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On eigenvalue pinching in positive Ricci curvature

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Abstract. We shall show that for manifolds with $\mathrm{Ric} \geq n-1$ the radius is close to π iff the $(n+1)$st eigenvalue is close to n. This extends results of Cheng and Croke which show that the diameter is close to π iff the first eigenvalue is close to n. We shall also give a new proof of an important theorem of Colding to the effect that if the radius is close to π, then the volume is close to that of the sphere and the manifold is Gromov-Hausdorff close to the sphere. From work of Cheeger and Colding these conditions imply that the manifold is diffeomorphic to a sphere.

1. Introduction

In order to better understand the results presented here we first cover a little bit of history about eigenvalues in positive Ricci curvature. Throughout the entire paper let us assume that M is a closed n-dimensional Riemannian manifold with Ricci curvature $\mathrm{Ric} \geq n-1$. The first result for such manifolds is due to Myers. In 1941 he showed that such manifolds have diameter $\mathrm{diam} M \leq \pi$, the diameter of the unit sphere S^n. In 1958 Lichnerowicz showed, using the Bochner formula, that the first eigenvalue $\lambda_1(M) \geq n$ (see [15] and also [4] for a different proof). Note that n is the first eigenvalue on the unit sphere and that it has multiplicity $n+1$ since any linear function on the ambient Euclidean space evidently yields an eigenfunction with eigenvalue n on the unit sphere. Obata, in 1962, showed that $\lambda_1(M)$ can only be n if M is the unit sphere (see [16]). Jumping ahead to 1975 we get to a result of S.Y. Cheng which qualitatively says that if $\mathrm{diam} M$ is close to π, then $\lambda_1(M)$ is close to n (see [7]). From this he concluded, in particular, that if $\mathrm{diam} M = \pi$, then $\lambda_1(M) = n$, and thus Obata's result shows that M is the unit sphere. Then in 1982 Croke showed a converse to Cheng's result,

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namely, that if $\lambda_1(M)$ is close to n, then diam$M$ is close to π (see [11] and also [3] for a sharper result).

Given that the multiplicity of λ_1 is $n+1$ for the unit sphere, it is natural to suppose that λ_{n+1} is close to n iff in some sense M spreads out in the same way the sphere does. Volume is a natural way in which to measure the way in which the sphere spreads out. Another is radius, rad$M$, which is the radius of the smallest closed metric ball covering the whole space. In the two papers [8] and [9], Colding, generalizing work of Otsu, Shiohama, and Yamaguchi (see [17] and [21]), showed that the following three conditions are equivalent:

1) vol$M$ is close to vol(S^n),
2) rad$M$ is close to π,
3) M is Gromov-Hausdorff close to S^n.

Furthermore, these conditions imply by a theorem of Cheeger-Colding (see [6]) that M is diffeomorphic to S^n. Note that condition 3 trivially implies 2 and that 2 follows from 1 using absolute volume comparison. The other implications, however, are highly nontrivial. In any case, we have three natural and equivalent ways in which to measure how a manifold spreads out like a sphere. The main object of this paper is to show an extension the results of Croke and Cheng, and also add to the above list of equivalent conditions a fourth.

Theorem 1.1. *If M is a complete Riemannian n-manifold with* Ric$M \geq n-1$, *then* rad$M$ *is close to π if and only if $\lambda_{n+1}(M)$ is close to n.*

In [9, Remark 4.21] Colding points out that his work can be used to generalize Croke's result. Although it is unclear what generalization is meant, Colding had something like the above theorem in mind. Our proof, however, does not depend on any of the Hessian estimates or new integral Toponogov-like theorems Colding developed. Rather we are actually able to use the techniques developed here to prove the above mentioned results of Colding. It is perhaps also interesting to point out that the Abresch-Gromoll maximum principle is used in an essential way in this paper, moreover, possibly for the first time on functions which are not related to excess functions (see [1]).

Using the results from [5] together with what we have developed here it is not hard to see that λ_k is close to n iff the space is Gromov-Hausdorff close to a k-fold sine suspension over some metric space. In fact our work shows that λ_k is close to n iff M contains a subset which is Gromov-Hausdorff close to S^{k-1}, the warped product rigidity then follows from [5].

The techniques developed here can with some extra work also be adapted to the situations discussed in [10]. Thus we can establish the volume convergence result (see [10, Theorem 0.1]) and the almost maximal volume result (see [10, Theorem 0.8]) of Colding without the use of his integral integral Toponogov-like theorems. The proofs will appear in a different paper.

We should also point out that the above theorem generalizes to the situation where the amount of curvature which lies below $n-1$ is small in L^p. That is, $\bar{k}(p, n-1)$ is small (see [20]).

The structure of the paper is as follows. We start in section 2 by explaining our conventions and proving the theorems of Lichnerowicz and Obata. These results are very easy to prove and form the motivation for our later analysis. In section 3 we explain some basic estimates for eigenfunctions with close to minimal eigenvalue. We then go on in section 4 to prove how large radius implies that λ_{n+1} is close to n. Section 5 is devoted to showing the opposite direction. In section 6 we show how having small λ_{n+1} means that the manifold is Gromov-Hausdorff close to the sphere and has almost maximal volume. In the last section we have collected some basic analytical facts which are important for our proofs. The reader is referred to [18] for basic material not explained here.

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2. Some background material

2.1. Some notation. Throughout we shall adopt the notation that $\tau(\varepsilon|n, p, h, ...)$ is a nonnegative function which depends on $\varepsilon, n, p, h, ...$ and which converges to zero as $\varepsilon \to 0$. These functions τ will always be explicit, but it eases the exposition to disregard the explicit nature of these functions. The convenience of this notation is that even though τ might change from line to line in a calculation it still maintains these basic features.

Another important convention is that L^p norms are always normalized, i.e.,

$$\|f\|_p = \left(\frac{1}{\mathrm{vol} M}\int_M |f|^p\right)^{1/p}.$$

From Hölders inequality it then follows that for $p \leq q$ we have $\|f\|_p \leq \|f\|_q$. Also $\lim_{p\to\infty}\|f\|_p = \|f\|_\infty$.

