BOTT'S LECTURE NOTES ON MORSE THEORY AT UCLA

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Abstract. These notes, typed in 2020, are based on lectures given by Raoul Bott on Morse Theory at UCLA in February 2002. However, the contents are reorganized and supplemented, since the original lectures, after almost two decades, are only preserved as handwritten notes that are not very clear. These notes cover the basics of classical Morse theory, its applications to compact Lie groups, which ultimately leads to the proof of the Bott periodicity theorem. Section two also includes a very brief introduction of Witten's alternative approach to Morse theory.

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1. Baby Morse Theory

In this section, we will go through the basics of Morse theory, which Bott calls "Baby Morse Theory". Some of the theorems mentioned will be proved in later sections. Good references include the first chapter of Milnor's *Morse theory* ([1]) as well as sections 1-3 in Bott's earlier lecture notes ([2]).

Definition 1.1. M is a compact n-dimensional manifold and $f : M \to \mathbb{R}$ is smooth. A point, p, is a called a **critical point** if $\frac{\partial f}{\partial x^i} |_p = 0$ in local coordinates. It is called **non-degenerate** if $\det(\frac{\partial^2 f}{\partial x^i \partial x^j} |_p) \neq 0$.

It is clear that non-degenerate critical points are isolated.

Definition 1.2. The **index,** $\lambda(p)$, of a non-degenerate critical point p is the number of negative eigenvalues of $(\frac{\partial^2 f}{\partial x^i \partial x^j} |_p)$, the Hessian of f at p.

Given a smooth function on a manifold M with non-degenerate critical points, the Morse inequalities relate the number of critical points of f to the Betti numbers of M.

Theorem 1.3. *If $M_t(f) = \sum_{p\,\text{critical}} t^{\lambda(p)}$, then we have the Morse inequalities*
$$M_t(f) - P_t(M,K) = (1+t)Q_K(t),$$
where $Q_K(t)$ is a polynomial with non-negative coefficients and
$$P_t(M,K) = \sum_k t^k \dim_K H^k(M,K).$$

Here $P_t(M,K)$ in the above theorem is referred to as the **Poincare series**. The following is another more explicit statement of the Morse inequalities (see [1] and [2]). In this statement, c_i, h_i, and q_i corresponds to the coefficients of $M_t(f)$, $P_t(M,K)$, and $Q_K(t)$, respectively.

Theorem 1.3'. *Let c_i be the number of critical points of index i of f and h_i the i-th Betti number of M. Then there exists a sequence of non-negative integers $q_{-1} = 0, q_0, q_1, ...$ such that*
$$c_i - h_i = q_i + q_{i-1}, \quad i = 0, 1, 2, ...$$
Therefore, there is a sequence of inequalities:
$$\begin{aligned} c_0 &\geq h_0 \\ c_1 - c_0 &\geq h_1 - h_0 \\ c_2 + c_1 - c_0 &\geq h_2 - h_1 + h_0 \\ &\vdots \end{aligned}$$

The Morse inequalities follow from the following assertions.

Theorem A. *Let $M_a = \{p \mid f(p) \leqslant a\}$, $M_b = \{p \mid f(p) \leqslant b\}$, $a < b$. If there are no critical points between a and b, then $M_a \simeq M_b$ are diffeomorphic.*

Theorem B. *If there exists one non-degenerate critical point p in $(a,b]$, then $M_b = M_a \cup e^{\lambda(p)}$.*

These two theorems are also due to Morse and need the following facts:

(1) Morse Lemma: Near a non-degenerate critical point of f, one can introduce a non-degenerate local coordinate such that
$$f = -x_1^2 - x_2^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_n^2.$$
(2) Existence Theorem on ODE: Put a Riemannian structure on M, change df to the gradient vector field ∇f characterized by $df(Y) = \langle \nabla f, Y\rangle$.

Theorem A is immediate if we follow the flow of ∇f. Theorem B can be intuitively understood by looking at the level sets of $f = f(p) - y^2 - x^2$ where $x^2 = x_1^2 + x_2^2 + \cdots + x_\lambda^2$, $y^2 = x_{\lambda+1}^2 + \cdots + x_n^2$, and p is the critical point. (Figure 1)

2. Smale's and Witten's approach to Morse Theory

In this approach, instead of a smooth function f, we consider its gradient vector field ∇f. On a compact manifold M, the function $f : M \to \mathbb{R}$ must have critical points (since max and min always exist). The gradient flow gives us a trajectory from one critical point, say p, to another critical point, say q. Physicists call this trajectory an "instanton" since a particle moving along this trajectory spends a lot

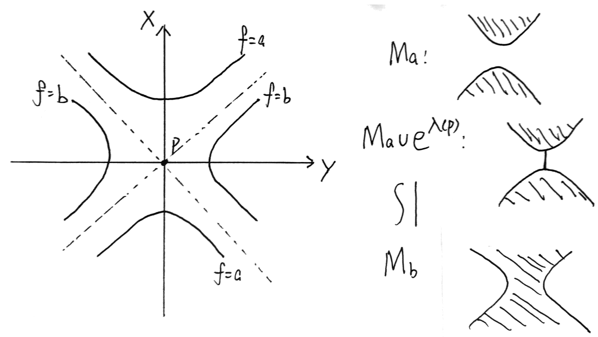

FIGURE 1. L: level sets of f near p; R: Attaching a $\lambda(p)$-cell to M_a

of time near p, whips across from p to q in an instant, and spends a lot of time near q.

At a critical point p, the flows starting at p form a cell W_p^- and those ending at p form a cell W_p^+. An example on the torus can be seen in figure 2.

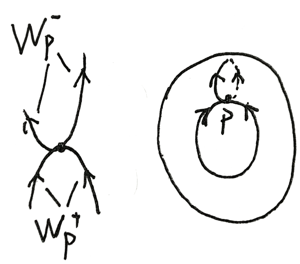

FIGURE 2.

What we would like to do is to show that we can deform the function so that the cell W_p^- has flows running into cell W_q^+ for $\lambda_q = \lambda_p + 1$ throughout the manifold, then we would have a CW-structure on M.

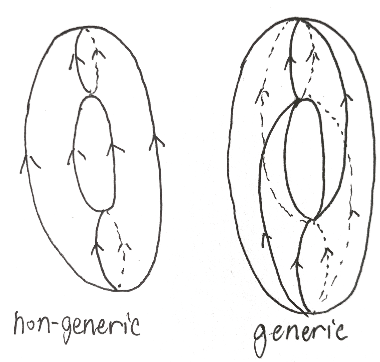

FIGURE 3.

Generically, we have the following. W_q^+, W_p^- intersect as generically as they can if and only if the dimension of $W_q^+ \cap W_p^-$ is $\lambda_q - \lambda_p + 1$. Back to the torus example, the height function does not give us a generic case. We get a generic case by perturbing the function a little bit. (figure 3)

In such a construction, we get a CW-structure on M, thus an algorithm for obtaining the cohomology of M, which also gives us the Morse inequalities.

Proposition 2.1. *We can get the cohomology, $H^*(M)$, of a manifold M as follows:*

(1) Choose a non-degenerate generic Morse function, f, such that the corresponding W_q^+, W_p^- intersect generically whenever $\lambda_q = \lambda_p + 1$.
(2) Construct the chain complex $C^f(M) = \bigoplus_{\text{crit. pt. } p} \mathbb{R}e_p$, where $e_p = W_p^+$ and dim $e_p = \lambda_p$.
(3) The differential operators $d : C_k \to C_{k-1}$ are defined by $de_p = \sum \pm e_q$, where the sum is over all q such that there exists instanton from q to p, i.e., $\lambda_q = \lambda_p + 1$.

Then $H^*(M) = H^*(C^f(M))$.

The above construction is mainly due to Smale. Witten further developed the theory using Hodge Theory.

Hodge Theory

First, we have the **de Rham complex**

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \ .$$

Nowadays, we view $\Omega^q(M)$ as the space of cross sections of exterior power of the cotangent bundle:

$$\Omega^q(M) = \Gamma\left(\bigwedge^q T^*M\right).$$

Since $d^2 = 0$, we get the de Rham cohomology

$$H^*_{\text{dR}}(M) = \frac{\operatorname{Ker} d}{\operatorname{Im} d}.$$

We can see that $H^*_{\text{dR}}(-)$ is a contravariant functor, and $H^*_{\text{dR}}(M)$ is finite dimensional for M compact.

Now, let's put a Riemannian metric (g_{ij}) on an oriented manifold M (i.e. positive definite inner product on T_*M) and define a global inner product:

$$\langle \omega, \eta \rangle = \int_M g(\omega, \eta)\text{vol} = \int_M \omega \wedge \eta.$$

Then we can the take adjoint of the de Rham differentials and get

$$\langle d^*\omega, \eta \rangle = \langle \omega, d\eta \rangle$$

This gives us

$$\Omega^0(M) \underset{d^*}{\overset{d}{\rightleftharpoons}} \Omega^1(M) \underset{d^*}{\overset{d}{\rightleftharpoons}} \Omega^2(M) \underset{d^*}{\overset{d}{\rightleftharpoons}} \cdots \underset{d^*}{\overset{d}{\rightleftharpoons}} \Omega^n(M)$$

Definition 2.2. The Laplacian is $\Delta := dd^* + d^*d$

Example 2.3. $M = \mathbb{R}$, $d : f \mapsto \frac{\partial f}{\partial x} dx$. Since $\int_0^1 (fg)' dx = 0$, we have $\int_0^1 f' g dx = -\int_0^1 f g' dx$. So $d^* : f dx \mapsto -\frac{\partial f}{\partial x}$.

