DEMYSTIFYING THE WEITZENBÖCK CURVATURE OPERATOR

PETER PETERSEN

Abstract. The goal is to define two concepts that make computations of Laplacians of Tensors considerably simpler. This allows us in almost no time to prove many major theorems that use the Bochner Technique.

Introduction

The object of this note is to better understand Lichnerowicz Laplacians for a tensor and show that the curvature term can be deconstructed fairly easily. The strategy comes from considering the Hodge Laplacian on forms. Weitzenböck realized, prior to Hodge's work, that the Hodge Laplacian can be decomposed into two terms, one is the connection Laplacian, the other a tensorial term that depends on the curvature of the manifold. This term is often called the Weitzenböck curvature operator on forms. This curvature operator will be extended to tensors. When this term is added to the connection Laplacian we obtain one version of what is called the Lichnerowicz Laplacian.

One step in our reduction is modeled on W.A. Poor's approach to the Hodge Laplacian, which in turn was inspired by work of Chern, and only relies on the canonical representation of the orthogonal group on tensors. This allows us to include all tensors and also take it one step further. There have been many other attempts to understand the Hodge Laplacian on forms. One particularly effective approach uses spin calculus as well as other added constructions.

Along the way we shall see that almost all curvature concepts: Ricci, sectional, isotropic, complex sectional curvatures etc are related to a Weitzenböck curvature of a articular type of tensor.

The *Weitzenböck curvature operator* on a tensor is defined by

$$\mathfrak{Ric}\left(T\right)\left(X_1,...,X_k\right)=\sum\left(R\left(e_j,X_i\right)T\right)\left(X_1,...,e_j,...,X_k\right)$$

We use a variation of the Ricci tensor to symbolize this as it is the Ricci tensor when evaluated on vector fields and 1-forms. The more standard notation involving W is too easily confused with the Weyl tensor

The *Lichnerowicz Laplacian* is defined as

$$\Delta_L T=\nabla^*\nabla T+c\mathfrak{Ric}\left(T\right)$$

for a suitable constant $c>0$. We shall see below that the Hodge Laplacian on forms is of this type with $c=1$. In addition, interesting information can also be extracted for symmetric $(0,2)$ tensors as well as the curvature tensor via this operator when we use $c=\frac{1}{2}$.

2000 *Mathematics Subject Classification.* 53C20.

Supported in part by NSF-DMS grant 1006677.

The Bochner technique works for tensors that lie in the kernel of some Lichnerowicz Laplacian

$$\Delta_L T = \nabla^*\nabla T + c\mathfrak{Ric}(T) = 0.$$

The idea is to use one of two maximum principles to show that T is parallel. In order to apply the maximum principle we need

$$g(\nabla^*\nabla T, T) \leq 0$$

which by the equation for T is equivalent to showing

$$g(\mathfrak{Ric}(T), T) \geq 0.$$

The two maximum principles that have been used most in the past are stated in the next lemma.

Lemma. *Let (M, g) be a complete Riemannian manifold and T a smooth tensor such that*

$$g(\nabla^*\nabla T, T) \leq 0.$$

If $|T|$ has a maximum or $|T| \in L^2$, then T is parallel.

Proof. Note that when M is closed both conditions on T are trivially satisfied.

In case $|T|$ has a maximum we can simply apply the maximum principle to the function $|T|^2$.

When M is closed and oriented the divergence theorem also offers an alternative proof by observing

$$\begin{aligned} 0 &\leq \int |\nabla T|^2 \, d\mathrm{vol} \\ &= \int g(\nabla^*\nabla T, T)\, d\mathrm{vol} \\ &\leq 0. \end{aligned}$$

This proof has been extended by Yau to the case where M is noncompact using the assumption $|T| \in L^2$ (see) □

The two assumptions we make about T

$$\Delta_L T = 0$$

and

$$g(\mathfrak{Ric}(T), T) \geq 0$$

require some discussion. The first assumption is usually implied by showing that some other naturally defined Laplacian Δ satisfies a Weitzenböck formula

$$\Delta = \nabla^*\nabla T + c\mathfrak{Ric}(T)$$

The fact that $\Delta T = 0$ might come from certain natural restrictions on the tensor or even as a consequence of having nontrivial topology. The second assumption

$$g(\mathfrak{Ric}(T), T) \geq 0$$

is often difficult to check and in many cases it took decades to figure what curvature assumptions gave the best results. The goal here is to first develop a different formula for $\mathfrak{Ric}(T)$ and second to change T in a suitable fashion so as to create a significantly simpler formula for $g(\mathfrak{Ric}(T), T)$. This formula will immediately show that $g(\mathfrak{Ric}(T), T)$ is nonnegative when the curvature operator is nonnegative. It will also make it very easy to calculate precisely what happens when T is a $(0,1)$ or

$(0,2)$ tensor. It is worthwhile mentioning that the original proofs of some of these facts were quite complicated and only developed long after the Bochner technique had been introduced.

