The Journal of Geometric Analysis
Volume 1, Number 4, 1991

Small Excess and Ricci Curvature

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Abstract. It is proved that a Riemannian n-manifold with Ricci curvature $\geq (n-1)$ and a lower injectivity radius bound is a sphere provided the diameter is sufficiently close to π.

1. Introduction

A complete Riemannian n-manifold M with Ricci curvature $\operatorname{ric}(M) \geq (n-1)$ has diameter $\operatorname{diam}(M) \leq \pi$ (see [M]) and $\operatorname{diam}(M) = \pi$ iff M is isometric to the unit n-sphere (Cheng's maximal diameter theorem, see [C]). It is therefore natural to ask whether there is some kind of diameter sphere theorem, like there is for sectional curvature (see [GrS]).

However, Anderson and Otsu have come up with some counterexamples.

Examples (see [A], [0]). For $n \geq 4$ there are closed Riemannian n-manifolds M with, say, $H_2(M) \neq 0$, which carry metrics g_ε such that $\operatorname{ric}(M, g_\varepsilon) \geq (n-1)$, $\operatorname{diam}(M, g_\varepsilon) \geq \pi - \varepsilon$, and $\liminf_{\varepsilon \to 0} \operatorname{vol}\,(M, g_\varepsilon) > 0$.

Thus one must have some additional assumptions in play before one can hope to prove any sphere theorems. There have been several such results (see [I], [S], [W1], [E]), all of which were generalized in [GP]. In all these papers the authors assume that there is a lower bound for the sectional curvature and then use critical point theory as in [GrS]. It was also shown in [GP] that this lower sectional curvature bound virtually makes away with the Ricci curvature condition as

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Math Subject Classification: 53C20.
Key Words and Phrases: Excess, Ricci curvature, sphere theorem.
The author was partially supported by the NSF and the Alfred P. Sloan Foundation.

ISSN 1050-6926

long as one has small excess. The excess $\mathrm{exc}(M)$ of a metric space is defined as

$$\mathrm{exc}(M) = \inf_{p,q\in M} \sup_{x\in M} (d(p,x) + d(x,q) - d(p,q)).$$

The importance of the diameter pinching condition — $\mathrm{ric}(M) \geq (n-1)$, $\mathrm{diam}(M) \geq \pi - \varepsilon$ — then lies in the fact that such manifolds satisfy $\mathrm{exc}(M) \leq \varphi(n,\varepsilon)$, where $\varphi(n,\varepsilon) \to 0$ as $\varepsilon \to 0$.

A manifold with excess zero must be a twisted sphere (i.e., the union of two disks). Under suitable regularity conditions one would therefore expect that spaces with small excess must be topologically equivalent to a sphere (see [GP] for some counterexamples). The purpose of this note is to show:

Theorem. *Let $n \geq 2$ be an integer and $i_0, k, V > 0$. There is an $\varepsilon = \varepsilon(n, i_0, k, V) > 0$ such that any closed connected Riemannian n-manifold M with* $\mathrm{ric}(M) \geq -k^2(n-1)$, $\mathrm{inj}(M) \geq i_0$, $\mathrm{vol}(M) \leq V$, *and* $\mathrm{exc}(M) \leq \varepsilon$ *is a twisted sphere.*

Remark. The upper volume bound is equivalent to an upper diameter bound given the Ricci curvature and injectivity radius conditions. ☐

The theorem immediately implies:

Corollary. *Given an integer $n \geq 2$ and $i_0 > 0$ there is an $\varepsilon = \varepsilon(n, i_0) > 0$ such that any Riemannian n-manifold M with* $\mathrm{ric}(M) \geq (n-1)$, $\mathrm{inj}(M) \geq i_0$, *and* $\mathrm{diam}(M) \geq \pi - \varepsilon$ *is a twisted sphere.*

Both the theorem and the corollary seem pretty optimal for $n \geq 4$ given the examples of Otsu and Anderson. For $n = 3$, however, Wu has shown in [W2] that one only needs to assume $\mathrm{vol}M \geq \nu > 0$ rather than $\mathrm{inj}(M) \geq i_0$. One might speculate whether or not the Ricci curvature assumption in the theorem is necessary. However, on the 2-torus we have:

Counterexample. For the construction we need one radially symmetric disc D that has totally geodesic boundary of length 2π and that is flat near the boundary, and also two identical flat cylinders C, where the base circle has length π. Now choose two antipodal points on ∂D, thus dividing the boundary into two arcs of equal length. Then identify each of these arcs to the top of each cylinder. This yields a surface $\tilde{C}$ that is a cylinder, and a C^∞ Riemannian manifold everywhere except at one point. This point is where the two antipodal points on ∂D get identified with themselves and to points on the boundary of the cylinders. The double $\tilde{C} \cup_{\partial \tilde{C}} \tilde{C}$ of this space is a metric space of excess zero. To see this just let p and q be the centers of the disks in the two copies of $\tilde{C}$, then clearly

$$d(p,x)+d(x,q)=d(p,q) \quad \text{for all } x.$$

One can smooth this space by making the two singular points saddle points and hence only introduce negative curvature. One can then easily get example of metrics on the 2-torus where the excess goes to zero, while the diameter, volume, and injectivity radius stay virtually constant.

The counterexample was developed in discussions with K. Grove. The author would also like to thank R. Greene for many helpful conversations on this paper.

2. Proof of the Theorem

It suffices to show that if M_i is a sequence of Riemannian n-manifolds satisfying $\operatorname{ric} M_i \geq -\ell^2(n-1)$, $\operatorname{inj}(M_i) \geq i_0$, $\operatorname{vol} M_i \leq V$, and $\operatorname{exc}(M_i) \longrightarrow 0$, then M_i is a twisted sphere for large i. Using the results obtained in [AC], we can assume that all the M_i's are $C^{1,\alpha}$-diffeomorphic to a fixed manifold M and that the corresponding pullback metrics g_i on M converge in the C^α, $\alpha < 1$ topology to a C^α Riemannian metric g on M. In [AC] it is furthermore proved that the square of the distance functions $d_i^2 : M \times M \longrightarrow \mathbb{R}$ for the g_i-metrics have uniform $C^{1,\alpha}$ bounds, as long as $d_i < i_0$. We can therefore also assume that d_i^2 converges in the $C^{1,\alpha}$ topology to the distance function d^2 for g, on some neighborhood of the diagonal in $M \times M$. Thus $\nabla_2 d_i$ (= gradient with respect to second variable) converges to $\nabla_2 d$, whenever this makes sense.

In order to proceed we need the following proposition. The notation is as above except we do not need to assume that $\operatorname{exc}(M, g_i) \longrightarrow 0$.

Proposition. *The metric space (M, g) has the following properties:*

(a) Every two points can be joined by a C^1 unit speed segment (= curve whose length equals the distance between its endpoint).

(b) If $c : [a,b] \longrightarrow M$ is a piecewise C^1 curve such that the length of c equals $d(c(a), c(b))$ then c is C^1.

(c) If c_i is a sequence of C^1 unit speed segments, then some subsequence converges to a C^1 unit speed segment in the C^1 topology.

Proof. (a) Fix $p, q \in M$. For each i there is a C^1 geodesic segment $c_i[0,1] \longrightarrow M$, $c_i(0) = p$, $c_i(1) = q$ for the g_i-metric. The Arzela–Ascoli Theorem then implies that some subsequence c_{i_j} converges to a curve $c : [0,1] \longrightarrow M$ such that the length of c equals $d(p,q)$. Now since the c_i's are geodesic segments they must satisfy $\dot{c}_i(t) = \nabla_2 d_i(c_i(s), c_i(t))$, for s, t sufficiently close. Thus $\dot{c}_{i_j}$ must converge uniformly to a continuous vectorfield X along c. Using the fundamental theorem of calculus in some coordinate system then yields

$$c_i(t) = c_i(s) + \int_s^t \nabla_2 d_i(c_i(s), c_i(b))\, dv.$$

The left-hand side converges to $c(t)$ and the right-hand side to $c(s) + \int_s^t X(v)\, dv$. Hence c is C^1 and $\dot{c} = X$. We can then reparametrize c to have unit speed.