2.2. Some linear algebra. If we have a linear map $L : V \to W$ between Euclidean spaces, then it has two natural norms. First, the Euclidean norm

$$|L| = \sqrt{\mathrm{tr}\,(L^* \circ L)}.$$

If we select orthonormal bases for V and W and write L as a matrix $\left(l_{ij}\right)$, then this is simply

$$|L| = \sqrt{\sum_{i,j} \left|l_{ij}\right|^2}.$$

Second, the operator norm

$$\|L\| = \max_{|v|=1} |L(v)| = \max_{|v|=1} \sqrt{\langle L^* \circ L(v), v\rangle}.$$

If we assume that $\dim V = n$, then these two norms are related as follows

$$\|L\|^2 \leq |L|^2 \leq n \|L\|^2.$$

Using that $L^* \circ L : V \to V$ is self-adjoint and positive semidefinite we can find eigenvalues $\mu_1^2 \leq \ldots \leq \mu_n^2$. In terms of these eigenvalues the above norms are

$$\begin{aligned} |L| &= \sqrt{\mu_1^2 + \cdots + \mu_n^2}, \\ \|L\| &= \mu_n. \end{aligned}$$

From this we get the following very useful observation. Suppose

$$n - \delta \leq |L|^2 \leq n \|L\|^2 \leq n + \delta,$$

then

$$\left|\det\left(L^* \circ L\right) - 1\right| \leq \tau(\delta|n).$$

Another linear algebra fact about the Euclidean norm we shall use is that if $L : V \to V$ and V has dimension n, then

$$0 \leq \left| L - \frac{1}{n}\mathrm{tr}L \cdot I \right|^2 = |L|^2 - \frac{1}{n}|\mathrm{tr}L|^2.$$

2.3. The spectrum of a manifold. For a function f on our closed Riemannian n-manifold M, we have the gradient ∇f, the Hessian $\nabla^2 f$, which we think of as a symmetric $(1, 1)$-tensor, and the trace $\mathrm{tr}\left(\nabla^2 f\right) = \Delta f$, which is the Laplacian of f. The identity $(1, 1)$-tensor is denoted I.

There is a divergent sequence of eigenvalues $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots.$, counted with multiplicities, for the Laplacian. We can select corresponding eigenfunctions f_i, i.e., $\Delta f_i = -\lambda_i f_i$. These form an orthogonal basis for $L^2(M)$. We normalize these functions so that

$$\frac{1}{\mathrm{vol}M}\int_M f_i^2 = \frac{1}{n+1} \quad \left(\text{actually } f_0 = \frac{1}{\sqrt{n+1}}\right).$$

Since they form an orthogonal basis for $L^2(M)$ they satisfy

$$\begin{aligned} \frac{1}{\mathrm{vol}M}\int_M f_i &= 0, \quad i \geq 1, \\ \frac{1}{\mathrm{vol}M}\int_M f_i f_j &= 0, \quad i \neq j, \\ \frac{1}{\mathrm{vol}M}\int_M g\left(\nabla f_i, \nabla f_j\right) &= 0, \quad i \neq j, \\ \frac{1}{\mathrm{vol}M}\int_M |\nabla f_i|^2 &= \frac{\lambda_i}{n+1}. \end{aligned}$$

Note that if we take the Fourier expansion of a function f

$$\begin{aligned} f &= \sum \alpha_i f_i, \\ \alpha_i &= \frac{n+1}{\operatorname{vol} M} \int_M f \cdot f_i, \end{aligned}$$

then

$$\frac{1}{\operatorname{vol} M} \int_M |f|^2 = \frac{1}{n+1} \sum \alpha_i^2.$$

So if

$$\frac{1}{\operatorname{vol} M} \int_M |f|^2 \approx \frac{1}{n+1},$$

then it must follow that

$$\sum \alpha_i^2 \approx 1.$$

2.4. The unit sphere. On the unit sphere $S^n \subset \mathbb{R}^{n+1}$ the eigenfunctions look particularly nice. Namely, they are simply the restriction of harmonic homogeneous polynomials on $\mathbb{R}^{n+1}$. In particular, we have that $\lambda_1 = n$ and that it has multiplicity $n+1$ as all linear functions on $\mathbb{R}^{n+1}$ restricted to S^n become eigenfunctions with eigenvalue n. If we select $f_i : S^n \to \mathbb{R}$ as the ith coordinate function, then we have that they conform to our conventions for eigenfunctions listed above.

$$\begin{aligned} \Delta f_i &= -n f_i, \\ \frac{1}{\operatorname{vol} S^n} \int_{S^n} f_i^2 &= \frac{1}{n+1}, \\ \frac{1}{\operatorname{vol} S^n} \int_{S^n} f_i f_j &= 0. \end{aligned}$$

What is more, they are naturally associated with distance functions on the sphere. If $x_0 = \left(x_0^1, ..., x_0^{n+1}\right) \in S^n$ and $d(x) = d(x, x_0)$ is the spherical distance between x and x_0, then we have

$$\begin{aligned} \cos d(x) &= x \cdot x_0 = \sum x_0^i f_i(x), \\ \Delta \cos d &= -n \cos d, \\ \frac{1}{\operatorname{vol} S^n} \int_{S^n} |\cos d|^2 &= \frac{1}{n+1}, \\ \frac{1}{\operatorname{vol} S^n} \int_{S^n} \cos d &= 0, \end{aligned}$$

and the x_0^is are the Fourier coefficients for $\cos d$.

In particular, we see that there are two ways in which to express the identity map on S^n. First, simply by evaluation of the eigenfunctions

$$\Phi(x_0) = \left(f_1\left(x_0\right), ..., f_{n+1}(x_0)\right).$$

Second, by computing the Fourier coefficients of $\cos d$

$$\Psi(x_0) = \left(\frac{n+1}{\operatorname{vol} S^n}\int_{S^n} f_1 \cos d, ..., \frac{n+1}{\operatorname{vol} S^n}\int_{S^n} f_{n+1}\cos d\right).$$

2.5. The results of Lichnerowicz and Obata. We shall now assume that $\operatorname{Ric} M \geq n-1$. It will be instructive to see how the Lichnerowicz/Obata theorems work. The Bochner formula applied to ∇f_1 says

$$\begin{aligned}\Delta\frac{1}{2}|\nabla f_1|^2 &= \left|\nabla^2 f_1\right|^2 - \lambda_1 |\nabla f_1|^2 + \operatorname{Ric}(\nabla f_1, \nabla f_1)\\ &\geq \frac{1}{n}|\Delta f_1|^2 + (n-1-\lambda_1)|\nabla f_1|^2\\ &= \frac{\lambda_1^2}{n}|f_1|^2 + (n-1-\lambda_1)|\nabla f_1|^2.\end{aligned}$$