1. Lichnerowicz Laplacians

In this section ....

Conventions

$$\left(\nabla^{*}T\right)\left(X_{2},...,X_{k}\right)=-\left(\nabla_{E_{i}}T\right)\left(E_{i},X_{2},...,X_{k}\right)$$

1.1. **Forms.** The first obvious case to try this philosophy on is that of the Hodge Laplacian on k-forms as we already know that harmonic forms compute the topology of the underlying manifold.

Theorem 1.1. (Weitzenböck, 1923) *For any form the Hodge Laplacian is the Lichnerowicz Laplacian with $c=1$. Specifically,*

$$\begin{aligned}\triangle\omega &= \left(d\delta+\delta d\right)\left(\omega\right)\\ &= \nabla^{*}\nabla\omega+\mathfrak{Ric}\left(\omega\right).\end{aligned}$$

Proof. We shall follow the proof discovered by W.A. Poor. To perform the calculations we need

$$\begin{aligned}\delta\omega\left(X_{2},...,X_{k}\right) &= -\sum\left(\nabla_{E_{i}}\omega\right)\left(E_{i},X_{2},...,X_{k}\right),\\ d\omega\left(X_{0},...,X_{k}\right) &= \sum\left(-1\right)^{i}\left(\nabla_{X_{i}}\omega\right)\left(X_{0},...,\hat{X}_{i},...,X_{k}\right)\end{aligned}$$

and employ the usual assumptions about all covariant derivatives of vector fields vanishing at a fixed point $p\in M$. We this in mind we get

$$\begin{aligned}d\delta\omega\left(X_{1},...,X_{k}\right) &= \sum\left(-1\right)^{i+1}\nabla_{X_{i}}\delta\omega\left(X_{1},...,\hat{X}_{i},...,X_{k}\right)\\ &= \sum\left(-1\right)^{i}\nabla_{X_{i}}\nabla_{E_{j}}\omega\left(E_{j},X_{1},...,\hat{X}_{i},...,X_{k}\right)\\ &= -\sum\nabla_{X_{i}}\nabla_{E_{j}}\omega\left(X_{1},...,E_{j},...,X_{k}\right)\end{aligned}$$

$$\begin{aligned}\delta d\omega\left(X_{1},...,X_{k}\right) &= -\sum\nabla_{E_{j}}d\omega\left(E_{j},X_{1},...,X_{k}\right)\\ &= -\sum\nabla_{E_{j}}\nabla_{E_{j}}\omega\left(X_{1},...,X_{k}\right)\\ &\quad -\sum\left(-1\right)^{i}\nabla_{E_{j}}\nabla_{X_{i}}\omega\left(E_{j},X_{1},...,\hat{X}_{i},...,X_{k}\right)\\ &= \left(\nabla^{*}\nabla\omega\right)\left(X_{1},...,X_{k}\right)\\ &\quad +\sum\nabla_{E_{j}}\nabla_{X_{i}}\omega\left(X_{1},...,E_{j},...,X_{k}\right)\end{aligned}$$

Thus

$$\begin{aligned}\triangle\omega &= \nabla^{*}\nabla\omega+\sum\left(R\left(E_{j},X_{i}\right)\omega\right)\left(X_{1},...,E_{j},...,X_{k}\right)\\ &= \nabla^{*}\nabla\omega-\sum\left(R\left(E_{j},X_{i}\right)\omega\right)\left(X_{1},...,E_{j},...,X_{k}\right)\\ &= \nabla^{*}\nabla\omega+\mathfrak{Ric}\left(\omega\right)\end{aligned}$$

□

1.2. **The Curvature Tensor.** We show that a suitably defined Laplacian is in fact a Lichnerowicz Laplacian. This Laplacian is a symmetrized version of $(\nabla_X (\nabla^* R))(Y,Z,W)$ so as to make it have the same symmetries as R. It appears as the right hand side in the formula below.

