MORSE THEORY AND IT'S APPLICATION TO HOMOTOPY THEORY

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Abstract. The following is a LaTeX version of Bott's 1960 book on Morse theory which is no longer in print. Some proofs are supplemented and some ambiguous notations are fixed.

Contents

1. Introduction

To get a first idea about Morse Theory, we consider a simple example. Let T be a 2-dimensional torus, resting on its tangent plane V as indicated in fig.1. The distance of the points of T from the tangent place V is a real analytic, hence smooth (C^∞) function f on T. We set

$$T^a = \{x \in T \mid f(x) \leq a\}$$

T^a is empty for $a < 0$, $\{p\}$ for $a = 0$, homeomorphic with a 2-cell for $0 < a < f(q)$, homeomorphic with he product of a circle and a line segment for $f(q) < a < f(p)$, is homeomorphic with the figure indicated in fig.2 for $f(q) < a < f(s)$, and the whole torus for $a \geq f(s)$.

As f grows from 0 to $f(s)$, T^a grows in successive steps from a point to the whole torus. From the topological point of view, something new comes at the level of p, q, r, and s. These points are the critical points of f, the points where $df = 0$.

Here, we touch on a first essential idea of Morse Theory. There is a close relation between the behavior of f as a smooth function, especially with respect to its critical points, and the topological structure of T. Let us make this more precise. For that

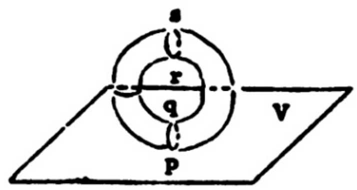

FIGURE 1.

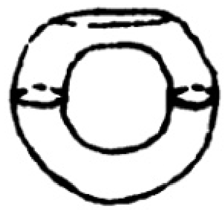

FIGURE 2.

purpose, we introduce the concept of attaching an r-cell to a topological space X. More generally, let Y be a second topological space, Z a subspace of Y and $f : Z \to X$ a continuous map. Let $\tilde{X}$ be the topological space obtained as follows: in the disjoint union of X and Y, we identify the points s of Z with their image $f(s)$ and provide the resulting space with the quotient topology. We say that $\tilde{X}$ is obtained from X by attaching Y to X according to the pair (Z, f). In particular, if y is an r-cell e_r and Z its boundary $\dot{e}_r$, we simply say that e_r is attached to X according to the map f and write $\tilde{X} = X \bigcup_f e_r$. Furthermore, we introduce the index of a non-degenerate critical point of a smooth function f defined on a smooth manifold M^n.

As we said above, a **critical point** of f is a point $x \in M^n$, such that at x, $df = 0$, or expressed in local coordinates $(x_1, ..., x_n)$:

$$\frac{\partial f}{\partial x_i} = 0, \qquad i = 1, ..., n.$$

A critical point x of f is called **non-degenerate** if

$$\mathbf{H}f = \left(\frac{\partial^2 f}{\partial x^i \partial x^j} \right)$$

is a nonsingular $n \times n$ matrix. It is easily proved that the number of positive and negative eigenvalues of $\mathbf{H}f$ is independent of the local coordinates. The last number is naturally called the **index** of x as a critical point of f. In the case of our example, the index of p, q, r, and s is 0, 1, 1, 2 respectively. From the homotopic point of view, the building up of t in successive steps can be described as follows: If there are no critical points between a and b then T^a is of the same homotopic type as T^b. If there is a single critical point of index k in this range, then T^b is obtained by attaching a k-cell.

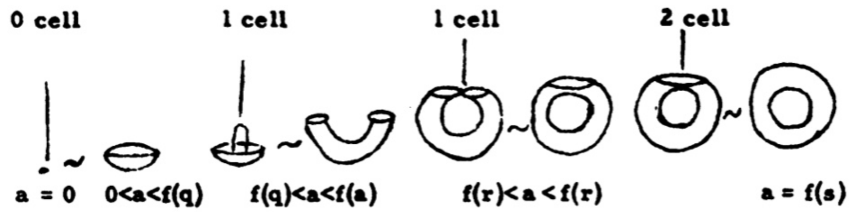

FIGURE 3. The building up of T by successive attaching of cells corresponding to the critical points of f

The main result of the simplest part of Morse theory states that the situation as described for the special function f on the special smooth manifold T is a general one. This is expressed by the following theorem.

Theorem 1.1. *Let f be a smooth function on the compact smooth manifold M, such that f has only non-degenerate critical points. Set $M^a = \{x \in M, f(x) \leq a\}$. Then, if $f^{-1}(a \leq s \leq b)$ does not contain any critical point of f, then M^a and M^b are homeomorphic. If $f^{-1}(a \leq s \leq b)$ contains only one critical point x of index λ, $a < f(x) < b$, then $M^b \sim M^a \bigcup e_\lambda$, e_λ being attached to M^a by a conveniently chosen map.*

As always, "smooth" means "C^∞" and "manifold" means "connected manifold".

Let the situation be as in the theorem, and let $a_i \in \mathbb{R}$, $i = 1, ..., m$ be the critical values of f, which we suppose to correspond 1-to-1 to the critical points of f and

$$M^{a_1} \subset M^{a_2} \subset ...M^{a_m}.$$

It can be shown, precisely in the same way as it can be shown that $\pi_k(S^n) = 0$ for $0 \leq k < n$, that $\pi_k(M^{a_i}, M^{a_{i-1}}) = 0$ for $0 \leq k \leq \lambda_i - 2$, where λ_i denotes the index of the critical point corresponding to a_i, also that the homomorphism

$$\pi_{\lambda_i - 1}(M^{a_{i-1}}) \to \pi_{\lambda_i - 1}(M^{a_i})$$

is onto. Thus from the exact homotopy sequence we can conclude:

$$\pi_{\lambda_i - 1}(M^{a_{i-1}}) = \pi_{\lambda_i - 1}(M^{a_i}) \qquad 0 \leq k \leq \lambda_i - 1.$$

Further, it is easy to see that

$$H^r(M^{a_i}, M^{a_{i-1}}; \mathbb{Z}) = 0 \qquad 0 \leq r \leq \lambda_i$$

and

$$H^r(M^{a_i}, M^{a_{i-1}}; \mathbb{Z}) = \mathbb{Z} \qquad r = 0, \lambda_i.$$

The second part of Morse theory is analogous to the first one, but deals with a different, more complicated situation.

Let M be a compact Riemannian manifold. Let P, Q be a couple of points on M. We denote by $\mu_{P,Q}(M)$ the set of sectionally smooth curves, joining P and Q, and parametrized proportionally to the length of arc by a parameter t, $0 \leq t \leq 1$.

Let $L(u)$ be the length of the curve $u \in \mu_{P,Q}(M)$. If we set for every pair of curves $u, v \in \mu_{P,Q}(M)$

$$\mathfrak{L}(u,v) = \max_{t\in[0,1]} \operatorname{dist}(u(t), v(t)) + |L(u) - L(v)|$$

then it is easily seen that $\mathfrak{L}(u,v)$ is a distance function on $\mu_{P,Q}(M)$. Now the result of Morse Theory in the previous theorem has an analogue in this case. The role of f by $\mathfrak{L}$ and the geodesics in $\mu_{P,Q}(M)$ take the place of the critical points. More precisely, the following theorem holds:

Theorem 1.2. *Let M be a compact Riemannian manifold, P and Q a pair of points on M, $\mu_{P,Q}(M)$ the metric space of sectionally smooth curves from P to Q, parametrized proportionally to length of arc. Set*

$$\Omega^c_{P,Q}(M) = \{u \in \mu_{P,Q}(M), \mathfrak{L})(u) \leq c\},$$

where $\mathfrak{L}(u)$ denotes the length of u.
If there is no geodesic of length l, $a \leq l \leq b$, then

$$\Omega^b_{P,Q}(M) \sim \Omega^a_{P,Q}(M).$$

If there is just one such geodesic g of length l, $a < l < b$, then

$$\Omega^b_{P,Q}(M) \sim \Omega^a_{P,Q}(M) \bigcup e_\lambda,$$

where λ denoting the number of conjugate points of P along g which lie between P and Q.

Since $\pi_k(\Omega_{P,Q}(M)) = \pi_{k+1}(M)$ ([1], p.55), this theorem yields information on the homotopy structure of M, provided that the "geodesic structure" of M is sufficiently known. This is the case for special kinds of manifolds, in particular symmetric spaces.

2. Morse Theory of Smooth Functions on a Manifold

We start with the first part of Theorem 1.1.

Theorem 2.1. *Let M be a compact smooth manifold and $f : M \to \mathbb{R}$ a smooth function on M. Set $M^a = \{x \in M, f(x) \leq a\}$. if $df(p) \neq 0$ for all points $p \in M$ with $a \leq f(p) \leq b$, then M^a and M^b are homeomorphic.*

We recall that a local 1-parameter group of diffeomorphisms of M is a (smooth) map

$$\Phi : M \times \mathbb{R} \to M,$$

such that

(1) $\Phi|_{M\times\{t\}}$ is a diffeomorphism of M;
(2) $\Phi(m, t_1 + t_2) = \Phi(\Phi(m, t_1), t_2)$ for $|t_1|$, $|t_2|$, and $|t_1| + |t_2| < \epsilon$.

If $p \in M$ is not a fixed point of Φ, then there passes just one orbit curve of Φ through p, Φ determines in a natural way a vector field $\dot{\Phi}$ on M by setting

$$D_{\dot{\Phi}(p)} g = (\dot{\Phi}(g))(p) := \lim_{t\to 0} \frac{g(\Phi(p,t)) - g(p)}{t},$$

$p \in M$, g a smooth function on M. $\dot{\Phi}$ is zero at the fixed points of ϕ and tangent to the orbits of ϕ at the other points of M.

Lemma 2.2. *Let M be a compact smooth manifold and V a vector field on M. Then there exists a (uniquely determined) local 1-parameter group Φ of diffeomorphisms of M such that $\dot{\Phi} = V$*

For the proof of Lemma 2.2, see [2, p.5]. We now give the proof of Theorem 2.1.

Proof. We introduce a Riemannian metric on M. This can be done for example in the following way. Let $\{U_1, ..., U_m\}$ be a finite open covering of M with coordinate systems, and $\{\varphi_1, ..., \varphi_m\}$ a partition of unity corresponding to $\{U_1, ..., U_m\}$. Let

$$x_1^i, ..., x_n^i$$

be coordinates in U_i and $x \in U_i$. for every pair (X, Y) of tangent vectors to M at x, we define $\varphi_i(X, Y)$ be setting

$$\left(\frac{\partial}{\partial x_k^i}, \frac{\partial}{\partial x_l^i} \right) = \delta_{kl}$$

and extending by linearity. Then

$$(X, Y) = \sum_{i=1}^{m} \varphi_i(X, Y)$$

defines a global Riemannian metric on M. This metric enables us to define a special vector field ∇f on M, the gradient of f, by

$$(\nabla f(p), Y(p)) = df|_p(Y(p))$$

Obviously, $\nabla f(p) = 0$ is equivalent to $df|_p = 0$. Now let ϕ be the local 1-parameter group of diffeomorphisms with the property that $\dot{\phi} = -\nabla f$. Since by assumption, $df|_p \neq 0$ for all $x \in M$ with $a \leq f(x) \leq b$, by Lemma 2.2, there passes through every point of this subset of M a unique integral curve of $-\nabla f$. This is also true if we replace a by a conveniently chosen $a' < a$. Since $f(m) - f(\phi(m, \frac{1}{2}\epsilon))$ is continuous, this function has a positive minimum on the compact set $\{x \in M, a' \leq f(x) \leq b\}$. Therefore, every integral curve of $-\nabla f$ which meets M^b meets $M^{a'}$ and also conversely.

A homeomorphism $h : M^b \to M^a$ is constructed as follows. For $y \in M^{a'}$, we set $h(y) = y$. Now let $y \in M^b - M^{a'}$. Suppose the integral curve of $-\nabla f$ through y intersects M^a, $M^{a'}$, and M^b in y_a, $y_{a'}$, and y_b respectively. We determine $h(y)$ on this curve C by setting

$$l(h(y)y_{a'}) = \frac{l(y_a y_{a'})}{l(y_b y_{a'})} l(y y_{a'})$$

where l denotes the length of arc along C. □

For the proof of the second part of Theorem 1.1, we need some lemmas.

Lemma 2.3 (Morse)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function on $\mathbb{R}^n$, such that $f(0) = 0$, $f_{x_i}(0) = 0$, $i = 1, ..., n$, and $\det(f_{x_i x_j}(0)) \neq 0$. Then in a neighborhood U of 0, coordinates $(y_1, ..., y_n)$ can be introduced such that in U*

$$f = -y_1^2 - y_2^2 - ... - y_\lambda^2 + y_{\lambda+1}^2 + ... + y_n^2,$$

where λ is the index of f at 0.

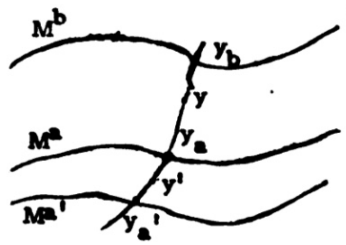

FIGURE 4.

Proof. We start with the construction of smooth functions $a_{ij}(x)$, $i,j = 1,...,n$, such that

$$f = \sum_{i,j=1}^{n} a_{ij}(x)x_i x_j,$$

where $a_{ij}(x) = a_{ji}(x)$. Using $f(0) = df(0) = 0$, we find in an elementary way:

$$\begin{aligned} f(x) &= \int_0^1 \frac{\partial}{\partial t}(f(xt))dt = \sum_{i=1}^{n} x_i \int_0^1 \partial_i(f(xt))dt \\ &= \sum_{i=1}^{n} x_i \left[\partial_i f(xt)(t-1)\right]_0^1 - \sum_{i=1}^{n} x_i \int_0^1 \frac{\partial}{\partial t}(\partial_i f(xt))(t-1)dt \\ &= \sum_{j=1}^{n}\sum_{i=1}^{n} \left(-\int_0^1 \partial_j(\partial_i f(xt)(t-1))dt\right) x_i x_j \\ &= \sum_{i,j=1}^{n} a_{ij}(x)x_i x_j. \end{aligned}$$

We may assume: $a_{11}(0) \neq 0$(at least one element $a_{ij} \neq 0$, in the case this element is a_{ii} with $i \neq 1$, we simply permute, in the case this element is A_{ij} with $i \neq j$, we introduce new coordinates $\bar{x_1}, ..., \bar{x_n}$ by setting $\bar{x_i} = x_i + x_j$, $\bar{x_j} = x_i - x_j$, $\bar{x_k} = x_k$ for$k \neq i, j$).

