The Ricci and Bianchi Identities

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The basic Ricci identity is simply one of the many ways of defining curvature. It applies universally to connections and curvatures of all bundles. The goal here is to show how it also leads to the Bianchi identities.

Covariant Derivatives

Throughout this paper assume that we have a Riemannian manifold (M, g) and a bundle $E \to M$ with a metric and compatible connection. The connection on M and E are both denoted ∇. Thus $\nabla_X Y$ and $\nabla_X s$ denote the covariant derivatives of a vector field Y on M and section s of E in the direction of X. In case E is a tensor bundle the connection ∇ is the one induced by the Riemannian (Levi-Civita) connection on M. Recall that the Riemannian connection is *torsion free*

$$\nabla_X Y - \nabla_Y X = L_X Y = [X, Y]$$

and *metric*

$$0 = (\nabla_X g)(Y, Z) = D_X (g(Y, Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z).$$

The last condition also says that the metric g is *parallel* or has *vanishing covariant derivative.*

Using the covariant derivative on M and E it is possible to define covariant derivatives of mixed "tensors" that involve both vector fields and sections. It is also possible to define covariant and Lie derivatives of multi-linear objects, e.g., we have the covariant derivative of the covariant derivative

$$(\nabla_X \nabla)_Y s = \nabla_X (\nabla_Y s) - \nabla_{\nabla_X Y} s - \nabla_Y (\nabla_X s).$$

Note however, that this is not tensorial in X!

It is important to realize that this is not the same as the *second covariant derivative* of s

$$\nabla^2_{X,Y} s \quad = \quad \nabla_X (\nabla_Y s) - \nabla_{\nabla_X Y} s.$$

The two concepts are related by

$$\nabla^2_{X,Y} s = (\nabla_X \nabla)_Y s + \nabla_Y (\nabla_X s).$$

The Ricci Identities

The *Ricci identity* is simply one way of defining the curvature of sections

$$\nabla^2_{X,Y} s - \nabla^2_{Y,X} s = R_{X,Y} s$$

and it clearly agrees with the standard definition

$$R_{X,Y} s = \nabla_X (\nabla_Y s) - \nabla_X (\nabla_Y s) - \nabla_{[X,Y]} s$$

if we use the definition of the second covariant derivative and that the connection is torsion free.

From this identity one gets iterated Ricci identities by taking one more derivative

$$\nabla^3_{X,Y,Z} s - \nabla^3_{Y,X,Z} s = R_{X,Y} \nabla_Z s - \nabla_{R_{X,Y} Z} s$$

and

$$\nabla^3_{X,Y,Z}s - \nabla^3_{X,Z,Y}s = (\nabla_X R)_{Y,Z}\, s + R_{Y,Z}\nabla_X s.$$

These follow from the various way one can iterate covariant derivatives:

$$\nabla^3_{X,Y,Z}s = \nabla^2_{X,Y}\left(\nabla_Z s\right) - \nabla_{\nabla^2_{X,Y}Z}s$$

and

$$\nabla^3_{X,Y,Z}s = \nabla_X\left(\nabla^2\right)_{Y,Z} s + \nabla^2_{Y,Z}\left(\nabla_X s\right)$$

and then using the Ricci identity.

The Bianchi Identities

The Lie derivative of the Riemannian (Levi-Civita) connection is defined as

$$\begin{aligned}
\left(L_X\nabla\right)_Y Z &= L_X\left(\nabla_Y Z\right) - \nabla_{L_X Y}Z - \nabla_Y L_X Z \\
&= \nabla_X\nabla_Y Z - \nabla_{\nabla_X Y}Z - \nabla_Y\nabla_X Z \\
&\quad -\nabla_{\nabla_Y Z}X + \nabla_{\nabla_Y X}Z + \nabla_Y\nabla_Z X \\
&= R_{X,Y}Z + \nabla^2_{Y,Z}X.
\end{aligned}$$