Theorem 1.2. *The curvature tensor R on a Riemannian manifold satisfies*

$$\begin{aligned} & (\nabla^*\nabla R)(X,Y,Z,W) + \frac{1}{2}\mathfrak{Ric}(R)(X,Y,Z,W) \\ = \; & \frac{1}{2}(\nabla_X \nabla^* R)(Y,Z,W) - \frac{1}{2}(\nabla_Y \nabla^* R)(X,Z,W) \\ & \frac{1}{2}(\nabla_Z \nabla^* R)(W,X,Y) - \frac{1}{2}(\nabla_W \nabla^* R)(Z,X,Y) \end{aligned}$$

Proof. By far the most important ingredient in the proof is that we have the second Bianchi identity at our disposal. We will begin the calculation by considering the (0,4)-curvature tensor R. Fix a point p, let X,Y,Z,W be vector fields with $\nabla X = \nabla Y = \nabla Z = \nabla W = 0$ at p and let E_i be normal coordinates at p. Then

$$\begin{aligned} (\nabla^*\nabla R)(X,Y,Z,W) &= -\sum_{i=1}^{n}\left(\nabla^2_{E_i,E_i}R\right)(X,Y,Z,W) \\ &= \sum_{i=1}^{n}\left(\nabla^2_{E_i,X}R\right)(Y,E_i,Z,W) + \left(\nabla^2_{E_i,Y}R\right)(E_i,X,Z,W) \\ &= \sum_{i=1}^{n}\left(\nabla^2_{X,E_i}R\right)(Y,E_i,Z,W) + \left(\nabla^2_{Y,E_i}R\right)(E_i,X,Z,W) \\ &\quad + \sum_{i=1}^{n}(R(E_i,X)(R))(Y,E_i,Z,W) + (R(E_i,Y)(R))(E_i,X,Z,W) \\ &= (\nabla_X \nabla^* R)(Y,Z,W) - (\nabla_Y \nabla^* R)(X,Z,W) \\ &\quad - \sum_{i=1}^{n}(R(E_i,X)(R))(E_i,Y,Z,W) + (R(E_i,Y)(R))(X,E_i,Z,W) \end{aligned}$$

where we note that the last two terms are half of the expected terms in $-\mathfrak{Ric}(R)(X,Y,Z,W)$.

Using that R is symmetric in the pairs X, Y and Z, W we then obtain

$$\begin{aligned}
(\nabla^*\nabla R)(X,Y,Z,W) &= \frac{1}{2}(\nabla^*\nabla R)(X,Y,Z,W) - \frac{1}{2}(\nabla^*\nabla R)(Z,W,X,Y) \\
&= \frac{1}{2}((\nabla_X\nabla^* R)(Y,Z,W) - (\nabla_Y\nabla^* R)(X,Z,W)) \\
&\quad +\frac{1}{2}((\nabla_Z\nabla^* R)(W,X,Y) - (\nabla_W\nabla^* R)(Z,X,Y)) \\
&\quad -\frac{1}{2}\sum_{i=1}^{n}(R(E_i,X)(R))(E_i,Y,Z,W) + (R(E_i,Y)(R))(X,E_i,Z,W) \\
&\quad -\frac{1}{2}\sum_{i=1}^{n}(R(E_i,Z)(R))(E_i,W,X,Y) + (R(E_i,W)(R))(Z,E_i,X,Y) \\
&= \frac{1}{2}((\nabla_X\nabla^* R)(Y,Z,W) - (\nabla_Y\nabla^* R)(X,Z,W)) \\
&\quad +\frac{1}{2}((\nabla_Z\nabla^* R)(W,X,Y) - (\nabla_W\nabla^* R)(Z,X,Y)) \\
&\quad -\frac{1}{2}\mathfrak{Ric}(R)(X,Y,Z,W)
\end{aligned}$$

□

One might expect that, as with the Hodge Laplacian, there should also be terms where one takes the divergence of certain derivatives of R. However, the second Bianchi identity shows that these terms already vanish for R. In particular, R is harmonic if it is divergence free: $\nabla^* R = 0$.