Firstly, let $a_{11}(0) > 0$. In a neighborhood of 0, we can set

$$\begin{aligned} \tilde{y_1} &= \sqrt{a_{11}}\left(x_1 + \frac{\sum a_{1\alpha}x_\alpha}{a_{11}}\right) \qquad && \alpha = 2, ..., n \\ \tilde{y_k} &= x_k && k = 2, ..., n \end{aligned}$$

where $\sqrt{a_{11}}$ denotes the positive root of a_{11}. It follows:

$$\sum a_{ij}x_i x_j = \tilde{y_1}^2 + \sum_{\alpha,\beta=2}^{n} b_{\alpha\beta}\tilde{y_\alpha}\tilde{v_\beta}.$$

$$d\tilde{y_1} = (d\sqrt{a_{11}})\left(x_1 + \frac{\sum a_{1\alpha}x_\alpha}{a_{11}}\right) + \sqrt{a_{11}}\left(dx_1 + d\left(\frac{\sum a_{1\alpha}x_\alpha}{a_{11}}\right)\right)$$

$$d\tilde{y_1}(0) = \sqrt{a_{11}}dx_1 + \sum c_\alpha dx_\alpha$$
$$d\tilde{y_\alpha}(0) = dx_\alpha, \quad \alpha = 2, ..., n$$
$$d\tilde{y_1} \wedge ... \wedge d\tilde{y_n} = \sqrt{a_{11}}dx_1 \wedge ... \wedge dx_n \neq 0$$

Hence we have $\det\left(\frac{\partial \tilde{y}}{\partial x}(0)\right) \neq 0$.

Since $\det(b_{\alpha\beta}(0)) = \det(a_{ij}(0)) \left(\det\frac{\partial \tilde{y}}{\partial x}(0)\right)^2$ and $\det(a_{ij}(0)) = \frac{1}{2^n}\det f_{x_i x_j}(0) \neq 0$, we also have $\det(b_{\alpha\beta}) \neq 0$

In the case that $a_{11} < 0$, we set

$$\sum a_{ij}x_i x_j = -y_1^2 + \sum_{\alpha,\beta} y^\alpha y^\beta.$$

By induction, we get functions $y_1, ..., y_n$ such that

$$f = \sum_{i=1}^n \epsilon_i y_i^2, \quad \epsilon_i = \pm 1,$$

and $dy_1 \wedge ... \wedge dy_n(0) \neq 0$. Since $(y_1, ..., y_n)$ can be used as coordinates in a neighborhood of 0, the number of negative ϵ_i's equals the index of f at 0. This completes the proof. □

Lemma 2.4. *Let X be a topological space, e_r an r-cell, I_s an s-cell, given by*

$$I_s = \{(t_1, ..., t_s) \in \mathbb{R}^s, 0 \leq t_i \leq 1\}$$

and let $e_r \times I_s$ be attached to X by a map $f : \dot{e}_r \times I_s \to X$. If we set $g = f|_{\dot{e}_r \times (0,...,0)}$, then $X \bigcup_f (e_r \bigcup I_s) \sim X \bigcup_g e_r$.

Proof. As illustrated in fig. 5, $X \bigcup_f (e_r \bigcup I_s)$ can be deformed into $X \bigcup_g e_r$ by a standard deformation. □

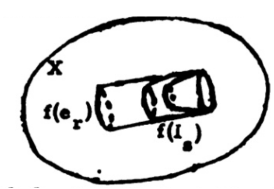

FIGURE 5.

Theorem 2.5. *Let M be a compact smooth manifold, $f : M \to \mathbb{R}$ a smooth function on M. Set $M^a = \{x \in M, f(x) \leq a\}$. If $f^{-1}(a \leq x \leq b)$ conatins exactly one non-degenerate critical point p of index λ, $a < f(p) < b$, then $M^b \sim M^a \bigcup e_\lambda$, where e_λ is attached to M^a by a conveniently chosen map.*

Proof. We may assume that $f(p) = 0$. From Theorem 2.1, it follows that it is sufficient to prove the existence of a number ϵ, $0 < \epsilon \leq b$, such that $M^\epsilon \sim M^a \bigcup e_\lambda$. By Lemma 2.3, there is a neighborhood U of p and local coordinates such that in U, f is given by

$$f = -y_1^2 - y_2^2 - ... - y_\lambda^2 + y_{\lambda+1}^2 + ... + y_n^2,$$

Furthermore, we may assume M to be provided with a Riemannian metric ds^2 which is Euclidean in U. If we set:

$$A^\epsilon = \{y \in M^\epsilon \bigcap U, y_1^2 + ... + y_\lambda^2 \leq \rho\}$$
$$M_*^\epsilon = \overline{M^\epsilon - A^\epsilon}$$

then $M^\epsilon = A^\epsilon \bigcup M_*^\epsilon$. For conveniently chosen $\epsilon > 0$, $\rho > 0$, this just mean that M^ϵ is obtained from M_*^ϵ by attaching a product $e_\lambda \times I_{n-\lambda}$ to M_* in the way described in Lemma 2.4. By this Lemma, we find

$$M^\epsilon \sim M_*^\epsilon \bigcup e_\lambda.$$

It can be shown in the same way as in the proof of Theorem 2.1 that M_*^ϵ and M^a are homeomorphic. It only has to be shown that the gradient of f is not zero and that any integral curve starting from the boundary of M^a meets the boundary of M_*^ϵ. Using the fact that ds^2 is Euclidean in U, this is easily proved. □

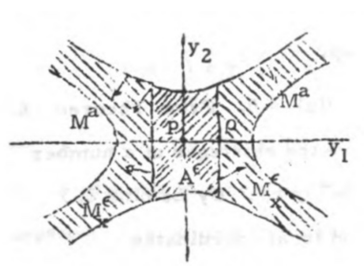

Figure 6. The case of $n = 2$, $\lambda = 1$

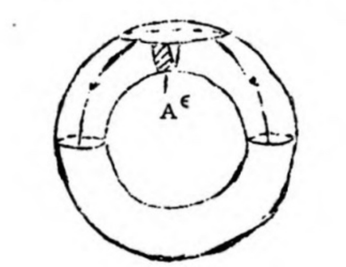

Figure 7. The case of the torus, considered in the Introduction

A smooth function f on a smooth manifold M is called non-degenerate if all critical points of f are non-degenerate; and such a function is called strictly non-degenerate if $f(p) \neq f(q)$ for every pair of critical points p and q. In Theorem 2.1 and Theorem 2.5, we only considered strictly non-degenerate functions.

It is natural to question whether there always exist non-degenerate and strictly non-degenerate functions on a given manifold. The answer is given by the following.

Theorem 2.6 (Morse-Thom)**.** *let M be a compact smooth manifold, $\mathfrak{Z}$ the space of smooth functions on M, provided with the compact open topology. Then the subset of non-degenerate functions is everywhere dense in $\mathfrak{Z}$.*

For the proof of this theorem, see [3], p.155.

An immediate consequence of Theorem 2.6 is the following theorem.

Theorem 2.7. *Let the situation be as in Theorem 2.6. Then the subset of strictly non-degenerate functions is also dense in* $\mathfrak{Z}$.

Proof. It is sufficient to prove the following local result:

Let f be a function on $\mathbb{R}^n$, $(x_1, ..., x_n)$ with only one non-degenerate critical point at $(0, ..., 0)$. Then there exists a function f' on $\mathbb{R}^n$ such that:

(1) $|f - f'| < \epsilon$ everywhere on $\mathbb{R}^n$ and $f = f'$ outside a neighborhood of $(0, ..., 0)$;
(2) f' also has only one (non-degenerate) critical point, and this is again the point $(0, ..., 0)$;
(3) $f'(0) = f(0) + \epsilon$.

Here ϵ denotes an arbitrarily but sufficiently small positive number. Set

$$A = \{(x_1, ..., x_n) \in \mathbb{R}^n, \sum_{i=1}^{n} x_i^2 \leq 1\}$$

$$B = \{(x_1, ..., x_n) \in \mathbb{R}^n, 1 \leq \sum_{i=1}^{n} x_i^2 \leq 2\}$$

$$C = \{(x_1, ..., x_n) \in \mathbb{R}^n, \sum_{i=1}^{n} x_i^2 \geq 2\}$$

Let g be a function on $\mathbb{R}^n$ with $g(x) = 0$ if $x \in C$, $0 \leq g(x) \leq 1$ if $x \in B$, and $g(x) = 1$ if $x \in A$. Taking for $f'(x)$ the function $f(x) + \epsilon g(x)$, $\epsilon > 0$, we can see that f' fulfills the conditions of (1), (2), (3) provided that ϵ is sufficiently small. The only thing to be verified is that f' has no critical points on B. But on B, we have uniform bounds α and β, $\alpha > 0$, such that $\|df\| \geq \alpha$ and $\|dg\| \leq \beta$. From

$$\|df'\| \geq \|df\| - \epsilon\|dg\| \geq \alpha - \epsilon\beta$$

it follows that f' has no critical points on B provided that ϵ is sufficiently small. Hence f' has only one (non-degenerate) critical point and that is the point 0. □

3. The Morse Inequalities

Let M be a compact manifold, which can be built up by successively attaching cells, int he way described in section 2. Then it can be shown (see [8]) that there is a CW-complex K such that its cells are in dimension preserving 1-1 correspondence with the attaching cells, and the homology of K is the homology of M (with respect to any group of coefficients). Accepting this result, the well known Morse inequalities follow immediately from the results of section 2 if we take $\mathbb{R}$ as a domain of coefficients. In fact, let

$$C = \sum_{i=0}^{n} C_i$$

the (naturally graded) vector space of chains of K,

$$Z = \sum_{i=1}^{n} Z_i$$

the space of cycles,

$$B = \sum_{i=1}^{n} B_i$$

the space of boundaries and

$$H = \sum_{i=1}^{n} H_i$$

the real homology group of K. By definition, we have the exact sequences:

(1) $0 \to Z \to C \xrightarrow{\delta} B \to 0$
(2) $0 \to B \to Z \to H \to 0$

where δ reduces the degree by 1. Set $\dim C_i = c_i$, $\dim Z_i = z_i$, $\dim B_i = b_i$, $\dim H_i = h_i$ (the i-th Betti number of K), $i = 0, 1, 2, \dots$. Combining (1) and (2), we find:

$$z_i + b_{i-1} = c_i, \quad b_i + h_i = z_i, \quad c_i - h_i = b_i + b_{i-1} \quad i = 0, 1, \dots \quad (b_{-1} = 0)$$

Since $b_i \geq 0$, $i = 0, 1, \dots$, these relations lead to the following sequence of inequalities:

$$\begin{gathered} c_0 \geq h_0 \\ c_1 - c_0 \geq h_1 - h_0 \\ c_2 + c_1 - c_0 \geq h_2 - h_1 + h_0 \\ \dots\dots \end{gathered}$$

Let f be a non-degenerate smooth function on the compact smooth manifold M. By Theorem 2.7, there is a strictly non-degenerate function f' on M such that f and f' have the same number of critical points of index i, $i = 0, 1, 2, \dots$. Combination of Theorem 2.5, the result stated at the beginning of this section, and the above set of inequalities leads to

Theorem 3.1. *Let f be a non-degenerate smooth function on the compact smooth manifold M. Let c_i be the number of critical points of index i of f and h_i, $i = 0, 1, 2, \dots$ the i-th Betti number of M. Then there exists a sequence of non-negative integers $b_{-1}, b_0, b_1, \dots$ such that*

$$c_i - h_i = b_i + b_{i-1} \quad i = 0, 1, 2, \dots$$

Therefore, there is a sequence of inequalities:

$$\begin{gathered} c_0 \geq h_0 \\ c_1 - c_0 \geq h_1 - h_0 \\ c_2 + c_1 - c_0 \geq h_2 - h_1 + h_0 \\ \dots\dots \end{gathered}$$

These inequalities are known as the Morse inequalities. They imply that a (non-degenerate) smooth function on a compact smooth manifold necessarily has a certain number of critical points of index i, $i = 0, 1, \dots$.

4. Manifolds Embedded in an Euclidean Space

In this section, the results of section 2 are applied to the special case that M^n is embedded smoothly in a Euclidean space $\mathbb{R}^{n+k}$, $(x_1, ..., x_{n+k})$ and f is the distance function from the points of M to a fixed point p, $p \in \mathbb{R}^{n+k}$. We denote this function by $l_p = l_p(x)$, $x \in M^n$.

Let $q \in M^n$ be a critical point of l_p. This is equivalent to saying that pq is perpendicular to the tangent space TM_q of M at q. Therefore, q is also a critical point of the function l_r, where r denotes any point of the line pq, $r \notin M^n$.

We choose a convenient coordinate system as follows. For q, we choose $q = 0 = (0, ..., 0)$ and $TM_0 = \{x_{n+1} = x_{n+2} = ... = x_{n+k} = 0\}$. In a neighborhood of 0, $x_1, ..., x_n$ can be used as a system of local coordinates on M, and in $\mathbb{R}^{n+k}$, M^n can be given locally by

$$x_{n+l} = g_l(x_1, ..., x_n), \quad l = 1, ..., k$$

Since $\overrightarrow{p0}$ is perpendicular to M^n, the coordinate of p can be taken as $(0, ..., 0, p_1, ..., p_k)$. Finally, we set $(0, ..., 0, tp_1, ..., tp_k) = tp$ for $0 < t$.

