is injective. When $p = k$ this follows from the construction of the dual. For $p < k$ select a nontrivial $[\omega \in H^p(S)]$ and using Poincaré duality $\tau \in H^{k-p}(S)$, such that $[\omega \wedge \tau] \in H^k(S)$ is nontrivial. This shows that $[\eta_S^N \wedge \pi^*(\omega) \wedge \pi^*(\tau)]$ is nontrivial. This in turn implies that $[\eta_S^N \wedge \pi^*(\omega)]$ is nontrivial. \square

The next goal is to find a characterization of η_S^U , this characterization is valid as long as $\pi : U \rightarrow S$ is merely a retraction with connected preimages, i.e., $\pi \circ i = id_S$, where $i : S \rightarrow U$ is the inclusion. However, we will only use it for tubular neighborhoods. The characterization makes it possible to construct the dual in many situations and also shows why the Thom isomorphism follows from corollary 7.6.6.

THEOREM 8.1.5. *The dual is characterized as a closed form with compact support that integrates to 1 along fibers $\pi^{-1}(p)$ for all $p \in S$. In particular, when U is a tubular neighborhood the dual generates the cohomology of the fibers $H_c^*(\pi^{-1}(p)) = H_c^*(\mathbb{R}^m)$.*

PROOF. The characterization requires a choice of orientation for the fibers. It is chosen so that $T_p\pi^{-1}(p) \oplus T_pS$ and T_pU have the same orientation. For $\omega \in \Omega^k(S)$ we note that

$\pi^*\omega$ is constant on $\pi^{-1}(p)$, $p \in S$. Therefore, if η is a closed compactly supported form that integrates to 1 along all fibers, then

$$\int_U \eta \wedge \pi^*\omega = \int_S \int_{\pi^{-1}(p)} \eta \wedge \pi^*\omega = \int_S \omega$$

as desired.

Conversely, we define

$$\begin{aligned} f &: S \rightarrow \mathbb{R}, \\ f(p) &= \int_{\pi^{-1}(p)} \eta_S^U \end{aligned}$$

and note that

$$\int_S \omega = \int_U \eta_S^U \wedge \pi^*\omega = \int_S f \omega$$

for all ω . Since the support of ω can be chosen to be in any open subset of S , this shows that $f = 1$ on S . \square

Unless explicitly stated, we assume that duals of submanifolds are calculated inside tubular neighborhoods. Given the structure of the dual on the fibers we shall generally use the notation η_S with the implicit assumption that it is defined in some tubular neighborhood of S . Note that tubular neighborhoods are constructed to be naturally diffeomorphic to a tube around the zero section of a normal bundle of $S \subset N$ (theorem 3.4.6). With that in mind it is natural to focus attention on oriented vector bundles $E \rightarrow S$ with oriented base S .

COROLLARY 8.1.6. *If $F : S' \rightarrow S$ is a map between closed oriented manifolds and $\pi : E \rightarrow S$ is an oriented m -dimensional vector bundle, then*

$$F^*(\eta_S^E) = \eta_{S'}^{F^*(E)}.$$

PROOF. The pullback vector bundle is given by

$$F^*(E) = \{(x, v) \in S' \times E \mid F(x) = \pi(v)\}$$

and thus has the same the same fibers as E . This also naturally orients $F^*(E)$. When restricting $F^*(\eta_S^E)$ to a fiber $F^*(E)_x \simeq E_{F(x)}$ we see that

$$\int_{F^*(E)_x} F^*(\eta_S^E) = \int_{E_{F(x)}} \eta_S^E = 1.$$

Theorem 8.1.5 then implies the claim. \square

COROLLARY 8.1.7. *If $F : M \rightarrow N$ is proper and transverse to S , then for suitable tubular neighborhoods we have*

$$[F^*(\eta_S)] = [\eta_{F^{-1}(S)}].$$

PROOF. We can assume that both M and N are embedded in Euclidean space so that the tangent spaces come with inner product structures. The key is simply to observe that if $T^\perp(S \subset N)$ is the normal bundle, then the pullback bundle F^*T^\perp is isomorphic to the normal bundle $T^\perp(F^{-1}(S) \subset M)$. Since F is transverse to S it follows that each fiber $T_x^\perp F^{-1}(S)$ is mapped to a subspace $DF(T_x^\perp F^{-1}(S))$ that is a complement to $T_{F(x)}S \subset T_{F(x)}N$. We can then orthogonally project it onto $T_{F(x)}^\perp S$ to obtain an isomorphism

$$\text{proj}_{T^\perp S} \circ DF|_{T^\perp F^{-1}(S)} : T_x^\perp F^{-1}(S) \rightarrow F^*(T_{F(x)}^\perp S).$$

This isomorphism is orientation preserving as the orientation on $T_x^\perp F^{-1}(S)$ is chosen to agree with the orientation for $DF(T_x^\perp F^{-1}(S))$ as in section 5.4.1. \square

A special interesting case of naturality occurs for submanifolds.

COROLLARY 8.1.8. *If $S_1^{k_1}, S_2^{k_2} \subset N$ are compact, transverse and oriented, then with suitable orientations on $S_1 \cap S_2$ the dual is given by*

$$[\eta_{S_1} \wedge \eta_{S_2}] = [\eta_{S_1 \cap S_2}].$$

PROOF. We have the inclusions $S_1 \cap S_2 \subset S_1 \subset U$ and $\eta_{S_1 \cap S_2} = i^*(\eta_{S_2})$ since the inclusion $i: S_1 \rightarrow N$ is transverse to S_2 . Thus for $\omega \in \Omega^{n-k_1-k_2}(N)$ we see that

$$\int_N \eta_{S_1} \wedge \eta_{S_2} \wedge \omega = \int_{S_1} i^*(\eta_{S_2} \wedge \omega) = \int_{S_1} \eta_{S_1 \cap S_2}^{S_1} \wedge i^*(\omega) = \int_{S_1 \cap S_2} \omega$$

showing that $\eta_{S_1} \wedge \eta_{S_2}$ represents the dual to $S_1 \cap S_2 \subset N$. \square

We can also apply the naturality of the dual to obtain a new formula for intersection numbers.

COROLLARY 8.1.9. *If $\dim M + \dim S = \dim N$, and $F: M \rightarrow N$ is proper, then*

$$I(F, S) = \int_M F^*(\eta_S) = \int_M F^*(\eta_S^N).$$

PROOF. We can assume that F is transverse to S as in corollary 8.1.7. Here $F^{-1}(S)$ is a finite collection of points and its normal bundle $F^{-1}(S)$ is simply the tangent spaces at these points. Similarly, the pullback bundle $F^*(T^\perp S)$ consists of finitely many vector spaces that can be identified with the tangent spaces to M via $\text{proj}_{T^\perp S} \circ DF$. The orientation of the fibers $T_x^\perp F^{-1}(S) \simeq F^*(T^\perp S)_x \simeq T_{F(x)}^\perp S$ are chosen so that the isomorphisms are orientation preserving. This might not agree with the orientation of $T_x M$ (as in section 5.4.1) thus assigning $\text{sign}_x = \pm 1$ as an orientation for each $x \in F^{-1}(S)$. The sum of these signs is precisely the intersection number. Next identify $T_x^\perp F^{-1}(S)$ with a tubular neighborhood $V \supset F^{-1}(S)$, i.e., a finite collection of pairwise disjoint discs V_x , and use proposition 1.4.20 to select a tubular neighborhood $U \supset S$ corresponding to $T^\perp S$ such that $F^{-1}(U) \subset V$. The orientation choice of x gives us the crucial difference between integrating $F^*(\eta_S^{T^\perp S})$ and $F^*(\eta_S^U)$:

$$\int_{T_x^\perp F^{-1}(S)} F^*(\eta_S^{T^\perp S}) = \text{sign}_x \int_{V_x} F^*(\eta_S^U)$$

and thus

$$I(F, S) = \int_V F^*(\eta_S^U).$$

Finally,

$$I(F, S) = \int_V F^*(\eta_S^U) = \int_M F^*(\eta_S^N),$$

since $\eta_S \in \Omega_c^m(U) \subset \Omega_c^m(N)$ can be used as a representative for $[\eta_S^N]$. \square

Note the the integral vanishes when F doesn't intersect S or when $[\eta_S^N] = 0$. The advantage of this formula is that the right-hand side can be calculated even when F isn't transverse to S . As both sides are invariant under proper homotopies of F this gives us a more general way of calculating intersection numbers.

8.2. The Hopf-Lefschetz Formulas

We are going to relate the Euler characteristic and Lefschetz numbers to the cohomology of the space.

THEOREM 8.2.1. (Hopf-Poincaré) *If M is a closed oriented n -manifold, then*

$$\chi(M) = I(\Delta, \Delta) = \sum (-1)^p \dim H^p(M).$$

PROOF. If we consider the map

$$\begin{aligned} (id, id) &: M \rightarrow \Delta, \\ (id, id)(x) &= (x, x), \end{aligned}$$

then the Euler characteristic can be computed as the intersection number

$$\begin{aligned} \chi(M) &= I(\Delta, \Delta) \\ &= I((id, id), \Delta) \\ &= \int_M (id, id)^* (\eta_\Delta^{M \times M}). \end{aligned}$$

Thus we need a formula for the Poincaré dual $\eta_\Delta = \eta_\Delta^{M \times M}$. To find this formula we use Künneth's formula for the cohomology of the product. To this end select a basis ω_i for the cohomology theory $H^*(M)$ as well as a dual basis τ_i , i.e.,

$$\int_M \omega_i \wedge \tau_j = \delta_{ij},$$

where the integral is assumed to be zero if the form $\omega_i \wedge \tau_j$ doesn't have degree n .