If we integrate this over M we get

$$\begin{aligned}0 &\geq \frac{1}{\operatorname{vol} M}\int\left(\frac{\lambda_1^2}{n}|f_1|^2 + (n-1-\lambda_1)|\nabla f_1|^2\right)\\ &= \frac{\lambda_1^2}{n}\frac{1}{n+1} + (n-1-\lambda_1)\frac{\lambda_1}{n+1}.\end{aligned}$$

Thus

$$\left(1-\frac{1}{n}\lambda_1\right)\lambda_1 \leq 0.$$

Since $\lambda_1 > 0$ this must mean that $\lambda_1 \geq n$. In case equality occurs all of the above inequalities must be equalities. In particular, we have equality in

$$0 \leq \left|\nabla^2 f_1 - \frac{1}{n}\Delta f_1 \cdot I\right|^2 = \left|\nabla^2 f_1\right|^2 - \frac{1}{n}|\Delta f_1|^2.$$

Thus

$$\nabla^2 f_1 + f_1 \cdot I = 0.$$

From this it follows that $f_1^2 + |\nabla f_1|^2$ is constant. Due to our normalizations, however, the normalized integral of this is 1, hence

$$f_1^2 + |\nabla f_1|^2 = 1.$$

This means that $\cos^{-1} f_1$ is well-defined and has $\left|\nabla \cos^{-1} f_1\right| \equiv 1$ away from the points where $f_1 = \pm 1$. Thus $\cos^{-1} f_1$ is a distance function to the point where $f_1 = 1$. Moreover, the equation $\nabla^2 f_1 + f_1 \cdot I = 0$ tells us that the Hessian of this distance function is precisely the same as it would be on the unit sphere. Hence M is the unit sphere.

3. Basic eigenfunction estimates

In Sections 3, 4, 5, and 6 we fix a closed Riemannian manifold (M, g) with $\mathrm{Ric} \geq (n-1)$. Fix some $\varepsilon > 0$ and assume that for $i = 1, ..., k$ we have $n \leq \lambda_i \leq n + \varepsilon$. We shall prove a couple of lemmas here about the eigenfunctions corresponding to these eigenvalues.

Lemma 3.1. *For $i = 1, ..., k$ and any $p \geq 1$ we have*

$$f_i^2 + |\nabla f_i|^2 \leq 1 + \tau(\varepsilon|n),$$

$$\left\| f_i^2 + |\nabla f_i|^2 - 1 \right\|_p \leq \tau(\varepsilon|n, p).$$

Proof. First observe that

$$\Delta f_i^2 = -2\lambda_i f_i^2 + 2|\nabla f_i|^2$$

and

$$\Delta |\nabla f_i|^2 = 2\left|\nabla^2 f_i\right|^2 - 2\lambda_i |\nabla f_i|^2 + 2\mathrm{Ric}(\nabla f_i, \nabla f_i)$$

Thus

$$\begin{aligned}
\Delta\left(f_i^2 + |\nabla f_i|^2\right) &= 2\left|\nabla^2 f_i\right|^2 - 2\lambda_i |\nabla f_i|^2 + 2\mathrm{Ric}(\nabla f_i, \nabla f_i) \\
&\quad -2\lambda_i f_i^2 + 2|\nabla f_i|^2 \\
&= 2\left(\left|\nabla^2 f_i\right|^2 - \frac{\lambda_i^2}{n} f_i^2\right) \\
&\quad +2\left(\mathrm{Ric}(\nabla f_i, \nabla f_i) - (n-1)|\nabla f_i|^2\right) \\
&\quad -2(\lambda_i - n)|\nabla f_i|^2 + 2\frac{\lambda_i}{n}(\lambda_i - n) f_i^2 \\
&= 2\left(\left|\nabla^2 f_i\right|^2 - \frac{(\Delta f_i)^2}{n}\right) \\
&\quad +2\left(\mathrm{Ric}(\nabla f_i, \nabla f_i) - (n-1)|\nabla f_i|^2\right) \\
&\quad -2(\lambda_i - n)|\nabla f_i|^2 + 2\frac{\lambda_i}{n}(\lambda_i - n) f_i^2 \\
&= 2\left|\nabla^2 f_i + \frac{\lambda_i}{n} f_i\right|^2 \\
&\quad +2\left(\mathrm{Ric}(\nabla f_i, \nabla f_i) - (n-1)|\nabla f_i|^2\right) \\
&\quad -2(\lambda_i - n)|\nabla f_i|^2 + 2\frac{\lambda_i}{n}(\lambda_i - n) f_i^2
\end{aligned}$$

Note that the two first terms are nonnegative and that the last two will → 0 provided f_i and ∇f_i are bounded. That ∇f_i is bounded follows from the gradient estimate (see Theorem 7.3). Then f_i is also bounded since we fixed its average. Thus

$$\Delta\left(f_i^2+|\nabla f_i|^2\right)\geq-\tau\left(\varepsilon|n\right).$$

The generalized maximum principle (see Theorem 7.1) then tells us that

$$\begin{aligned} f_i^2+|\nabla f_i|^2 &\leq \frac{1}{\operatorname{vol}M}\int_M\left(f_i^2+|\nabla f_i|^2\right)+\tau\left(\varepsilon|n\right)\\ &=\frac{1}{n+1}+\frac{\lambda_i}{n+1}+\tau\left(\varepsilon|n\right)\\ &=1+\tau\left(\varepsilon|n\right). \end{aligned}$$

Since $f_i^2+|\nabla f_i|^2$ is bounded above by a number which is close to its average it follows that

$$\left\|f_i^2+|\nabla f_i|^2-1\right\|_1=\tau\left(\varepsilon|n\right).$$

Since $\left|f_i^2+|\nabla f_i|^2-1\right|$ is bounded by 1 for small ε it follows that

$$\begin{aligned} \left\|f_i^2+|\nabla f_i|^2-1\right\|_p &\leq \left\|\left|f_i^2+|\nabla f_i|^2-1\right|^{p-1}\right\|_\infty\left(\left\|f_i^2+|\nabla f_i|^2-1\right\|_1\right)^{\frac{1}{p}}\\ &\leq\left(\left\|f_i^2+|\nabla f_i|^2-1\right\|_1\right)^{\frac{1}{p}}\\ &=\tau\left(\varepsilon|n,p\right). \end{aligned}$$

□

Note that examples of Anderson show that the L^p bound cannot be made into an L^∞ bound as the gradient of f_i might be zero near the place where the function itself is zero. The way in which this could happen is if one has some nontrivial topology near the zero set for f_i and examples from [2] show that could happen. In case one has the stronger condition $\sec\geq 1$, however, the results of [17] indicate that one should be able to get an L^∞ bound. Still, something like a smoothed version of $\sum_{\sin}S_k^n$, a sine suspension over a sphere of curvature $k>1$, might give counter examples to that as well.