Let, as usual, $\mathbf{H}l_{tp}(0)$ denote the Hessian of l_{tp} at 0. A straightforward calculation gives:

$$\mathbf{H}l_{tp}(0) = \frac{1}{t\|p\|} I - \left(\sum_{l=1}^{k} \frac{p_l}{\|p\|} \frac{\partial^2 g_l}{\partial x_i \partial x_j}(0) \right),$$

where I denotes the $n \times n$ identity matrix and $\|p\| = (\sum_{l+1}^{k} p_l^2)^{1/2}$. Using a well known result on quadratic forms (see [4], p. 158), we conclude that it is possible to choose a base in M_0, such that

$$\frac{1}{\|p\|} I \qquad \text{and} \qquad -\sum_{l=1}^{k} \frac{p_l}{\|p\|} \frac{\partial^2 g_l}{\partial x_i \partial x_j}(0)$$

are simultaneously reduced to diagonal form, in such a way that

$$\frac{1}{\|p\|} I$$

is reduced to the identity itself. With respect to this base, $\mathbf{H}l_{tp}(0)$ is given by a matrix

$$\mathbf{H}l_{tp}(0) = \begin{pmatrix} a_{11} + \frac{1}{t} & & 0 \\ & \ddots & \\ 0 & & a_{nn} + \frac{1}{t} \end{pmatrix}.$$

We see

(1) the coefficients in the diagonal are strictly decreasing functions of t;
(2) only for a finite number of values of t, $t_1, ..., t_m$, $\mathbf{H}l_{tp}(0)$ is non-singular, and t is a degenerate critical point of l_{tp};
(3) for $0 < t \ll 1$, $\mathbf{H}l_{tp}(0)$ is positive definite.

It follows that the index of $\mathbf{H}l_{tp}(0)$ is a decreasing (integer valued) function of $\frac{1}{t}$, which only jumps at the points $t_1 p, ..., t_m p$, and at these points this jump just equals $\nu(\mathbf{H}l_{t_i p}(0))$ = the dimension if the nullity of $\mathbf{H}l_{t_i p}(0)$, $i + 1, ...m$. Since by (3), the index of $\mathbf{H}l_{tp}(0)$ is zero in a neighborhood of 0, we can state

Theorem 4.1. *Let the smooth manifold M be embedded smoothly in the Euclidean space R. For the point $p \in R$, $p \notin M$, let $l_p(x)$ denote the distance from the points*

$x \in M$ to p. Let q be a critical point of $l_p(x)$ (degenerate or not). Then the index of the Hessian $\mathbf{H}l_p(q)$ is given by

$$\operatorname{index} \mathbf{H}l_p(q) = \sum_{0<t<1} \nu(\mathbf{H}l_{(1-t)p+tq}).$$

Setting $L_p(x) = l_p^2(x)$, we immediately deduce from

$$\frac{\partial^2 L_r}{\partial x_i \partial x_j}(0) = 2l_r \frac{\partial^2 l_r}{\partial x_i \partial x_j}(0), \qquad i,j = 1,...,n \qquad r \notin M,$$

Corollary 4.2. *The statement of Theorem 4.1 remains true if we replace $l_p(x)$ by $L_p(x)$.*

<u>*Example:*</u> Let M be the unit sphere S^2 embedded in R^3. Let p be a point in $\mathbb{R}^3$, $p \notin S^2$. The critical points of L_p are obviously the intersection points q_1 and q_2 of $p0$ with S^2. In figure 8, $\mathbf{H}L_p(q_1)$ has index 0 (L_x is degenerate for no point between p and q_1) and $\mathbf{H}L_p(q_2)$ has index 2, for there is only one point x between p and q_2 such that $\mathbf{H}L_x(q_2)$ is degenerate, which is 0 and $\nu(\mathbf{H}L_0(q_2)) = 2$.

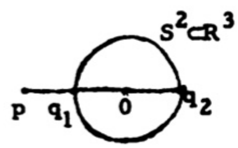

FIGURE 8.

Let $(x_1, ..., x_n)$ be a system of coordinates in $\mathbb{R}^n$, and let the straight line $g = g(t)$ be given by

$$x_i = p_i + tq_i, \qquad i = 1, ..., n.$$

A **variation of g** will be a smooth family of lines

$$V(s,t) : x_i(s) = p_i(s) + tq_i(s), \qquad -\infty < s < \infty,$$

with $p_i(0 = p_i)$ and $q_i(0) = q_i$, $i + 1, ..., n$.

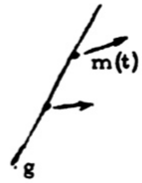

FIGURE 9.

A variation of g induces in a natural way a vector field $J(t)$ along g:

$$J(t) = \left.\frac{dp_i(s)}{ds}\right|_{s=0} + \left.\frac{dq_i(s)}{ds}\right|_{s=0}$$

Such a field, induced by a variation of g will be called a **Jacobi field along g**. Jacobi fields are also characterized by the property $\frac{d^2J}{dt^2} = 0$ all along g. It follows, that the Jacobi fields along g form a vector space of dimension $2n$, which will be denoted by J_g.

Let M^m be a proper smooth submanifold of $\mathbb{R}^n$, and let g be perpendicular to M at $g(0)$. A **variation of g relative to M** is a variation $V(s,t)$ of g with the following properties:

(1) $V(s,0) \in M$
(2) all lines $V(s,t)$ are perpendicular to M at $V(s,0)$.

FIGURE 10.

Let $J_g(M)$ be the set of Jacobi fields along g which are induced by variations of g relative to M. In the sequel to this section, it will become obvious that $J_g(M)$ is a linear subspace of J_g. So far, at every point $g(t_0)$ of g, there is a natural restriction homomorphism

$$r_{t_0} : J_g(M) \to T_{g(t_0)}R.$$

Definition 4.3. $g(t_0)$, $t_0 \neq 0$, is called a **focal segment** of M of multiplicity ν, $\nu > 0$, if and only if $\dim \operatorname{Ker} r_{t_0} = \nu$.

The fundamental result of this section is the following

Theorem 4.4. *Let the notations be as introduced before. Then $g(t_0)$ is a focal segment of M of multiplicity ν if and only if $g(0)$ is a degenerate critical point of $L_{g(t_0)}(x)$ of nullity ν.*

Proof. Let ν be the nullity of $L_{g(t_0)}(x)$. We may assume $g(t_0)$ to be the point $(0,...,0)$. If $(u_1,...,u_m)$ is a system of local coordinates on M in a neighborhood of $g(0)$, and as usual

$$L = \sum_{i=1}^{n} x_i^2(u_\alpha),$$

then the number ν equals m minus the rank of the quadratic form

$$\mathbf{H}L|_{g(0)} = \sum_{\alpha,\beta=1}^{m} \left.\frac{\partial^2 L}{\partial u_\alpha \partial u_\beta}\right|_{g(0)} U^\alpha V^\beta,$$

where $U = (U^\alpha)$ and $V = (V^\beta)$ are tangent vectors to M at $g(0)$. Setting

$$\frac{\partial x_i}{\partial u_\alpha} = \Lambda^i_\alpha \quad \text{and} \quad \frac{\partial}{\partial u_\beta}(\Lambda^i_\alpha) = \Lambda^i_{\alpha,\beta}, \qquad i = 1, ..., n; \quad \alpha, \beta = 1, ..., m,$$

we find:

$$\frac{\partial L}{\partial u_\alpha} = 2\sum_{i=1}^n x_i(u_\alpha)\frac{\partial x_i}{\partial u_\alpha} = 2\sum_{i=1}^n x_i \Lambda^i_\alpha$$

$$\frac{\partial^2 L}{\partial u_\alpha \partial u_\beta} = 2\sum_{i=1}^n \Lambda^i_\alpha \Lambda^i_\beta + 2\sum_{i=1}^n x_i \Lambda^i_{\alpha,\beta}$$

If we denote by (U, V) the inner product, induced by the embedding of M in $\mathbb{R}^n$, and set

$$n(U, V) = \sum_{\alpha,\beta=1}^m \sum_{i=1}^n \frac{x_i}{l}\Lambda^i_{\alpha,\beta} U^\alpha V^\beta,$$

where l denotes the distance from $g(0)$ to 0, we find

$$\mathbf{H}L|_{g(0)}(U, V) = 2\{(U, V) + l \cdot n(U, V)\}.$$

Let $V(s, t) = p(s) + tq(s)$ be a variation of g relative to M. From the definition of $J_g(M)$, it follows immediately

$$(1) \quad p(s) \in M;$$

$$(2) \quad \sum_{i=1}^n q_i(s)\Lambda^i_\alpha(p(s)) = 0, \alpha = 1, ..., m \text{ for all values of } s.$$

Differentiation of (2) with respect to s gives

$$(3) \quad \sum_{i=1}^n \frac{dq_i}{ds}\Lambda^i_\alpha(p(s)) + \sum_{\beta=1}^m \sum_{i=1}^n q_i(s)\Lambda^i_{\alpha,\beta}(p(s))\frac{\partial u_\beta}{\partial s}(p(s)) = 0, \quad \alpha = 1, ..., m$$

where $u_\beta = u_\beta(s)$ is the curve on M, described by $x_i = p_i(s)$. Setting in particular $s = 0$, we get

$$(4) \quad \sum_{i=1}^n \frac{dq_i}{ds}(0)\Lambda^i_\alpha(g(0)) + \sum_{\beta=1}^m \sum_{i=1}^n \frac{p_i(0)}{l}\Lambda^i_{\alpha,\beta}(g(0))\frac{\partial u_\beta}{\partial s}(g(0)) = 0, \quad \alpha = 1, ..., m.$$

The Jacobi field along g induced by $V(s, t)$ can be written as

$$\eta_i(t) = \eta_i(0) + t\dot{\eta}_i(0), \qquad i = 1, ..., n$$

with

$$\left.\frac{dp_i}{ds}\right|_{s=0} = \eta_i(0)$$

$$\left.\frac{dq_i}{ds}\right|_{s=0} = \dot{\eta}_i(0), \qquad i = 1, ..., n.$$

Obviously, $\eta(0) \in T_{g(0)}M$ and $\eta_\beta = \frac{\partial u_\beta}{\partial s}$, $\beta = 1, ..., m$. Now let $U = (U^\alpha)$ be any tangent vector to M^m in $g(0)$. Multiplying (4) with U^α, we get by summation:

$$(\dot{\eta(0)}, U) - n(\eta(0), U) = 0.$$

Conversely, let $\eta(t)$ be a Jacobi field along g, such that

(a) $\eta(0) \in T_{g(0)}M$;

(b) $(\dot{\eta}(0), U) - n(\eta(0), U) = 0$ for all $U \in T_{g(0)}M$.

Let $p_i(s)$ be a curve on M with $p_i(0) = p_i$, and

$$\left.\frac{dp_i}{ds}\right|_{s=0} = \eta_i(0)$$

Consider (3) as a system of differential equations for $q_1, ..., q_n$ with s as independent variable. There exists a solution with

$$q_i(0) = \eta_i(0) \quad \text{and} \quad \left.\frac{dq_i}{ds}\right|_{s=0} = \dot{\eta}_i(0), \qquad i = 1, ..., n,$$

since by (b), (4) is satisfied for these values of $q_i(0)$ and $\left.\frac{dq_i}{ds}\right|_{s=0}$. Let $q_i(s)$ be such a solution. For this solution, we have:

$$\sum_{i=1}^{n} q_i(s)\Lambda_\alpha^i(p(s)) = \text{constant}.$$

Since

$$\sum_{i=1}^{n} q_i(0)\Lambda_\alpha^i(p(0)) = 0,$$

this constant is zero. It follows that there exists an element of $J_g(M)$ inducing $\eta(t)$ along g. Therefore, the properties (a) and (b) are characteristic for the elements of $J_g(M)$.

The elements $\eta \in J_g(M)$, for which $r_{t_0}(\eta)(0) = 0$ are characterized as those elements of J_g, which satisfy the following conditions:

(a) $\eta(0) \in T_{g(0)}M$;
(b) $(\eta(0), U) + l \cdot n(\eta(0), U) = 0$ for all $U \in T_{g(0)}M$;
(c) $\eta(0) + l \cdot \dot{\eta}(0) = 0$.

From this, it is immediate that

$$\dim \operatorname{Ker} r_{t_0} = \nu(\mathbf{H}L_{g(0)}(p)) = \nu.$$

This proves Theorem 4.4. □

Remark 4.5. If we define a linear transformation

$$T_* : T_{g(0)}M \to T_{g(0)}M$$

by setting $(T_*U, V) = n(U, V)$, the conditions (a) and (b) for the elements of $J_g(M)$ can be replaced by the following ones:

(a') $\eta(0) \in T_{g(0)}M$;
(b') $\dot{\eta}(0) - T_*(\eta(0)) \in T_{g(0)}M^\perp$ (the orthogonal complement of $T_{g(0)}M$ in $T_{g(0)}R$).

This leads to $m + (n - m) = n$ linearly independent conditions for an element of J_g to be in $J_g(M)$. Therefore:

$$\dim J_g(M) = n.$$

From this it follows easily that $\dim \operatorname{Ker} r_{t_0} = \nu$ is equivalent to

$$\dim(TR_{g(t_0)}/r_{t_0}(J_g(M))) = \nu.$$

Combination of Corollary 4.2 and Theorem 4.4 gives

Theorem 4.6. *Let M be a proper smooth submanifold of $\mathbb{R}^n$. Let $a \in \mathbb{R}^n$, $a \notin M$, and let $b \in M$ be a critical point of the function $L_a(x)$. Let $\nu(t)$ be the multiplicity of the focal segment, and zero otherwise. Then the index of b equals*

$$\sum_{0<t<1} \nu(t).$$

5. Topology of Flag Manifolds

This section is devoted to a sketch of some typical topological applications of elementary Morse theory, as this theory was developed in the preceding sections.