Theorem 2.4 (Hodge)**.** *We have the following decomposition:*

$$\Omega^q(M) = \bigoplus_\lambda \Omega^q_\lambda(M),$$

where $\Omega^q_\lambda(M) = \{q\text{–forms } \phi : \Delta\phi = \lambda\phi\}$*, and* λ*'s range over a discrete set.*

Corollary 2.5. *The total cohomology is the direct sum of cohomologies over* $\lambda' s$*, i.e.* $H^* = \bigoplus H^*_\lambda$*, where* H^*_λ *is the cohomology of the chain complex*

$$\Omega^0_\lambda(M) \xrightarrow{d_\lambda} \Omega^1_\lambda(M) \xrightarrow{d_\lambda} \Omega^2_\lambda(M) \xrightarrow{d_\lambda} \cdots \xrightarrow{d_\lambda} \Omega^n_\lambda(M) \; .$$

Proof. Since $\Delta = dd^* + d^*d$ we have $\Delta d = d\Delta = dd^*d$, so d preserves eigenspaces. □

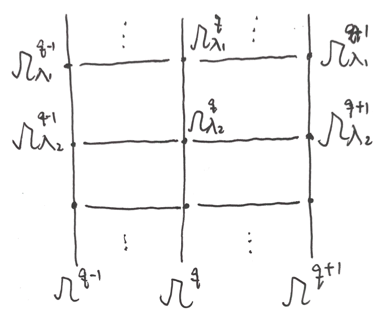

FIGURE 4. Hodge decomposition

Corollary 2.6. $H^*(M) \simeq \mathcal{H}^*(M) = \Omega^*_0$ *where* $\mathcal{H}^*$ *denotes the subcomplex of* Ω^* *of harmonic forms.*

Proof. By Corollary 2.5, we only need to compute H^*_λ for each λ. For $\lambda > 0$,

$$\Delta_\lambda \varphi = \lambda\varphi \Rightarrow d\frac{d^*}{\lambda} + \frac{d^*}{\lambda} d = 1.$$

If φ is closed, i.e. $d\varphi = 0$, then

$$d(\frac{d^*}{\lambda}\varphi) = \varphi$$

so φ is also exact, thus $H^*_\lambda = 0$ for all $\lambda > 0$. □

Having $(\Omega^*(M), d)$, the de Rham complex, we now deform the differential to $d_s = e^{-sf} d e^{sf}$, which doesn't change the cohomology, i.e. $H^*_s(M) = H^*(M)$. Applying Hodge theory, we get the corresponding Laplacian Δ_s.

Example 2.7. Consider $M = S^1$. As in example 2.3, we have

$$d\phi = \phi' \qquad d^*\phi = -\phi'$$

then

$$d_s\phi = e^{-sf} d e^{sf} \phi = e^{-sf}(sf' e^{sf} + e^{sf}\phi') = \partial + sf'.$$

So $d_s = \partial + sf'$ and similarly, $d_s^* = -\partial + sf'$. Consider the function $f = c + \frac{x^2}{2}$ at a minimum (which is a critical point), we have

$$\Delta_s^0 = d_s^* \circ d_s = (\partial + sf')(-\partial + sf') = -\partial^2 + s^2x^2 - sf''$$
$$\Delta_s^1 = d_s \circ d_s^* = (-\partial + sf')(\partial + sf') = -\partial^2 + s^2x^2 + sf''$$

where $f'' = 1$. To physicists, $H_s = -\partial^2 + s^2x^2$ is the quantum mechanical harmonic oscillator, which has a known spectrum: for $s > 0$, $\mathrm{Spec}(H_s) = s, 3s, 5s, ...$, thus

$$\mathrm{Spec}(\Delta_s^0) = 0, 2s, 4s, ... \qquad \mathrm{Spec}(\Delta_s^1) = 2s, 4s, 6s, ...$$

Similarly, at a maximum of the function f, we have

$$\mathrm{Spec}(\Delta_s^0) = 2s, 4s, 6s, ... \qquad \mathrm{Spec}(\Delta_s^1) = 0, 2s, 4s, ...$$

Recall that the cohomology are the harmonic parts, we can see that the 0-th cohomology is induced by functions "concentrated" near the minima and the 1-st cohomology is induced by 1-forms "concentrated" near the maxima.

The above situation is not a coincidence, and in fact, we have the eigenspace of Δ_s^k induced by k-forms "concentrated" near critical points of index k. In fact, for s very large, the complex $(\Omega_s^{a*}(M), d_s)$, which is $(\Omega_s^*(M), d_s)$ restricted to eigenspaces with eigenvalue less than some $a > 0$, is exactly the complex constructed in proposition 2.1 with basis in terms of critical points. Then, by $H^*(M) = H_s^{a*}(M)$, we have a similar result to proposition 2.1 which gives us a ways to compute $H^*(M)$ in terms of critical points, so now we can prove the Morse inequalities.

Proof. We show Witten's proof of Morse inequalities. We have the two exact sequences:

$$0 \longrightarrow Z \longrightarrow C \xrightarrow{d} B \longrightarrow 0 \qquad 0 \longrightarrow B \longrightarrow Z \longrightarrow H \longrightarrow 0$$

where Z is the cell complex, B is the boundary, C is the cycle and H is the cohomology. Counting in each dimension, we have for their corresponding polynomials

$$Z_t + \frac{B_t}{t} = C_t, \qquad Z_t = B_t + H_t.$$

Subtracting, we have

$$C_t - H_t = \frac{B_t}{t} + B_t = \frac{B_t}{t}(1+t).$$

with positive integer coefficients, which implies Morse inequalities. $\square$

A richer explanation of this section can be found in Bott's other lecture notes which this document also uses as a reference. ([3])

3. More on Baby Morse Theory

We now return to the classical approach. In the 1940's, people did Morse Theory with the Eilenberg-Steenrod axioms:

(0) H^* is contravariant on spaces.
(1) $H^*(p) = \mathbb{R}$ for a point p.
(2) For $X = U \cup V$, we have the Mayer-Vietoris sequence

$$... \to H^*(X) \to H^*(U) \oplus H^*(V) \to H^*(U \cap V) \to H^{*+1}(X) \to ...$$

(3) Homotopic maps induce the same map in cohomology.

When we attach e^λ to X, we have

$$X \cap e^\lambda \simeq S^{\lambda-1} \to X \sqcup e^\lambda \simeq X \sqcup p \to X \cup e^\lambda.$$

Then we have the Mayer-Vietoris Sequence in dimension λ:

$$0 \to H^{\lambda-1}(X \cup e^\lambda) \to H^{\lambda-1}(X) \oplus H^{\lambda-1}(p) \to$$

$$H^{\lambda-1}(S^{\lambda-1}) \to H^\lambda(X \cup e^\lambda) \to H^\lambda(X) \oplus H^\lambda(p) \to 0,$$

which simplifies to

$$0 \to H^{\lambda-1}(X \cup e^\lambda) \to H^{\lambda-1}(X) \xrightarrow{\alpha^*} \mathbb{R} \to H^\lambda(X \cup e^\lambda) \to H^\lambda(X) \to 0.$$

Let ΔH^λ be the change of the Poincare series in degree λ by attaching e^λ to X, then

(1) If α^* is onto, then $H^\lambda(X) = H^\lambda(X \cup e^\lambda)$, $\Delta H^\lambda = 0$, $\Delta H^{\lambda-1} = -t^{\lambda-1}$.
(2) If α^* is zero, then $H^{\lambda-1}(X) = H^{\lambda-1}(X \cup e^\lambda)$, $\Delta H^\lambda = t^\lambda$, $\Delta H^{\lambda-1} = 0$.

Then the change in poincare series: $\Delta P_t = t^\lambda$ or $\Delta P_t = -t^{\lambda-1}$, while $\Delta M_t = t^\lambda$. So $\Delta(M_t - P_t) = 0$ or $\Delta(M_t - P_t) = t^{\lambda-1} + t^\lambda$, both of which have non-negative coefficients. Thus we have proved the Morse inequalities, as we recall:

Theorem 1.3. *For the Morse function $M_t(f) = \sum_p t^{\lambda(p)}$, we have the Morse inequalities*

$$M_t(f) - P_t(M, K) = (1+t)Q_K(t),$$

where $Q_K(t)$ is a polynomial with non-negative coefficients and $P_t(M,K)$ is the Poincare series:

$$P_t(M,K) = \sum_k t^k \dim_K H^k(M,K)$$

By now, we have introduced three approaches to obtain the Morse inequalities. The first by Smale; the second by Witten; and the third by Morse, as given above. Now we give a corollary useful for computing homology.

Corollary 3.1. *Let the notations be the same as those in Theorem 1.3.*

(1) ***Lacunary Principle****: If $M_t(f) = \sum m_\lambda(f) t^\lambda$, and $m_\lambda(f) m_{\lambda+1}(f) = 0$ for all λ, then $M_t(f) = P_t(M)$.*
(2) ***Completion Principle****: If to each critical point p, we can assign a compact submanifold N_p such that p is a non-degenerate maximum for $f|_{N_p}$, then $P_t(M) = M_t(f)$ and the fundamental classes of the $H_*(N_p)$'s form a homology basis of $H_*(M)$ over $\mathbb{Z}/2\mathbb{Z}$. If all N_p's are orientable, then the result also holds over $\mathbb{Z}$.*

Proof. The Lacunary Principle is an immediate consequence of the Morse inequalities. It applies whenever there are no critical points with adjacent indices.