By Künneth's theorem $\pi_1^*(\omega_i) \wedge \pi_2^*(\tau_j)$ is a basis for $H^*(M \times M)$. The dual basis is up to a sign given by $\pi_1^*(\tau_k) \wedge \pi_2^*(\omega_l)$ as we can see by calculating

$$\begin{aligned} & \int_{M \times M} \pi_1^*(\omega_i) \wedge \pi_2^*(\tau_j) \wedge \pi_1^*(\tau_k) \wedge \pi_2^*(\omega_l) \\ &= (-1)^{\deg \tau_j \deg \tau_k} \int_{M \times M} \pi_1^*(\omega_i) \wedge \pi_1^*(\tau_k) \wedge \pi_2^*(\tau_j) \wedge \pi_2^*(\omega_l) \\ &= (-1)^{\deg \tau_j (\deg \tau_k + \deg \omega_l)} \int_{M \times M} \pi_1^*(\omega_i) \wedge \pi_1^*(\tau_k) \wedge \pi_2^*(\omega_l) \wedge \pi_2^*(\tau_j) \\ &= (-1)^{\deg \tau_j (\deg \tau_k + \deg \omega_l)} \left(\int_M \omega_i \wedge \tau_k \right) \left(\int_M \omega_l \wedge \tau_j \right) \\ &= (-1)^{\deg \tau_j (\deg \tau_k + \deg \omega_l)} \delta_{ik} \delta_{lj} \end{aligned}$$

Clearly this vanishes unless $i = k$ and $l = j$.

This can be used to compute η_Δ for $\Delta \subset M \times M$. We assume that

$$\eta_\Delta = \sum c_{ij} \pi_1^*(\omega_i) \wedge \pi_2^*(\tau_j).$$

On one hand

$$\begin{aligned} & \int_{M \times M} \eta_\Delta \wedge \pi_1^*(\tau_k) \wedge \pi_2^*(\omega_l) \\ &= \sum c_{ij} \int_{M \times M} \pi_1^*(\omega_i) \wedge \pi_2^*(\tau_j) \wedge \pi_1^*(\tau_k) \wedge \pi_2^*(\omega_l) \\ &= \sum c_{ij} (-1)^{\deg \tau_j (\deg \tau_k + \deg \omega_l)} \delta_{ki} \delta_{jl} \\ &= c_{kl} (-1)^{\deg \tau_l (\deg \tau_k + \deg \omega_l)} \end{aligned}$$

While on the other hand the fact that $(id, id) : M \rightarrow \Delta$ is a map of degree 1 tells us that

$$\begin{aligned}
 \int_{M \times M} \eta_{\Delta} \wedge \pi_1^*(\tau_k) \wedge \pi_2^*(\omega_l) &= \int_{\Delta} \pi_1^*(\tau_k) \wedge \pi_2^*(\omega_l) \\
 &= \int_M (id, id)^*(\pi_1^*(\tau_k) \wedge \pi_2^*(\omega_l)) \\
 &= \int_M \tau_k \wedge \omega_l \\
 &= (-1)^{\deg(\tau_k) \deg(\omega_l)} \delta_{kl}.
 \end{aligned}$$

Thus

$$c_{kl} (-1)^{\deg \tau_l (\deg \omega_l + \deg \tau_k)} = (-1)^{\deg \tau_k \deg \omega_l} \delta_{kl}$$

or in other words $c_{kl} = 0$ unless $k = l$ and in that case

$$\begin{aligned}
 c_{kk} &= (-1)^{\deg \tau_k (2 \deg \omega_k + \deg \tau_k)} \\
 &= (-1)^{\deg \tau_k \deg \tau_k} \\
 &= (-1)^{\deg \tau_k}.
 \end{aligned}$$

This yields the formula

$$\eta_{\Delta} = \sum (-1)^{\deg \tau_i} \pi_1^*(\omega_i) \wedge \pi_2^*(\tau_i).$$

The Euler characteristic can now be computed as follows

$$\begin{aligned}
 \chi(M) &= \int_M (id, id)^*(\eta_{\Delta}^{M \times M}) \\
 &= \int_M (id, id)^*\left(\sum (-1)^{\deg \tau_i} \pi_1^*(\omega_i) \wedge \pi_2^*(\tau_i)\right) \\
 &= \sum (-1)^{\deg \tau_i} \int_M \omega_i \wedge \tau_i \\
 &= \sum (-1)^{\deg \tau_i} \\
 &= \sum (-1)^p \dim H^p(M).
 \end{aligned}$$

□

A generalization of this leads us to a similar formula for the Lefschetz number of a map $F : M \rightarrow M$.

THEOREM 8.2.2. (Lefschetz) *If $F : M \rightarrow M$, then*

$$L(F) = I(\text{graph}(F), \Delta) = \sum (-1)^p \text{tr}(F^* : H^p(M) \rightarrow H^p(M)).$$

PROOF. This time we use the map $(id, F) : M \rightarrow \text{graph}(F)$ sending x to $(x, F(x))$ to compute the Lefschetz number

$$\begin{aligned}
I(\text{graph}(F), \Delta) &= \int_M (id, F)^* \eta_\Delta \\
&= \int_M (id, F)^* \left(\sum (-1)^{\deg \tau_i} \pi_1^*(\omega_i) \wedge \pi_2^*(\tau_i) \right) \\
&= \sum (-1)^{\deg \tau_i} \int_M \omega_i \wedge F^* \tau_i \\
&= \sum (-1)^{\deg \tau_i} \int_M \omega_i \wedge F_{ij} \tau_j \\
&= \sum (-1)^{\deg \tau_i} F_{ij} \delta_{ij} \\
&= \sum (-1)^{\deg \tau_i} F_{ii} \\
&= \sum (-1)^p \text{tr}(F^* : H^p(M) \rightarrow H^p(M)).
\end{aligned}$$

□

The definition $I(\text{graph}(F), \Delta)$ for the Lefschetz number is not consistent with [Guillemin-Pollack]. But, if we use their definition, then the formula we just established would have a sign $(-1)^{\dim M}$ on it. This is a very common confusion in the general literature.

8.3. Examples of Lefschetz Numbers

It is in fact true that $\text{tr}(F^* : H^p(M) \rightarrow H^p(M))$ is always an integer, but, to see this requires that we know more algebraic topology. In the cases we study here this can be established directly. Two cases where we do know this to be true are when $p = 0$ or $p = \dim M$ and M is compact, connected and oriented, in those cases

$$\begin{aligned}
\text{tr}(F^* : H^0(M) \rightarrow H^0(M)) &= 1, \\
\text{tr}(F^* : H^n(M) \rightarrow H^n(M)) &= \deg F.
\end{aligned}$$

8.3.1. Spheres and Real Projective Spaces. The simplicity of the cohomology of spheres and odd dimensional projective spaces now immediately give us the Lefschetz number in terms of the degree.

When $F : S^n \rightarrow S^n$ we have $L(F) = 1 + (-1)^n \deg F$. This confirms that any map without fixed points must be homotopic to the antipodal map and therefore have degree $(-1)^{n+1}$.

When $F : \mathbb{R}P^{2n+1} \rightarrow \mathbb{R}P^{2n+1}$ we have $L(F) = 1 - \deg(F)$. This also conforms with our feeling for what happens with orthogonal transformations. Namely, if $F \in Gl_{2n+2}^+(\mathbb{R})$, then it is possible to not have a fixed point as $F : \mathbb{R}^{2n+2} \rightarrow \mathbb{R}^{2n+2}$ might not have an eigenvector. On the other hand, if $F \in Gl_{2n+2}^-(\mathbb{R})$, then there should be at least two fixed points.

The even dimensional version $F : \mathbb{R}P^{2n} \rightarrow \mathbb{R}P^{2n}$ is a bit trickier as the manifold isn't orientable and thus our above approach doesn't work. However, as the only nontrivial cohomology group is when $p = 0$ we would expect the mod 2 Lefschetz number to be 1 for all F . When $F \in Gl_{2n+1}(\mathbb{R})$, this is indeed true as such maps have an odd number of real eigenvalues. For general F we can lift to a map $\tilde{F} : S^{2n} \rightarrow S^{2n}$ satisfying the symmetry condition

$$\tilde{F}(-x) = \pm \tilde{F}(x).$$

The sign \pm must be consistent on the entire sphere. If it is $+$ then we have that $\tilde{F} \circ A = \tilde{F}$, where A is the antipodal map. This shows that $\deg \tilde{F} \cdot (-1)^{2n+1} = \deg \tilde{F}$, and hence that $\deg \tilde{F} = 0$. In particular, \tilde{F} and also F must have a fixed point. If the sign is $-$ and we assume that \tilde{F} doesn't have a fixed point, then the homotopy to the antipodal map

$$H(x, t) = \frac{(1-t)\tilde{F}(x) - tx}{|(1-t)\tilde{F}(x) - tx|}$$

must also be odd

$$\begin{aligned} H(-x, t) &= \frac{(1-t)\tilde{F}(-x) - t(-x)}{|(1-t)\tilde{F}(-x) - t(-x)|} \\ &= -\frac{(1-t)\tilde{F}(x) - t(x)}{|(1-t)\tilde{F}(x) - t(x)|} \\ &= -H(x, t). \end{aligned}$$

This implies that F is homotopic to the identity on $\mathbb{R}P^{2n}$ and thus $L(F) = L(id) = 1$.