As usual, we denote by $U(n)$ the unitary group in n complex variables. $U(n_1) \times U(n_2) \times ... \times U(n_k)$, $n_1 + ... + n_k = n$, can be considered as a subgroup of $U_(n)$ in a canonical way. The quotient space is the complex flag manifold of type $(n_1, ..., n_k)$, to be denoted as $W(n_1, ..., n_k)$. For $k = 2$, we get a Grassmann variety, in particular for $n_1 = 1$, $n_2 = n - 1$, a complex projective space.

As a first application, we shall sketch the proof of the following

Theorem 5.1. *$H^i(W(n_1, ..., n_k), \mathbb{Z}) = 0$ for i odd, and $W(n_1, ..., n_k)$ has no torsion.*

The tangent space to $U(n)$ as the identity I of $U(n)$ we denote as R. As a real vector space, it can be identified with the vector space of skew-symmetric complex $n \times n$ matrices, i.e., those matrices (a_{ij}) with $a_{ij} = -\bar{a}_{ji}$. If we set

$$(X, Y) = -\mathrm{Tr}(X \cdot Y)$$

for every pair of elements X, Y in R, we immediately see that we get a positive definite Riemannian metric $\mathfrak{m}$ on R, which makes R into an Euclidean space. $U(n)$ operates on R by the adjoint action $\mathrm{Ad}_{U(n)}$, defined by

$$\mathrm{Ad}_U X = UXU^{-1}, \quad U \in U(n), \quad X \in R.$$

From a standard property of the trace, we readily deduce that $\mathrm{Ad}_{U(n)}$ leaves $\mathfrak{m}$ invariant. Therefore, $\mathrm{Ad}_{U(n)}$ gives a representation of $U(n)$ as a group of orthogonal transformations of R.

We intend to study the orbits of $U(n)$ in R. Let $X \in R$. It is easy to see that the subgroup of $U(n)$ leaving X invariant is of the type $U(n_1) \times ... \times U(n_k)$, $n_1 + ... + n_k = n$. In fact, let V be an n-dimensional complex vector space, on which $U(n)$ operates as the group of unitary matrices and R as the group of skew symmetric matrices. In the same way as in the real case for symmetric matrices, there is an orthogonal decomposition of V, such that X leaves the factors of this decomposition invariant. The elements of $U(n)$ which leaves X invariant by the adjoint actions are those unitary matrices which leave these factors invariant. Therefore, for a suitable base, the subgroup of $U(n)$ leaving X invariant is a group $U(n_1) \times ... \times U(n_k)$, $n_1 + ... + n_k = n$ canonically embedded in $U(n)$. The orbit of X is the flag manifold $W(n_1, ..., n_k)$. It is clear that all types of flag manifolds appear in this way.

Let $X \in R$. The orbit of $\mathrm{Ad}_{U(n)}$ through X will be denoted by M_X.

Lemma 5.2. *Let M be any orbit of $U(n)$ in R, $A \in R$, a point of an orbit of maximal dimension, and let AB be orthogonal to M at B. A segment BC of the* *line AB is a focal segment of multiplicity ν of $L_B(M)$ if and only if* $\dim M_A = \dim M_C = \nu$

This lemma is a very special case of a more general theorem, which states, for example, the same results for the case that $U(n)$ is replaced by an arbitrary connected Lie group.

The proof of Lemma 5.2 is very analogous to the considerations in section 9 and will therefore be omitted.

From Lemma 5.2, Theorem 4.4, and the fact that the dimension of a complex flag manifold is always even, we conclude that the index of every critical point of a function $L_P(M)$ is always even. Applying the results of section 3 to a general point P, we obtain the statement of Theorem 5.1 (The existence of such a point P can be proved easily in this case; it follows also from the following considerations).

As a second example, we sketch how elementary Morse theory can be used to obtain the Betti numbers of a complex flag manifold.

It follows from the definition of $\mathfrak{m}$ that the Jacobi bracket $[X, Y] = XY = YX$, $X, Y \in R$ has the property

$$(1) \qquad ([X, Y], Z) = (X, [Y, Z])$$

Lemma 5.3. *The tangent space T_X to M_X at X is given by*

$$(a) \qquad T_X = \{Z = [Y, X], Y \in R\}$$

and the normal space to M_X at X by

$$(b) \qquad N_X = \{Y \in R, [Y, X] = 0\}.$$

Proof. (a) For every element $Y \in R$, e^{tY} is a 1-parameter family subgroup of $U(n)$, to which Y is the tangent vector in I. We get all vectors Z of T_X by taking

$$(2) \qquad Z = \lim_{t \to 0} \frac{\mathrm{Ad}_{e^{tY}} X - X}{t}$$

for all elements $Y \in R$. But instead of (2), we can write

$$\begin{aligned} Z &= \lim_{t \to 0} \frac{(1 + tY + ...)X(1 - tY + ...) - X}{t} \\ &= \lim_{t \to 0} \frac{t(YX - XY) + t^2(...)}{t} \\ &= YX - XY = [Y, X]. \end{aligned}$$

(b) $[X, Y] = 0$ is equivalent to $(A, [X, Y]) = 0$ for all $A \in R$. This on its turn is by (1) equivalent to $([A, X], Y) = 0$ for all $A \in R$ and, by (a), just means that $Y \in N_X$. □

Lemma 5.4. *If at one of its points, a line is perpendicular to an orbit, then that line is perpendicular to all orbits which it intersects.*

Proof. Let the line $B + tA$ be perpendicular to M_B at B. By Lemma 5.3, this means that $([X, B], A) = 0$ for all $X \in R$. Since $([X, A], A) = (X, [A, A]) = 0$ for all $X \in R$, we find: $([X, B + tA], A) = ([X, B], A) + t([X, A], A) = 0$ for all $X \in R$, and this is precisely the formal expression of our assertion. □

With respect to a suitable basis in V, every element of R can be written as

$$\begin{pmatrix} i\theta_1 & a_{12} & \cdots & a_{1n} \\ -\bar{a}_{12} & i\theta_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ -\bar{a}_{1n} & \cdots & \cdots & i\theta_n \end{pmatrix}$$

then

$$R = h \oplus e_{12} \oplus ... \oplus e_{(n+1)n} = h \oplus \sum e_{kl}$$

where

$$h = \left\{ \begin{pmatrix} i\theta_1 & & 0 \\ & \ddots & \\ 0 & & i\theta_n \end{pmatrix} \right\}, \quad e_{kl} = \left\{ \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & a_{kl} & \vdots \\ \vdots & -\bar{a}_{kl} & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix} \right\}, \quad k \neq 1$$

Let h_* denote the subset of h for which the real number $\theta_1, ...\theta_n$ are all different. In geometric language, h_* consists of those points in the vector space h, which do not lie on any of the hyperplanes $\theta_i - \theta_j = 0$, $i, j = 1, ..., n$, $i < j$. Obviously, h_* consists of "almost all" points of h.

Lemma 5.5. *If $X \in h_*$, then $N_X = h$.*

Proof. It is readily verified that in this case, $[X, \sum e_{kl}] = \sum e_{kl}$. From $R = h \oplus \sum e_{kl}$,we deduce $[X, R] = [X, h] \oplus [X, \sum e_{kl}] = \sum e_{kl}$, since $[X, h] = 0$ by definition of the bracket. By lemma 5.3, $[X, R] = T_X$, $N_X = h$. □

Lemma 5.6. *(a) Let M be any orbit of $\mathrm{Ad}_{U(n)}$ in R and $P \in h_*$. Then the critical points of the function $L_P(M)$ are precisely the points of $M \cap h$. (b) In particular, these points are independent of the choice of P in h_*.*

Proof. (a) Let $\mathfrak{L}$ be a critical point of $L_P(M)$. The line $P\mathfrak{L}$ is perpendicular to M at $\mathfrak{L}$. From Lemma 5.4, it follows that $P\mathfrak{L}$ is perpendicular to M_P at P. Since $P \in h_*$, and , by Lemma 5.5, we also have the direction of $P\mathfrak{L}$ lies in h, so we see that $P\mathfrak{L}$ lies in h, and in particular that $\mathfrak{L} \in h$.

(b) For every point $X \in h$, we have $[X, h] = 0$. If $\mathfrak{L} \in M \cap h$, then every line through $\mathfrak{L}$ in h is perpendicular to M at $\mathfrak{L}$ (Lemma 5.3), in particular $P\mathfrak{L}$. Since M is always compact, $L_P(M)$ certainly has critical points. Therefore, every orbit intersects h. □

Let $P \in h_*$ and $\mathfrak{L}$ a (non-degenerate) critical point of $L_P(M)$. By Theorem 4.4 and the index $\lambda(\mathfrak{L})$ of $\mathfrak{L}$ is given by

$$\lambda(\mathfrak{L}) = \sum_i \nu(F_i),$$

where the points F_i are the focal points on the segment $P\mathfrak{L}$ and $\nu(F_i)$ are their multiplicities as focal points. From Lemma 5.2, we see that the focal points F_i are the points where the segment $P\mathfrak{L}$ intersects orbits of lower dimension, and furthermore that, if F is such a point, then

$$\nu(F) = \dim M_P - \dim M_F.$$

But since $P\mathfrak{L}$ lies in h, these points F_i are precisely the points of intersection with the hyperplanes $\theta_i - \theta_j = 0$, $i < j$. It is obvious, that $\dim M_P - \dim M_F$ equals two times the number of $\theta_i - \theta_j = 0$, $i < j$, which vanish at F.

The problem to determine the Betti numbers of a flag manifold is now reduced to a problem of Euclidean geometry. In fact, let M be any orbit, that is some type of flag manifold. The intersection points $\mathfrak{L}_1, ..., \mathfrak{L}_5$ of M with h can be determined explicitly by an algebraic procedure. Taking a general point P in h_*, the intersections of the segments $P\mathfrak{L}_1, ..., P\mathfrak{L}_5$ with the hyperplanes $\theta_i - \theta_j = 0$, $i < j$, can be determined also, and therefore the index of every critical point of $L_P(M)$. Since the homology of M vanishes in odd dimensions, the i-th Betti number of M is just the number of critical points $\mathfrak{L}_j$ of index i.

Needless to say that similar methods apply to many other situations.

6. The Structure of the Space $\Omega_{p,q}(M)$

Let M be a compact Riemannian manifold and p, q a couple of points on M. We denote $\Omega_{p,q}(M)$ the set of sectionally smooth curves joining p with q, i.e. the set of piecewise smooth mappings $f : [0,1] \to M$, with $f(0) = p$, $f(1) = q$, parametrized proportionally to the length of arc (constant speed); by $\mathfrak{L}(c)$, $c \in \Omega_{p,q}(M)$ the length of c; by $\Omega^a_{p,q}(M)$ the subset of $\Omega_{p,q}(M)$ determined by

$$\Omega^a_{p,q}(M) = \{c \in \Omega_{p,q}(M) | \mathfrak{L}(c) \leq a\};$$

by $\rho(x,y)$ the distance of pair of points x, y on M, i.e. the infimum of $\mathfrak{L}(d)$, $d \in \Omega_{x,y}(M)$. A simple argument (see [1], pg. 45) shows that

$$\rho(c,c') = \max_{0 \leq t \leq 1} \rho(c(t), c'(t)) + |\mathfrak{L}(c) - \mathfrak{L}(c')|$$

is a distance function on $\Omega_{p,q}(M)$. With respect to the topology induced by this metric, $\mathfrak{L}$ is a continuous function on $\Omega_{p,q}(M)$. We consider $\Omega_{p,q}(M)$ provided with this topology.

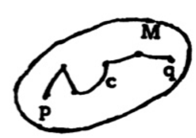

Figure 11.

$\rho(x,y)$ is a continuous, but in general not a differentiable function on $M \times M$. However, there exists a number $\rho > 0$ such that for $\rho(x,y) < \rho$, there is just one geodesic arc from x to y, of length $\rho(x,y)$. For this ρ, $\rho^2(x,y)$, restricted to the part of $M \times M$ determined by $\rho(x,y) < \rho$, is a differentiable function of (x,y). ([1], pg. 49).

In the sequel, $\Omega_{p,q}(M)$ will be considered for a fixed pair of points p, q on M. So we can write Ω instead of $\Omega_{p,q}(M)$ and Ω^a instead of $\Omega_{p,q}(M)^a$. Let n be such that $b = \frac{a^2}{n+1} < \rho^2$. If we set $M \times ... \times M (n \text{ times}) = M_*$ and

$$\rho^2(p,x_1) + ... + \rho^2(x_n,q) = \phi(x_1,...,x_n) = \phi(x),$$

then ϕ is a differentiable function on M^b_*, where M^b_* is determined by

$$M^b_* = \{x \in M_*, \phi(x) \le b\}.$$

The following theorem relates Ω^a very strongly with M^b_*.

Theorem 6.1. *Ω^a and M^b_* are of the same homotopy type, i.e. there exist maps $\alpha : \Omega^a \to M^b_*$ and $\beta : M^b_* \to \Omega^a$, such that $\beta \circ \alpha$ and $\alpha \circ \beta$ are homotopic with the identity map of Ω^a and M^b_* respectively.*

Proof. The proof is carried out in four steps.

(a) Definition of α.

For $c \in \Omega^a$, we define

$$\alpha(c) = \{c(t_1), ..., c(t_n)\}, \quad t_i = \frac{i}{n+1}, \quad i = 1, ..., n.$$

We have to verify: $\alpha(c) \subset M^b_*$. The length of the arc $c(t_i)c(t_{i+1})$ of c equals $\frac{\mathfrak{L}(c)}{n+1}$, hence

$$\rho(c(t_i), c(t_{i+1})) \le \frac{\mathfrak{L}(c)}{n+1}.$$

From the definition of ϕ, it follows that

$$\phi(\alpha(c)) \le (n+1)\frac{\mathfrak{L}^2(c)}{(n+1)^2} \le \frac{a^2}{n+1} = b.$$

So for every

$$c \in \Omega^a,\ \alpha(c) \in M^b_* \text{ or } \alpha(\Omega^a) \subset M^b_*.$$

(b) Definition of β.