We now generalize the above estimate for eigenfunctions to certain linear combinations of these eigenfunctions.

Lemma 3.2. *Let* $\phi=\sum_{i=1}^k\alpha_i f_i$ *with* $\left|\sum\alpha_i^2-1\right|\leq\tau\left(\varepsilon|n\right)$, *then*

$$\begin{aligned} \phi^2+|\nabla\phi|^2&\leq1+\tau\left(\varepsilon|n\right),\\ \left\|\phi^2+|\nabla\phi|^2-1\right\|_p&\leq\tau\left(\varepsilon|n,p\right). \end{aligned}$$

Proof. First we have

$$\frac{1}{\mathrm{vol}M}\int_M \phi = 0,$$

$$\begin{aligned}\left|\frac{1}{\mathrm{vol}M}\int_M \phi^2 - \frac{1}{n+1}\right| &= \left|\sum \frac{\alpha_i\alpha_j}{\mathrm{vol}M}\int_M f_i f_j - \frac{1}{n+1}\right| \\ &= \left|\sum \frac{\alpha_i^2}{\mathrm{vol}M}\int_M f_i^2 - \frac{1}{n+1}\right| \\ &= \left|\frac{1}{n+1}\sum \alpha_i^2 - \frac{1}{n+1}\right| \\ &\le \tau(\varepsilon|n),\end{aligned}$$

$$\begin{aligned}\left|\frac{1}{\mathrm{vol}M}\int_M |\nabla\phi|^2 - \frac{n}{n+1}\right| &= \left|\sum \frac{\alpha_i\alpha_j}{\mathrm{vol}M}\int_M g\left(\nabla f_i, \nabla f_j\right) - \frac{n}{n+1}\right| \\ &= \left|\sum \frac{\alpha_i^2}{\mathrm{vol}M}\int_M |\nabla f_i|^2 - \frac{n}{n+1}\right| \\ &= \left|\sum \alpha_i^2 \frac{\lambda_i}{n+1} - \frac{n}{n+1}\right| \\ &\le \tau(\varepsilon|n),\end{aligned}$$

and finally

$$\begin{aligned}\Delta\phi &= \sum \alpha_i \Delta f_i \\ &= -n\phi - \sum \alpha_i (\lambda_i - n) f_i \\ &\ge -n\phi - \tau(\varepsilon|n).\end{aligned}$$

Moreover

$$\begin{aligned}\Delta\left(\phi^2 + |\nabla\phi|^2\right) &= 2\phi\Delta\phi + 2|\nabla\phi|^2 + 2\left|\nabla^2\phi\right|^2 + 2g(\nabla\phi, \nabla\Delta\phi) \\ &\quad +2\mathrm{Ric}(\nabla\phi, \nabla\phi) \\ &= 2\left(\left|\nabla^2\phi\right|^2 - \frac{1}{n}(\Delta\phi)^2\right) \\ &\quad +2\left(\mathrm{Ric}(\nabla\phi, \nabla\phi) - (n-1)|\nabla\phi|^2\right) - \tau(\varepsilon|n) \\ &\ge -\tau(\varepsilon|n).\end{aligned}$$

Thus we have again by the generalized maximum principle (see Theorem 7.1)

$$\phi^2 + |\nabla\phi|^2 \le 1 + \tau(\varepsilon|n).$$

And also that

$$\left\|\phi^2 + |\nabla\phi|^2 - 1\right\|_p \le \tau(\varepsilon|n, p).$$

□

Now define $\Phi_k : M \to \mathbb{R}^k$ by $\Phi_k(x) = (f_1(x), ..., f_k(x))$. The differential of this map is denoted $d\Phi_k : TM \to \mathbb{R}^k$. Note that it is the standard differential, not the derivative $D\Phi_k : TM \to T\mathbb{R}^k$. This map satisfies

Lemma 3.3.

$$|\Phi_k| \leq 1 + \tau(\varepsilon|n)$$

$$\|d\Phi_k\| \leq 1 + \tau(\varepsilon|n)$$

Proof. We just showed that for all choices of α_i with $\left|\sum \alpha_i^2 - 1\right| \leq \tau(\varepsilon|n)$ we must have

$$\phi^2 = \left|\sum \alpha_i f_i(x)\right|^2 \leq 1 + \tau(\varepsilon|n).$$

For a fixed $x_0 \in M$, where $0 < |\Phi_k(x_0)|^2 = \sum f_i^2(x_0)$, we can therefore use

$$\alpha_i = \frac{f_i(x_0)}{\sqrt{\sum f_i^2(x_0)}}$$

to see that

$$|\Phi_k(x_0)|^2 = \sum f_i^2(x_0) \leq 1 + \tau(\varepsilon|n).$$

For the second inequality first note that, if $|v| = 1$ and $\left|\sum \beta_i^2 - 1\right| \leq \tau(\varepsilon|n)$ then have from 3.2 that

$$\begin{aligned}\left|\sum \beta_i \, df_i(v)\right| &= \left|\sum \beta_i g(\nabla f_i, v)\right| \\ &= \left|g\left(\sum \beta_i \nabla f_i, v\right)\right| \\ &\leq \left|\sum \beta_i \nabla f_i\right| \\ &\leq 1 + \tau(\varepsilon|n).\end{aligned}$$

Thus, if $\sum |df_i(v)|^2 > 0$, we can use

$$\beta_i = \frac{df_i(v)}{\sqrt{\sum |df_i(v)|^2}}$$

to conclude that

$$\sum |df_i(v)|^2 \leq 1 + \tau(\varepsilon|n).$$

This implies that

$$\|d\Phi_k\| \leq 1 + \tau(\varepsilon|n).$$

□

4. Large radius

In this section we shall show that "long" distance functions are C^0 close to the part of its Fourier expansion $\sum_{i=1}^{k}\alpha_i f_i$ where only the eigenfunctions corresponding to eigenvalues close to n occur. This gives the following extension of Cheng's estimate for the first eigenvalue in positive Ricci curvature.