1.3. **Symmetric** $(0,2)$ **Tensors.** Let h be a symmetric $(0,2)$ tensor. We can define a connection dependent exterior derivative $d^\nabla h$ as follows

$$d^\nabla h(X,Y,Z) = (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z).$$

While this definition is a bit mysterious it does occur naturally in differential geometry. Originally it comes from considering the second fundamental II for an immersed hypersurface $M^n \to \mathbb{R}^{n+1}$. In this case the Codazzi-Mainardi equations can be expressed as

$$d^\nabla \mathrm{II} = 0.$$

The second natural situation is the Ricci tensor where the second Bianchi identity implies

$$\left(d^\nabla \mathrm{Ric}\right)(Z,W,Y) = (\nabla^* R)(Y,Z,W).$$

Using this exterior derivative we obtain a formula that is similar to what we saw for forms and the curvature tensor.

Theorem 1.3. *Any symmetric* $(0,2)$ *tensor* h *on a Riemannian manifold satisfies*

$$(\nabla_X\nabla^* h)(X) + \left(\nabla^* d^\nabla h\right)(X,X) = (\nabla^*\nabla h)(X,X) + \frac{1}{2}(\mathfrak{Ric}(h))(X,X)$$

Proof. Observe that on the left hand side the terms are

$$(\nabla_X\nabla^* h)(X) = -\left(\nabla^2_{X,E_i} h\right)(E_i,X)$$

and

$$\begin{aligned}\left(\nabla^* d^\nabla h\right)(X,X) &= -\left(\nabla_{E_i} d^\nabla h\right)(E_i,X,X)\\ &= -\left(\nabla^2_{E_i,E_i} h\right)(X,X) + \left(\nabla^2_{E_i,X} h\right)(E_i,X)\end{aligned}$$

Adding these we obtain

$$\begin{aligned}\left(\nabla_X \nabla^* h\right)(X) + \left(\nabla^* d^\nabla h\right)(X,X) &= \left(\nabla^*\nabla h\right)(X,X) + \left(\nabla^2_{E_i,X} h\right)(E_i,X) - \left(\nabla^2_{X,E_i} h\right)(E_i,X)\\ &= \left(\nabla^*\nabla h\right)(X,X) + \left(R\left(E_i,X\right)h\right)(E_i,X)\end{aligned}$$

Using that h is symmetric we finally conclude that

$$\left(R\left(E_i,X\right)h\right)(E_i,X) = \frac{1}{2}\left(\mathfrak{Ric}\left(h\right)\right)(X,X)$$

thus finishing the proof. □

A symmetric $(0,2)$ tensor is called a Codazzi tensor if $d^\nabla h$ vanishes and harmonic if in addition it is divergence free. This characterization can be simplified slightly.

Proposition 1.4. *A symmetric $(0,2)$ tensor is harmonic iff it is a Codazzi tensor and has constant trace.*

Proof. In general we have that

$$\begin{aligned}\left(\nabla^* h\right)(X) &= -\left(\nabla_{E_i} h\right)(E_i,X)\\ &= -\left(\nabla_{E_i} h\right)(X,E_i)\\ &= -\left(\nabla_X h\right)(E_i,E_i) + \left(d^\nabla h\right)(X,E_i,E_i)\\ &= -D_X\left(\mathrm{tr} h\right) + \left(d^\nabla h\right)(X,E_i,E_i)\end{aligned}$$

Thus Codazzi tensors are divergence free iff their trace is constant. □

This shows that hypersurfaces that have constant mean curvature have harmonic second fundamental form. This fact has been exploited by both Lichnerowicz and Simons. For the Ricci tensor to be harmonic it suffices to assume that it is Codazzi, but this in turn is a strong condition as it is the same as saying that the full curvature tensor is harmonic.

Corollary 1.5. *The Ricci tensor is harmonic iff the curvature tensor is harmonic.*

Proof. We know that the Ricci tensor is a Codazzi tensor precisely when the curvature tensor has vanishing divergence. The contracted Bianchi identity together with the above proposition then tells us

$$\begin{aligned}2D_X\mathrm{scal} &= -\left(\nabla^*\mathrm{Ric}\right)(X)\\ &= D_X\left(\mathrm{trRic}\right)\\ &= D_X\left(\mathrm{scal}\right)\end{aligned}$$

Thus the scalar curvature must be constant and the Ricci tensor divergence free. □

2. Natural Derivations

In differential geometry there are many natural derivations on tensors coming from various combinations of derivatives. We shall attempt to tie these together in a natural and completely algebraic fashion by using that all $(1,1)$ tensors naturally act as derivations on tensors.