We now prove the Completion Principle.

Set $f(p) = a$. Since p is a non-degenerate (also assumed unique) maximum of $f|_{N_p}$, by the Morse Lemma, we have local coordinates $x = (x_1, ..., x_\lambda)$ of N_p near p, such that

$$f|_{N_p}(x) = a - x_1^2 - ... - x_\lambda^2.$$

We can extend x to coordinates of M near p, $(x_1, ..., x_\lambda, x_{\lambda+1}, ..., x_n)$ such that

$$f(x) = a - x_1^2 - ... - x_\lambda^2 + x_{\lambda+1}^2 + ... + x_n^2.$$

Then by Morse theory, we have

$$(N_p)^a = (N_p)^{a-\epsilon} \cup e^\lambda$$
$$M^a = M^{a-\epsilon} \cup e^\lambda$$

where the boundary of the λ-cell, $\dot{e}^\lambda$ is $\{x_1^2 + ... + x_\lambda^2 = \epsilon, x_{\lambda+1} = ... = x_n = 0\}$. Since $f(p) = a$ is the unique maximum of $f|_{N_p}$, e^λ corresponds to the nontrivial top homology class of $H_*(N_p)$, so $\dot{e}^\lambda$ is homologically trivial in N_p, thus also homologically trivial in M. Therefore it also induces a homology class in $H_\lambda(M)$.

Recall that the CW-complex is built up by attaching one cell at each critical point. When a cell is attached during this process, it immediately induces a nontrivial homology class as shown above. Since $\dot{e}^\lambda$ is homologically trivial for every cell attached later, this homology class remains nontrivial in $H_\lambda(M)$. □

If the conditions of the lacunary principle or the completion principle are satisfied, we see that the Morse inequalities become equalities, namely

$$M_t(f) = P_t(M).$$

In this case, we call f a **perfect Morse function**.

Example 3.2. Find the $\mathbb{Z}/2\mathbb{Z}$ homology of the real projective space $\mathbb{R}P^n$. (Refer to the $\mathbb{C}P^n$ case in Milnor's *Morse Theory*)

We think of $\mathbb{R}P^n$ as $S^n/\sim$, or equivalence classes of $(n+1)$-tuples $(x_0, ..., x_n)$ of real numbers, with $\sum |x_i|^2 = 1$. Denote the equivalence classes of $(x_0, ...x_n)$ as $(x_0 : ... : x_n)$. Then define a Morse function $f : \mathbb{R}P^n \to \mathbb{R}$ by

$$f(x_0 : ... : x_n) = \sum c_j |x_j|^2,$$

where $c_0 < c_1 ... < c_n$ are distinct real constants. The fact that they are distinct guarantees that the function has non-degenerate critical points.

To determine the critical points of f, we first observe that

$$\mathbb{R}P^n = \bigcup U_j$$

where $U_j = \{(x_0 : ... : x_n) \in \mathbb{R}P^n, x_j \neq 0\}$ and on each U_j, we have coordinate functions $x_{j0}, ..., x_{jj-1}, x_{jj+1}, ..., x_{jn} : U_j \to \mathbb{R}$

$$x_{ji} = |x_j| \frac{x_i}{x_j}, \quad x_{ji}^2 = x_i^2$$

that taken together map U_j to the unit ball in $\mathbb{R}^n$. In these coordinates, we can express $f|_{U_j}$ as

$$f = c_j x_j^2 + \sum_{i \neq j} c_i x_i^2 = c_j (1 - \sum_{i \neq j} x_i^2) + \sum_{i \neq j} c_i x_{ji}^2 = c_j + \sum_{i \neq j} (c_i - c_j) x_{ji}^2$$

Obviously, the only critical point occurs when $x_{ji} = 0$ for all i and has index $(n-j)$. So the only critical points of f are $p_0 = (1 : 0 : ... : 0)$, $p_1 = (0 : 1 : 0 : ... : 0)$,..., $p_n = (0 : ... : 0 : 1)$ and the index of p_j is $(n-j)$. This implies that $\mathbb{R}P^n$ has one cell in each dimension from 0 to n, so we need to use the Completion principle.

Define $N_j = \{(x_0 : ... : x_n) \in \mathbb{R}P^n, x_{j+1} = ... = x_n = 0\}$, then $N_n = \mathbb{R}P^n$ and for all $j \neq n$, $N_j \subset U_n$. Then we have

(1) p_n is the non-degenerate maximum of $f|_{N_n} = f$;

(2) p_j is the non-degenerate maximum of

$$f|_{N_j} = f|_{U_n \bigcap N_j} = \left(c_n + \sum_{i \neq n} (c_i - c_n) x_{ni}^2 \right) \Bigg|_{N_j} = c_n + \sum_{i<j} (c_i - c_n) x_{ni}^2.$$

Thus by the Completion principle, p_j corresponds to the fundamental class of $H_*(N_j; \mathbb{Z}/2\mathbb{Z})$, and the fundamental classes form a homology basis of $H_*(\mathbb{R}P^2; \mathbb{Z}/2\mathbb{Z})$, so $H_*(\mathbb{R}P^2; \mathbb{Z}/2\mathbb{Z}) = \bigoplus_j \mathbb{Z}/2\mathbb{Z}_{(j)}$.

Until now, we only applied Morse theory to compute homology. We would also like to know the change of the homotopy groups, π_*, when attaching an λ-cell. To achieve this, we need the following theorem on the homotopy group of spheres.

Theorem 3.3. *$\pi_q(S^n) = 0$, $q < n$, $\pi_q(X) = \pi_q(X \cup e^\lambda)$ for $q < \lambda - 1$.*

However in 1949, we only knew:

- $\pi_q(S^n) = 0$ for $q < n$;
- $\pi_n(S^n) = \mathbb{Z}$;
- $\pi_{n+1}(S^n) = \mathbb{Z}_2$ for $n \geq 2$;
- $\pi_{n+2}(S^n) = \mathbb{Z}_2$ (Whitehead)
- $\pi_3(S^2) = \mathbb{Z}$ (Hopf)

The last bullet point is a result of the famous Hopf fibration $p : S^3 \twoheadrightarrow S^2$. View S^3 as (z_1, z_2) where $|z_1|^2 + |z_2|^2 = 1$ in $\mathbb{C}^2 = \mathbb{R}^4$, S^2 as (z, x) where $|z|^2 + x^2 = 1$ in $\mathbb{C} \times \mathbb{R} = \mathbb{R}^3$. Then p is given by

$$(z_1, z_2) \mapsto (z_1 z_2^*, |z_1|^2 - |z_2|^2).$$

In this fibration, the fibers are circles, and any two of them link. The definition of fiber bundle is in the next section.

4. Homotopy Groups of Compact Lie Groups

In this section, we'll introduce the most important results of applying Morse theory to the homotopy groups of compact Lie groups. The proof will be outlined in later sections. We start by defining a fiber bundle. (Reference: [4])

Definition 4.1. A **fiber bundle** structure on a total space E, with fiber F, base space X, consists a projection map $\pi : E \to X$ such that each point of X has a neighborhood U for which there is a local trivialization $\phi_U : \pi^{-1}(U) \to U \times F$ making the diagram commute

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\varphi_U"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl] \\ & U & \end{tikzcd}$$

As an easy consequence of the "short exact sequence" $F \longrightarrow E \xrightarrow{\pi} X$, we have the long exact sequence:

Theorem 4.2. *We have the following long exact sequence*

$$\to \pi_k(F) \to \pi_k(E) \to \pi_k(X) \xrightarrow{\delta} \pi_{k-1}(F) \to \pi_{k-1}(E) \to \pi_{k-1}(X) \to .$$

where δ is the holonomy map.

Corollary 4.3. *Consider the fiber bundle* $Y \longrightarrow X \times Y \xrightarrow{\pi} X$*, then we have*

$$\pi_k(X \times Y) = \pi_k(X) \oplus \pi_k(Y).$$

To understand δ, let's first look at the following example.

Example 4.4. Let S^1 be the unit sphere in the complex plane. We have a two-sheeted fiber bundle $F \longrightarrow E = S^1 \xrightarrow{\pi} X = S^1$, $\pi(z) = z^2$, F is two points.

Pick an element γ from $\pi_1(X)$, which is a loop around the circle once, with both endpoints p. Lifting it to $\pi_1(E)$, we have a curve with two different endpoints, $p_1^*, p_2^* \in \pi^{-1}(p) = F$. Then $\delta(\gamma) = p_1^* - p_2^*$.

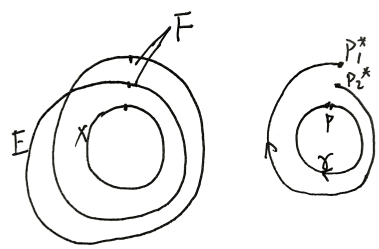

Figure 5. Example 4.4

For higher δ, pick an element $\gamma : S^k \to X$ from $\pi_k(X)$. We can view γ as a map from the closed $k-$disk D^k to X such that the boundary, which is homeomorphic to S^{k-1}, is mapped to a single point, say x_0. Lift γ to $\gamma^* : D \to E$, then $\gamma^*|_{\partial D} : S^{k-1} \to \pi^{-1}(x_0) = F$ is an element of $\pi_{k-1}(F)$. We define as $\delta(\gamma) := \gamma^*|_{\partial D}$.