8.3.2. Tori. Next let us consider $M = T^n$. The torus is a product of n circles. If we let $d\theta$ be a generator for $H^1(S^1)$ and $d\theta_i = \pi_i^*(d\theta)$, where $\pi_i : T^n \rightarrow S^1$ is the projection onto the i^{th} factor, then example 7.2.7 or Künneth's formula (theorem 7.6.1) tells us that $H^p(T^n)$ has a basis of the form $d\theta_{i_1} \wedge \cdots \wedge d\theta_{i_p}$, $i_1 < \cdots < i_p$. Thus F^* is entirely determined by knowing what F^* does to $d\theta_i$. We write $F^*(d\theta_i) = \alpha_{ij}d\theta_j$. The action of F^* on the basis $d\theta_{i_1} \wedge \cdots \wedge d\theta_{i_p}$, $i_1 < \cdots < i_p$ is

$$\begin{aligned} F^*(d\theta_{i_1} \wedge \cdots \wedge d\theta_{i_p}) &= F^*(d\theta_{i_1}) \wedge \cdots \wedge F^*(d\theta_{i_p}) \\ &= \alpha_{i_1 j_1} d\theta_{j_1} \wedge \cdots \wedge \alpha_{i_p j_p} d\theta_{j_p} \\ &= (\alpha_{i_1 j_1} \cdots \alpha_{i_p j_p}) d\theta_{j_1} \wedge \cdots \wedge d\theta_{j_p} \end{aligned}$$

this is zero unless j_1, \dots, j_p are distinct. Even then, these indices have to be reordered thus introducing a sign. Note also, that there are $p!$ ordered j_1, \dots, j_p that when reordered to be increasing are the same. To find the trace we are looking for the "diagonal" entries, i.e., those j_1, \dots, j_p that when reordered become i_1, \dots, i_p . If $S(i_1, \dots, i_p)$ denotes the set of permutations of i_1, \dots, i_p then we have shown that

$$\text{tr} F^*|_{H^p(T^n)} = \sum_{i_1 < \cdots < i_p} \sum_{\sigma \in S(i_1, \dots, i_p)} \text{sign}(\sigma) \alpha_{i_1 \sigma(i_1)} \cdots \alpha_{i_p \sigma(i_p)}.$$

This leads us to the formula

$$L(F) = \sum_{p=0}^n (-1)^p \sum_{i_1 < \cdots < i_p} \sum_{\sigma \in S(i_1, \dots, i_p)} \text{sign}(\sigma) \alpha_{i_1 \sigma(i_1)} \cdots \alpha_{i_p \sigma(i_p)}.$$

We claim that this can be simplified considerably by making the observation

$$\begin{aligned} \det(\delta_{ij} - \alpha_{ij}) &= \sum_{\sigma \in S(1, \dots, n)} \text{sign}(\sigma) (\delta_{1\sigma(1)} - \alpha_{1\sigma(1)}) \cdots (\delta_{n\sigma(n)} - \alpha_{n\sigma(n)}) \\ &= \sum_{\sigma \in S(1, \dots, n)} \text{sign}(\sigma) (-1)^p \alpha_{i_1 \sigma(i_1)} \cdots \alpha_{i_p \sigma(i_p)} \delta_{i_{p+1} \sigma(i_{p+1})} \cdots \delta_{i_n \sigma(i_n)}, \end{aligned}$$

where in the last sum $\{i_1, \dots, i_p, i_{p+1}, \dots, i_n\} = \{1, \dots, n\}$. Since the terms vanish unless the permutation fixes i_{p+1}, \dots, i_n we have shown that

$$L(F) = \det(\delta_{ij} - \alpha_{ij}).$$

Finally we claim that the $n \times n$ matrix $[\alpha_{ij}]$ has integer entries. To see this first lift F to $\tilde{F} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and think of $T^n = \mathbb{R}^n / \mathbb{Z}^n$ where \mathbb{Z}^n is the usual integer lattice. Let e_i be the canonical basis for \mathbb{R}^n and observe that $e_i \in \mathbb{Z}^n$. The fact that \tilde{F} is a lift of a map in T^n means that $\tilde{F}(x + e_i) - \tilde{F}(x) \in \mathbb{Z}^n$ for all x and $i = 1, \dots, n$. Since \tilde{F} is continuous we see that

$$\tilde{F}(x + e_i) - \tilde{F}(x) = \tilde{F}(e_i) - \tilde{F}(0) = Ae_i \in \mathbb{Z}^n$$

For some $A = [a_{ij}] \in \text{Mat}_{n \times n}(\mathbb{Z})$. We can then construct a linear homotopy

$$H(x, t) = (1-t)\tilde{F}(x) + t(Ax).$$

This defines a homotopy on T^n since:

$$\begin{aligned} H(x + e_i, t) &= (1-t)\tilde{F}(x + e_i) + tA(x + e_i) \\ &= (1-t)(\tilde{F}(x) + Ae_i) + t(Ax + Ae_i) \\ &= (1-t)(\tilde{F}(x)) + t(Ax) + Ae_i \\ &= H(x, t) + Ae_i \end{aligned}$$

showing that F is homotopic to the linear map A on T^n . This means that $F^* = A^*$. Since $A^*(d\theta_i) = a_{ji}d\theta_j$, we have shown that $[\alpha_{ij}]$ is an integer valued matrix.

8.3.3. Complex Projective Space. The cohomology groups of $\mathbb{P}^n = \mathbb{C}\mathbb{P}^n$ vanish in odd dimensions and are one dimensional in even dimensions. The trace formula for the Lefschetz number therefore can't be too complicated. It turns out to be even simpler and completely determined by the action of the map on $H^2(\mathbb{P}^n)$, analogously with what happened on tori. To establish this we need to show that any generator $[\omega] \in H^2(\mathbb{P}^n)$ has the property that $[\omega^k] \in H^{2k}(\mathbb{P}^n)$ is a generator (see also example 7.2.6 for a different proof). We can use induction on n to show this. Fix $\mathbb{P}^{n-1} \subset \mathbb{P}^n$ and recall from section 7.2 that $H^{2k}(\mathbb{P}^n) \rightarrow H^{2k}(\mathbb{P}^{n-1})$ is an isomorphism for $k \leq n-1$. We can now use the induction hypothesis to claim that $[\omega^k|_{\mathbb{P}^{n-1}}] \in H^{2k}(\mathbb{P}^{n-1})$ are nontrivial for $k \leq n-1$. This in turn shows that $[\omega^k] \in H^{2k}(\mathbb{P}^n)$ are nontrivial for $k \leq n-1$. Finally, since the duality pairing

$$\begin{aligned} H^2(\mathbb{P}^n) \times H^{2(n-1)}(\mathbb{P}^n) &\rightarrow H^{2n}(\mathbb{P}^n), \\ ([\omega_1], [\omega_2]) &\mapsto [\omega_1 \wedge \omega_2] \end{aligned}$$

is nondegenerate it follows that $[\omega^n] = [\omega \wedge \omega^{n-1}] \in H^{2n}(\mathbb{P}^n)$ is a generator.

Now let $F : \mathbb{P}^n \rightarrow \mathbb{P}^n$ and define λ by $F^*(\omega) = \lambda\omega$. Then $F^*(\omega^k) = \lambda^k\omega^k$ and

$$L(F) = 1 + \lambda + \dots + \lambda^n.$$

If $\lambda = 1$ this gives us $L(F) = n + 1$, which was the answer we got for maps from $Gl_{n+1}(\mathbb{C})$. In particular, the Euler characteristic $\chi(\mathbb{P}^n) = n + 1$. When $\lambda \neq 1$, the formula simplifies to

$$L(F) = \frac{1 - \lambda^{n+1}}{1 - \lambda}.$$

Since λ is real we note that this can't vanish unless $\lambda = -1$ and $n + 1$ is even. Thus all maps on \mathbb{P}^{2n} have fixed points, just as on $\mathbb{R}\mathbb{P}^{2n}$. On the other hand \mathbb{P}^{2n+1} does admit a map without fixed points, it just can't come from a complex linear map. Instead we just select a real linear map without fixed points that still yields a map on \mathbb{P}^{2n+1}

$$I\left([z^0 : z^1 : \dots]\right) = [-\bar{z}^1 : \bar{z}^0 : \dots].$$

If I fixes a point then

$$\begin{aligned} -\lambda \bar{z}^1 &= z^0, \\ \lambda \bar{z}^0 &= z^1 \end{aligned}$$

which implies

$$-|\lambda|^2 z^i = z^i$$

for all i . Since this is impossible the map does not have any fixed points.

Finally we should justify why λ is an integer. Let $F_1 = F|_{\mathbb{P}^1} : \mathbb{P}^1 \rightarrow \mathbb{P}^n$ and observe that

$$\lambda[\omega] = [F^*(\omega)] = [F_1^*(\omega)].$$

We now claim that F_1 is homotopic to a map $\mathbb{P}^1 \rightarrow \mathbb{P}^1$. To see this note that $F_1(\mathbb{P}^1) \subset \mathbb{P}^n$ is compact and has measure 0 by Sard's theorem. Thus we can find $p \notin \text{im}(F_1) \cup \mathbb{P}^1$. This allows us to deformation retract $\mathbb{P}^n - p$ to a $\mathbb{P}^{n-1} \supset \mathbb{P}^1$. This \mathbb{P}^{n-1} might not be perpendicular to p in the usual metric, but, one can always select a metric where p and \mathbb{P}^1 are perpendicular and then use the \mathbb{P}^{n-1} that is perpendicular to p . Thus $F_1 : \mathbb{P}^1 \rightarrow \mathbb{P}^n$ is homotopic to a map $F_2 : \mathbb{P}^1 \rightarrow \mathbb{P}^{n-1}$. We can repeat this argument until we obtain a map $F_n : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ homotopic to the original F_1 . This implies that

$$\lambda[\omega] = [F^*(\omega)] = [F_n^*(\omega)]$$

and consequently $\lambda = \deg(F_n)$.