If we set $p = x_0$ and $q = x_{n+1}$, from the definition of ϕ, we derive immediately for $x = (x_1, ..., x_n) \in M^b_*$:

$$\sum_{i=0}^{n} \rho^2(x_i, x_{i+1}) \le b, \text{ or, for } i = 0, ..., n : \rho(x_i + x_{i+1}) \le \sqrt{b} = \frac{a}{\sqrt{n+1}} < \rho.$$

Hence for $i = 0, ..., n$, x_i and x_{i+1} can be joined by a unique geodesic arc. The union of these arcs determines an element of Ω, which, by definition, is $\beta(x)$. We only have to verify that $\beta(x) \in \Omega^a$, i.e. that $\mathfrak{L}(\beta(x)) \le a$. This is an immediate consequence of the Schwarz inequality:

$$\mathfrak{L}(\beta(x)) = \sum_{i=0}^{n} \rho(x_i + x_{i+1}) \le \sqrt{n+1}\sqrt{\phi(x)} \le \sqrt{n+1}\frac{a}{\sqrt{n+1}} = a.$$

(c) Construction of a deformation $D_\tau : \Omega^a \to \Omega^a$, $0 \le \tau \le 1$, with $D_0 =$ identity and $D_1 = \beta \circ \alpha$.

Let $c \in \Omega^a$ and let for $i = 0, ..., n+1$, $x_i = c(t_i)$, $t_i = \frac{i}{n+1}$. Set $x_i^\tau = c((1-\tau)t_i + \tau t_{i+1})$ and define $D_\tau(c)$ as the (conveniently parametrized) element of Ω^a, consisting of the (unique determined) geodesic arc px_0^τ, the arc $x_0^\tau x_1$ of c, the geodesic arc $x_1 x_1^\tau, ...$, the arc $x_n^\tau q$ of c. Obviously, D_0 is the identity and $D_1 = \beta \circ \alpha$. It has to be verified that D_τ is actually a deformation. This is straightforward but not completely trivial. We refer to [1], pg. 51.

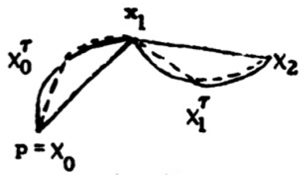

FIGURE 12.

(d) Construction of a deformation $\Delta_\tau : M_*^b \to M_*^b$, with $\Delta_0 =$ identity and $\Delta_1 = \alpha \circ \beta$.

For $x \in M_*^b$ let $s_i \in [0,1]$ be such that $\beta(x)(s_i) = x_i$, $i = 0, ..., n+1$. Set $\beta(x)(t_i) = y_i$, $\beta(x)((1-\tau)s_i + \tau t_i) = x_i^\tau$, $x^\tau = (x_1^\tau, ..., x_n^\tau)$ and $\Delta_\tau(x) = x^\tau$. We have to verify $x^\tau \in M_*^b$.

Since the distance from x_i^τ to x_{i+1}^τ at most equal the distance from x_i^τ to x_{i+1}^τ along $\beta(x)$, we find

$$\begin{aligned}\rho(x_i^\tau, x_{i+1}^\tau) &\leq \mathfrak{L}(\beta(x))[(1-\tau)s_{i+1} + \tau t_{i+1} - (1-\tau)s_i - \tau t_i] \\ &= \mathfrak{L}(\beta(x))[(1-\tau)(s_{i+1} - s_i) + \tau(t_{i+!} - t_i)]\end{aligned}$$

If we set the distance from x_i^τ to x_{i+1}^τ along $\beta(x)$ as δ_i, this inequality becomes

$$\rho(x_i^\tau, x_{i+1}^\tau) \leq (1-\tau)\delta_i + \tau \frac{\mathfrak{L}(\beta(x))}{n+1}.$$

From this follows, by the definition of ϕ:

$$\begin{aligned}\phi(x^\tau) &= \sum_{i=0}^{n} \rho^2(x_i^\tau, x_{i+1}^\tau) \\ &\leq (1-\tau)^2 \left(\sum_{i=0}^{n} \delta_i^2\right) + \tau^2 \frac{\mathfrak{L}^2(\beta(x))}{n+1} + 2\tau(1-\tau)\frac{\mathfrak{L}(\beta(x))}{n+1}\left(\sum_{i=0}^{n} \delta_i\right) \\ &= (1-\tau)^2\phi(x) + \tau^2 \frac{\mathfrak{L}^2(\beta(x))}{n+1} + 2\tau(1-\tau)\frac{\mathfrak{L}^2(\beta(x))}{n+1} \\ &= \phi(x) + \left(\phi(x) - \frac{\mathfrak{L}^2(\beta(x))}{n+1}\right)(1-\tau)^2 - \left(\phi(x) - \frac{\mathfrak{L}^2(\beta(x))}{n+1}\right).\end{aligned}$$

Since, by Schwarz inequality, $\phi(x) - \frac{\mathfrak{L}^2(\beta(x))}{n+1} \geq 0$, we find for $0 \leq \tau \leq 1$:

$$\phi(x^\tau) \leq \phi(x) \leq b.$$

This means that $x^\tau \in M_*^b$.

Δ_0 is the identity and $\Delta_1 = \alpha \circ \beta$ by definition. For the straightforward proof that Δ_τ is actually a deformation, we again refer to [1].

□

7. THE INDEX THEOREM

We use the notations of Section 6. On M_*^b, we shall study the function, which some may refer as the energy function:

$$\phi(x) = \sum_{i=0}^{n} \rho^2(x_i, x_{i+1})$$

Theorem 7.1. *$x = (x_1, ..., x_n) \in M_*^b$ is a critical point of ϕ if and only if*
- *(i) $px_1x_2..x_nq$ is a geodesic from p to q;*
- *(ii) $\rho(p, x_1) = \rho(x_1, x_2) = ... = \rho(x_nq)$.*

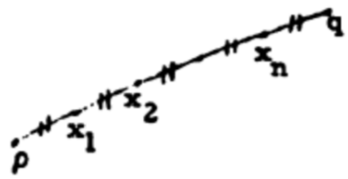

FIGURE 13.

Proof. We assume the First Variation Formula, which can be stated in the following way (see [3]). Let $a, b \in M$, $a \neq b$, with $\rho(a, b) < \rho$ (see section 6), let g be the unique geodesic segment from a to b. X_0 and X_1 are the unit tangent vector to g at a and b, respectively. U_0 and U_1 are tangent vectors to M at a and b, respectively. The function $\rho(x, y)$ is differentiable in a neighborhood of (a, b) on $M \times M$. So we can speak of the directional derivative of $\rho(x, y)$ in the direction of the tangent vector (U_0, U_1) at (a, b) as

$$\langle X_1, U_1\rangle - \langle X_0, U_0\rangle.$$

Furthermore, if we fix a and consider the function $\rho(a, x)$, $x \neq a$, on M, the directional derivative in direction U_1 is simply $\langle X_1, U_1\rangle$.

We denote by s_i the length of the geodesic segment g_i from x_i to x_{i+1}. On g_i, denote the unit tangent vectors to g_i at x_i, x_{i+1} by s_i^-, s_i^+, respectively, $i = 1, ..., n$. Then the directional derivative of ϕ in direction $(U_1, ..., U_n)$ is:

$$2s_0\langle s_0^+, U_1\rangle - 2s_1\langle s_1^-, U_1\rangle + 2s_1\langle s_1^+, U_2\rangle - ... - 2s_n\langle s_n^-, U_n\rangle.$$

For this expression, we can write:

$$2[\langle(s_0s_0^+ - s_1s_1^-), U_1\rangle + \langle(s_1s_1^+ - s_2s_2^-), U_2\rangle + ... + \langle(s_{n-1}s_{n-1}^+ - s_ns_n^-), U_n\rangle]$$

x is a critical point of ϕ, if and only if this expression vanishes for all tangent vectors $(U_1, ...U_n)$. This means precisely that x is a critical point of ϕ, if and only if $px_1x_2...x_n$ is a geodesic from p to q and $\rho(p, x_1) = \rho(x_1, x_2) = ... = \rho(x_n, q)$. □

Remark 7.2. If we consider the continuous function $L = \sum_{i=0}^n \rho(x_i, x_{i+1})$, then L is differentiable in a neighborhood of all points $x = (x_1, ..., x_n) \in M_*^b$ with $x_i \neq x_j$ for all $i \neq j$. In the same way as in the proof of Theorem 7.1, we see that such a point x is a critical point of L, if and only if $px_1x_2...x_nq$ is a geodesic from p to q.

The index theorem gives an expression for the index of a critical point of ϕ. To derive this expression, we need several preliminaries.

Lemma 7.3. *Let A be a quadratic form on the real vectors space Y. then the index of A can be characterized as the maximal dimension of a subspace of V on which the restriction of A is positive.*

Proof. Let d de this dimension, λ the index of A, and $\epsilon_1, ..., \epsilon_n$ a coordinate system on V, such that A is given by

$$\sum_{i=1}^{\lambda} \epsilon_i^2 + \sum_{j=\lambda+1}^{n} \epsilon_j^2.$$

Since A is negative definite on the subspace of V spanned by $x_1, ..., x_\lambda$, $d \geq \lambda$. Conversely, suppose A to be negative definite on a subspace W of V of dimension $d' > \lambda$. W has a subspace of dimension at least 1 in common with the subspace spanned by $x_{\lambda+1}, ..., x_n$ on which A is semi-positive definite. This is impossible, thus $\lambda \geq d$. Combination gives $\lambda = d$. □

Lemma 7.4. *Let $A(t)$, $0 \leq t \leq 1$, be a continuous family of quadratic forms on the n-dimensional real vector space V^n, with the following properties:*

(i) $A(t_1) \leq A(t_2)$ *for* $t_1 \leq t_2$*;*
(ii) $A(t)$ *is degenerate for a finite number of t-values,* $0 < t < 1$*:* $\alpha_1, ..., \alpha_r$ *and possibly for* $t = 0$*, but not for* $t = 1$*.*

Then: $\lambda(A(0)) - \lambda(A(1)) = \sum_{k=1}^{r} \nu(A(\alpha_k))$.

Proof. Let us look at the point α_i. From condition (i) and Lemma 7.2, we deduce

$$(2) \qquad \lambda(A(t)) \leq \lambda(A(\alpha_i)) \qquad \text{for } t \geq \alpha_i.$$

On the other hand, if $A(\alpha_i)$ is negative definite on the subspace W of V^n, then $A(t)$ is negative definite on W for $|t - \alpha_i| < \epsilon$. From Lemma 7.2, it follows

$$(3) \qquad \lambda(A(t)) \geq \lambda(A(\alpha_i)) \qquad \text{for } |t - \alpha_i| < \epsilon.$$

Combining (2) and (3) we find:

$$(4) \qquad \lambda(A(t)) = \lambda(A(\alpha_i)) \qquad \text{for } \alpha_i \leq t \leq \alpha_i + \epsilon.$$

Denoting by $\mu(A(t))$ the number of positive terms in a reduction of $A(t)$ to diagonal form, we prove in the same way for $\alpha_i - \epsilon < t < \alpha_i$:

$$(5) \qquad \mu(A(t')) = \mu(A(\alpha_i)).$$

Combination of (4) and (5) gives:

$$\lambda(A(t') - \lambda(A(t)) = n - \mu(A(t')) - \lambda(A(t)) = n - \mu(A(\alpha_i))\lambda(A(\alpha_i)) = \nu(A(\alpha_i))$$

the nullity of $A(\alpha_i)$. Application of this last result to the points $\alpha_1, ..., \alpha_r$ clearly gives the statement of the lemma. □

Let $g(t)$ be a constant-speed geodesic on M. A family of geodesics $g_\alpha(t)$, $-\infty < \alpha < \infty$, all parametrized proportionally to arc-length from $g_\alpha(0)$, is called a **variation** of $g(t)$ if

(i) $g_\alpha(t)$ depends differentiably on α;
(ii) $g_0(t) = g(t)$.

A variation of $g(t)$ induces a vector field $U(t)$ along g by

$$U(t) = \left. \frac{\partial V(\alpha, t)}{\partial \alpha} \right|_{\alpha=0}.$$

A vector field along g, induced by a variation of g is called a **Jacobi field** along g. Clearly, this definition generalizes the concept of Jacobi field along a straight line in a Euclidean space, as defined in Section 4. A vector field along the segment s of g is called a Jacobi field along s, if and only if it is the restriction to s of a Jacobi field along g. If a and b are the endpoints of s, the vector space of Jacobi fields along g which vanishes at both a and b is denoted by $\bigwedge_{ab}^{s}$.

We admit the following properties of Jacobi fields which are either trivial or can be proved by standard methods.

(i) Let $U'(t)$ be the covariant derivative of $U(t)$ along g. Then there is one and only one Jacobi field along g with given initial values $U(t_0)$ and $U'(t_0)$;

Proof sketch. Jacobi fields are determined by the Jacobi equation, which is a second-order differential equation, so (i) is trivial. □

(ii) If the segment $g(t)$, $0 \le t \le 1$, of g lies completely in a neighborhood in which every two points can be joined by a unique geodesic arc, then there is one and only one Jacobi field along $g(t)$ with prescribed boundary values at $g(0)$ and $g(1)$;

Proof sketch. g is the unique geodesic from $g(0)$ to $g(1)$, so $g(0)$ and $g(1)$ are in the injective radius of each other. In particular, they are not conjugate. Then for $U(g(0)) = 0$, the linear transformation $T_{g(0)}M \to T_{g(1)}M$, $\dot{U}(g(0)) \mapsto U(g(1))$ is non-singular. So combining with (i), there exists unique Jacobi field with $U(g(0)) = 0$, $U(g(1)) = b$. Similarly, there exists unique Jacobi field with $U(g(0)) = a$, $U(g(1)) = 0$. Summing up these two Jacobi fields yields the desired Jacobi field. □

(iii) If the Jacobi field $U(t)$ is perpendicular to $g(t)$ for $t = t_1$ and for $t = t_2$, $t_1 \ne t_2$, then $U(t)$ is perpendicular to $g(t)$ for all values of t.