Theorem 4.1. *For every* $\varepsilon > 0$ *there is a* $\delta = \delta(\varepsilon, n)$ *such that any Riemannian n-manifold with* $\mathrm{Ric}M \geq n-1$ *and* $\mathrm{rad}M \geq \pi - \delta$ *satisfies* $\lambda_{n+1}(M) \leq n+\varepsilon$.

Fix $\varepsilon > 0$ as in the previous section and let $i = 1, ..., k$ be the indices corresponding to eigenvalues in the interval $[n, n+\varepsilon]$. In [8, Lemma 1.10] Colding establishes that long distance functions are L^2 close to the appropriate part of their Fourier expansions. This proof is straightforward and uses only relative volume comparison together with Cheng's theorem that there are eigenvalues in the interval $[n, n+\varepsilon]$. Specifically we have

Lemma 4.2. *There is a* $\delta = \delta(\varepsilon, n) > 0$ *such that any distance function* $d(x) = d(x, x_0)$ *with* $\max d \geq \pi - \delta$ *satisfies*

$$\left|\frac{1}{\mathrm{vol}M}\int \cos d\right| \leq \tau(\varepsilon|n),$$

$$\left|\frac{1}{\mathrm{vol}M}\int |\cos d|^2 - \frac{1}{n+1}\right| \leq \tau(\varepsilon|n),$$

$$\left\|\cos d - \sum_{i=1}^{k}\alpha_i f_i\right\|_2 \leq \tau(\varepsilon|n),$$

$$\left|\sum_{i=1}^{k}\alpha_i^2 - 1\right| \leq \tau(\varepsilon|n),$$

where $\sum_{i=0}^{\infty}\alpha_i f_i$ *is the Fourier expansion of* $\cos d$.

We can strengthen this lemma as follows (see also [5, Lemma 6.15] for a similar result).

Lemma 4.3. *With notation as in the above lemma we have*

$$\left|\cos d - \sum_{i=1}^{k}\alpha_i f_i\right| \leq \tau(\varepsilon|n).$$

Proof. From the previous section (see Lemma 3.2) we know that $f = \sum_{i=1}^{k}\alpha_i f_i$ is bounded in L^∞ by $1+\tau(\varepsilon|n)$. So, of course, is $\cos d$. Thus closeness between these two functions in L^2 implies closeness in L^p for any $p > 2$ as well. Specifically

$$\|\cos d - f\|_p \leq \tau (\varepsilon|n, p) .$$

The Laplacian of the difference $f - \cos d$ satisfies

$$\begin{aligned}
\Delta (f - \cos d) &= \Delta f - \Delta \cos d \\
&\leq - \sum_{i=1}^{k} \alpha_i \lambda_i f_i + n \cos d \\
&= -n (f - \cos d) - \sum_{i=1}^{k} \alpha_i (\lambda_i - n) f_i \\
&= \phi + \tau (\varepsilon|n) ,
\end{aligned}$$

where $||\phi||_p \leq \tau (\varepsilon|n, p)$. Note that this inequality still holds in the barrier sense at points where $\cos d$ is not differentiable. Now use the generalized maximum principle (see Theorem 7.1) on $\cos d - f$ for some fixed $p > n/2$ to conclude that

$$\begin{aligned}
f - \cos d \geq & \left(\frac{1}{\mathrm{vol} M} \int (f - \cos d) \right) \\
& - C (n, p) \left(\|n (f - \cos d)\|_p + \left\| \sum_{i=1}^{k} \alpha_i (\lambda_i - n) f_i \right\|_p \right) \\
\geq & -\tau (\varepsilon|n) - \tau (\varepsilon|n, p) \\
= & -\tau (\varepsilon|n) .
\end{aligned}$$

Using this estimate in the above inequality for the Laplacian then yields an inequality of the form

$$\Delta (f - \cos d) \leq \tau (\varepsilon|n) .$$

Since we also know that $f - \cos d$ attains values close to zero (from $\frac{1}{\mathrm{vol} M} \int (f - \cos d)$ being small) and also that the gradient of $f - \cos d$ is bounded by $2 + \tau (\varepsilon|n)$, it follows from the Abresch-Gromoll maximum principle (see Theorem 7.2) that

$$f - \cos d \leq \tau (\varepsilon|n) .$$

Thus showing that

$$|f - \cos d| \leq \tau (\varepsilon|n) .$$

□

With this lemma behind us we can now prove the above theorem.

Proof of Theorem 4.1 and half of Theorem 1.1. Suppose that $\mathrm{rad} M \geq \pi - \delta(\varepsilon, n)$ so that the above lemma works for all distance functions. Then consider the map

$$\Psi_k(x_0) = (\alpha_1, ..., \alpha_k)$$

which takes x_0 to the Fourier coefficients for $\cos d$ corresponding to $i = 1, ..., k$. From the above (see Lemma 4.2) we know that the image of this map is close to the unit sphere in $\mathbb{R}^k$. More precisely

$$\left| |\Psi_k|^2 - 1 \right| \leq \tau(\varepsilon|n).$$

Moreover,

$$\begin{aligned} |\Psi_k(x_0) - \Phi_k(x_0)|^2 &= \sum_{i=1}^{k} |\alpha_i - f_i(x_0)|^2 \\ &= \sum_{i=1}^{k} |\alpha_i|^2 + \sum_{i=1}^{k} |f_i(x_0)|^2 - 2\sum_{i=1}^{k} \alpha_i f_i(x_0) \\ &\leq 2 + \tau(\varepsilon|n) - 2f(x_0) \\ &\leq 2 + \tau(\varepsilon|n) - 2\cos 0 + \tau(\varepsilon|n) \\ &\leq \tau(\varepsilon|n). \end{aligned}$$