The next two examples show two different approaches to finding a specific form ω . The first example is an abstract construction that yields a unique form, the second offers a concrete calculation of the form in coordinates.

EXAMPLE 8.3.1. The form ω that generates $H^2(\mathbb{P}^2)$ can be constructed to have the property that $\int_{\mathbb{P}^1} \omega = 1$ for all $\mathbb{P}^1 \subset \mathbb{P}^n$. Recall that we showed in example 7.2.6 that the space of $U(n+1)$ invariant 2-forms is 1-dimensional. So it is clear that we can find $\omega \in \Omega_{U(n+1)}^2(\mathbb{P}^n)$ such that $\int_{\mathbb{P}^1} \omega = 1$ for a specific $\mathbb{P}^1 \subset \mathbb{P}^n$. However, $U(n+1)$ also acts transitively on the space of \mathbb{P}^1 s in \mathbb{P}^n . Specifically, a \mathbb{P}^1 corresponds to a complex subspace of dimension 2 in \mathbb{C}^{n+1} and for any two such subspaces there is a unitary transformation that takes one into the other. This shows that our chosen 2-form also integrates to 1 on all other $\mathbb{P}^1 \subset \mathbb{P}^n$.

EXAMPLE 8.3.2. With a bit of complex analysis notation we obtain a more concrete construction.

Using the submersion $\mathbb{C}^{n+1} - \{0\} \rightarrow \mathbb{P}^n$ that sends (z^0, \dots, z^n) to $[z^0 : \dots : z^n]$ we should be able to construct ω on $\mathbb{C}^{n+1} - \{0\}$. A bit of auxiliary notation is needed to define the desired 2-form ω on $\mathbb{C}^{n+1} - \{0\}$:

$$\begin{aligned} dz^i &= dx^i + \sqrt{-1}dy^i, \\ d\bar{z}^i &= dx^i - \sqrt{-1}dy^i, \\ \frac{\partial f}{\partial z^i} &= \frac{1}{2} \left(\frac{\partial f}{\partial x^i} - \sqrt{-1} \frac{\partial f}{\partial y^i} \right), \\ \frac{\partial f}{\partial \bar{z}^i} &= \frac{1}{2} \left(\frac{\partial f}{\partial x^i} + \sqrt{-1} \frac{\partial f}{\partial y^i} \right) \\ \partial f &= \frac{\partial f}{\partial z^i} dz^i, \\ \bar{\partial} f &= \frac{\partial f}{\partial \bar{z}^i} d\bar{z}^i. \end{aligned}$$

The factor $\frac{1}{2}$ and strange signs ensure that the complex differentials work as one would think

$$\begin{aligned} dz^j \left(\frac{\partial}{\partial z^i} \right) &= \frac{\partial z_j}{\partial z^i} = \delta_i^j = \frac{\partial \bar{z}_j}{\partial \bar{z}^i} = d\bar{z}^j \left(\frac{\partial}{\partial \bar{z}^i} \right), \\ dz^j \left(\frac{\partial}{\partial \bar{z}^i} \right) &= 0 = d\bar{z}^j \left(\frac{\partial}{\partial z^i} \right) \end{aligned}$$

More generally we can define $\partial\omega$ and $\bar{\partial}\omega$ for complex valued forms by simply computing ∂ and $\bar{\partial}$ of the coefficient functions just as the local coordinate definition of d , specifically

$$\begin{aligned} \partial(f dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q}) &= \partial f \wedge dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q}, \\ \bar{\partial}(f dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q}) &= \bar{\partial} f \wedge dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q}. \end{aligned}$$

With this definition we see that

$$\begin{aligned} d &= \partial + \bar{\partial}, \\ \partial^2 &= \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial = 0 \end{aligned}$$

and the Cauchy-Riemann equations for holomorphic functions can be stated as

$$\bar{\partial}f = 0.$$

Working on $\mathbb{C}^{n+1} - \{0\}$ define

$$\begin{aligned} \Phi(z) &= \log |z|^2 \\ &= \log(z^0 \bar{z}^0 + \cdots + z^n \bar{z}^n) \end{aligned}$$

and

$$\omega = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\Phi.$$

As $|z|^2$ is invariant under $U(n+1)$ the form ω will also be invariant. If we multiply $z \in \mathbb{C}^{n+1} - \{0\}$ by a nonzero scalar λ then

$$\begin{aligned} \Phi(\lambda z) &= \log(|\lambda z|^2) = \log|\lambda|^2 + \log|z|^2 \\ &= \log|\lambda|^2 + \Phi(z) \end{aligned}$$

so when taking derivatives the constant $\log|\lambda|^2$ disappears. This shows that the form ω becomes invariant under multiplication by complex scalars and so defines a form on \mathbb{P}^n . That said, it is not possible to define Φ on all of \mathbb{P}^n . We give a local coordinate representation below. It is called the potential, or Kähler potential, of ω . Note that the form is exact on $\mathbb{C}^{n+1} - \{0\}$ since

$$\partial\bar{\partial} = (\partial + \bar{\partial})\bar{\partial} = d\bar{\partial}.$$

To show that ω is a nontrivial element of $H^2(\mathbb{P}^n)$ it suffices to show that $\int_{\mathbb{P}^1} \omega \neq 0$. By deleting a point from \mathbb{P}^1 we can coordinatize it by \mathbb{C} . Specifically we consider

$$\mathbb{P}^1 = [z^0 : z^1 : 0 : \cdots : 0],$$

and coordinatize $\mathbb{P}^1 - \{[0 : 1 : 0 : \cdots : 0]\}$ by $z \mapsto [1 : z : 0 : \cdots : 0]$. Then

$$\begin{aligned}
\omega &= \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log(1 + z\bar{z}) \\
&= \frac{\sqrt{-1}}{2\pi} \left(\partial \left(\frac{z d\bar{z}}{1 + |z|^2} \right) \right) \\
&= \frac{\sqrt{-1}}{2\pi} \left(\frac{\partial(z d\bar{z})}{1 + |z|^2} - \left(\partial(1 + |z|^2) \right) \wedge \frac{z d\bar{z}}{(1 + |z|^2)^2} \right) \\
&= \frac{\sqrt{-1}}{2\pi} \left(\frac{dz \wedge d\bar{z}}{1 + |z|^2} - (\bar{z} dz) \wedge \frac{z d\bar{z}}{(1 + |z|^2)^2} \right) \\
&= \frac{\sqrt{-1}}{2\pi} \left(\frac{dz \wedge d\bar{z}}{1 + |z|^2} - \frac{|z|^2 dz \wedge d\bar{z}}{(1 + |z|^2)^2} \right) \\
&= \frac{\sqrt{-1}}{2\pi} \frac{dz \wedge d\bar{z}}{(1 + |z|^2)^2} \\
&= \frac{\sqrt{-1}}{2\pi} \frac{d(x + \sqrt{-1}y) \wedge d(x - \sqrt{-1}y)}{(1 + x^2 + y^2)^2} \\
&= \frac{\sqrt{-1}}{2\pi} \frac{2\sqrt{-1} dy \wedge dx}{(1 + x^2 + y^2)^2} \\
&= \frac{1}{\pi} \frac{dx \wedge dy}{(1 + x^2 + y^2)^2} \\
&= \frac{1}{\pi} \frac{r dr \wedge d\theta}{(1 + r^2)^2}
\end{aligned}$$

If we delete the π in the formula this is the volume form for the sphere of radius $\frac{1}{2}$ in stereographic coordinates, or the volume form for that sphere in Riemann's conformally

flat model. Specifically,

$$\begin{aligned}
\int_{\mathbb{P}^1} \omega &= \int_{\mathbb{P}^1 - \{[0:1:0:\dots:0]\}} \omega \\
&= \int_{\mathbb{C}} \frac{1}{2\pi\sqrt{-1}} \frac{d\bar{z} \wedge dz}{(1+|z|^2)^2} \\
&= \int_{\mathbb{R}^2} \frac{1}{\pi} \frac{dx \wedge dy}{(1+x^2+y^2)^2} \\
&= \frac{1}{\pi} \int_0^\infty \int_0^{2\pi} \frac{r dr \wedge d\theta}{(1+r^2)^2} \\
&= \int_0^\infty \frac{2r dr}{(1+r^2)^2} \\
&= 1.
\end{aligned}$$

This tells us that the concretely defined form is the unique form described abstractly in the previous example.

We can more generally calculate ω in the coordinates $z = (z^1, \dots, z^n) \in \mathbb{C}^n$ corresponding to points $[1 : z^1 : \dots : z^n] \in \mathbb{P}^n$.

$$\begin{aligned}
\omega &= \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log(1 + z^1 \bar{z}^1 + \dots + z^n \bar{z}^n) \\
&= \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log(1 + |z|^2) \\
&= \frac{\sqrt{-1}}{2\pi} \left(\partial \left(\frac{\bar{\partial} |z|^2}{1 + |z|^2} \right) \right) \\
&= \frac{\sqrt{-1}}{2\pi} \left(\frac{\partial \bar{\partial} |z|^2}{1 + |z|^2} - \frac{\partial |z|^2 \wedge \bar{\partial} |z|^2}{(1 + |z|^2)^2} \right) \\
&= \frac{\sqrt{-1}}{2\pi (1 + |z|^2)^2} \left((1 + |z|^2) \partial \bar{\partial} |z|^2 - \partial |z|^2 \wedge \bar{\partial} |z|^2 \right)
\end{aligned}$$

and in coordinates

$$\begin{aligned}
\omega &= \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log(1 + |z|^2) \\
&= \frac{\sqrt{-1}}{2\pi} \frac{\partial^2 \log(1 + |z|^2)}{\partial z^i \partial \bar{z}^j} dz^i \wedge d\bar{z}^j \\
&= \frac{\sqrt{-1}}{2\pi} F_{i\bar{j}} dz^i \wedge d\bar{z}^j.
\end{aligned}$$

Here the matrix $[F_{i\bar{j}}]$ is Hermitian and in fact positive definite. The entries are given by

$$F_{i\bar{j}} = \frac{(1 + |z|^2) \delta_{ij} - z^j \bar{z}^i}{(1 + |z|^2)^2}.$$

Here $[z^j \bar{z}^i] = z \cdot z^*$, where z^* is the adjoint of the column matrix z . The kernel of $z \cdot z^*$ consists of all the vectors orthogonal to z and z is an eigenvector with eigenvalue $|z|^2$. This gives the eigenspace decomposition for $[F_{i\bar{j}}]$. Specifically, $n-1$ eigenvectors with eigenvalue $\frac{1}{1+|z|^2}$ and one eigenvector with eigenvalue $\frac{1}{(1+|z|^2)^2}$. Thus $\det[F_{i\bar{j}}] = (1+|z|^2)^{-n-1}$.