2.1. **Endomorphisms as Derivations.** The goal is to show that $(1,1)$ tensors naturally act as derivations on the space of all tensors.

We use the natural homomorphism

$$Gl(V) \to Gl(T(V))$$

where $T(V)$ is the space of all tensors over the vector space V. This respects the natural grading of tensors: The subspace of (s,t)-tensors is spanned by

$$v_1 \otimes \cdots \otimes v_s \otimes \phi_1 \otimes \cdots \otimes \phi_t$$

where $v_1, ..., v_s \in V$ and $\phi_1, ..., \phi_t : V \to \mathbb{R}$ are linear functions. The natural homomorphism acts trivially on scalars, on vectors

$$g \cdot v = g(v),$$

on 1-forms

$$g \cdot \phi = \phi \circ g^{-1},$$

and on general tensors:

$$\begin{aligned} & g \cdot (v_1 \otimes \cdots \otimes v_s \otimes \phi_1 \otimes \cdots \otimes \phi_t) \\ = \; & g(v_1) \otimes \cdots \otimes g(v_s) \otimes \left(\phi_1 \circ g^{-1}\right) \otimes \cdots \otimes \left(\phi_t \circ g^{-1}\right) \end{aligned}$$

The derivative of this action yields a linear map

$$\mathrm{End}(V) \to \mathrm{End}(T(V))$$

which for each $L \in \mathrm{End}(V)$ induces a derivation on $T(V)$. Specifically if $L \in \mathrm{End}(V)$, then

$$Lv = L(v)$$

on vectors, while on 1-forms

$$L\phi = -\phi \circ L$$

and general tensors

$$\begin{aligned} & L(v_1 \otimes \cdots \otimes v_s \otimes \phi_1 \otimes \cdots \otimes \phi_t) \\ = \; & L(v_1) \otimes \cdots \otimes v_s \otimes \phi_1 \otimes \cdots \otimes \phi_t \\ & + \cdots \\ & + v_1 \otimes \cdots \otimes L(v_s) \otimes \phi_1 \otimes \cdots \otimes \phi_t \\ & - v_1 \otimes \cdots \otimes v_s \otimes (\phi_1 \circ L) \otimes \cdots \otimes \phi_t \\ & - \cdots \\ & - v_1 \otimes \cdots \otimes v_s \otimes \phi_1 \otimes \cdots \otimes (\phi_t \circ L) \end{aligned}$$

As the natural derivation comes from an action that preserves symmetries of tensors we immediately obtain.

Proposition 2.1. *The linear map*

$$\begin{aligned} \mathrm{End}(V) & \to \mathrm{End}(T(V)) \\ L & \to LT \end{aligned}$$

is a Lie algebra homomorphism that preserves symmetries on tensors.

We also need to know how this derivation interacts with an inner product. The inner product on $T(V)$ is given by declaring

$$e_{i_1} \otimes \cdots \otimes e_{i_p} \otimes e^{j_1} \otimes \cdots \otimes e^{j_q}$$

an orthonormal basis when $e_1, ..., e_n$ is an orthonormal basis for V and $e^1, ..., e^n$ the dual basis for V^*.

Proposition 2.2. *If V has an inner product, then*

The adjoint of $L : V \to V$ extends to become the adjoint for $L : T(V) \to T(V)$.

If $L \in \mathfrak{so}(V)$, i.e., L is skew-adjoint, then L commutes with type change of tensors.

If $L \in \mathfrak{so}(V)$, then L commutes with contractions of symmetric tensors.

2.2. **Derivatives.** Note that both the Lie derivative L_U and the covariant derivative ∇_U act as derivations on tensors. However, these operations are non-trivial on functions. Therefore, they are not of the type we just introduced above.

Proposition 2.3. *If we think of ∇U as the $(1,1)$ tensor $X \to \nabla_X U$, then*

$$L_U = \nabla_U - (\nabla U)$$

Proof. It suffices to check that this identity holds on vector fields and functions. On functions it reduces to the definition of directional derivatives, on vectors from the definition of Lie brackets. □

This proposition indicates that one can make sense of the expression $\nabla_T U$ where T is a tensor and U a vector field. It has in other places been named $A_X T$, but as that now generally has been accepted as the A-tensor for a Riemannian submersion we have not adopted the older notation.