Proof sketch. By $\ddot{g} = 0$ and $\ddot{U} + R(U, \dot{g})\dot{g} = 0$, we have

$$\frac{d}{dt}\langle \dot{U}, \dot{g}\rangle = \langle \ddot{U}, \dot{g}\rangle = \langle R(U, \dot{g})\dot{g}, \dot{g}\rangle = 0.$$

Thus $\langle \dot{U}, \dot{g}\rangle = \frac{d}{dt}\langle U, \dot{g}\rangle$ is constant, then $\langle U, \dot{g}\rangle$ is linear. Since $\langle U, \dot{g}\rangle$ is 0 for $t = t_1$ and for $t = t_2$, $t_1 \ne t_2$, it must be constant. □

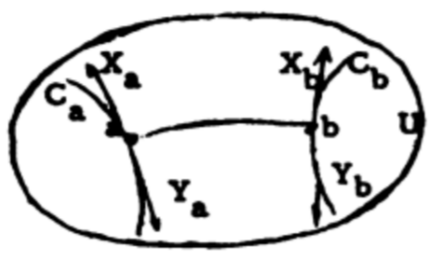

FIGURE 14.

Let $U \subset M$ be an open set in which every two points can be joined by a unique geodesic arc. Let $a, b \in U$, $a \ne b$, and let $g(t)$, $0 \le t \le 1$, be the constant-speed geodesic from a to b. The function $l = \rho(x, y)$ is a differentiable function on $U \times U$ in a neighborhood of (a, b). Let C_a and C_b be differentiable cells of codimension 1, passing through a and b, respectively, and perpendicular to $g(t)$ at a and b, respectively. $C = C_a \times C_b$ is a submanifold of $U \times U$. Let l^* be the restriction of l

to C. It follows from (4), that (a,b) is a critical point of l^*, and the Hessian $\mathbf{H}l^*_{a,b}$ is well-defined.

We admit the following expression for $\mathbf{H}l^*_{a,b}$ (this is the so-called **Second Variation Formula**, see [3]):

$$(6) \qquad XY\mathbf{H}l^*_{a,b} = (X_b, Y'_b) - (X_a, Y'_a) + n_b(X_b, Y_b) - n_a(X_a, Y_a),$$

where the symbols have the following meaning: X_a and Y_a are tangent vectors to C_a at a, X_b and Y_b are tangent vectors to C_b at b. (X_a, X_b) and (Y_a, Y_b) can be considered as tangent vectors of C at (a,b). By property (ii) of Jacobi fields, there is a unique Jacobi field $Y(t)$ along $g(t)$ with $Y(0) = Y_a$ and $Y(1) = Y_b$. $Y'(t)$ denotes the covariant derivative of Y along $g(t)$. Finally, $n_a(X_a, Y_a)$ and $n_b(X_b, Y_b)$ denote the second fundamental forms at a and b with respect to $g(t)$, evaluated at X_a, Y_a and X_b, Y_b respectively. For a definition of this for which plays no further role in these notes, we refer to [5], pg. 257, and the reference given there.

Furthermore, if we fix a and consider the function $l = \rho(a, x)$, $x \neq a$, and also its restriction l^* to C_b, then b is a critical point of l^*, and

$$(7) \qquad X_b Y_b \mathbf{H}l^* = (X_b, Y'_b) + n_b(X_b, Y_b),$$

where Y'_b is the covariant derivative of the Jacobi field which is 0 at a and Y_b at b.

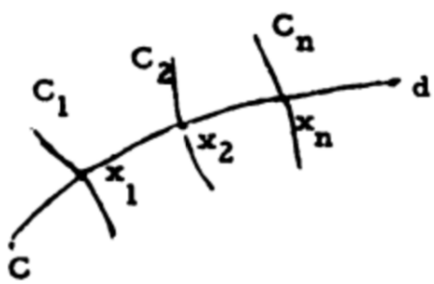

FIGURE 15. Caption

Again, we use the notations of section 6, but replace p, q by c, d. Suppose $cx_1x_2...x_nd$ is a geodesic from c to d such that $x_i \neq x_j$ for $i \neq j$. From the remark after Theorem 7.1, it follows that $x = (x_1, ..., x_n) \in M^b_*$ is a critical point of the function

$$L = \sum_{i=0}^{n} \rho(x_i, x_{i+1}).$$

Let C_i, $i = 1, ..., n$ be differentiable cells of codimension 1, passing through x_i and orthogonal to $g(t)$ at x_i. We set $C_1 \times ... \times C_n = N$ and consider N as a submanifold of M^b_*. Clearly, $x \in N$. Let L^* be the restriction of L to N. Obviously, x is a critical point of L^*. From (6) and (7), we deduce the following expression for the Hessian $\mathbf{H}L^*_x$:

$$(8) \qquad UV\mathbf{H}L^*_x = \sum_{i=1}^{n} U_i(V_i^+ - V_i^-).$$

where $U = (U_1, ..., U_n)$, $V = (V_1, ..., V_n)$, U_i and V_i being tangent vectors to C_i in x_i, $i = 1, ..., n$. V_i^- and V_i^+ denote the values at x_i of the covariant derivative along

$x_{i-1}x_i$ and x_ix_{i+1}, respectively, of the Jacobi field determined by V_{i-1}, V_i and V_i, V_{i+1}, respectively, with $V_0 = V_{n+1} = 0$.

If V is in the null space of $\mathbf{H}L^*$, then it follows from (8) and property (i) that the Jacobi fields determined by V_0 and V_1 along rx_1, by V_1 and V_2 along x_1x_2,... form a global Jacobi field along $g(t)$, $0 \le t \le 1$, which vanishes at r and s. Conversely, by (8) and property (iii), such a global field determines a vector V in the null space of $\mathbf{H}L^*$.

Proposition 7.5. *Let $g(t)$, $0 \le t \le 1$, be a geodesic on M, $g(0) = r$, $g(1) = s$. Let $x = (x_1, ..., x_n) \in M_*^b$ be a subdivision of $g(t)$, $x_0 = r$, $x_{n+1} = s$ and $x_i \ne x_j$ for $i \ne j$. Let L_x^* be defined as above. Then $\nu(\mathbf{H}L_x^*) = \dim \bigwedge_{rs}^{g(t)}$.*

Again we use the notations of Section 6. Let $g(t)$, $0 \le t \le 1$, be a constant-speed geodesic, with $g(0) = p$, $g(1) = q$. Let $g(t)$ be subdivided by $x = (x_1, .., x_n) \in M_*^b$, set $p = x_0$, $q = x_{n+1}$ and suppose $x_i \ne x_j$ for $i \ne j$ for $i, j = 1, ..., n+1$.

We claim that there are only a finite number of t-values, $0 < t < 1$, which we call $\alpha_1, ..., \alpha_r$, such that $\dim \bigwedge_{pg(\alpha_i)}^{g} \ne 0$. (see [3] for the proof)

Let $g(s_1)$ be a point of $g(t)$ between x_n and q, $s_1 \ne \alpha_i$, $i = 1, ..., r$. For $s_1 \le t \le 1$, we define the functions $L(t)$ and $L^*(t)$ in the same way with respect to p and $g(t)$ as the functions L and L^* are defined with respect to p and q.

From the remark after Theorem 7.1, it follows that x is a critical point of $L(t)$ and therefore of $L^*(t)$ for all t, $s_1 \le t \le 1$. Using the triangle inequality, we find for $t_1 \le t_2$:

$$L(t_2)(x') \le L(t_1)(x') + \rho(g(t_1), g(t_2))$$

for x' in a neighborhood of x. Since

$$L(t_2)(x) = L_x(t_1)(x) + \rho(g(t_1), g(t_2))$$

we find

$$\mathbf{H}L(t_1)|_x \ge \mathbf{H}L(t_2)|_x.$$

By restriction (Lemma 7.2):

$$\mathbf{H}L^*(t_1)|_x \ge \mathbf{H}L^*(t_2)|_x.$$

Applying Lemma 7.3 we find:

$$\lambda(\mathbf{H}L^*|_x) = \lambda(\mathbf{H}L^*(s_1)|_x) + \sum_{s_1 \le t \le 1} \nu(\mathbf{H}L^*(t)(x)),$$

and by Proposition 7.4:

$$\lambda(\mathbf{H}L^*|_x) = \lambda(\mathbf{H}L^*(s_1)|_x) + \sum_{s_1 \le t \le 1} \dim \bigwedge\nolimits_{pg(t)}^{g} . \tag{9}$$

Now we replace the subdivision x of $g(t)$ by a subdivision $y = (y_1, ..., y_n) \in M_*^b$ of the segment $pg(s_1)$ of $g(t)$, in such a way that the t-value of y_n is smaller than that the t-value of x_n.

On M_*^b there is a path Y from x to y such that $L^*(s)$ non-degenerate at all of its points. Since $\mathbf{H}L^*(s)$ depends on s continuously and is non-degenerate for all the points of Y, we find

$$\lambda(\mathbf{H}L_y^*(s)) = \lambda(\mathbf{H}L_x^*(s)).$$

We choose a point $g(s_2) \ne \alpha_1, ..., \alpha_r$ between y_n and $g(s_1)$, and have the same reasoning for the segment $g(s_1)g(s_2)$ as we did above for the segment $g(s_1)q$. We can go this way, such that after a finite number of steps we arrive at a point s_l on

$g(t)$, lying close enough to p that the function $L(s_k)$ has an absolute, non-degenerate minimum at the subdivision $z = (z_1, ..., z_n) \in M^b_*$, obtained by repeated application of our procedure. It follows that $\mathbf{H}L_z(s_k)$ and therefore $\mathbf{H}L^*_z(s_k)$ is positive definite; i.e., $\lambda(\mathbf{H}L^*_z(s_k)) = 0$. Repeated application of (9) finally gives

$$\lambda(\mathbf{H}L^*(x)) = \sum_{0<t<1} \dim \textstyle\bigwedge^g_{pg(t)} . \tag{10}$$

As always, let $g(t)$ be a geodesic on M, with $g(0) = p$, $g(1) = q$, and let $\dim \bigwedge^g_{pq} \neq 0$. Let $x = (x_1, ..., x_n) \in M^b_*$ be a subdivision of $g(t)$ with the property:

$$\rho(p, x_1) = \rho(x_1, x_2) = ... = \rho(x_n, q).$$

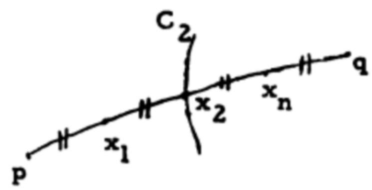

FIGURE 16.

According to Theorem 7.l1 and the remark at the end of it, x is a critical point L (which is a function differentiable in neighborhood of x on M^b_*) and of ϕ.

Again, let C_i, $i = 1, ..., n$ be differentiable cells, perpendicular to $g(t)$ in x_i, and $N = C_1 \times ... \times C_n$. Let L^* and ϕ^* be the restrictions of L and ϕ to N respectively. x is a critical point of both L^* and ϕ^*. We intend to prove the following relations:

$$\lambda(\mathbf{H}\phi_x) = \lambda(\mathbf{H}L_x) = \lambda(\mathbf{H}L^*_x) = \lambda(\mathbf{H}\phi^*_x). \tag{11}$$

For the proof, we need the following more general lemma.

Let M be a differentiable manifold and N a differentiable submanifold of M. Let $p \in N$ be a critical point of the differentiable function f on M. Then p is also a critical point of the function $f^* = f|_N$. We will compare the indices at p only, the subscript p will be omitted.

Lemma 7.6. *Under the conditions just described,* $\lambda(\mathbf{H}f^*) \leq \lambda(\mathbf{H}f)$*. Further, if there exists a differentiable map* $\tau : U \to N$ *of some neighborhood* U *of* p *on* M *into* N *such that:*

(i) $\tau(p) = p$;
(ii) $d(\tau|_N)$ *is an automorphism of the tangent space* T_pN *of* N *at* p;
(iii) $f(\tau(x)) \leq f(x)$ *for all* $x \in U$;

then $\lambda(\mathbf{H}f^*) = \lambda(Hf)$.

Proof. We may identify $\mathbf{H}f^*_p$ with the restriction of $\mathbf{H}f_p$ to N_p. It follows trivially from lemma 7.2 that under a restriction the index of a quadratic form can only decrease. Hence $\lambda(\mathbf{H}L) \geq \lambda(\mathbf{H}f^*)$. Now let $\tau^*(\mathbf{H}f)$ be the quadratic form on M_p defined by

$$(\tau^*(\mathbf{H}f))(x, y) = \mathbf{H}f(d\tau(x), d\tau(y)).$$

We can define $\tau^*(\mathbf{H}f^*)$ similarly since $d\tau(N_p) \subset N_p$. We have the following sequence:

$$\lambda(\tau^*(\mathbf{H}f^*)) = \lambda(\mathbf{H}f^*) \leq \lambda(\mathbf{H}f) \leq \lambda(\tau^*(\mathbf{H}f)). \tag{12}$$

The first relation follows from the fact that $d\tau|_{N_p}$ is an automorphism of N_p; that is, from condition (ii). The second we already used. The third is a consequence of condition (iii). Indeed, let $F(x) = f(x) - f(\tau(x))$, $x \in U$. Then $F \geq 0$, with $F(p) = dF(p) = 0$. Hence $\mathbf{H}F_p$ is non-negative. But this Hessian is precisely $\mathbf{H}f - \tau^*(\mathbf{H}f)$, whence $\mathbf{H}f - \tau^*(\mathbf{H}f)$, and this clearly implies $\lambda(\mathbf{H}f) \leq \lambda(\tau^*(\mathbf{H}f))$. On the other hand, we also have

$$(13) \qquad \lambda(\tau^*(\mathbf{H}f^*)) = \lambda(\tau^*(\mathbf{H}f)).$$

To see this remark that $\tau^*(\mathbf{H}f^*)$ is just the restriction of $\tau^*(\mathbf{H}f^{)}$ to N_p. Hence

$$(14) \qquad \lambda(\tau^*(\mathbf{H}f^*)) \leq \lambda(\tau^*(\mathbf{H}f)).$$

Suppose A is a subspace on which $\tau^*(\mathbf{H}f)$ is negative definite. Then A does not intersect the kernel of $d\tau$, therefore $B = d\tau(A) \subset N_p$ has the same dimension as A. By definition, $\tau^*(\mathbf{H}f^*)$ is negative definite on B and since $\dim B = \dim A$, we find:

$$(15) \qquad \lambda(\tau^*(\mathbf{H}f^*)) \geq \lambda(\tau^*(\mathbf{H}f)).$$

Combining (14) and (15), we get (13).