Thus Φ_k maps not only almost into the unit ball as in Lemma 3.3 but close to the unit sphere. In other words,

$$\left| |\Phi_k|^2 - 1 \right| \leq \tau(\varepsilon|n).$$

This, however, implies that

$$\begin{aligned} 0 &= \left| \frac{k}{n+1} - \frac{1}{\mathrm{vol} M} \int \sum_{i=1}^{k} f_i^2 \right| \\ &= \left| \frac{k}{n+1} - \frac{1}{\mathrm{vol} M} \int |\Phi_k|^2 \right| \\ &\geq \left| \frac{k}{n+1} - 1 \right| - \tau(\varepsilon|n). \end{aligned}$$

Thus, in fact, $k = n + 1$, provided the radius of the manifold is close to π. In other words, we have the maximal number of eigenvalues close to n. □

5. Small eigenvalues

We shall now show that if we have $n + 1$ eigenvalues which are almost minimal, then the radius is large. Precisely we claim

Theorem 5.1. *Let M be a closed Riemannian n-manifold with* $\mathrm{Ric}M \geq n-1$. *If* $\lambda_{n+1}(M) \leq n+\varepsilon$, *then* $\mathrm{rad}M \geq \pi - \tau(\varepsilon|n)$.

First we show that Φ_{n+1} maps close to the unit sphere. Throughout we assume $\varepsilon > 0$ is fixed and that $\lambda_{n+1}(M) \leq n+\varepsilon$.

Lemma 5.2.

$$\left||\Phi_{n+1}|^2 - 1\right| \leq \tau(\varepsilon|n).$$

Proof. First observe that

$$\frac{1}{\mathrm{vol}M}\int_M |\Phi_{n+1}|^2 = \frac{1}{\mathrm{vol}M}\int_M \sum_{i=1}^{n+1} f_i^2 = 1.$$

Moreover, we know from Lemma 3.3 that

$$|\Phi_{n+1}|^2 = \sum_{i=1}^{n+1} f_i^2 \leq 1 + \tau(\varepsilon|n).$$

Therefore, it follows that

$$\left\||\Phi_{n+1}|^2 - 1\right\|_p \leq \tau(\varepsilon|n, p).$$

Now compute the Laplacian of $|\Phi_{n+1}|^2$

$$\begin{aligned}
\Delta \sum_{i=1}^{n+1} f_i^2 &= \sum_{i=1}^{n+1}\left(-2\lambda_i f_i^2 + 2|\nabla f_i|^2\right) \\
&= \sum_{i=1}^{n+1}\left(-2(n+1)f_i^2 + 2(n-\lambda_i)f_i^2 + 2\left(f_i^2 + |\nabla f_i|^2\right)\right) \\
&= \sum_{i=1}^{n+1}\left(2 - 2(n+1)f_i^2 + 2(n-\lambda_i)f_i^2 + 2\left(f_i^2 + |\nabla f_i|^2 - 1\right)\right) \\
&= 2(n+1)\left(1 - \sum_{i=1}^{n+1} f_i^2\right) \\
&\quad + \sum_{i=1}^{n+1} 2(n-\lambda_i)f_i^2 + \sum_{i=1}^{n+1} 2\left(f_i^2 + |\nabla f_i|^2 - 1\right).
\end{aligned}$$

Here each term is small in L^p thus we must have from the generalized maximum principle (see Theorem 7.1) used on both $|\Phi_{n+1}|^2$ and $-|\Phi_{n+1}|^2$ that

$$\left|\sum_{i=1}^{n+1} f_i^2 - 1\right| \leq \tau(\varepsilon|n).$$

□

With this we can now prove the theorem.

Proof of Theorem 5.1 and the other half of Theorem 1.1. Using that

$$\left|1-\sum_{i=1}^{n+1} f_i^2\right| \leq \tau\left(\varepsilon|n\right)$$

we can for each fixed x_0 choose

$$\alpha_i = f_i\left(x_0\right),$$

then $f = \sum \alpha_i f_i$ has almost maximal value at x_0. By decreasing these coefficients slightly we can assume that $|f|^2 + |\nabla f|^2 \leq 1$ and $f\left(x_0\right) \geq 1-\tau\left(\varepsilon|n\right)$. We now claim that f is close to $\cos d\left(x, x_0\right)$. First observe that $\cos^{-1} f$ is well-defined and has $\left|\nabla \cos^{-1} f\right| \leq 1$. Moreover, $0 \leq \cos^{-1} f\left(x_0\right) \leq \tau\left(\varepsilon|n\right)$. Thus it follows that

$$\begin{aligned}\cos^{-1} f\left(x\right) &\leq \cos^{-1} f\left(x_0\right) + d\left(x, x_0\right)\\ &\leq \tau\left(\varepsilon|n\right) + d\left(x, x_0\right).\end{aligned}$$

Inverting this yields

$$f\left(x\right) \geq \cos d\left(x, x_0\right) - \tau\left(\varepsilon|n\right).$$

It then follows as in Lemma 4.3 that

$$\Delta\left(f-\cos d\right) \leq \tau\left(\varepsilon|n\right)$$

The Abresch-Gromoll maximum principle (see Theorem 7.2) can then be used as in the proof of Theorem 4.1 (here we use that $f-\cos d$ is small at x_0 not that the average is small as we don't know that this is true) to conclude that

$$\left|f-\cos d\right| \leq \tau\left(\varepsilon|n\right).$$

Given that $\int f = 0$ it must follow that $\left|\frac{1}{\mathrm{vol} M}\int \cos d\right| \leq \tau\left(\varepsilon|n\right)$. However, relative volume comparison shows that this can only happen if, in fact, $d\left(\cdot, x_0\right)$ is close to π somewhere. Since x_0 was arbitrary this means that the radius of the manifold is close to π. □

6. Convergence

In this section we shall reprove Colding's results from [8] and [9] with different techniques. We shall assume that our Riemannian manifold has radius close to π and that λ_{n+1} is close to n. The idea is to construct a degree ±1 map onto the unit sphere which is a Gromov-Hausdorff approximation.

Using that we know $|\Phi_{n+1}| \approx 1$ let us redefine this map so that it actually goes into the unit sphere

$$\Phi(x) = \frac{\Phi_{n+1}(x)}{|\Phi_{n+1}(x)|}.$$

Our first claim is that this map gives a Gromov-Hausdorff approximation to a subset of the unit sphere.