2.3. **Curvatures.** For any tensor T we define the curvature of T as follows

$$\begin{aligned} R_{X,Y}T &= (\nabla_X (\nabla_Y T)) - (\nabla_Y (\nabla_X T)) - \left(\nabla_{[X,Y]} T\right) \\ &= \nabla^2_{X,Y} T - \nabla^2_{Y,X} T \end{aligned}$$

Note that $R_{X,Y}$ acts as a derivation on tensors. Moreover as the Hessian of a function is symmetric

$$\nabla^2_{X,Y} f = \nabla^2_{Y,X} f$$

it follows that it acts trivially on functions. This shows that the Ricci identity holds

$$R_{X,Y} = R(X,Y)$$

where on the right hand side we think of $R(X,Y)$ as a $(1,1)$ tensor acting on tensors. As an example note that when T is a $(0,k)$ tensor then

$$\begin{aligned} (R_{X,Y}T)(X_1, \ldots, X_k) &= (R(X,Y)T)(X_1, \ldots, X_k) \\ &= -T(R(X,Y)X_1, \ldots, X_k) \\ &\quad \vdots \\ &\quad -T(X_1, \ldots, R(X,Y)X_k) \end{aligned}$$

3. Computing $g(\mathfrak{Ric}(T), T)$

The various types of derivations introduced in the previous section enable us to better understand the curvature expressions $g(\mathfrak{Ric}(T), T)$. We start by extending Poor's result for forms to general tensors. We then make a change to the tensor T that further simplifies this expression.

3.1. Simplification of $\mathfrak{Ric}(T)$.

Since $R(X,Y) : T_pM \to T_pM$ is always skew-symmetric it can be decomposed using an orthonormal basis of skew-symmetric transformations $\Xi_\alpha \in \mathfrak{so}(T_pM)$. A tricky point enters our formulas at this point. It comes from the fact that if v and w are orthonormal, then $v \wedge w \in \Lambda^2 T_pM$ is a unit vector, while the corresponding skew symmetric operator, a rotation of $\pi/2$ in span $\{v, w\}$, has Euclidean norm $\sqrt{2}$. To avoid confusion and unnecessary factors we assume that $\mathfrak{so}(T_pM)$ is endowed with the metric that comes from $\Lambda^2 T_pM$. With that in mind we have

$$\begin{aligned} R(X,Y) &= g(R(X,Y), \Xi_\alpha)\Xi_\alpha \\ &= g(\mathfrak{R}(X \wedge Y), \Xi_\alpha)\Xi_\alpha \\ &= g(\mathfrak{R}(\Xi_\alpha), X \wedge Y)\Xi_\alpha \\ &= -g(R(\Xi_\alpha)X, Y)\Xi_\alpha \end{aligned}$$

where the last line is due to the convention

$$g((x \wedge y)(v), w) = -g(x \wedge y, v \wedge w).$$

This allows us to rewrite the Weitzenböck curvature operator.

Lemma 3.1. *For any $(0,k)$ tensor T*

$$\begin{aligned} \mathfrak{Ric}(T) &= -\sum R(\Xi_\alpha)(\Xi_\alpha T), \\ \Delta_L T &= \nabla^*\nabla T - c\sum R(\Xi_\alpha)(\Xi_\alpha T) \end{aligned}$$

Proof. This is a straightforward calculation:

$$\begin{aligned} \mathfrak{Ric}(T)(X_1, ..., X_k) &= \sum (R(e_j, X_i)T)(X_1, ..., e_j, ..., X_k) \\ &= -\sum g(R(\Xi_\alpha)e_j, X_i)(\Xi_\alpha T)(X_1, ..., e_j, ..., X_k) \\ &= -\sum (\Xi_\alpha T)(X_1, ..., g(R(\Xi_\alpha)e_j, X_i)e_j, ..., X_k) \\ &= \sum (\Xi_\alpha T)(X_1, ..., R(\Xi_\alpha)X_i, ..., X_k) \\ &= -\sum (R(\Xi_\alpha)(\Xi_\alpha T))(X_1, ..., X_i, ..., X_k) \end{aligned}$$

□

At first sight we have replaced a simple sum over j and i with a possibly more complicated sum. The next result justifies the reformulation.