We now turn our attention to the theory of Lie groups.

Corollary 4.5. *Let G be a Lie group, K a closed subgroup of G, and G/K the space of cosets. Then we get the fiber bundle*

$$\begin{array}{c} G \\ {\scriptstyle K}\big\downarrow \\ G/K \end{array}$$

Example 4.6. In case of the special orthogonal groups, consider the fiber bundle

$$\begin{array}{c} SO(n+1) \\ {\scriptstyle SO(n)}\big\downarrow \\ SO(n+1)/SO(n) = S^n \end{array}$$

It gives us the long exact sequence

$$... \longrightarrow \pi_{k+1}(S^n) \longrightarrow \pi_k(SO(n)) \longrightarrow \pi_k(SO(n+1)) \longrightarrow \pi_k(S^n) \longrightarrow ... \,.$$

So for $k < n-1$, we have

$$0 \longrightarrow \pi_k(SO(n)) \xrightarrow{\sim} \pi_k(SO(n+1)) \longrightarrow 0 \,.$$

Therefore, increasing n for fixed k, $\pi_k(SO(n))$ stabilize as n become large enough. We can then define the k-th homotopy group of the infinite orthogonal group

$$\pi_k(SO) := \pi_k(SO(n)) \qquad \text{for} \qquad n > k+1.$$

Example 4.7. Similarly, in case of special unitary groups, by the fiber bundle

$$\begin{array}{c} SU(n+1) \\ {\scriptstyle SU(n)}\Big\downarrow \\ SU(n+1)/SU(n) = S^{2n+1} \end{array},$$

we can see that $\pi_k(SU(n))$ stabilizes and defines $\pi_k(SU)$.

Example 4.8. In case of compact symplectic groups, we also have the fiber bundle

$$\begin{array}{c} Sp(n+1) \\ {\scriptstyle Sp(n)}\Big\downarrow \\ Sp(n+1)/Sp(n) = S^{4n+1} \end{array}.$$

Then $\pi_k(Sp(n))$ also stabilizes and we can define $\pi_k(SU)$.

Notice that $Sp(n) \subset U(2n)$ is the quaternionic unitary group, or the fixed-point set of the involution $A \mapsto JA^{-1}$ on $SU(2n)$ where $J = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$.

We now introduce the famous Bott Periodicity Theorem. A sketch of the proof will be given in later sections:

Theorem 4.9 (Bott)**.** *The homotopy groups of the classic groups are periodic:*
(1) $\pi_k(U) \cong \pi_{k+2}(U)$
(2) $\pi_k(O) \cong \pi_{k+4}(Sp)$
(3) $\pi_k(Sp) \cong \pi_{k+4}(O)$
By (2) and (3), $\pi_k(O) \cong \pi_{k+8}(O)$ *and* $\pi_k(Sp) \cong \pi_{k+8}(Sp)$.

In general, a sequence of Lie groups $K \subset H \subset G$ gives the fiber bundle

$$\begin{array}{c} G/K \\ {\scriptstyle H/K}\Big\downarrow \\ G/H \end{array}$$

If we consider the sequence $U(m) \subset U(n) \times U(m) \subset U(n+m)$, then we have

$$\begin{array}{c} U(n+m)/U(n) \\ {\scriptstyle U(m)}\Big\downarrow \\ U(n+m)/U(n) \times U(m) \end{array}$$

Also, we have the bundle

$$U(n) \longrightarrow U(n+m) \longrightarrow U(n+m)/U(n),$$

Applying π_k to this bundle, we have

$$\pi_k(U(n+m)/U(n)) = 0 \qquad \text{for } k \ll n$$

as the homotopy groups of the unitary group stabiles. If we apply π_k to the earlier bundle, we have for $k \ll n$:

$$0 \longrightarrow \pi_{k+1}\left(\frac{U(n+m)}{U(n) \times U(m)}\right) \xrightarrow{\sim} \pi_k(U(m)) \longrightarrow \pi_k\left(\frac{U(n+m)}{U(n)}\right) = 0$$

Corollary 4.10. *For $k \ll n$, we have*

$$\pi_k(U(m)) \cong \pi_{k+1}\left(\frac{U(n+m)}{U(n) \times U(m)}\right),$$

where $\frac{U(n+m)}{U(n)\times U(m)}$ is the Grassmanian of n-planes in the $n+m$ complex vector space. Sending $m \to \infty$, we have

$$\pi_k(U) = \pi_{k+1}(BU(n)),$$

where $BU(n)$ is the classifying space for $U(n)$ or the set of Grassmannian n-dimensional subspaces in an infinite-dimensional complex Hilbert space.

5. The Billiard-ball Problem

In this section, we discuss an application of Morse theory. Think of a mass bouncing inside a domain with elastic reflections off the boundary. The angle between the trajectory and the tangent line of the boundary on both sides of the reflection must be the same. We are interested in the case of the mass tracing a closed trajectory, along which the mass reflects off the boundary finite times, say k. We say such a trajectory has period k. Figure 6 shows the case when $k = 3$. (For more on the Billiard-Ball problem, refer to [5]).

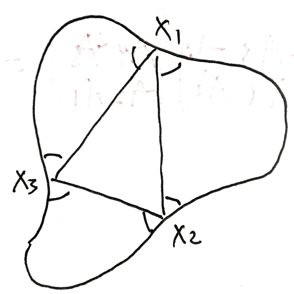

Figure 6.

Question: How many closed trajectories of period $k = 3$ are there in a domain bounded by the boundary X?

Consider the function

$$l = |x_1 - x_2| + |x_2 - x_3| + |x_3 - x_1|$$

If we vary one point, say x_2 along the tangent space of the boundary, then

$$\begin{aligned}\frac{\partial l}{\partial x_2} &= \frac{d}{dx_2}(\langle x_1 - x_2, x_1 - x_2\rangle^{\frac{1}{2}} + \langle x_3 - x_2, x_3 - x_2\rangle^{\frac{1}{2}}) \\ &= \left\langle \mu, \frac{x_1 - x_2}{\sqrt{|x_1 - x_2|}} \right\rangle + \left\langle \mu, \frac{x_3 - x_2}{\sqrt{|x_3 - x_2|}} \right\rangle\end{aligned}$$

where μ is any unit tangent vector in the tangent space of the boundary at x_2. Then

$$\frac{\partial l}{\partial x_2} = 0 \quad \Leftrightarrow \quad \left\langle \mu, \frac{x_1 - x_2}{\sqrt{|x_1 - x_2|}} + \frac{x_3 - x_2}{\sqrt{|x_3 - x_2|}} \right\rangle = 0$$

Notice that $\frac{x_1-x_2}{\sqrt{|x_1-x_2|}}+\frac{x_3-x_2}{\sqrt{|x_3-x_2|}}$ is the angle bisector of the directions from x_2 to x_1 and x_3. Therefore the above condition is exactly saying that this angle bisector is a normal vector of the tangent plane, which means x_2 is a reflection point of the trajectory. The same argument applies for x_1, x_3. Combining them gives us:

$$(x_1, x_2, x_3) \text{ is a critical point of } l \quad \Leftrightarrow \quad x_1x_2x_3 \text{ determines a period-3 trajectory}$$

At this point, for $k \geq 3$, it is very hard to proceed, so we consider the simpler case when $k = 2$. First, l is a real-valued function on $S^n \times S^n/\mathbb{Z}_2 = S^n * S^n$, the symmetric product of two S^n's. S^n comes from the fact that the boundary is set to be a closed convex hypersurface of $\mathbb{R}^{n+1}$. We quotient $S^n \times S^n$ by $\mathbb{Z}_2$ since swapping the two reflection points doesn't change the trajectory they represent. We can then apply Morse Theory.

Example 5.1. Suppose the boundary is an n-dimensional ellipsoid embedded in $\mathbb{R}^{n+1}$. Then the critical points of the l are exactly the pair of diagonals, which are the minimums, and pair of antipodals along the $n+1$ axes of the ellipsoid. The indices of the antipodals as critical points are $2n, 2n-1, 2n-2, ..., n$.

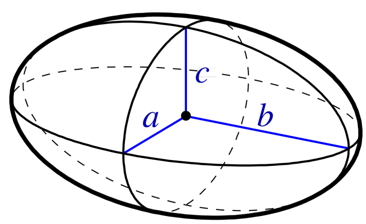

FIGURE 7.

If we want to obtain the homology of $S^n * S^n$, we have to apply the completion principle since the indices of the critical points are consecutive, but it is not clear how to apply them. However, since we already know that only the chords passing through the center are relevant trajectories, we can restrict the length function l to these situations. Each chord can be determined by one of its intersections with the ellipsoid. Antipodal points determine the same chord, so l becomes a function from $\mathbb{R}P^n$ to $\mathbb{R}$ which is exactly the function constructed in example 3.2. Then by example 3.2, the completion principle applies to $l : \mathbb{R}P^n \to \mathbb{R}$, and thus also applies to $l : S^n * S^n \to \mathbb{R}$.

Example 5.2. Suppose the boundary is S^n embedded in $\mathbb{R}^{n+1}$. Then the critical points of $l : S^n * S^n \to \mathbb{R}$ are the diagonals, which are the minimum, or the antipodal pairs, which are the maximums. However, the critical points are all degenerate. In fact, the minimums form a submanifold which can be viewed as S^n and the maximums form a submanifold which can be viewed as $\mathbb{R}P^n$.