Lemma 6.1.

$$|\Phi(x) \cdot \Phi(x_0) - \cos d(x, x_0)| \le \tau(\varepsilon|n),$$

$$||d\Phi - d\Phi_{n+1}|| \le \tau(\varepsilon|n),$$

$$||d\Phi|| \le 1 + \tau(\varepsilon|n),$$

Proof. The first condition is obvious from the above since $\Phi_{n+1}(x) \cdot \Phi_{n+1}(x_0) = \sum_{i=1}^{n+1} f_i(x) f_i(x_0)$ and $|\Phi_{n+1}| \approx 1$. The second condition comes about as follows. Since $\sum f_i^2 = 1 + \tau(\varepsilon|n)$ we have from the inequality (see Lemma 3.2)

$$\left|\sum \alpha_i f_i(x)\right|^2 + \left|\sum \alpha_i \nabla f_i(x)\right|^2 \le 1 + \tau(\varepsilon|n)$$

that

$$\left|\sum f_i(x) f_i(x)\right|^2 + \left|\sum f_i(x) \nabla f_i(x)\right|^2 \le 1 + \tau(\varepsilon|n).$$

Thus

$$\left|\sum f_i \nabla f_i\right|^2 \le \tau(\varepsilon|n)$$

or in other words

$$\|\Phi_{n+1} \cdot d\Phi_{n+1}\|^2 \le \tau(\varepsilon|n).$$

The chain rule now tells us that

$$d\Phi = |\Phi_{n+1}|^{-1} d\Phi_{n+1} - |\Phi_{n+1}|^{-3/2} \Phi_{n+1}(\Phi_{n+1} \cdot d\Phi_{n+1}).$$

So using $|\Phi_{n+1}| \approx 1$ we get the desired estimate for $\|d\Phi - d\Phi_{n+1}\|$. This together with the fact that $\|d\Phi_{n+1}\| \le 1 + \tau(\varepsilon|n)$ (see Lemma 3.3) then implies $\|d\Phi\| \le 1 + \tau(\varepsilon|n)$. □

It now remains to show that M has almost maximal volume and maps onto the unit sphere. It suffices to show that the image of Φ is all of the sphere as we have already shown that $\|d\Phi\| \le 1 + \tau(\varepsilon|n)$. In fact, we shall show that Φ has degree ± 1, this will certainly show that Φ is onto.

Lemma 6.2. $\Phi : M \to S^n$ *has degree* ± 1.

Proof. For simplicity let us first assume that M is oriented. Then the degree can be computed as

$$\deg(\Phi) = \frac{1}{\operatorname{vol} M} \int_M \det d\Phi.$$

Thus, we need only show that $\frac{1}{\operatorname{vol} M} \int_M \det d\Phi$ is within distance $2/3$ of ± 1.

Observe that the Euclidean norm of $d\Phi_{n+1}$ satisfies

$$\sum_{i=1}^{n+1} |\nabla f_i|^2 = |d\Phi_{n+1}|^2 \leq n \, \|d\Phi_{n+1}\|^2 \leq n + \tau(\varepsilon|n).$$

However, we also have that

$$\begin{aligned}\frac{1}{\operatorname{vol} M} \int_M \sum_{i=1}^{n+1} |\nabla f_i|^2 &= \sum_{i=1}^{n+1} \frac{\lambda_i}{n+1} \\ &= \sum_{i=1}^{n+1} \frac{n}{n+1} + \tau(\varepsilon|n) \\ &= n + \tau(\varepsilon|n).\end{aligned}$$

Thus it must follow that

$$\left\| |d\Phi_{n+1}|^2 - n \right\|_1 \leq \tau(\varepsilon|n).$$

Now divide M into two complimentary parts

$$A_\delta = \left\{ x \in M : \left| |d\Phi_{n+1}|^2 - n \right| \leq \delta \right\}$$

and its complement $M - A_\delta$. For fixed δ we shall assume that ε is so small that all of the above quantities $\tau(\varepsilon|n) < \delta$. We can now estimate the volume of A_δ as follows.

$$\begin{aligned}\frac{\operatorname{vol}(M - A_\delta)}{\operatorname{vol} M} \cdot \delta &\leq \frac{1}{\operatorname{vol} M} \int_{M - A_\delta} \left| |d\Phi_{n+1}|^2 - n \right| \\ &\leq \left\| |d\Phi_{n+1}|^2 - n \right\|_1 \\ &= \tau(\varepsilon|n)\end{aligned}$$

Hence

$$\frac{\operatorname{vol}(M - A_\delta)}{\operatorname{vol} M} = \tau(\varepsilon|n, \delta).$$

Thus the set A_δ has relatively large measure, moreover, on this set we have

$$n - \delta \leq |d\Phi_{n+1}|^2 \leq n \, \|d\Phi_{n+1}\|^2 \leq n + \tau(\varepsilon|n).$$

From this we conclude that

$$\left|\det\left(\left(d\Phi_{n+1}\right)^*\left(d\Phi_{n+1}\right)\right)-1\right|\leq\tau\left(\delta|n\right)\text{ on }A_\delta.$$

Since we have

$$\|d\Phi_{n+1}-d\Phi\|\leq\tau\left(\varepsilon|n\right),$$

this implies that

$$\left||\det d\Phi|^2-1\right|\leq\tau\left(\delta|n\right)\text{ on }A_\delta.$$

We can suppose that the orientation on M is chosen so that $\det d\Phi>0$ on A_δ. As we certainly have that $|\det d\Phi|^2$ is bounded by $1+\tau\left(\varepsilon|n\right)$ on M it follows that

$$\begin{aligned}&\left|\left(\frac{1}{\operatorname{vol}M}\int_M\det d\Phi\right)-1\right|\\&\qquad\leq\frac{1}{\operatorname{vol}M}\int_{A_\delta}|\det d\Phi-1|+\frac{1}{\operatorname{vol}M}\int_{M-A_\delta}|\det d\Phi-1|\\&\qquad\leq\frac{1}{\operatorname{vol}M}\int_{A_\delta}|\det d\Phi-1|+\frac{\operatorname{vol}\left(M-A_\delta\right)}{\operatorname{vol}M}\left(2+\tau\left(\varepsilon|n\right)\right)\\&\qquad\leq\tau\left(\delta|n\right)+\tau\left(\varepsilon|n,\delta\right).\end{aligned}$$

Thus by first fixing δ so that $\tau\left(\delta|n\right)<1/3$ and then ε so that $\tau\left(\varepsilon|n,\delta\right)<1/3$ as well we must have

$$\left|\left(\frac{1}{\operatorname{vol}M}\int_M\det d\Phi\right)-1\right|<2/3.$$

But this implies that Φ has degree 1.