Corollary 3.2. *If $\mathfrak{R} \geq 0$, then $g(\mathfrak{Ric}(T), T) \geq 0$.*

Proof. Select the orthonormal basis Ξ_α to consist of eigenvectors for $\mathfrak{R}$, i.e., $\mathfrak{R}(\Xi_\alpha) = \lambda_\alpha \Xi_\alpha$. When taking inner products with T and using that $R(\Xi_\alpha)$ is skew symmetric we obtain

$$\begin{aligned} -\sum g\left(R\left(\Xi_\alpha\right)\left(\Xi_\alpha T\right), T\right) &= \sum g\left(\Xi_\alpha T, R\left(\Xi_\alpha\right) T\right) \\ &= \sum \lambda_\alpha \left|\Xi_\alpha T\right|^2 \end{aligned}$$

This shows that the curvature term is nonnegative when the curvature operator is nonnegative. □

This corollary immediately implies:

Theorem 3.3. (D. Meyer 1971 and D. Meyer-Gallot 1975) *Let (M,g) be a closed Riemannian manifold. If the curvature operator is nonnegative, then all harmonic forms are parallel. Moreover, when the curvature operator is positive the only parallel p-forms have $p = 0, n$.*

Proof. The first statement is immediate given the Weitzenböck formula for forms. For the second part we note that when the curvature operator is positive then the formula

$$0 = g\left(\mathfrak{Ric}\left(\omega\right), \omega\right) = \sum \lambda_\alpha \left|\Xi_\alpha \omega\right|^2$$

shows that $\Xi_\alpha \omega = 0$ for all α. Hence by linearity $L\omega = 0$ for all skew-symmetric L. If we assume $k < n$ and select L so that $L(e_i) = 0$ for $i < k$, $L(e_k) = e_{k+1}$, then

$$0 = (L\omega)(e_1, ..., e_k) = -\omega(e_1, ..., e_{k-1}, e_{k+1})$$

Since the basis was arbitrary this shows that $\omega = 0$. □

Next we present a similar result for the curvature tensor.

Theorem 3.4. (Tachibana, 1974) *Let (M,g) is a closed Riemannian manifold. If the curvature operator is nonnegative and $\nabla^* R = 0$, then $\nabla R = 0$. If in addition the curvature operator is positive, then (M,g) has constant curvature.*

Proof. We know from above that

$$\nabla^*\nabla R + \frac{1}{2}\mathfrak{Ric}\left(R\right) = 0$$

So if the curvature operator is nonnegative, then $\nabla R = 0$.

Moreover, when the curvature operator is positive it follow,s as in the case of forms, that $LR = 0$ for all $L \in \mathfrak{so}(T_pM)$. This condition implies, as we shall show below, that $R(x,y,y,z) = 0$ and $R(x,y,v,w) = 0$ when the vectors are perpendicular. This in turn shows that any bi-vector $x \wedge y$ is an eigenvector for $\mathfrak{R}$, but this can only happen if $\mathfrak{R} = kI$ for some constant k.

To show that the mixed curvatures vanish first select L so that $L(y) = 0$ and $L(x) = z$, then

$$\begin{aligned} 0 &= LR(x,y,y,x) = -R(L(x), y, y, x) - R(x, y, y, L(x)) \\ &= -2R(x,y,y,z). \end{aligned}$$

Polarizing in $y = v + w$, then shows that

$$R(x,v,w,z) = -R(x,w,v,z)$$

The Bianchi identity then implies

$$\begin{aligned} R(x,v,w,z) &= R(w,v,x,z) - R(w,x,v,z) \\ &= -2R(w,x,v,z) \\ &= 2R(x,w,v,z) \\ &= -2R(x,v,w,z) \end{aligned}$$

showing that $R(x,v,w,z) = 0$. □

3.2. **Simplification of $g(\mathfrak{Ric}(T), T)$.** The goal is now to work out the formula for 1-forms and general $(0,2)$ tensors. In both cases one easily recovers the results that are already known for such tensors.