To deal with such a situation, we define a non-degenerate critical manifold, generalizing the concept of a non-degenerate critical point.

Definition 5.3. Let f be a smooth function on the manifold M. The connected submanifold N of M will be called a **non-degenerate critical manifold** of f, if

(1) N is a closed manifold of critical points of f;

(2) for all $x \in N$, the null space of the Hessian of f_x is precisely the tangent space to N.

We can also extend the notion of the index of a critical point. Assume N is connected and consider the normal bundle $\nu(N) \longrightarrow N$. Then the Hessian of f defines a self-adjoint endomorphism of the normal bundle . $\nu_N = \nu_N^+ \oplus \nu_N^-$. Then the index of the critical manifold N is simply the fiber dimension of λ_N of ν_N^-.

Now, we would like to know how a critical manifold N counts in the Poincare series of M. We have the following result:

Proposition 5.4. *A non-degenerate critical manifold N counts as $t^{\lambda_N} \cdot P_t(N)$*

This is a consequence of the following theorem:

Theorem 5.5. *Let f be a smooth function on M such that for $a \leq x \leq b$, there is only one critical value $x = c$, $a < c < b$. Suppose furthermore that $f^{-1}(c)$ is the non-degenerate critical manifold N. If M^b is compact, then:*

$$M_b = M_a \cup \text{ the disk bundle of } \nu_N^-$$

6. More on Lie Groups

Let G be a compact Lie group. Consider left-invariant vector fields. The set of left-invariant vector fields on G is called $\mathfrak{g}$, the Lie algebra of G.

Since Ad_g defined by $g \mapsto ghg^{-1}$ is a an automorphism of G, by taking its derivative at the origin e, we have $\mathrm{Ad}_g : X \mapsto gXg^{-1}$, an automorphism of T_eG. This is the adjoint action of G on its Lie algebra $\mathfrak{g}$.

Definition 6.1. The group homomorphism $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$, $g \mapsto \mathrm{Ad}_g$ is called the adjoint representation of G.

We also have the Lie bracket operation on $\mathfrak{g}$: $[X, Y] = XY - YX$, which is also left-invariant. The Lie brackets satisfy the following axioms:

- (anticommutativity) $[X, Y] = -[Y, X]$
- (Jacobi identity) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$, or ad_X is a derivation where $\mathrm{ad}_X = [X, -]$.

The Lie bracket can also be obtained as the derivative of $g \mapsto \mathrm{Ad}_g$;

$$\begin{aligned}
\lim_{t\to 0} \frac{\mathrm{Ad}_{e^{tY}} X - X}{t} &= \lim_{t\to 0} \frac{(1 + tY + ...)X(1 - tY + ...) - X}{t} \\
&= \lim_{t\to 0} \frac{t(YX - XY) + t^2(...)}{t} \\
&= YX - XY = [Y, X] = \mathrm{ad}_Y X.
\end{aligned}$$

Example 6.2. Let $G = SO(3)$, then $\mathfrak{g} = \mathbb{R}^3$. The Lie bracket would just be the cross-product: $[X, Y] = \vec{X} \times \vec{Y}$. $SO(3)$ acts on $\mathbb{R}^3$ by rotations and the orbits of are just the 2-spheres centered at the origin. Some consequences follow:

(1) Any linear function $f : \mathbb{R}^3 \to \mathbb{R}$ restricts to a perfect Morse function on the orbits (each orbit), even for the center.
(2) All the indices of the critical points of such functions are even.

In fact, (1) and (2) are properties of the adjoint representation in any compact connected Lie group.

Theorem 6.3. *(1) and (2) hold for any compact connected Lie group.*

(1) Any linear function $f : \mathfrak{g} \to \mathbb{R}$ restricts to a perfect Morse function on the orbits of Ad.
(2) All the indices of the critical points of such functions are even.

In the theorem above, compactness is necessary since it allows us to find a bi-invariant Riemannian structure on G by averaging an arbitrary left-invariant metric over the right translations. Correspondingly, this implies that there exists an Ad-invariant inner product on $\mathfrak{g}$, which we denote as $(\quad,\quad)$. This inner product makes ad skew-adjoint since $t \mapsto (\mathrm{Ad}_{e^{tY}}(X), \mathrm{Ad}_{e^{tY}}(Z))$ is constant and its derivative at $t = 0$ is

$$([Y, X], Z) + (X, [Y, Z]) = (\mathrm{ad}_Y X, Z) + (X, \mathrm{ad}_Y Z).$$

The canonical form of a skew-adjoint map consists of its kernel and squares of the form

$$\begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix}^2 = -\begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda^2 \end{pmatrix}.$$

In particular the **Killing form**

$$\langle X, Y\rangle := \mathrm{trace}(\mathrm{ad}_X \cdot \mathrm{ad}_Y)$$

becomes non-positive and $\mathrm{trace}(\mathrm{ad}_X \cdot \mathrm{ad}_X) = 0$ only when X lies in the center of $\mathfrak{g}$.

Take $X \in \mathfrak{g}$, $X \neq 0$. Let $\mathfrak{g}_X = \{Y \in \mathfrak{g} : [X, Y] = 0\} = \ker \mathrm{ad}_X$. Also denote $\mathfrak{g}^X = \{[X, Y] : Y \in \mathfrak{g}\} = \mathrm{im}\, \mathrm{ad}_X$, then we have the following theorem:

Theorem 6.4. *The following is a short exact sequence:*

$$0 \longrightarrow \mathfrak{g}_X \hookrightarrow \mathfrak{g} \xrightarrow{\mathrm{ad}_X} \mathfrak{g}^X \longrightarrow 0$$

where

$$\mathfrak{g} = \mathfrak{g}_X \oplus \mathfrak{g}^X$$

is an orthogonal decomposition that is invariant under ad_U *for all* $U \in \mathfrak{g}_X$*. Moreover*

$$\mathfrak{g} = \cup_{g\in G} \mathrm{Ad}_g \mathfrak{g}_X,$$

i.e., any element in $\mathfrak{g}$ *can be conjugated to an element in* $\mathfrak{g}_X$*.*

Proof. Since $(\quad,\quad)$ is invariant under ad_X, we have

$$(\mathfrak{g}_X, \mathfrak{g}^X) = (\mathfrak{g}_X, [X, \mathfrak{g}]) = ([g_X, X], \mathfrak{g}) = 0.$$

So

$$\mathfrak{g} = \mathfrak{g}_X \oplus \mathfrak{g}^X.$$

Next ad_U invariance follows from

$$\mathrm{ad}_U([X, Y]) = [\mathrm{ad}_U X, Y] + [X, \mathrm{ad}_U Y] = [X, \mathrm{ad}_U Y].$$

To prove the last statement fix $Y \in \mathfrak{g} - \mathfrak{g}_X$ and let g_0 be a maximum point for

$$g \mapsto (Y, \mathrm{Ad}_g X)$$

the differential of this map at g is given by

$$Z \mapsto (Y, \mathrm{ad}_Z \mathrm{Ad}_g X) = (\mathrm{ad}_{\mathrm{Ad}_g X} Y, Z).$$

Thus $\mathrm{ad}_{\mathrm{Ad}_{g_0} X} Y = 0$ showing that $Y \in \mathfrak{g}_{\mathrm{Ad}_{g_0} X} = \mathrm{Ad}_{g_0} \mathfrak{g}_X$. □

Consider the orbit of Ad in $\mathfrak{g}$ containing X. The tangent space of the orbit at X is by definition:

$$\left\{\lim_{t\to 0}\frac{\mathrm{Ad}_{e^{tY}}X - X}{t} = -\mathrm{ad}_X Y \mid Y \in \mathfrak{g}\right\}$$

So in fact, $\mathfrak{g}^X$ is just the tangent space of the orbit of Ad containing X at X.

Note that every element in $\mathfrak{g}$ is contained in a maximal Abelian subalgebra. Such subalgebras are called **Cartan subalgebras**. All such subalgebras are in fact conjugate to each other and their dimension is called the **rank of G**.

Theorem 6.5. *For an open and dense set of $X \in \mathfrak{g}$ the subalgebra $\mathfrak{g}_X$ is a Cartan subalgebra.*

Proof. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra.

First note that if $X_1, ..., X_k$ is a basis for $\mathfrak{h}$, then $Y \in \mathfrak{h} - \mathfrak{g}_{X_1} \cap \cdots \cap \mathfrak{g}_{X_k}$ commutes with the basis elements and thus maximality of the Cartan algebra shows that such Y cannot exist. So $\mathfrak{h} = \mathfrak{g}_{X_1} \cap \cdots \cap \mathfrak{g}_{X_k}$.