We can now calculate

$$\begin{aligned}\omega^n &= \left(\frac{\sqrt{-1}}{2\pi}\right)^n (F_{i\bar{j}} dz^i \wedge d\bar{z}^j)^n \\ &= \left(\frac{\sqrt{-1}}{2\pi}\right)^n (F_{i_1\bar{j}_1} \cdots F_{i_n\bar{j}_n} dz^{i_1} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge dz^{i_n} \wedge d\bar{z}^{j_n})\end{aligned}$$

Now note that this vanishes unless all of the indices i_1, \dots, i_n , as well as j_1, \dots, j_n , are distinct. After rearranging we obtain

$$\begin{aligned}\omega^n &= \left(\frac{\sqrt{-1}}{2\pi}\right)^n \text{sign}(i_1, \dots, i_n) \text{sign}(j_1, \dots, j_n) F_{i_1\bar{j}_1} \cdots F_{i_n\bar{j}_n} dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n \\ &= \left(\frac{\sqrt{-1}}{2\pi}\right)^n n! \det[F_{i\bar{j}}] dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n \\ &= \frac{n!}{\pi^n (1+|z|^2)^{n+1}} dx^1 \wedge dy^1 \wedge \cdots \wedge dx^n \wedge dy^n\end{aligned}$$

and

$$\int_{\mathbb{P}^n} \omega^n = \int_{\mathbb{P}^n - \mathbb{P}^{n-1}} \omega^n = \int_{\mathbb{R}^{2n}} \pi n! \left(\frac{1}{\pi(1+|z|^2)}\right)^{n+1} dx^1 \wedge dy^1 \wedge \cdots \wedge dx^n \wedge dy^n > 0.$$

This shows that ω^n is a volume form and that $\omega^k \in H^{2k}(\mathbb{P}^n)$ is a generator for all $k = 0, \dots, n$.

8.4. Exercises

- (1) Show that a map $F : S^2 \times S^2 \rightarrow \mathbb{C}\mathbb{P}^2$ has even degree. Hint: Use suitable $\omega \in \Omega^2(\mathbb{C}\mathbb{P}^2)$ and $\omega' \in \Omega^2(S^2)$ that can be combined to generate volume forms on $\mathbb{C}\mathbb{P}^2$ and $S^2 \times S^2$.
- (2) Show that there are classes in $H^1(T^2)$ which are not dual to closed 1-dimensional submanifolds of T^2 .
- (3) Show that if $E \rightarrow M$ is a rank k vector bundle, then $H_c^*(M) \simeq H_c^{*+m}(E)$, provided M and E are oriented manifolds. Show that the Möbius band is a counterexample in case E is not orientable.
- (4) Show that $\chi_M = p_M(-1)$.

Characteristic Classes

This final section introduces the Euler class and shows how it can be used to construct characteristic classes.

9.1. The Euler Class

We are interested in studying duals and in particular Euler classes in the special case where we have a vector bundle $\pi : E \rightarrow M$ and M is thought of a submanifold of E by embedding it into E via the zero section. The total space E is assumed oriented in such a way that a positive orientation for the fibers together with a positive orientation of M gives us the orientation for E . The dimensions are set up so that the fibers of $E \rightarrow M$ have dimension m .

The dual $\eta_M^E \in H_c^m(E)$ is in this case usually called the Thom class of the bundle $E \rightarrow M$. The embedding $M \subset E$ is proper so by restriction to M this dual defines a class $[e(E)] = i^*(\eta_M^E) \in H^m(M)$ called the Euler class (note that we only defined duals to closed submanifolds so $H_c(M) = H(M)$.) Since all sections $s : M \rightarrow E$ are homotopy equivalent we see that $e(E) = s^*\eta_M$. This immediately proves a very interesting theorem.

THEOREM 9.1.1. *If a bundle $\pi : E \rightarrow M$ has a nowhere vanishing section then $e(E) = 0$.*

PROOF. Let $s : M \rightarrow E$ be a section and consider $C \cdot s$ for a large constant C . Then the image of $C \cdot s$ must be disjoint from the compact support of η_M and hence $s^*(\eta_M) = 0$. \square

This Euler class is also natural

PROPOSITION 9.1.2. *Let $F : N \rightarrow M$ be a map that is covered by a vector bundle map $\bar{F} : E' \rightarrow E$, i.e., \bar{F} is a linear orientation preserving isomorphism on fibers. Then*

$$e(E') = F^*(e(E)).$$

An example is the pull-back vector bundle is defined by

$$F^*(E) = \{(p, v) \in N \times E \mid \pi(v) = F(q)\}.$$

Reversing orientation of fibers changes the sign of η_M^E and hence also of $e(E)$. Using $F = id$ and $\bar{F}(v) = -v$ yields an orientation reversing bundle map when k is odd, showing that $e(E) = 0$. Thus we usually only consider Euler classes for even dimensional bundles.

The Euler class can also be used to detect intersection numbers. In case M and the fibers have the same dimension, we can define the intersection number $I(s, M)$ of a section $s : M \rightarrow E$ with the zero section or simply M . The formula is

$$\begin{aligned} I(s, M) &= \int_M s^*(e(E)) \\ &= \int_M e(E) \end{aligned}$$

since all sections are homotopy equivalent to the zero section.

In the special case of the tangent bundle to an oriented manifold M we already know that the intersection number of a vector field X with the zero section is the Euler characteristic. Thus

$$\chi(M) = I(X, M) = \int_M e(TM)$$

This result was first proven by Hopf and can be used to compute χ using a triangulation. This is explained in [Guillemin-Pollack] and [Spivak].

THEOREM 9.1.3. *The Euler class is characterized by*

$$\eta_M^E \wedge \pi^*(e_M^E) = \eta_M^E \wedge \eta_M^E \in H_c^{2m}(E).$$

In particular, $e_M^E = 0$ if m is odd.

PROOF. Since $\pi^*(e_M^E)$ and η_M^E represent the same class in $H^m(E)$ we have that

$$\pi^*(e_M^E) - \eta_M^E = d\omega.$$

Then

$$\begin{aligned} \eta_M^E \wedge \pi^*(e_M^E) - \eta_M^E \wedge \eta_M^E &= \eta_M^E \wedge (d\omega) \\ &= d(\eta_M^E \wedge \omega). \end{aligned}$$

Since $\eta_M^E \wedge \omega$ is compactly supported this shows that $\eta_M^E \wedge \pi^*(e_M^E) = \eta_M^E \wedge \eta_M^E$.

Moreover, as the map

$$\begin{aligned} H^m(M) &\rightarrow H_c^{2m}(E), \\ e &\mapsto \eta_S^E \wedge \pi^*(e) \end{aligned}$$

is injective, it follows that that the relation $\eta_S^E \wedge \pi^*(e) = \eta_S^E \wedge \eta_S^E$ implies that $e = e_M^E$. In particular, $e_S = 0$ when $\eta_S^E \wedge \eta_S^E = 0$. This applies to the case when m is odd as

$$\eta_S^E \wedge \eta_S^E = -\eta_S^E \wedge \eta_S^E.$$

□

The Euler class has other natural properties when we do constructions with vector bundles.

THEOREM 9.1.4. *Given two vector bundles $E \rightarrow M$ and $E' \rightarrow M$, the Whitney sum has Euler class*

$$e(E \oplus E') = e(E) \wedge e(E').$$

PROOF. As we have a better characterization of duals we start with a more general calculation.