This Lie derivative is tensorial in Y, Z as well as symmetric. The symmetry comes from taking Lie derivatives of the identity:

$$L_Y Z = \nabla_Y Z - \nabla_Z Y.$$

Thus

$$\left(L_X L\right)_Y Z = \left(L_X\nabla\right)_Y Z - \left(L_X\nabla\right)_Z Y$$

but here the left hand side vanishes due to the *Jacobi identity*

$$\begin{aligned}
\left(L_X L\right)_Y Z &= L_X\left(L_Y Z\right) - L_{L_X Y}Z - L_Y L_X Z \\
&= [X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] \\
&= 0.
\end{aligned}$$

The *first Bianchi identity* now follows from the Ricci and Jacobi identities in the following way:

$$\begin{aligned}
0 &= \left(L_X\nabla\right)_Y Z - \left(L_X\nabla\right)_Z Y \\
&= R_{X,Y}Z + \nabla^2_{Y,Z}X - R_{X,Z}Y - \nabla^2_{Z,Y}X \\
&= R_{X,Y}Z + R_{Z,X}Y + R_{Y,Z}X.
\end{aligned}$$

The *second Bianchi identity* similarly follows by using the first Bianchi identity and the Ricci identities for third covariant derivatives. First note that

$$\left(\nabla_X R\right)_{Y,Z} s = \nabla^3_{X,Y,Z}s - \nabla^3_{X,Z,Y}s - \nabla^3_{Y,Z,X}s + \nabla^3_{Z,Y,X}s + \nabla_{R_{Y,Z}X}s.$$

We then note that if we add over the cyclic permutations of X, Y, Z then the 12 third covariant derivatives cancel and the 3 remaining terms cancel due to the first Bianchi identity

$$
\begin{aligned}
(\nabla_X R)_{Y,Z}\, s + (\nabla_Z R)_{X,Y}\, s + (\nabla_Y R)_{Z,X}\, s \\
&= \nabla^3_{X,Y,Z} s - \nabla^3_{X,Z,Y} s - \nabla^3_{Y,Z,X} s + \nabla^3_{Z,Y,X} s + \nabla_{R_{Y,Z} X} s \\
&+ \nabla^3_{Z,X,Y} s - \nabla^3_{Z,Y,X} s - \nabla^3_{X,Y,Z} s + \nabla^3_{Y,X,Z} s + \nabla_{R_{X,Y} Z} s \\
&+ \nabla^3_{Y,Z,X} s - \nabla^3_{Y,X,Z} s - \nabla^3_{Z,X,Y} s + \nabla^3_{X,Z,Y} s + \nabla_{R_{Z,X} Y} s \\
&= \nabla_{R_{X,Y} Z + R_{Z,X} Y + R_{Y,Z} X} s \\
&= 0.
\end{aligned}
$$

The Extended Jacobi Identity

Finally we mention that the Jacobi identity naturally extends to tensors in the following fashion. We can always tale Lie derivatives of tensors $L_X T$. The natural extension then says

$$L_X L_Y T - L_Y L_X T = L_{[X,Y]} T$$

When T is a function this is the definition of the Lie bracket, when T is a vector field it is the Jacobi identity. With a bit of work it is not hard to show that this holds on all tensors. Probably the simplest approach is to show that it also holds for 1-forms and then on tensor products $f\omega_1 \otimes \cdots \otimes \omega_s \otimes X_1 \otimes \cdots \otimes X_t$, where f is a function, ω_i 1-forms, and X_j vector fields, by using that Lie derivatives satisfies Leibniz' rule

$$L_Z (T_1 \otimes T_2) = (L_Z T_1) \otimes T_2 + T_1 \otimes L_Z T_2.$$

This version of the Jacobi identity can then be rewritten in the terminology we used above:

$$(L_X L)_Y\, T = L_X L_Y T - L_{L_X Y} T - L_Y L_X T = 0.$$