We next claim that for an open dense set of $(t_1, ..., t_k) \in \mathbb{R}^k$ we have

$$\mathfrak{g}_{Y_1} \cap \cdots \cap \mathfrak{g}_{Y_k} = \mathfrak{g}_{\sum t_i Y_i}$$

as long as $Y_1, ..., Y_k$ commute, which is obviously the case in a Cartan subalgebra. Clearly the inclusion $\subset$ always holds and the righthand side has minimal dimension for an open dense set $(t_1, ..., t_k) \in \mathbb{R}^k$. Thus we only need to establish equality for one such $(t_1, ..., t_k)$ where no coordinate vanishes. Moreover, by induction on k we see that it suffices to show the claim for $k = 2$. If $Z \in \mathfrak{g}_{t_1Y_1+t_2Y_2} - (\mathfrak{g}_{Y_1} \cap \mathfrak{g}_{Y_2})$, then we have

$$0 \neq t_1 [Y_1, Z] = -t_2 [Y_2, Z] \in \mathfrak{g}^{Y_1} \cap \mathfrak{g}^{Y_2}$$

and

$$\mathrm{ad}_{t_1Y_1+t_2Y_2}([Y_1, Z]) = [\mathrm{ad}_{t_1Y_1+t_2Y_2}Y_1, Z]+[Y_1, \mathrm{ad}_{t_1Y_1+t_2Y_2}Z] = [Y_1, \mathrm{ad}_{t_1Y_1+t_2Y_2}Z] = 0.$$

We note that since Y_1, Y_2 commute also $\mathrm{ad}_{Y_1}, \mathrm{ad}_{Y_2}$ commute. Thus $\mathfrak{g}^{Y_1} \cap \mathfrak{g}^{Y_2}$ is invariant under both $\mathrm{ad}_{Y_1}, \mathrm{ad}_{Y_2}$ and in particular also for $\mathrm{ad}_{t_1Y_1+t_2Y_2}$ for all t_1, t_2. So as long as $\mathrm{ad}_{t_1Y_1+t_2Y_2}$ is invertible on $\mathfrak{g}^{Y_1} \cap \mathfrak{g}^{Y_2}$, such Z doesn't exist, so we have

$$\mathfrak{g}_{t_1Y_1+t_2Y_2} = \mathfrak{g}_{Y_1} \cap \mathfrak{g}_{Y_2}.$$

Such nonzero scalars t_1, t_2 clearly exist and in fact form an open and dense set in $\mathbb{R}^2$.

This shows that $\mathfrak{h} = \mathfrak{g}_X$ for an open dense set of $X \in \mathfrak{h}_* \subset \mathfrak{h}$. By theorem 6.4 we have $\mathfrak{g} = \cup_{g\in G}\mathrm{Ad}_g\mathfrak{h}$ so $\cup_{g\in G}\{\mathrm{Ad}_g\mathfrak{h}_*\}$ is an open dense set consisting of X where $\mathfrak{g}_X$ is a Cartan subalgebra. □

Definition 6.6. We say that X is **regular** if $\mathfrak{g}_X$ is a Cartan subalgebra. The subgroup, $e^{\mathfrak{g}_X}$, is Abelian and has dimension equal to the rank, it is a **maximal torus** in G as any maximal Abelian subgroup of a compact group is compact.

Let $N \subset \mathbb{R}^n$ and p a point not in N. Look at the function l_p^2 where l_p is the distance function from the point p. Then $q \in N$ is a critical point of $l_p^2|_N$ if p is a focal point of N at q and the index of the critical point is the sum of the multiplicities of the focal points on the segment pq.

Example 6.7. Let N be the unit sphere S^2 embedded in $\mathbb{R}^3$. Let p be a point in $\mathbb{R}^3$, $p \notin S^2$. The critical points of l_p^2 are obviously the intersection points q_1 and q_2 of the line $p0$ with S^2. $\mathrm{Hess}l_p^2(q_1)$ has index 0 (l_x^2 is degenerate, for no point between p and q_1, i.e. has no focal point) and $\mathrm{Hess}l_p^2(q_2)$ has index 2, for there is only one point x between p and q_2 such that $\mathrm{Hess}l_x^2(q_2)$ is degenerate, which is 0 and $\nu(\mathrm{Hess}l_0^2(q_2)) = 2$, i.e., q_2 is a focal point of $l_0^2|_N$ with multiplicity 2.

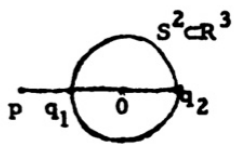

FIGURE 8.

Let $\mathcal{O}_Y$ be an orbit of Ad, say gYg^{-1}, $Y \neq 0$. We've seen that the tangent space of the orbit at Y is $\mathfrak{g}^Y$.

Take a regular X and consider square of the distance function l_X^2 on the orbit. For any point $Y' \in \mathcal{O}_Y$, we have

$$l_X^2(Y') = |Y' - X|^2 = |Y'|^2 - 2\langle Y', X\rangle + |X|^2$$

We want to look for the critical points of this function, or the focal points of $\mathcal{O}_Y$. Notice that Y is a critical point when the line YX is perpendicular to $\mathcal{O}_Y$ at Y and we have the following lemma:

Lemma 6.8. *If at one of its points, a line is perpendicular to an orbit, then that line is perpendicular to all orbits which it intersects.*

Proof. Let the line $X + tZ$ be perpendicular to $\mathcal{O}_X$ at X. Since the tangent space is $\mathfrak{g}^X$, this means that $([A, X], Z) = 0$ for all $A \in \mathfrak{g}$. Since $([A, Z], Z) = (A, [Z, Z]) = 0$ for all $A \in \mathfrak{g}$, we find: $([A, X + tZ], Z) = ([A, X], Z) + t([A, Z], Z) = 0$ for all $A \in \mathfrak{g}$. Then the line is perpendicular to $\mathcal{O}_{X+tZ}$ at $X + tZ$ for all t. □

Let's see some examples.

Example 6.9. Let G be $SU(3)$, then $\mathfrak{g}$ is the space of skew-adjoint complex matrices A, such that $A + \bar{A}^t = 0$ and $\mathrm{trace}A = 0$. Consider the subset of $\mathfrak{g}$:

$$\mathfrak{h} = \left\{ \begin{pmatrix} i\theta_1 & 0 & 0 \\ 0 & i\theta_2 & 0 \\ 0 & 0 & i\theta_3 \end{pmatrix} \mid \theta_1 + \theta_2 + \theta_3 = 0 \right\}.$$

This is a Cartan subalgebra. Let $\mathfrak{h}_*$ denote the subset of $\mathfrak{h}$ for which $\theta_1, \theta_2, \theta_3$ are all different. In geometric language, $\mathfrak{h}_*$ consists of those points in the vector space $\mathfrak{h}$, which do not lie on any of the hyperplanes $\theta_i - \theta_j = 0$, $i, j = 1, 2, 3$, $i < j$. Obviously, $\mathfrak{h}_*$ consists of "almost all" points of $\mathfrak{h}$ it is also the set of regular vectors from theorem 6.5.

We would like to visualize $\mathfrak{h}$, especially the "irregular" set where the critical points lie. $\mathfrak{h}$ is two dimensional, thus a plane. All $Y \in \mathfrak{h}$ such that $\theta_1 = \theta_2$ form a line in this plane. Accordingly, we draw three lines and have the following diagram:

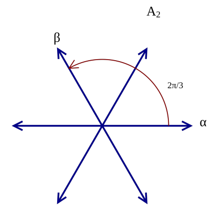

FIGURE 9.

The intersection of the three lines corresponds to the point Y where $\theta_1 = \theta_2 = \theta_3 = 0$. The Weyl group is the permutation group on 3 elements. In general, such diagrams are constructed as follows.

Let's recall the short exact sequence

$$0 \longrightarrow \mathfrak{g}_X \hookrightarrow \mathfrak{g} \xrightarrow{ad_X} \mathfrak{g}^X \longrightarrow 0$$

Fix a regular X, set $\mathfrak{h} = \mathfrak{g}_X$. Set $\mathfrak{m} = \mathfrak{g}^X$, then we have

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$$

which is the more standard notation of the decomposition mentioned earlier. By lemma 6.8 and the statement about critical points just prior to that lemma, we can see that given a generic $P \in \mathfrak{h}_*$, the critical points of $l_P^2|_{\mathcal{O}}$ are exactly the points in $\mathcal{O} \cap \mathfrak{h}$ and these points are independent of the choice of $P \in \mathfrak{h}_*$.

To find these critical points geometrically we construct the infinitesimal diagram, or the root system of G. First we have an Euclidean space with dimension equal the rank of G. Consider the set $\{\dim \mathfrak{g}_Y - \dim \mathfrak{h}\}$ for all $Y \in \mathfrak{h}$ and draw the positive ones in the Euclidean space. The hyperplanes correspond to the lines in figure 9 and are called root spaces. The critical points then lie on these root spaces. The Weyl group of G is the subgroup of the isometry group of the root system which is generated by reflections through the hyperplanes orthogonal to the roots.

Example 6.10. (1) $SU(2)$: We have

$$\mathfrak{h} = \left\{ \begin{pmatrix} i\theta_1 & 0 \\ 0 & i\theta_2 \end{pmatrix}, \theta_1 + \theta_2 = 0 \right\}$$

So the rank is 1. Here $\dim \mathfrak{g}_Y > 1$ only when $\theta_1 = \theta_2 = 0$, so we only draw one point on a line. The Weyl group is $\mathbb{Z}_2$.

(2) $SO(4)$, $SO(5)$, G_2 all have rank 2, see figure 10, 11, 12.

On the Lie algebra level, only things that commute are in the Cartan subalgebra $\mathfrak{h}$. On the Lie group level, only things that commute are in the maximal torus $e^{\mathfrak{h}} = T$.

Consider G/T, the "flag variety" of G. The Poincare series of G/T can be calculated by the following theorem.