In case M isn't orientable, its orientation covering will obviously still satisfy the radius and eigenvalue conditions. One can then proceed as above to establish the result for the orientation cover and then show that this gives a contradiction for the nonorientable manifold.

Another, more direct, method is to use the above proof to show that some regular value has only one preimage. Since the map is a Gromov-Hausdorff approximation points in the preimage are distance $\tau\left(\varepsilon|n\right)$ apart form each other. Now suppose that A_δ contains a metric ball B of size $\tau\left(\delta|n\right)$. Then Φ restricted to B has an explicit upper bounds on the order $1+\tau\left(\delta|n\right)$ for both $\|d\Phi\|$ and $\left\|(d\Phi)^{-1}\right\|$. The inverse function theorem then tells us that the preimage of the center of B has only one point in the preimage contained in a slightly smaller ball with the same center which is also still of size $\tau\left(\delta|n\right)$. On the other hand, for small enough ε all preimage points must be in this ball. Thus we have exhibited a regular point with just one preimage point. This means, even in the nonorientable case, that the $\mathbb{Z}_2$ degree is 1 and hence the map is onto. It still remains to be seen that A_δ contains the desired metric ball. However, $\|d\Phi\|\leq 1-\delta$ on $M-A_\delta$. Thus

$S^n - \Phi(M - A_\delta)$ must contain a metric ball of size $\tau(\delta|n)$. Using that Φ is a Gromov-Hausdorff approximation then shows that A_δ contains a ball of size $\tau(\delta|n)$ as well (note that we are now using the assertion that Φ has image which is $\tau(\varepsilon|n)$ dense in M). □

7. Some important inequalities

For the convenience of the reader and for purposes of generalizing our results to the setting of integral curvature we list here some analytical facts in a more general setting. Except for minor modifications these results are not original to this article. Let M be a Riemannian n-manifold. Functions on M satisfy a Sobolev inequality of the form

$$\left\| f - \frac{1}{\mathrm{vol} M} \int f \right\|_{\frac{2n}{n-2}} \leq C \|\nabla f\|_2 .$$

Gallot has shown that there is a uniform bound for C for manifolds with $\mathrm{Ric} \geq (n-1)\kappa$ and $\mathrm{diam} M \leq D$ (see [13] and references therein). More generally it was shown in [20] that there are unifom bounds for C provided $\bar{k}(p,\kappa)$ is small and $\mathrm{diam} M \leq D$, here $\bar{k}(p,\kappa)$ measures the amount of Ricci curvature that lies below $(n-1)k$. For the purposes of our applications a slightly weaker Sobolev constant actually suffices and Gallot in [12] showed that that Sobolev constant can be bounded provided $\bar{k}(p,\kappa)$ is small and $\mathrm{diam} M \leq D$.

The first maximum principle we need is a standard tool in PDE theory and is established using Moser iteration (see e.g., [14, Chapter 8].)

Theorem 7.1. *Let u be a function on M such that $\Delta u \geq -f$, where f is nonnegative, then*

$$u(x) \leq C^2 \cdot K(n,p) \cdot \|f\|_p + \frac{1}{\mathrm{vol} M} \int_M u,$$

where K is a constant that only depends on n and $p > n/2$.

The other maximum principle we need is a slight generalization of the Abresch-Gromoll estimate. In [19, Theorem 4.2] this was established for excess functions. However, it turns out that we need it here in its most general form. Let $G(r) : [0,R] \to [0,\infty]$ be the unique function satisfying

$$\begin{aligned} G(r) &> 0 \quad \text{for } 0 < r < D, \\ G'(r) &< 0 \quad \text{for } 0 < r < D, \\ G(R) &= 0, \\ G'' + \frac{n-1}{r} G' &= 1. \end{aligned}$$

Therefore, if d is a distance function on $\mathbb{R}^n$ we have that $\Delta (G \circ d) = 1$. Moreover, if d is a distance function on a closed manifold M where $\bar{k} (p) \leq \varepsilon$ is small, then we have that $\Delta (G \circ d) \geq 1 - \psi G'$, where $||\psi||_{2p} \leq O (\varepsilon | n, p)$. The result we need is listed below and is proved as in [19, Theorem 4.2].

Theorem 7.2. *Let* $\varepsilon, a > 0$ *and* $p > n/2$ *be given. There exist* $b = b (n, p, \varepsilon, a) > 0$ *such that any function* $u : B (x_0, R) \to [0, \infty]$ *on a complete manifold with* $\bar{k} (p, R) \leq b$ *satisfying* $u (x_0) < bG (R/2)$, $|\nabla u| \leq a$, $\Delta u \leq \phi$, *where* $\|\phi\|_p \leq b$ *will have the property that*

$$\sup_{B(x_0,R)} u \leq \varepsilon.$$

The last result is a gradient estimate for eigenfunctions. The proof of this is identical to the proof of the maximum principle in Theorem 7.1 after an application of the Bochner formula (see e.g., [14, Chapter 8]).

Theorem 7.3. *Let* f *be an eigenfunction with eigenvalue* λ *on* M, *then*

$$|\nabla f (x)| \leq K \left(\bar{k} (p), C, \lambda\right) \cdot \|f\|_2,$$

where $p > n/2$ *and* K *is a constant depending on the variables listed.*

Proof. The Bochner formula for ∇f is

$$\Delta \frac{1}{2} |\nabla f|^2 = |\mathrm{Hess} f|^2 - \lambda |\nabla f|^2 + \mathrm{Ric} (\nabla f, \nabla f).$$

Using the notation $\rho = |\min \{0, \mathrm{Ric}_-\}|$, where Ric_- denotes the lowest eigenvalue for the Ricci tensor, and Kato's inequality we get

$$\Delta |\nabla f| \geq - (\lambda + \rho) |\nabla f|.$$

Iteration now yields a bound of the form

$$|\nabla f (x)| \leq K \left(\bar{k} (p), C, \lambda\right) \cdot \|\nabla f\|_2.$$

Since

$$\|\nabla f\|_2 = \sqrt{\lambda} \|f\|_2$$

we obtain the desired estimate. □

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