Let $\pi : E \rightarrow M$ and $\pi' : E' \rightarrow M'$ be bundles and consider the product bundle $\pi \times \pi' : E \times E' \rightarrow M \times M'$. With this we have the projections $\pi_1 : E \times E' \rightarrow E$ and $\pi_2 : E \times E' \rightarrow E'$. Restricting to the zero sections gives the projections $\pi_1 : M \times M' \rightarrow M$ and $\pi_2 : M \times M' \rightarrow M'$. We claim that

$$\eta_{M \times M'} = (-1)^{n \cdot m'} \pi_1^*(\eta_M) \wedge \pi_2^*(\eta_{M'}) \in H_c^{m+m'}(E \times E').$$

Note that since the projections are not proper it is not clear that $\pi_1^*(\eta_M) \wedge \pi_2^*(\eta_{M'})$ has compact support. However, the support must be compact when projected to E and E' and thus be compact in $E \times E'$. To see the equality we select volume forms $\omega \in H^n(M)$ and

$\omega' \in H^{n'}(M')$ that integrate to 1. Then $\pi_1^*(\omega) \wedge \pi_2^*(\omega')$ is a volume form on $M \times M'$ that integrates to 1. Thus it suffices to compute

$$\begin{aligned}
& \int_{E \times E'} \pi_1^*(\eta_M) \wedge \pi_2^*(\eta_{M'}) \wedge (\pi \times \pi')^*(\pi_1^*(\omega) \wedge \pi_2^*(\omega')) \\
&= \int_{E \times E'} \pi_1^*(\eta_M) \wedge \pi_2^*(\eta_{M'}) \wedge \pi_1^*(\pi^*(\omega)) \wedge \pi_2^*((\pi')^*(\omega')) \\
&= (-1)^{n \cdot m'} \int_{E \times E'} \pi_1^*(\eta_M) \wedge \pi_1^*(\pi^*(\omega)) \wedge \pi_2^*(\eta_{M'}) \wedge \pi_2^*((\pi')^*(\omega')) \\
&= (-1)^{n \cdot m'} \left(\int_E \eta_M \wedge \pi^*(\omega) \right) \left(\int_{E'} \eta_{M'} \wedge (\pi')^*(\omega') \right) \\
&= (-1)^{n \cdot m'}.
\end{aligned}$$

When we consider Euler classes this gives us

$$e(E \times E') = \pi_1^*(e(E)) \wedge \pi_2^*(e(M')) \in H_c^{m+m'}(M \times M').$$

The sign is now irrelevant since $e(M') = 0$ if m' is odd.

The Whitney sum $E \oplus E' \rightarrow M$ of two bundles over the same space is gotten by taking direct sums of the vector space fibers over points in M . This means that $E \oplus E' = (id, id)^*(E \times E')$ where $(id, id) : M \rightarrow M \times M$ since

$$(id, id)^*(E \times E') = \{(p, v, v') \in M \times E \times E' : \pi(v) = p = \pi'(v')\} = E \oplus E'.$$

Thus we get the formula

$$e(E \oplus E') = e(E) \wedge e(E').$$

□

This implies

COROLLARY 9.1.5. *If a bundle $\pi : E \rightarrow M$ admits an orientable odd dimensional sub-bundle $F \subset E$, then $e(E) = 0$.*

PROOF. We have that $E = F \oplus E/F$ or if E carries an inner product structure $E = F \oplus F^\perp$. Now orient F and then E/F so that $F \oplus E/F$ and E have compatible orientations. Then $e(E) = e(F) \wedge e(E/F) = 0$. □

Note that if there is a nowhere vanishing section, then there is a 1 dimensional orientable subbundle. So this recaptures our earlier vanishing theorem. Conversely any orientable 1 dimensional bundle is trivial and thus yields a nowhere vanishing section.

A meaningful theory of invariants for vector bundles using forms should try to avoid odd dimensional bundles altogether. The simplest way of doing this is to consider vector bundles where the vector spaces are complex and then insist on using only complex and Hermitian constructions. This will be investigated further below.

The trivial bundles $\mathbb{R}^m \oplus M$ all have $e(\mathbb{R}^m \oplus M) = 0$. This is because these bundles are all pull-backs of the bundle $\mathbb{R}^m \oplus \{0\}$, where $\{0\}$ is the 1 point space.

To compute $e(\tau(\mathbb{P}^n))$ recall that $\tau(\mathbb{P}^n)$ is the conjugate of $\mathbb{P}^{n+1} - \{p\} \rightarrow \mathbb{P}^n$ which has dual $\eta_{\mathbb{P}^n} = \omega$. Since conjugation reverses orientation on 1 dimensional bundles this shows that $e(\tau(\mathbb{P}^n)) = -\omega$.

Since $\chi(\mathbb{P}^n) = n+1$ we know that $e(T\mathbb{P}^n) = (n+1)\omega^n$.

We go on to describe how the dual and Euler class can be calculated locally. Assume that M is covered by sets U_k such that $E|_{U_k}$ is trivial and that there is a partition of unit λ_k relative to this covering.

First we analyze what the dual restricted to the fibers might look like. For that purpose we assume that the fiber is isometric to \mathbb{R}^m . We select a volume form $\psi \in \Omega^{m-1}(S^{m-1})$ that integrates to 1 and a bump function $\rho : [0, \infty) \rightarrow [-1, 0]$ that is -1 on a neighborhood of 0 and has compact support. Then extend ψ to $\mathbb{R}^m - \{0\}$ and consider

$$d(\rho\psi) = d\rho \wedge \psi.$$

Since $d\rho$ vanishes near the origin this is a globally defined form with total integral

$$\begin{aligned} \int_{\mathbb{R}^m} d\rho \wedge \psi &= \int_0^\infty d\rho \int_{S^{m-1}} \psi \\ &= (\rho(\infty) - \rho(0)) \\ &= 1. \end{aligned}$$

Each fiber of E carries such a form. The bump function ρ is defined on all of E by $\rho(v) = \rho(|v|)$, but, the “angular” form ψ is not globally defined. As we shall see, the Euler class is the obstruction for ψ to be defined on E . Over each U_k the bundle is trivial so we do get a closed form $\psi_k \in \Omega^{m-1}(S(E|_{U_k}))$ that restricts to the angular form on fibers. As these forms agree on the fibers the difference depends only on the footpoints:

$$\psi_k - \psi_l = \pi^* \phi_{kl},$$

where $\phi_{kl} \in \Omega^{m-1}(U_k \cap U_l)$ are closed. These forms satisfy the cocycle conditions

$$\begin{aligned} \phi_{kl} &= -\phi_{lk}, \\ \phi_{ki} + \phi_{il} &= \phi_{kl}. \end{aligned}$$

Now define

$$\varepsilon_k = \sum_i \lambda_i \phi_{ki} \in \Omega^{m-1}(U_k)$$

and note that the cocycle conditions show that

$$\begin{aligned} \varepsilon_k - \varepsilon_l &= \sum_i \lambda_i \phi_{ki} - \sum_i \lambda_i \phi_{li} \\ &= \sum_i \lambda_i (\phi_{ki} - \phi_{li}) \\ &= \sum_i \lambda_i \phi_{kl} \\ &= \phi_{kl}. \end{aligned}$$

Thus we have a globally defined form $e = d\varepsilon_k$ on M since $d(\varepsilon_k - \varepsilon_l) = d\phi_{kl} = 0$. This will turn out to be the Euler form

$$e = d\left(\sum_i \lambda_i \phi_{ki}\right) = \sum_i d\lambda_i \wedge \phi_{ki}.$$

Next we observe that

$$\pi^* \varepsilon_k - \pi^* \varepsilon_l = \psi_k - \psi_l$$

so

$$\psi = \psi_k - \pi^* \varepsilon_k$$

defines a form on E . This is our global angular form. We now claim that

$$\begin{aligned}\eta &= d(\rho\psi) \\ &= d\rho \wedge \psi + \rho d\psi \\ &= d\rho \wedge \psi - \rho\pi^* d\varepsilon_k \\ &= d\rho \wedge \psi - \rho\pi^* e\end{aligned}$$

is the dual. First we note that it is defined on all of E , is closed, and has compact support. It yields e when restricted to the zero section as $\rho(0) = -1$. Finally when restricted to a fiber we can localize the expression

$$\eta = d\rho \wedge \psi_k - d\rho \wedge \pi^* \varepsilon_k - \rho\pi^* e.$$

But both $\pi^* \varepsilon_k$ and $\pi^* e$ vanish on fibers so η , when restricted to a fiber, is simply the form we constructed above whose integral was 1. This shows that η is the dual to M in E and that e is the Euler class.

We are now going to specialize to complex line bundles with a Hermitian structure on each fiber. Since an oriented Euclidean plane has a canonical complex structure this is the same as studying oriented 2-plane bundles. The complex structure just helps in setting up the formulas.

The angular form is usually denoted $d\theta$ as it is the differential of the locally defined angle. To make sense of this we select a unit length section $s_k : U_k \rightarrow S(E|_{U_k})$. For $v \in S(E|_{U_k})$ the angle can be defined by

$$v = h_k(v) s_k = e^{\sqrt{-1}\theta_k} s_k.$$

This shows that the angular form is given by

$$\begin{aligned}d\theta_k &= -\sqrt{-1} \frac{dh_k}{h_k} \\ &= -\sqrt{-1} d \log h_k.\end{aligned}$$

Since we want the unit circles to have unit length we normalize this and define

$$\psi_k = -\frac{\sqrt{-1}}{2\pi} d \log h_k.$$

On $U_k \cap U_l$ we have that

$$h_l s_l = v = h_k s_k$$

So

$$(h_l)^{-1} h_k s_k = s_l.$$

But $(h_l)^{-1} h_k$ now only depends on the base point in $U_k \cap U_l$ and not on where v might be in the unit circle. Thus

$$\pi^* g_{kl} = g_{kl} \circ \pi = h_k (h_l)^{-1}$$

where $g_{kl} : U_k \cap U_l \rightarrow S^1$ satisfy the cocycle conditions

$$\begin{aligned}(g_{kl})^{-1} &= g_{lk} \\ g_{ki} g_{il} &= g_{kl}.\end{aligned}$$

Taking logarithmic differentials then gives us

$$\begin{aligned} -\frac{\sqrt{-1}}{2\pi} \pi^* \frac{dg_{kl}}{g_{kl}} &= -\frac{\sqrt{-1}}{2\pi} \pi^* d \log(g_{kl}) \\ &= \left(-\frac{\sqrt{-1}}{2\pi} d \log(h_k) \right) - \left(-\frac{\sqrt{-1}}{2\pi} d \log(h_l) \right) \\ &= \left(-\frac{\sqrt{-1}}{2\pi} \frac{dh_k}{h_k} \right) - \left(-\frac{\sqrt{-1}}{2\pi} \frac{dh_l}{h_l} \right). \end{aligned}$$

Thus

$$\begin{aligned} \varepsilon_k &= -\frac{\sqrt{-1}}{2\pi} \sum_i \lambda_i d \log(g_{ki}), \\ \psi &= \left(-\frac{\sqrt{-1}}{2\pi} \frac{dh_k}{h_k} \right) - \pi^* \varepsilon_k \\ e &= d\varepsilon_k \\ &= d \left(\frac{\sqrt{-1}}{2\pi} \sum_i \lambda_i d \log(g_{ki}) \right) \\ &= \frac{\sqrt{-1}}{2\pi} \sum_i d\lambda_i \wedge d \log(g_{ki}) \end{aligned}$$

This can be used to prove an important result.