Pick a generic point $X \in \mathfrak{h}$ such that it is not on a plane in the diagram. Then define the function $\lambda : \mathfrak{h} \to \mathbb{Z}$ by $\lambda(Y) =$ twice the number of hyperplanes crossed by the straight line segment from X to Y for all $Y \in \mathfrak{h}$. Obviously, λ is constant

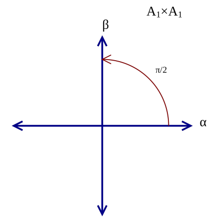

FIGURE 10. $SO(4)$

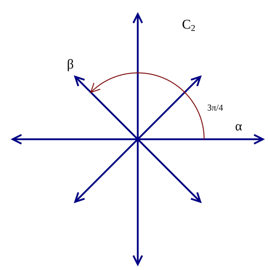

FIGURE 11. $SO(5)$

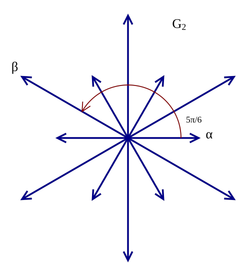

FIGURE 12. G_2

on each region divided by the hyperplanes. Call these regions the cells, and $\lambda(\alpha)$ the value of λ in the cell Δ_α, then we have

Theorem 6.11. *The Poincare series of G/T is given by*

$$P_t(G/T) = \sum_\alpha t^{\lambda(\alpha)}$$

where Δ runs over the cells of the diagram.

Example 6.12. When $G = SO(4)$, we have the diagram as shown in figure 13. So the Poincare series is

$$P_t(G/T) = 1 + 2t^2 + t^4 = (1 + t^2)^2.$$

The cohomology is that of $S^2 \times S^2$

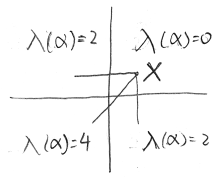

FIGURE 13.

Example 6.13. $G = G_2$, then we have the diagram as shown in figure 14. So the Poincare series is

$P_t(G/T) = 1 + 2t^2 + 2t^4 + 2t^6 + 2t^8 + 2t^{10} + t^{12} = (1 + t^2)(1 + t^2 + t^4 + t^6 + t^8 + t^{10})$

The cohomology is that of $S^2 \times \mathbb{C}P^5$

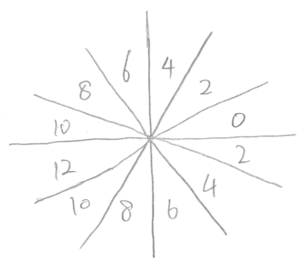

FIGURE 14.

We now consider a more global point of view. Recall: $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and look at $\mathrm{Ad}|_T$. $\mathfrak{h}$ is invariant under Ad, so Ad acts on $\mathfrak{m}$, and $\mathfrak{m}$ is a module for T. Moreover, $\mathfrak{m}$ is the direct sum of 2-dimensional representations. We may write $\mathfrak{m} = \bigoplus E_\alpha$, where E_α are planes. The diagram of G is just T with the sets T_α marked where T_α is kernel of representation on E_α.

Now we think of the corresponding diagram in $\mathfrak{h}$. Since $\exp : \mathfrak{h} \to T$ is the universal covering of the torus, we have a lattice. The lattice points are the kernel of the covering map, or the center of the Weyl group. The resulting diagram looks like a series of the previous diagrams one centered at each lattice point of a torus.

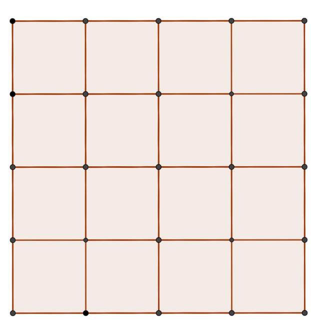

FIGURE 15. SO(4)

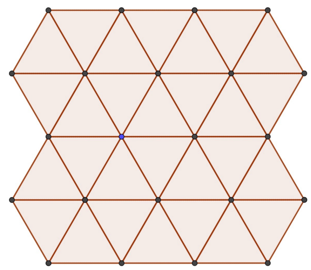

FIGURE 16. SU(3)

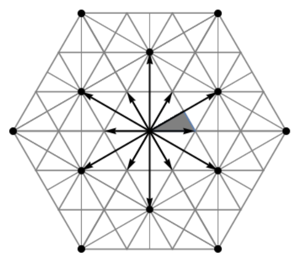

FIGURE 17. G_2

Example 6.14. $SO(4), SU(3), G_2$: see figure 15, 16, 17.

Starting from a lattice point, we call the section between two adjacent hyperplanes a fundamental chamber. We can also define λ on these diagrams as in the previous cases. Then we have the following theorem:

Theorem 6.15. *The Poincare series of the loop space is:*

$$P_t(\Omega G) = \sum_{\alpha} t^{\lambda(\alpha)}$$

α is over the cells of the fundamental chamber.

Example 6.16. Consider $SU(3)$, then we have the fundamental chamber as in figure 18. Then the Poincare series is

$$P_t(\Omega SU(3)) = 1 + t^2 + 2t^4 + 2t^6 + 3t^8 + ... = \frac{1}{1-t^2}\frac{1}{1-t^4},$$

which as we shall see in the next section is related to the loop spaces for S^3 and S^5. From this example, it's also clear that one consequence of the theorem is that the homology ΩG has no torsion.

We also note that

$$P_t(SU(3)/T) = 1 + 2t^2 + 2t^4 + t^6 = (1+t^2)(1+t^2+t^4).$$

So $SU(3)/T$ has the same homology as $S^2 \times \mathbb{C}P^2$. Note that we have a natural fibration

$$T \to S^3 \times S^5 \to S^2 \times \mathbb{C}P^2$$

which indicates the connection between the two Poincare series.

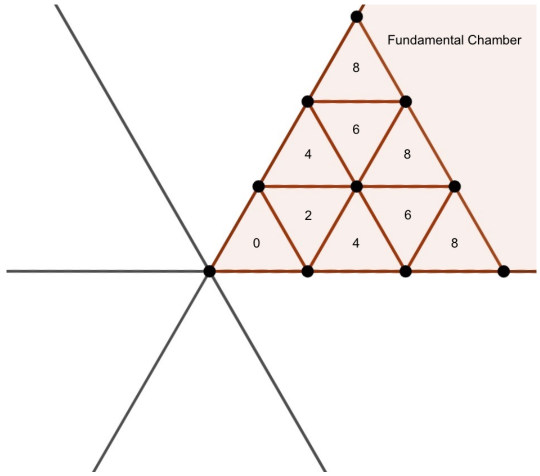

Figure 18.

Example 6.17. Consider G_2, then we have the fundamental chamber as in figure 19. Then the Poincare series is

$$P_t(\Omega G_2) = 1+t^2+t^4+t^6+t^8+2t^{10}+2t^{12}+2t^{14}+2t^{16}+2t^{18}+3t^{20}... = \frac{1}{1-t^2}\frac{1}{1-t^{10}},$$

which as we shall see in the next section is related to the loop spaces for S^3 and S^{11}. From example 6.13, we also note that

$$P_t(G_2/T) = 1+2t^2+2t^4+2t^6+2t^8+2t^{10}+t^{12} = (1+t^2)(1+t^2+t^4+t^6+t^8+t^{10}).$$

So G_2/T has the same homology as $S^2\times\mathbb{C}P^5$. Note that we have a natural fibration

$$T \to S^3 \times S^{11} \to S^2 \times \mathbb{C}P^5$$

which indicates the connection between the two Poincare series.

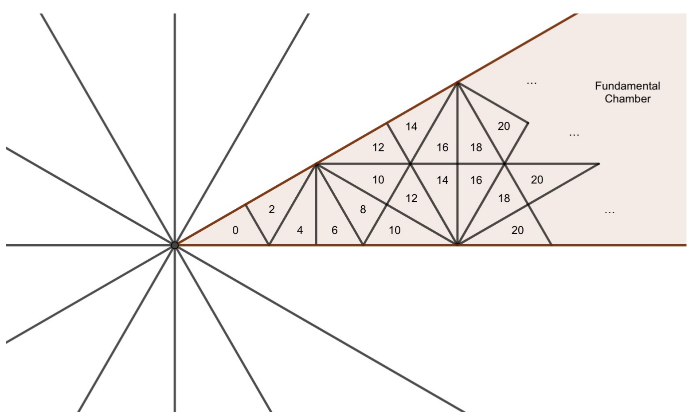

Figure 19.

Theorems 6.11 and 6.15 are the key results of Bott's famous 1956 paper. ([6])

7. Morse Theory Proper

In order to understand the homotopy of a manifold, we do Morse theory on its path space. A natural choice of the Morse function is the arclength functional. However, the arclength functional is invariant under change of parametrization, so minima, if they exist, do not come with a fixed parameter. This problem can be overcome by considering the energy functional.

Definition 7.1. Let M be a connected complete Riemannian manifold, define the **path space** of M as

$$E = \{\text{piecewise smooth maps } c : [0,1] \to M\}.$$

Define the **path space from** a **to** b as

$$\Omega(a,b) = \{\text{piecewise smooth maps } c : [0,1] \to M, c(0) = a, c(1) = b\}.$$

Definition 7.2. Given a piecewise smooth path $c : [0,1] \to M$, define the **energy functional** $S : E \to \mathbb{R}$ as

$$S(c) = \frac{1}{2}\int_0^1 \langle \dot{c(t)}, \dot{c(t)} \rangle dt.$$

Consider the fiber bundle: $\Omega(a,b) \to E \to M \times M$, we have the following proposition:

Proposition 7.3. *$\pi_k(\Omega(a,b)) = \pi_{k+1}(M)$ for any $a, b \in M$.*

Proof Sketch. This follows from the long exact sequence of homotopy groups for $\Omega(a,b) \to E \to M \times M$ by noting that E is contractible.