(b) Let $c : [a, b] \longrightarrow M$ be a segment in (M, g) that is C^1 on $[a, t_0]$ and $[t_0, b]$, and parametrized by arclength. Choose $s < t_0 < t$ where s and t are very close. Clearly $t - s = d(c(s), c(t))$. Since c is differentiable at t we then have

$$\begin{aligned} 1 &= g(\nabla_2 d(c(s), c(t)), \dot{c}(t)) \\ &\leq |\nabla_2 d(c(s), c(t)|\, |\dot{c}(t)| = 1 \end{aligned}$$

showing that $\dot{c}(t) = \nabla_2 d(c(s), c(t))$ for all $t \in (t_0, t_0 + \varepsilon)$ for some $\varepsilon > 0$. The same is obviously also true for $t \in (-\varepsilon + t_0, t_0)$. This shows that c must be C^1 also at t_0.

(c) Suppose c_i are C^1 unit speed geodesics in (M, g). The Arzela–Ascoli Theorem implies that c_i (sub) converges to a segment c. Using the fact that $\dot{c}_i(t) = \nabla_2 d(c_i(s), c_i(t))$ and that $\nabla_2 d$ varies continuously, shows, as in (a), that c is a C^1 unit speed geodesic and that if $c_{i_j} \longrightarrow c$ then also $\dot{c}_{i_j} \longrightarrow \dot{c}$. $\square$

We can now return to the proof of our theorem. Since the distance functions d_i converge to d and $\mathrm{exc}(M, g_i) \longrightarrow 0$, we get that $\mathrm{exc}(M) = 0$. Thus we can find $p, q \in M$ such that

$$d(p, x) + d(x, q) = d(p, q) \quad \text{for all} \quad x \in M.$$

From (a) we know that there are C^1 unit speed segments from p to x and from x to q. The juxtaposition of these two segments then yields a piecewise C^1 segment from p to q through x. We can then conclude from (b) that for any $x \in M - \{p, q\}$ there is a C^1 unit speed segment from p to q through x. If there were two such segments c_1 and c_2 with $c_1(t) = c_2(t) = x$, then also $\dot{c}_1(t) = \dot{c}_2(t)$. For otherwise we could construct a piecewise C^1 segment from p to q that was not differentiable at t. By defining $V(x) = \dot{c}_1(t)$ we then get a well-defined unit speed vectorfield on $m - \{p, q\}$. We claim that V is continuous. To see this let c_i be a sequence of C^1 unit speed segments from p to q, such that $x_i = c_i(t_i) \longrightarrow x \in M - \{p, q\}$. Using (c) we get a subsequence c_{i_j} converging to a C^1 unit speed segment c, which goes from p to q and $c(t) = x$ where $t = \lim t_{i_j}$. But then $V(x_{i_j}) = \dot{c}_{i_j}(t_{i_j}) \longrightarrow \dot{c}(t) = V(x)$. This shows that every subsequence of $\{V(x_i)\}$ has a subsequence that converges to $V(x)$. Therefore $V(x_i)$ must converge to $V(x)$.

This vectorfield $V(x)$ is a generalized gradient for $x \longrightarrow d(p, x)$. We can therefore argue as in [GS] and get a C^∞ unit speed vectorfield $V_\varepsilon(x)$ on $M - \{p, q\}$ such that $\sphericalangle_g(V_\varepsilon(x), V(x)) < \varepsilon \ll 1$ for all $x \in M - \{p, q\}$. This implies that the distance function $x \longrightarrow d(p, x)$ must increase at a definite rate along integral curves for $V_\varepsilon(x)$ (see also [GS]). We have then shown that all open metric balls $B(p, r)$ where $0 < r < d(p, q)$ are diffeomorphic. For very small r we know from [AC] that $B(p,r)$ is an open disk, hence all such balls are disks. By symmetry all the balls $B(q,r)$ are also disks. In this way we have exhibited M as the union of two open differentiable (C^1) disks.

Remark. Z. Shen has kindly pointed that in [CH] it is shown that geodesics in a C^α, $\alpha < 1$, Riemannian manifold are $C^{1,\alpha}$. Thus (a) in the proposition is automatically satisfied.

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Received May 30, 1991

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