LEMMA 9.1.6. *Let $E \rightarrow M$ and $E' \rightarrow M$ be complex line bundles, then*

$$\begin{aligned} e(\text{hom}(E, E')) &= -e(E) + e(E'), \\ e(E \otimes E') &= e(E) + e(E'). \end{aligned}$$

PROOF. Note that the sign ensures that the Euler class vanishes when $E = E'$.

Select a covering U_k such that E and E' have unit length sections s_k respectively t_k on U_k . If we define $L_k \in \text{hom}(E, E')$ such that $L_k(s_k) = t_k$, then h_k is a unit length section of $\text{hom}(E, E')$ over U_k . The transition functions are

$$\begin{aligned} g_{kl}s_k &= s_l, \\ \bar{g}_{kl}t_k &= t_l. \end{aligned}$$

For $\text{hom}(E, E')$ we see that

$$\begin{aligned} L_l(s_k) &= h_k(g_{lk}s_l) \\ &= g_{lk}L_l(s_l) \\ &= g_{lk}t_l \\ &= g_{lk}\bar{g}_{kl}t_k \\ &= (g_{kl})^{-1}\bar{g}_{kl}t_k \end{aligned}$$

Thus

$$L_l = (g_{kl})^{-1}\bar{g}_{kl}L_k.$$

This shows that

$$\begin{aligned}
e(\text{hom}(E, E')) &= -\frac{\sqrt{-1}}{2\pi} \sum_i d\lambda_i \wedge d \log \left((g_{ki})^{-1} \bar{g}_{ki} \right) \\
&= \frac{\sqrt{-1}}{2\pi} \sum_i d\lambda_i \wedge d \log(g_{ki}) - \frac{\sqrt{-1}}{2\pi} \sum_i d\lambda_i \wedge d \log(\bar{g}_{ki}) \\
&= -e(E) + e(E').
\end{aligned}$$

The proof is similar for tensor products using

$$\begin{aligned}
s_l \otimes t_l &= (g_{kl} s_k) \otimes (\bar{g}_{kl} t_k) \\
&= g_{kl} \bar{g}_{kl} (s_k \otimes t_k).
\end{aligned}$$

□

9.2. Characteristic Classes

All vector bundles will be complex and for convenience also have Hermitian structures. Dimensions etc will be complex so a little bit of adjustment is sometimes necessary when we check where classes live. Note that complex bundles are always oriented since $Gl_m(\mathbb{C}) \subset Gl_{2m}^+(\mathbb{R})$.

We are looking for a characteristic class $c(E) \in H^*(M)$ that can be written as

$$\begin{aligned}
c(E) &= c_0(E) + c_1(E) + c_2(E) + \cdots, \\
c_0(E) &= 1 \in H^0(M), \\
c_1(E) &\in H^2(M), \\
c_2(E) &\in H^4(M), \\
&\vdots \\
c_m(E) &\in H^{2m}(M), \\
c_l(E) &= 0, l > m
\end{aligned}$$

For a 1 dimensional or line bundle we simply define $c(E) = 1 + c_1(E) = 1 + e(E)$. There are two more general properties that these classes should satisfy. First they should be natural in the sense that

$$c(E) = F^*(c(E'))$$

where $F: M \rightarrow M'$ is covered by a complex bundle map $E \rightarrow E'$ that is an isomorphism on fibers. Second, they should satisfy the product formula

$$\begin{aligned}
c(E \oplus E') &= c(E) \wedge c(E') \\
&= \sum_{p=0}^{m+m'} \sum_{i=0}^p c_i(E) \wedge c_{p-i}(E')
\end{aligned}$$

for Whitney sums.

There are two approaches to defining $c(E)$. In [Milnor-Stasheff] an inductive method is used in conjunction with the Gysin sequence for the unit sphere bundle. This approach is explained in the next section. The other method is more abstract, clean, and does not use the Hermitian structure. It is analogous to the construction of splitting fields in Galois theory and is due to Grothendieck.

First we need to understand the cohomology of $H^*(\mathbb{P}(E))$. Note that we have a natural fibration $\pi : \mathbb{P}(E) \rightarrow M$ and a canonical line bundle $\tau(\mathbb{P}(E))$. The Euler class of the line bundle is for simplicity denoted

$$e = e(\tau(\mathbb{P}(E))) \in H^2(\mathbb{P}(E)).$$

The fibers of $\mathbb{P}(E) \rightarrow M$ are \mathbb{P}^{m-1} and we note that the natural inclusion $i : \mathbb{P}^{m-1} \rightarrow \mathbb{P}(E)$ is also natural for the tautological bundles

$$i^*(\tau(\mathbb{P}(E))) = \tau(\mathbb{P}^{m-1})$$

thus showing that

$$i^*(e) = e(\tau(\mathbb{P}^{m-1})).$$

As $e(\tau(\mathbb{P}^{m-1}))$ generates the cohomology of the fiber we have shown that the Leray-Hirsch formula for the cohomology of the fibration $\mathbb{P}(E) \rightarrow M$ can be applied. Thus any element $\omega \in H^*(\mathbb{P}(E))$ has an expression of the form

$$\omega = \sum_{i=0}^m \pi^*(\omega_i) \wedge e^{m-i}$$

where $\omega_i \in H^*(M)$ are unique. In particular we can write:

$$\begin{aligned} 0 &= (-e)^m + \pi^*(c_1(E)) \wedge (-e)^{m-1} + \cdots + \pi^*(c_{m-1}(E)) \wedge (-e) + \pi^*(c_m(E)) \\ &= \sum_{i=0}^m \pi^*(c_i(E)) \wedge (-e)^{m-i} \end{aligned}$$

This means that $H^*(\mathbb{P}(E))$ is an extension of $H^*(M)$ with a unique monic polynomial

$$p_E(t) = t^m + c_1(E)t^{m-1} + \cdots + c_{m-1}(E)t + c_m(E)$$

such that $p_E(-e) = 0$. Moreover, the total Chern class is defined as

$$p_E(1) = c(E) = 1 + c_1(E) + \cdots + c_m(E).$$

The reason for using $-e$ rather than e is that $-e$ restricts to the form ω on the fibers of $\mathbb{P}(E)$.

THEOREM 9.2.1. *Assume that we have vector bundles $E \rightarrow M$ and $E' \rightarrow M'$ both of rank m , and a smooth map $F : M \rightarrow M'$ that is covered by a bundle map that is fiberwise an isomorphism. Then*

$$c(E) = F^*(c(E')).$$

PROOF. We start by selecting a Hermitian structure on E' and then transfer it to E by the bundle map. In that way the bundle map preserves the unit sphere bundles. Better yet, we get a bundle map

$$\pi^*(E) \rightarrow (\pi')^*(E')$$

that also yields a bundle map

$$\tau(\mathbb{P}(E)) \rightarrow \tau(\mathbb{P}(E')).$$

Since the Euler classes for these bundles is natural we have

$$F^*(e') = e$$

and therefore

$$\begin{aligned} 0 &= F^* \left(\sum_{i=0}^m c_i(E') \wedge (-e')^{m-i} \right) \\ &= \sum_{i=0}^m F^* c_i(E') \wedge (-e)^{m-i} \end{aligned}$$

Since $c_i(E)$ are uniquely defined by

$$0 = \sum_{i=0}^m c_i(E) \wedge (-e)^{m-i}$$

we have shown that

$$c_i(E) = F^* c_i(E').$$

□

The trivial bundles $\mathbb{C}^m \oplus M$ all have $c(\mathbb{C}^m \oplus M) = 1$. This is because these bundles are all pull-backs of the bundle $\mathbb{C}^m \oplus \{0\}$, where $\{0\}$ is the 1 point space.

To compute $e(\tau(\mathbb{P}^n))$ recall that $\tau(\mathbb{P}^n)$ is the conjugate of $\mathbb{P}^{n+1} - \{p\} \rightarrow \mathbb{P}^n$ which has dual $\eta_{\mathbb{P}^n} = \omega$. Since conjugation reverses orientation on 1 dimensional bundles this shows that $e(\tau(\mathbb{P}^n)) = -\omega$.

The Whitney sum formula is established by proving the splitting principle.