Alternately observe that when $k = 0$, the two sides of the equation are just two different ways of writing the fundamental group of M up to conjugation. In general, take $a = b$ then $\Omega(a,b) = \Omega(M)$ is the loop space of M, then

$$\pi_k(\Omega(a,b)) = [S^k, \Omega M] \overset{*}{=} [\Sigma S^k, M] = [S^{k+1}, M] = \pi_{k+1}(M)$$

where ΣS^k is the suspension of S^k and $*$ is the Eckmann-Hilton duality. □

Having this proposition, we now understand why doing Morse theory on the path space gives us the homotopy groups of the manifold. Now consider the energy functional restricted to $\Omega(a,b)$, then we have

- Theorems A and B hold for $S|_{\Omega(a,b)}$.
- The critical points of S on $\Omega(a,b)$ are the geodesics from a to b.
- The index theorem holds.

Theorem 7.4 (The Index Theorem)**.** *Let $g : [0,1] \to M$ be a geodesic from a to b. Then g is a critical point of $S|_{\Omega(a,b)}$ and the index $\lambda(g)$ is the number of conjugate points $g(t)$ with $0 < t < 1$ counted with its multiplicity.*

Example 7.5. Let $M = S^2$, we know that the geodesics are the paths along great circles and two points are conjugate if they are antipodal points or the same point. In both cases, they are conjugate with multiplicity 1.

Suppose a and b are not antipodals, then the geodesics from a to b are $g_0, g_1, g_2, ...$ with $0, 1, 2, ...$ conjugate points respectively.

By the index theorem, these geodesic are critical points of $S|_{\Omega(a,b)}$ with indices $0, 1, 2, ...$ Therefore, by theorems A and B, we have

$$\Omega(a,b) = e^0 \cup e^1 \cup e^2 \cup ...$$

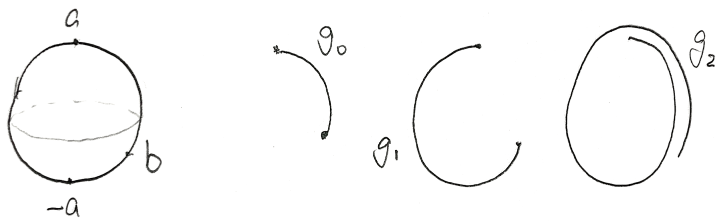

FIGURE 20.

We can then apply the completion principle. For example, we can consider the closed submanifold N_1 of $\Omega(a,b)$ consisting of paths that are in the intersection of a plane with S^2. With a and b fixed on the plane, the plane can only rotate with the line segment ab as an axis, so N_1 is diffeomorphic to S^1. Then N_1 is a closed submanifold with g_1 as the non-degenerate maximum of $S|_{N_1}$.

The Poincare series then follows:

$$P_t(\Omega S^2) = 1 + t + t^2 + ... = \frac{1}{1-t}.$$

Example 7.6. For $M = S^n$, everything is the same except that all conjugate points have multiplicity $n-1$, so we have

$$\Omega S^n = e^0 \cup e^{n-1} \cup e^{2(n-1)} \cup ... \qquad P_t(\Omega S^n) = \frac{1}{1-t^{n-1}}.$$

Moreover, since $e^0 \cup e^{n-1} = S^{n-1}$, we have

$$\Omega S^n = S^{n-1} \cup e^{2(n-1)} \cup ...$$

So $\pi_k(S^{n-1}) = \pi_{k+1}(\Omega S^n)$ for $k \leq 2n-4$. Combining with proposition 7.3 we have the following theorem:

Theorem 7.7 (The Freudenthal Suspension Theorem)**.** *The homotopy group $\pi_k(S^{n-1})$ is isomorphic to $\pi_{k+1}(S^n)$ for $k \leq 2n-4$.*

We can also prove the theorem by starting with $\Omega(a,b)$ where a and b are antipodal points. Then the minimal set of $S|_{\Omega(a,b)}$ forms a critical submanifold diffeomorphic to S^{n-1} (see figure 20) and all other critical points have index at least $2(n-1)$, which also gives us

$$\Omega S^n = S^{n-1} \cup e^{2(n-1)} \cup ...$$

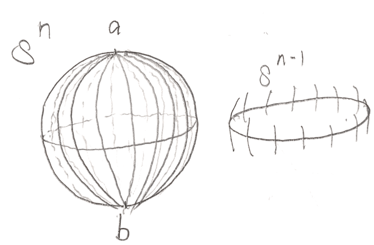

FIGURE 21.

The Freudenthal suspension theorem can be generalized to the following:

Theorem 7.8. *The homotopy groups $\pi_{n+k}(S^n)$ stabilize for $n \geq k+2$.*

We call these homotopy groups the stable homotopy groups of the spheres.

8. Proof of the Periodicity Theorem

Proposition 8.1. *In a compact Lie group C, let T be a maximal torus. Let $p \in T$ be a generic point. Then every geodesic in a bi-invariant metric, that starts at the identity and ends at p, lies entirely in T.*

In the universal cover, the geodesics lie in the universal cover of T, which is an Euclidean space. So the geodesics are straight lines.

Proof. Since we've seen in section 6 that the tangent space of the maximal torus and the tangent space of the orbit are orthogonal complements, the following lemma would be sufficient to prove the result:

Lemma 8.2. *Suppose s is a geodesic segment that is perpendicular to, L, an orbit of a group of isometries G, then s is perpendicular to all orbits it intersects*

Consider the orbit space C/G, which may not be a Riemannian manifold, but is still a metric space and has the concept of minimizing geodesics. Given a geodesic s in C perpendicular to an orbit L, it is then a lift of a minimizing geodesic in C/G, which is called the "horizontal lift". Then s must be a minimizing geodesic between any orbits, which could only be the case when it is orthogonal to all the orbits. (Reference: [8]) □

Now we try to prove the periodicity theorem for the unitary group. First, we start with $U(2n)$. In fact, it's sufficient to consider $SU(2n)$, and the geodesics in this case are more obvious. Take $a = e$, $b = -e$, both in the maximal torus of $SU(2n)$. By proposition 8.1, all geodesics in $\Omega(a, b)$ are on the maximal torus. Consider the geodesic:

$$\begin{pmatrix} e^{it} & & & & & \\ & \ddots & & & & \\ & & e^{it} & & & \\ & & & e^{-it} & & \\ & & & & \ddots & \\ & & & & & e^{-it} \end{pmatrix}, \qquad 0 \leq t \leq \pi.$$

It can be easily verified that this geodesic does not cross any root spaces, and thus has index 0. In fact, all minimal geodesics are of this form up to permutations along the diagonal. So the minimum critical manifold is $U(2n)/U(n) \times U(n)$. It can be proved that all other geodesics cross some root spaces, and have indices at least $2n + 2$. Therefore, the loop space of $SU(2n)$ is

$$\frac{U(2n)}{U(n) \times U(n)} \cup \ldots(\text{cells of dimension} \geq 2n+2).$$

It follows that for $k \ll n$:

$$\pi_{k+1}(U(2n)) = \pi_k(\Omega U(2n)) = \pi_k(U(2n)/U(n) \times U(n)) = \pi_{k-1}(U(n))$$

where the last equality is by the exact homotopy sequence of the fiber bundle

$$U(n) \to U(2n)/U(n) \to U(2n)/U(n) \times U(n).$$

Thus $\pi_k(U) = \pi_{k+2}(U)$ (see [2] for a more detailed proof).

More generally, we consider the case when the manifold is not necessarily a group, but a symmetric space of the form G/K where G is a Lie group and the stabilizer K is the fixed point set of an involution σ. Symmetric spaces also have maximal tori, so the method we used before can also be applied.

Let $G = SO(2n)$, $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Consider the path from $e = I$ to J, the minimal geodesics form a submanifold SO/U while other critical points are of higher indices, so we have

$$\pi_k(\Omega SO) = \pi_{k+1}(SO/U)$$

In general, if M is a symmetric space, then the minimal geodesics also form a symmetric space, so we can apply a similar method inductively and get:

- $\pi_k(\Omega SO) = \pi_{k+1}(SO/U)$
- $\pi_k(\Omega SO/U) = \pi_{k+1}(U/Sp)$
- $\pi_k(\Omega U/Sp) = \pi_{k+1}(Sp/Sp \times Sp)$
- $\pi_k(\Omega Sp/Sp \times Sp) = \pi_{k+1}(Sp)$
- $\pi_k(\Omega Sp) = \pi_{k+1}(Sp/U)$
- $\pi_k(\Omega Sp/U) = \pi_{k+1}(U/O)$
- $\pi_k(\Omega U/O) = \pi_{k+1}(O/O \times O)$
- $\pi_k(\Omega O/O \times O) = \pi_{k+1}(O)$

Combining the eight equalities, we have the periodicity theorem:

Theorem 8.3 (Bott)**.** *The homotopy groups of the classic groups are periodic:*

(1) $\pi_k(U) \cong \pi_{k+2}(U)$
(2) $\pi_k(O) \cong \pi_{k+4}(Sp)$
(3) $\pi_k(Sp) \cong \pi_{k+4}(O)$

By (2) and (3), $\pi_k(O) \cong \pi_{k+8}(O)$ *and* $\pi_k(Sp) \cong \pi_{k+8}(Sp)$*.*

9. Acknowledgement

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