From (13), it follows that the inequalities in (12) must in fact be equalities. In particular, $\lambda(\mathbf{H}f) = \lambda(\mathbf{H}f^*)$. This completes the proof of Lemma 7.5, and now we have all the necessary preparations for the proof of (11). □

Proof of (11). Let $F = \phi - \frac{1}{n+1}L^2$. By the Cauchy inequality $F \geq 0$. Further, near x, F is differentiable and $F(x) = 0$. Hence $dF(x) = 0$, and the Hessian of F is not negative at x. Thus $\mathbf{H}\phi_x \geq \frac{1}{n+1}\mathbf{H}L^2$, whence

$$(16) \qquad \lambda(\mathbf{H}\phi_x) \leq \lambda(\mathbf{H}L_x).$$

Our next steps is to show that $\lambda(\mathbf{H}L_x) = \lambda(\mathbf{H}L_x^*)$ by means of Lemma 7.5. We have to construct a map $\tau : U \to N$ of a suitable neighborhood U of x with the properties (i), (ii), and (iii) of Lemma 7.5. For thsi purpose, let ϵ, $0 < \epsilon < \frac{L(x)}{2(n+1)}$, be a real number and let C_i^ϵ be the parallel translates of C_i along $g(t)$ by a distance ϵ, in the direction from p to q. Similarly, let $C_i^{-\epsilon}$ be the ϵ-translate along $g(t)$ in the opposite direction. The geodesic g meets $C_i^{\pm\epsilon}$ transversely. Hence there is a neighborhood U'_ϵ of x, such that for all $y \in U'_\epsilon$ the polygon $\beta(y)$ (Section 6) also meets the $C_i^{\pm\epsilon}$ transversely, say at points $\eta_i^{\pm\epsilon}(y)$. These points, taken in the obvious order $p, \eta_1^-, \eta_1^+, \eta_2^-, ...$, defines a polygon $\eta_\epsilon(y)$. We now define the components of $\tau_\epsilon(y)$ to be the intersection of $\eta_\epsilon(y)$ with C_i. The transformation τ_ϵ is well defined and differentiable on some smaller vicinity U_ϵ of x, again because $\tau_\epsilon(x) = x$, so that $\eta_\epsilon(y)$ will meet the C_i's transversely for y close enough to x.

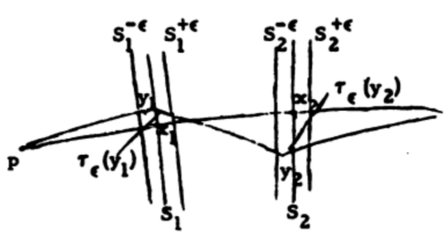

FIGURE 17.

In figure 17, the situation is described graphically. τ_ϵ maps U into N, further $\tau_\epsilon(x) = x$ and $L(\tau(y)) \leq L(y)$ as follows from the triangle inequality. This all requirements of Lemma 7.5 are satisfied except possibly (ii). We therefore still have to show that $\tau_\epsilon|_N$ has a non-singular differential for $0 < \epsilon < \frac{L(x)}{2(n+1)}$, and clearly depends continuously on ϵ. However, for $\epsilon = 0$, $\tau_0|_N$ is the identity. Hence for ϵ sufficiently small, $d(\tau_\epsilon|_N)$ is non-singular. Applying Lemma 7.5, we have established

$$(17) \qquad \lambda(\mathbf{H}L_x) = \lambda(\mathbf{H}L_x^*).$$

From (7) we deuce

$$(18) \qquad UV\mathbf{H}\phi_x^* = \frac{2L(x)}{n+1}\sum_{i=1}^{n} U_i(V_i^+ - V_i^),$$

where the symbols have the same meaning as in (8). From (8) and (18) it follows

$$(19) \qquad \lambda(\mathbf{H}L_x^*) = \lambda(\mathbf{H}\phi_x^*).$$

Finally, the first part of Lemma 7.5 gives:

$$(20) \qquad \lambda(\mathbf{H}\phi_x^*) \leq \lambda(\mathbf{H}\phi_x)$$

Combining (16), (17), (19), (20), we proved (11). □

From (10) and (11), we deduce the Index Theorem.

Theorem 7.7 (The Index Theorem)**.** *Let $g(t)$, $0 \leq t \leq 1$, be a geodesic on M, $g(0) = p$, $g(1) = q$ and let $x = (x_1, ..., x_n) \in M_*^b$ be the subdivision of $g(t)$ given by $\rho(p, x_1) = \rho(x_1, x_2) = ... = \rho(x_n, q)$. Then x is a critical point of the function*

$$\phi = \rho^2(p, x_1) + \rho^2(x_1, x_2) + ... + \rho^2(x_n, q),$$

and

$$\lambda(\mathbf{H}\phi_x) = \sum_{0<t<1} \dim \textstyle\bigwedge^g_{pg(t)},$$

where $\bigwedge^g_{pg(t)}$ denotes the vector space of Jacobi fields along $g(t)$, vanishing both at p and at $g(t)$.

8. Critical Manifolds

Definition 8.1. Let f be a smooth function on the manifold M. The connected submanifold V of M will be called a **non-degenerate critical manifold** of f, if

(1) every point $x \in V$ is a critical point of f;
(2) for all $x \in V$, the null space of $\mathbf{H}f_x$ is precisely the tangent space to V.

This definition generalizes the concept of a non-degenerate critical point.

From the definition, it follows that also the index $\lambda(\mathbf{H}f_x)$ is the same for all points $x \in V$. Therefore, we can speak of the index of f on V. It will be denoted by $\lambda_f(V)$ or simply by $\lambda(V)$ if there is no danger of confusion.

Lemma 8.2. *Let f be a smooth function on M, and $p \in M$ a critical point of f. If in a convenient neighborhood of p, we set $\rho(p, y) = \phi(y)$ and $\langle d\phi, df\rangle = F$, then p is a critical point of F and $\mathbf{H}F_p = 4\mathbf{H}f_p$.*

Proof. By direct verification. □

Lemma 8.3. *Let f be a smooth function on M which assumes its absolute minimum on the non-degenerate critical manifold* $V \subset M$*. If there are no other critical points of f on* M^a *and if* M^a *is compact, then* $M^a \sim V$*.*

Proof. Let $p \in V$, and let f' be the restriction of f to the "geodesic plane" perpendicular to V at p. The gradient of f' is transverse to an ϵ-sphere in this plane of ϵ small enough. This follows from lemma 8.2, since $\mathbf{H}f'_p$ is non-degenerate by the definition of a non-degenerate critical manifold. From the compactness of V, we deduce the existence of a global tubular neighborhood of N of V in M such that ∇f is transverse to the boundary of N. In the same way as in the proof of theorem 2.1, a deformation of M^a in N can be constructed. Combining this with the obvious fact that $N^\epsilon \sim V$, we get the statement of the lemma. □

Lemma 8.4. *Let f be a smooth function on M such that for* $a \le x \le b$*, there is only one critical value* $x = c$, $a < c < b$*. Suppose furthermore that* $f^{-1}(c)$ *is the non-degenerate critical manifold V. If* M^b *is compact, then:*

$$M^b \sim M^a \cup e_{\lambda_1} \cup \ldots \cup e_{\lambda_r}.$$

with $\lambda_i \ge \lambda(V)$*.*

Proof. Let N^{ϵ_1} and N^{ϵ_2} be closed, sufficient;y small tubular neighborhood of V in M, of radius ϵ_1, ϵ_2, respectively, $\epsilon_1 < \epsilon_2$. Let $\phi(x)$ be a smooth function on M^b with the following properties:

$$\phi(x) = 1 \qquad \text{on } N^{\epsilon_1}$$
$$\phi(x) = 0 \qquad \text{outside } N^{\epsilon_2}$$

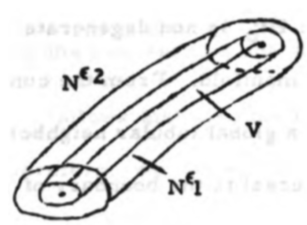

FIGURE 18.

Let $\pi : N^{\epsilon_2} \to V$ be the fibre projection and g a (strongly) non-degenerate function on V. We set

$$f' = f + \alpha\rho(x)\pi^*(g), \qquad \alpha > 0,$$

and assert that for α sufficiently small, f' has non-degenerate critical points only, all lying on V and precisely the critical points of g. We consider $f^{-1}(a < x < b) \subset M$. Outside N^{ϵ_2}, f' has no critical point. Between N^{ϵ_1} and N^{ϵ_2} and onto their boundaries, we have:

$$|df| > A, \quad d|\rho(x)\pi^*(g)| < B,$$

and therefore:

$$|df'| \ge |df| - \alpha d|\rho(x)\pi^*(g)| > A - \alpha B > 0$$

for α sufficiently small. By looking at a restriction to a fibre of N^{ϵ_1}, we see that on N^{ϵ_1}, f' has only critical points on V and there only at the critical points of g. In these points, the null space of $\mathbf{H}f$ is just the tangent space to V (by definition) and that of $\pi^*(g)$, the tangent space to the fibre of N^{ϵ_1} through that point (by direct verification). It follows that the critical points are non-degenerate and that their indices are at least $\lambda(f)_p = \lambda(V)$.

□

9. The Stable Homotopy Groups of the Unitary Group

In this final section, we shall apply the results of the preceding sections to the calculation of the stable homotopy group of the unitary groups. This means the following: The exact homotopy sequence of the fibering

$$U(n)/U(n-1) = S^{2n-1}$$

gives us

$$\pi_k(U(n)) = \pi_k(U(n-1)), \qquad 0 \leq k \leq 2n-2,$$

from which it follows that

$$\pi_k(U(n)) = \pi_k(U(m)), \qquad n, m \geq \frac{k+2}{2}.$$

This group is called the k-th stable homotopy group of the unitary group(s). It will be denoted by $\pi_k(U)$.

We start with a simple result concerning Jacobi fields on a compact connected Lie group, considered as a Riemannian manifold by providing it with a left and right invariant Riemannian structure. Since we need only the result for $SU(n)$, we shall state and prove it for this group, but the general case is no more difficult. The reason, that we consider $SU(n)$ rather than $U(n)$ is that $\Omega(SU(n))$ is connected but $\Omega(U(n))$ is not. The way in which we use this will become clear in the sequel.

The Lie algebra $\mathfrak{L}$ of $SU(n)$ is provided with a metric $\mathfrak{m}'$ in a natural way, namely with the restriction to $\mathfrak{L}$ of the metric $\mathfrak{m}$, introduced in Section 5.

For $Y \in \mathfrak{L}$,

$$V(\rho, t) = e^{\rho Y} \cdot e^{tX} \cdot e^{-\rho Y}, \qquad -\infty < \rho < +\infty$$

defines a variation of the geodesic $g(t) = e^{tX}$, $X \in \mathfrak{L}$. On $g(t)$, we consider in particular the segment $s : 0 \leq t \leq 1$, starting at the identity e of $SU(n)$ and ending at $a = g(1)$. $V(\rho, t)$ induces a Jacobi field along $g(t)$, given by

$$(1) \qquad U_Y(t) = \frac{\partial}{\partial \rho}(e^{\rho Y} \cdot g(t) \cdot e^{-\rho Y})\bigg|_{\rho=0}.$$

Writing instead of $e^{\rho Y} \cdot g(t) \cdot e^{-\rho Y}$, $e^{\rho Y} \cdot e^{-\mathrm{Ad}_{g(t)}\rho Y} \cdot g(t)$, we see that

$$(2) \qquad U_Y(t) = (Y - \mathrm{Ad}_{g(t)}(Y))g(t).$$

The result we shall prove is, that all elements of $\bigwedge_{ea}^{g}$ can be written in the form (1) for a suitable choice of Y.

Let $\mathscr{J}$ denote the vector space of Jacobi fields along s, which vanishes at e (but not necessarily at a). As remarked in Section 7, a Jacobi field along $g(t)$ is completely determined by its value at e and the value of its covariant derivative along $g(t)$ at e. From this, it is obvious that $\dim \mathscr{J} = \dim SU(n) = \dim \mathfrak{L}$.

Let $P \subset \mathfrak{L}$ be the subspace of $\mathfrak{L}$, consisting of all those elements of $\mathfrak{L}$ which induce the zero Jacobi field along s. From (2) and from

$$[X, Y] = \lim_{t \to 0} \frac{\mathrm{Ad}_{e^{tX}} Y - Y}{t},$$

it follows that

$$(3) \qquad P = \{Y \in \mathfrak{L}, [X, Y] = 0\}.$$

Consider the sequence

$$(4) \qquad 0 \longrightarrow P \xrightarrow{\alpha} \mathfrak{L} \oplus P \xrightarrow{\beta \oplus \gamma} \mathscr{J} \longrightarrow 0,$$

where α denotes the injection of P into $\mathfrak{L}$, β the homomorphism which attaches to $Y \in \mathfrak{L}$, $U_Y \in \mathscr{J}$, and where γ is defined as follows: $\gamma(Z)$, $Z \in P$, will be the Jacobi field along $g(t)$ induced by the variation

$$(5) \qquad W(\rho, t) = e^{t(\rho Z + X)}, \qquad -\infty < \rho < +\infty$$

of $g(t)$. Since $W(\rho, 0) = e$ for all ρ, $\gamma(Z) \in \mathscr{J}$ for all $Z \in P$.