THEOREM 9.2.2. *If a bundle $\pi : E \rightarrow M$ splits $E = L_1 \oplus \cdots \oplus L_m$ as a direct sum of line bundles, then*

$$c(E) = \prod_{i=1}^m (1 + e(L_i)).$$

PROOF. We pull back all classes to E without changing notation. We know that $c(E) = p_E(1)$ so it suffices to identify p_E with the monic polynomial of degree m defined by $p(t) = \prod_{i=1}^m (t + e(L_i))$. To prove this we need to show that

$$p(-e) = \prod_{i=1}^m (-e + e(L_i)) = 0.$$

Note that we can identify $-e + e(L_i)$ with the Euler class of $\text{hom}(\tau, L_i)$. With that in mind:

$$\begin{aligned} \prod_{i=1}^m (-e + e(L_i)) &= e \left(\bigoplus_{i=1}^m \text{Hom}(\tau, L_i) \right) \\ &= e(\text{Hom}(\tau, L_1 \oplus \cdots \oplus L_m)) \\ &= e(\text{Hom}(\tau, E)) \\ &= e(\text{Hom}(\tau, \tau \oplus \tau^\perp)) \\ &= e(\text{Hom}(\tau, \tau)) \wedge e(\text{Hom}(\tau, \tau^\perp)) \\ &= 0. \end{aligned}$$

Where the last equality follows from the fact that $\text{Hom}(\tau, \tau)$ has the identity map as a nowhere vanishing section. □

The splitting principle can be used to compute $c(T\mathbb{P}^n)$. First note that $T\mathbb{P}^n \simeq \text{Hom}(\tau(\mathbb{P}^n), \tau(\mathbb{P}^n)^\perp)$. Thus

$$\begin{aligned} T\mathbb{P}^n \oplus \mathbb{C} &= \text{Hom}(\tau(\mathbb{P}^n), \tau(\mathbb{P}^n)^\perp) \oplus \mathbb{C} \\ &= \text{Hom}(\tau(\mathbb{P}^n), \tau(\mathbb{P}^n)^\perp) \oplus \text{Hom}(\tau(\mathbb{P}^n), \tau(\mathbb{P}^n)) \\ &= \text{Hom}(\tau(\mathbb{P}^n), \tau(\mathbb{P}^n)^\perp \oplus \tau(\mathbb{P}^n)) \\ &= \text{Hom}(\tau(\mathbb{P}^n), \mathbb{C}^{n+1}) \\ &= \text{Hom}(\tau(\mathbb{P}^n), \mathbb{C}) \oplus \cdots \oplus \text{Hom}(\tau(\mathbb{P}^n), \mathbb{C}). \end{aligned}$$

Thus

$$\begin{aligned} c(T\mathbb{P}^n) &= c(T\mathbb{P}^n \oplus \mathbb{C}) \\ &= (1 + \omega)^{n+1}. \end{aligned}$$

This shows that

$$c_i(T\mathbb{P}^n) = \binom{n+1}{i} \omega^i$$

which conforms with

$$e(T\mathbb{P}^n) = c_n(T\mathbb{P}^n) = (n+1)\omega^n.$$

We can now finally establish the Whitney sum formula.

THEOREM 9.2.3. *For two vector bundles $E \rightarrow M$ and $E' \rightarrow M$ we have*

$$c(E \oplus E') = c(E) \wedge c(E').$$

PROOF. First we repeatedly projectivize so as to create a map $\tilde{N} \rightarrow M$ with the property that it is an injection on cohomology and the pull-back of E to \tilde{N} splits as a direct sum of line bundles. Then repeat this procedure on the pull-back of E' to \tilde{N} until we finally get a map $F: N \rightarrow M$ such that F^* is an injection on cohomology and both of the bundles split

$$\begin{aligned} F^*(E) &= L_1 \oplus \cdots \oplus L_m, \\ F^*(E') &= K_1 \oplus \cdots \oplus K_{m'} \end{aligned}$$

The splitting principle together with naturality then implies that

$$\begin{aligned} F^*(c(E \oplus E')) &= c(F^*(E \oplus E')) \\ &= c(L_1) \wedge \cdots \wedge c(L_m) \wedge c(K_1) \wedge \cdots \wedge c(K_{m'}) \\ &= c(F^*(E)) \wedge c(F^*(E')) \\ &= F^*c(E) \wedge F^*c(E') \\ &= F^*(c(E) \wedge c(E')). \end{aligned}$$

Since F^* is an injection this shows that

$$c(E \oplus E') = c(E) \wedge c(E').$$

□

9.3. The Gysin Sequence

This sequence allows us to compute the cohomology of certain fibrations where the fibers are spheres. As we saw above, these fibrations are not necessarily among the ones where we can use the Hirsch-Leray formula. This sequence uses the Euler class and will recapture the dual, or Thom class, from the Euler class.

We start with an oriented vector bundle $\pi : E \rightarrow M$. It is possible to put a smoothly varying inner product structure on the vector spaces of the fibration, using that such bundles are locally trivial and gluing inner products together with a partition of unity on M . The function $E \rightarrow \mathbb{R}$ that takes v to $|v|^2$ is then smooth and the only critical value is 0. As such we get a smooth manifold with boundary

$$D(E) = \{v \in E : |v| \leq 1\}$$

called the disc bundle with boundary

$$S(E) = \partial D(E) = \{v \in E : |v| = 1\}$$

being the unit sphere bundle and interior

$$\text{int}D(E) = \{v \in E : |v| < 1\}.$$

Two different inner product structures will yield different disc bundles, but, it is easy to see that they are all diffeomorphic to each other. We also note that $\text{int}D(E)$ is diffeomorphic to E , while $D(E)$ is homotopy equivalent to E . This gives us a diagram

$$\begin{array}{ccccccc} \rightarrow & H_c^p(\text{int}D(E)) & \rightarrow & H^p(D(E)) & \rightarrow & H^p(S(E)) & \rightarrow & H_c^{p+1}(\text{int}D(E)) & \rightarrow \\ & \downarrow & & \uparrow & & \downarrow & & \uparrow & \\ \rightarrow & H_c^p(E) & \rightarrow & H^p(E) & \rightarrow & H^p(S(E)) & \dashrightarrow & H_c^{p+1}(E) & \rightarrow \end{array}$$

where the vertical arrows are simply pull-backs and all are isomorphisms. The connecting homomorphism

$$H^p(S(E)) \rightarrow H_c^{p+1}(\text{int}D(E))$$

then yields a map

$$H^p(S(E)) \dashrightarrow H_c^{p+1}(E)$$

that makes the bottom sequence a long exact sequence. Using the Thom isomorphism

$$H^{p-m}(M) \rightarrow H_c^p(E)$$

then gives us a new diagram

$$\begin{array}{ccccccc} \rightarrow & H^{p-m}(M) & \xrightarrow{e \wedge} & H^p(M) & \rightarrow & H^p(S(E)) & \dashrightarrow & H^{p+1-m}(M) & \rightarrow \\ & \downarrow \eta_M \wedge \pi^*(\cdot) & & \downarrow & & \downarrow & & \downarrow & \\ \rightarrow & H_c^p(E) & \rightarrow & H^p(E) & \rightarrow & H^p(S(E)) & \rightarrow & H_c^{p+1}(E) & \rightarrow \end{array}$$

Most of the arrows are pull-backs and the vertical arrows are isomorphisms. The first square is commutative since $\pi^*i^*(\eta_M) = \pi^*(e)$ is represented by η_M in $H^m(E)$. This is simply because the zero section $I : M \rightarrow E$ and projection $\pi : E \rightarrow M$ are homotopy equivalences. The second square is obviously commutative. Thus we get a map

$$H^p(S(E)) \dashrightarrow H^{p+1-m}(M)$$

making the top sequence exact. This is the Gysin sequence of the sphere bundle of an oriented vector bundle. The connecting homomorphism which lowers the degree by $m-1$ can be constructed explicitly and geometrically by integrating forms on $S(E)$ along the unit spheres, but, we won't need this interpretation.

The Gysin sequence also tells us how the Euler class can be used to compute the cohomology of the sphere bundle from M .

To come full circle with the Leray-Hirsch Theorem we now assume that $E \rightarrow M$ is a complex bundle of complex dimension m and construct the projectivized bundle

$$\mathbb{P}(E) = \{(p, L) \mid L \subset \pi^{-1}(p) \text{ is a 1 dimensional subspace}\}$$

This gives us projections

$$S(E) \rightarrow \mathbb{P}(E) \rightarrow M.$$

There is also a tautological bundle

$$\tau(\mathbb{P}(E)) = \{(p, L, v) \mid v \in L\}.$$

The unit-sphere bundle for τ is naturally identified with $S(E)$ by

$$\begin{aligned} S(E) &\rightarrow S(\tau(\mathbb{P}(E))), \\ (p, v) &\rightarrow (p, \text{span}\{v\}, v). \end{aligned}$$

This means that $S(E)$ is part of two Gysin sequences. One where M is the base and one where $\mathbb{P}(E)$ is the base. These two sequences can be connected in a very interesting manner.

If we pull back E to $\mathbb{P}(E)$ and let

$$\tau^\perp = \{(p, L, w) \mid w \in L^\perp\}$$

be the orthogonal complement then we have that

$$\pi^*(e(E)) = e(\pi^*(E)) = e(\tau(\mathbb{P}(E))) \wedge e(\tau^\perp) \in H^*(\mathbb{P}(E)).$$

Thus we obtain a commutative diagram

$$\begin{array}{ccccc} & & H^{p-2}(\mathbb{P}(E)) & \xrightarrow{e(\tau) \wedge \cdot} & H^p(\mathbb{P}(E)) & & \\ \searrow & & \uparrow e(\tau^\perp) \wedge \pi^*(\cdot) & & \uparrow \pi^* & \searrow & \\ H^{p-1}(S(E)) & & & & & & H^p(S(E)) \\ \nearrow & & & & & \nearrow & \\ & & H^{p-2m}(M) & \xrightarrow{e(E) \wedge \cdot} & H^p(M) & & \end{array}$$

What is more we can now show in two ways that

$$\text{span}\{1, e, \dots, e^{m-1}\} \otimes H^*(M) \rightarrow H^*(\mathbb{P}(E))$$

is an isomorphism. First we can simply use the Leray-Hirsch result by noting that the classes $1, e, \dots, e^{m-1}$ when restricted to the fibers are the usual cohomology classes of the fiber \mathbb{P}^m . Or we can use diagram chases on the above diagram.

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