We shall prove that the sequence (4) is exact. For this, it is sufficient to prove: $\beta(Y) + \gamma(Z) = 0$, $Y \in \mathfrak{L}$, $Z \in P$ implies: $Y \in P$, $Z = 0$. Indeed, if we prove this, then $\beta \oplus \gamma$ is automatically surjective, since dim kernel$(\beta \oplus \gamma)) =$ dim P, dim image$(\beta \oplus \gamma)) =$ dim $\mathfrak{L} +$ dim $P -$ dim $P =$ dim $\mathfrak{L} =$ dim $\mathscr{J}$.

Now, (3) allows us to write instead of (5),

$$W(\rho, t) = e^{t\rho Z} \cdot e^{tX},$$

$$(6) \qquad \frac{\partial}{\partial \rho} W(\rho, t)\bigg|_{\rho=0} = tZe^{tX}.$$

If $\beta(Y) + \gamma(Z) = 0$, then, by (2) and (6):

$$(7) \qquad Y - \mathrm{Ad}_{e^{tX}}(Y) + tZ = 0, \qquad 0 \leq t \leq 1.$$

Denoting by (,) the inner product with respect to $\mathfrak{m}'$, we find:

$$(Y, Z) - (\mathrm{Ad}_{e^X}(Y), Z) + (Z, Z) = 0.$$

Since $(Y, Z) = (\mathrm{Ad}_{e^X}(Y), \mathrm{Ad}_{e^X}(Z))$, this leads to

$$(8) \qquad \mathrm{Ad}_{e^X}(Y)\{\mathrm{Ad}_{e^X}(Z) - Z\} + (Z, Z) = 0.$$

Z is in P, therefore by (2), the first term on the left side of (8) vanishes and we find: $(Z, Z) = 0$, hence $Z = 0$, hence, by (7), $Y \in P$. This proves the exactness of (4).

Finally, to prove that the elements of $\bigwedge_{ea}^{g}$ can be written in the form of (1), it is sufficient to prove that $(\beta \oplus \gamma)^{-1}(\bigwedge_{ea}^{g}) \subset \mathfrak{L}$. But the pair (Y, Z) $(Y \in \mathfrak{L}, Z \in P)$ lies in the inverse image if and only if (7) holds for $t = 1$. This gives again $Z = 0$.

Proposition 9.1. *Let $SU(n)$ be provided with a left and right invariant Riemannian structure, let $g(t)$, $0 \leq t \leq 1$, $g(0) = e$, $g(1) = p$ be a geodesic segment on $SU(n)$.*

Then every element of $\bigwedge_{ep}^{g}$ is induced by a variation of $g(t)$ of the form

$$e^{\rho Y} \cdot g(t) \cdot e^{-\rho Y}, \qquad -\infty < \rho < +\infty,$$

where Y denotes an appropriate element of the Lie algebra of $SU(n)$.

Our argument gives the same result for any compact Lie group, but we do not need this fact, in its turn a very special case of the "variational completeness" of symmetric spaces (see [6]; also [5] for the proof given above).

We consider the situation of Section 7, with $M = SU(2m)$, $p = e$ and $q = -e$. According to Theorem 7.1, the critical points of ϕ on $M_n^b = M_*^b$ correspond to the n-tuple $(x_1, ..., x_n)$, such that $ex_1x_2...x_n(-e)$ is a geodesic segment on $SU(2m)$, and such that $\rho(e, x_1) = \rho(x_1, x_2) = ... = \rho(x_n, -e)$. Let $y = (y_1, ..., y_n)$ be such an n-tuple. Then all the transforms of y by the natural operation of $SU(2m)$ on M_* (induced by the adjoint action of $SU(2m)$ on itself) are also critical points of ϕ. It follows that the critical points of ϕ on $N_*^b = M_*^b$ minus its boundary consist of a series of compact submanifolds of N_*^b, on each of which ϕ is constant.

We shall prove that these critical manifolds are non-degenerate in the sense of Section 8.

Let

$$X = \begin{pmatrix} i\alpha_1 & & 0 \\ & \ddots & \\ 0 & & i\alpha_{2m} \end{pmatrix}$$

be an element of the Lie algebra of $U(2m)$, considered as the space of skew-symmetric matrices (see Section 5). X lies in the Lie algebra of $SU(2m)$ if and only if

$$\sum_{k=1}^{2m} \alpha_k = 0.$$

In that case, the geodesic

$$e^{tX} = \begin{pmatrix} e^{i\alpha_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{i\alpha_{2m} t} \end{pmatrix}$$

lies on the maximal torus D, consisting of the diagonal matrices of $SU(2m)$.

Now let C be a critical manifold of ϕ on N_*^b, let $x \in C$, and let g be a geodesic $ex_1x_2...x_n(-e)$. There is an element $\omega \in SU(2m)$, such that $\omega x_1 \omega^{-1} \in D$. $\omega g \omega^{-1}$ passes through $\omega x_1 \omega^{-1}$ lying on D, so $\omega g \omega^{-1} \subset D$. Therefore, every critical manifold contains at least one point $x = (x_1, ..., x_n)$, such that $ex_1...x_n(-e) \subset D$.

To prove our statement, we have to show, that for a point p on a critical manifold C of ϕ, the zero space of $\mathbf{H}\phi_p$ is precisely the tangent space to C at p. A trivial calculation shows that every tangent vector to C at p certainly lies in the zero space. It therefore suffices to show that $\nu(\mathbf{H}\phi_p) = \dim C$. By the same kind of argument as used in Section 7 to prove that $\lambda(\mathbf{H}\phi_p) = \lambda(\mathbf{H}\phi_p^*)$, it can be proved (in the general case) that $\nu(\mathbf{H}\phi_p) = \nu(\mathbf{H}\phi_p^*)$. By Proposition 7.5, this nullity equals $\dim \bigwedge_{e(-e)}^s$, where s denotes the geodesic segment $ex_1...x_n(-e)$. If we denote by G the subgroup of $SU(2m)$, leaving s fixed, we have: $\dim C = \dim SU(2m) - \dim G$. On the other hand, if follows from Proposition 9.1, that $\dim \bigwedge_{e(-e)}^s = \dim \mathfrak{L} - \dim P$. Since $\dim \mathfrak{L} = \dim SU(2m)$, and since it follows easily from (3) that $\dim P = \dim G$, we have proved our theorem that C is a non-degenerate critical manifold.

From the proof, it follows also that to determine the indices of the critical manifold C, it is sufficient to determine the indices of geodesic segments $g(t)$, $0 \leq t \leq 1$, from e to $-e$, lying on D and considered as representing a critical point x of ϕ on N_*^b. By Theorem 7.7 (the index theorem), this index equals

$$\sum_{0<t<1} \dim \bigwedge_{eg(t)}^{(g)},$$

provided that $\dim \bigwedge_{eg(t)}^{(g)}$ differs from 0 only for a finite number of t-values, $0 \leq t \leq 1$. The argument, giving us the equality

$$\dim \bigwedge_{e(-e)}^{(s)} = \dim SU(2m) - \dim G,$$

tells us also, that

$$\dim \bigwedge_{eg(t)}^{(g)} = \dim G_{g(t)} - \dim G,$$

where $G_{g(t)}$ denotes the subgroup of $SU(2m)$, leaving the point $g(t)$ fixed.

Let the segment $s : 0 \leq t \leq 1$ on $g(t)$ be given by

$$g(t) = \begin{pmatrix} e^{2\pi i \alpha_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{2\pi i \alpha_{2m} t} \end{pmatrix}$$

Since $g(t) \subset SU(2m)$,

$$\sum_{k=1}^{2m} \alpha_k = 0,$$

and since $g(1) = -e$,

$$\alpha_k = \frac{2\beta_k - 1}{2}, \qquad k = 1, ..., 2m, \quad \beta_k \text{ integer}.$$

Conversely, every sequence of rational numbers $\alpha_1, ..., \alpha_{2m}$, fulfilling these conditions gives us a geodesic segment from e to $-e$ on D.

If we take $\alpha_1 = ... = \alpha_m = \frac{1}{2}, \alpha_{m+1} = ... = \alpha_{2m} = -\frac{1}{2}$, then it can be checked easily that for all values of t, $0 < t < 1$, $\dim G_{g(t)} = \dim G$. Therefore, this α-sequence gives a geodesic $ex_1...x_n(-e)$, such that $\lambda(x) = \lambda(C_1) = 0$, $x \in C_1$. Consequently, on this C_1, ϕ assumes its absolute minimum a_0.

$$C_1 = SU(2m)/S(U(m) \times U(m)) = U(2m)/(U(m) \times U(m))$$

is a Grassmann variety.

If we take for example the sequence

$$\alpha_1 = \frac{3}{2}, \alpha_2 = ... = \alpha_{m-1} = \frac{1}{2}, \alpha_m = ... = \alpha_{2m} = -\frac{1}{2},$$

then on the corresponding geodesic, there is for $0 < t < 1$ just one t-value $t_0 = \frac{1}{2}$, such that $\dim \bigwedge_{eg(\frac{1}{2})}^{(g)} \neq 0$ (there is just one value of t, $0 < t < 1$, namely $t = \frac{1}{2}$, such that $\frac{3}{2}t$ and $-\frac{1}{2}t$ are congruent modulo 1).

$$\begin{aligned} \dim \bigwedge_{eg(\frac{1}{2})}^{(g)} &= \dim S(U(m+2) \times U(m-2)) - \dim S(U(1) \times U(m-2) \times U(m+1)) \\ &= 2m + 2. \end{aligned}$$

It can be checked that the index of all geodesic g corresponding to α-sequence in which not only $\alpha_k = \pm\frac{1}{2}$ appears, is at least this number. In fact, it is sufficient to remark that in such a case, there is at least one point $g(t_0)$ with $\dim \bigwedge_{eg(t_0)}^{g} \neq 0$, $0 < t_0 < 1$ (and on the other hand only a finite number), and this dimension equals

$$S(U(n_1) \times ... \times U(n_k)) - S(U(m_1) \times ... \times U(m_l)),$$

where $m_1 + ... + m_l = 2m$ is a refinement of $n_1 + ... + n_k = 2m$. Then this expression is at least $2m + 2$. (see Milnor's *Morse Theory*, p.131)

Since two geodesic segments on D from e to $-e$ and corresponding to α-sequences with all $\alpha_k = \pm\frac{1}{2}$ clearly lies in the same critical manifold C, we deduce from Theorem 6.1, Lemma 8.2, Lemma 8.3 and the results above that for every $a > a_0$

$$\Omega^a_{e(-e)}(SU(2m)) = SU(2m)/S(U(m) \times U(m)) \cup e_{d_1} \cup e_{d_2} \cup ... \cup e_{d_i}$$

with $d_1, d_2, ..., d_i \geq 2m+2$, and i depending on a. Therefore

$$\begin{aligned}\pi_k(\Omega_{e(-e)}(SU(2m))) &= \pi_k(SU(2m)/S(U(m) \times U(m))) \\ &= \pi_k(U(2m)/U(m) \times U(m)), \qquad 3 \leq k \leq 2m,\end{aligned}$$

Since, as stated at the start of Section 1,

$$\pi_k(\Omega_{e(-e)}(SU(2m))) = \pi_{k+1}(SU(2m)),$$

we can conclude:

$$\pi_{k+1}(SU(2m)) = \pi_{k+1}(U(2m)) = \pi_k(U(2m)/U(m) \times U(m)), \qquad 3 \leq k \leq 2m. \tag{9}$$

At the very beginning of this section, we remarked that

$$\pi_k(U(2m)/U(m)) = 0$$

Using the exact homotopy sequence of the fibering

$$(U(2m)/U(m), U(2m)/U(m) \times U(m), U(m))$$

we find:

$$\pi_k(U(2m)/U(m) \times U(m)) = \pi_{k-1}(U(m)), \qquad 1 \leq k \leq 2m-2. \tag{10}$$

Combining (9) and (10), we get for large m:

$$\pi_{k+1}(U(2m)) = \pi_{k-1}(U(m)), \qquad k \geq 3.$$

Since $\pi_0(U) = \pi_2(U) = 0$ and $\pi_1(U) = \pi_3(U) = \mathbb{Z}$ ([7], p. 132), we have

$$\pi_{k+2}(U) = \pi_k(U), \qquad k \geq 0,$$

and explicitly

$$\begin{aligned}\pi_{2k}(U) &= 0, \\ \pi_{2k+1}(U) &= \mathbb{Z}, \qquad k = 0, 1, 2, ...\end{aligned}$$

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10. References

[1] H. Seifert and W. Threlfall: *Variationsrechnung im Grossen*, Teubner, Leipzig 1938.

[2] K. Nomizu: *Lie Groups and Differential Geometry*, Publications of the Mathematical Society of Japan, vol. 2, Tokyo 1956.

[3] M. Morse: *The Calculus of Variations in the Large*, A. M. S. Colloquium Publications, vol. 18, New York 1934.

[4] W. Graeub: *Linear Algebra*, Die Grundlehren der Mathematichen Wissenschaften, vol. 97, Springer, Berlin 1958.

[5] R. Bott: *An Applicaition of the Morse Theory to the Topology of Lie Groups*, Bulletin dela Societe Mathematique de France, vol. 84(1956), pp. 251-282.

[6] R. Bott and H. Sameloon: *Application of the Theory of Morse to Symmetric Spaces*, American Journal of Mathematics, vol. 80(1958), pp. 964-1029.

[7] N. Steenrod: *The Topology of Fibre Bundles*. Princeton 1951.

For the materials of these notes, see also

[8] R. Bott: *The Stable Homotopy Groups of the Classical Groups*, Annuls of Mathematics (2), vol. 70(1959), pp. 313-337.