Having redefined the Ricci curvature of tensors, we take it a step further and also discard the orthonormal basis Ξ_α. To assist in this note that a $(0,k)$-tensor T can be changed to a tensor $\hat{T}$ with values in $\Lambda^2 TM$. Implicitly this works as follows

$$g\left(L, \hat{T}(X_1, ..., X_k)\right) = (LT)(X_1, ..., X_k) \text{ for all } L \in \mathfrak{so}(TM) = \Lambda^2 TM.$$

Lemma 3.5. *For any $(0,k)$ tensor T we have*

$$g(\mathfrak{Ric}(T), T) = g\left(\mathfrak{R}\left(\hat{T}\right), \hat{T}\right).$$

Proof. This is a straight forward calculation

$$\begin{aligned} g(\mathfrak{Ric}(T), T) &= \sum g(\Xi_\alpha T, R(\Xi_\alpha) T) \\ &= \sum (\Xi_\alpha T)(e_{i_1}, ..., e_{i_k}) (R(\Xi_\alpha) T)(e_{i_1}, ..., e_{i_k}) \\ &= \sum g\left(\Xi_\alpha, \hat{T}(e_{i_1}, ..., e_{i_k})\right) g\left(R(\Xi_\alpha), \hat{T}(e_{i_1}, ..., e_{i_k})\right) \\ &= \sum g\left(\mathfrak{R}\left(g\left(\Xi_\alpha, \hat{T}(e_{i_1}, ..., e_{i_k})\right)\Xi_\alpha\right), \hat{T}(e_{i_1}, ..., e_{i_k})\right) \\ &= \sum g\left(\mathfrak{R}\left(\hat{T}(e_{i_1}, ..., e_{i_k})\right), \hat{T}(e_{i_1}, ..., e_{i_k})\right) \\ &= g\left(\mathfrak{R}\left(\hat{T}\right), \hat{T}\right) \end{aligned}$$

□

This new expression for $g(\mathfrak{Ric}(T), T)$ is clearly nonnegative when the curvature operator is nonnegative. In addition it also occasionally allows us to show that it is nonnegative under less restrictive hypotheses.

We obtain the well known result by Bochner that Ricci curvature controls 1-forms and vector fields.

Proposition 3.6. *If ω is a 1-form and X the dual vector field, then*

$$\hat{\omega}(Z) = Z \wedge X$$

and

$$g(\mathfrak{R}(\hat{\omega}), \hat{\omega}) = \mathrm{Ric}(X, X).$$

Proof. In this case

$$\begin{aligned} (L\omega)(Z) &= -\omega(L(Z)) \\ &= -g(X, L(Z)) \\ &= g(L, Z \wedge X) \end{aligned}$$

so

$$\hat{\omega}(Z) = Z \wedge X.$$

This shows that the curvature term in the Bochner formula becomes

$$\begin{aligned}
-\sum g\left(R\left(\Xi_\alpha\right)\left(\Xi_\alpha\omega\right),\omega\right) &= \sum g\left(\Xi_\alpha\omega, R\left(\Xi_\alpha\right)\omega\right) \\
&= \sum g\left(\hat{\omega}\left(E_i\right), \mathfrak{R}\left(\hat{\omega}\left(E_i\right)\right)\right) \\
&= \sum g\left(\mathfrak{R}\left(E_i \wedge X\right), E_i \wedge X\right) \\
&= \sum R\left(X, E_i, E_i, X\right) \\
&= \operatorname{Ric}\left(X, X\right)
\end{aligned}$$

□

More generally one can show that if ω is a p-form and

$$g\left(\Omega\left(X_1, ..., X_{p-1}\right), X_p\right) = \omega\left(X_1, ..., X_p\right),$$

then

$$\hat{\omega}\left(X_1, ..., X_p\right) = \sum_{i=1}^{p} (-1)^{p-i} X_i \wedge \Omega\left(X_1, ..., \hat{X}_i, ..., X_p\right).$$

Moreover, note that $\hat{\omega}$ can only vanish if ω vanishes.

Next we focus on understanding $\mathfrak{Ric}\left(\hat{h}\right)$ for any $(0,2)$-tensor. Given a $(0,2)$-tensor h there is a corresponding $(1,1)$-tensor called H

$$h(z, w) = g(H(z), w)$$

The adjoint of H is denoted H^*.

Proposition 3.7. *Let H be a $(1,1)$-tensor, then*

$$\hat{h}(z, w) = -H(z) \wedge w + z \wedge H^*(w)$$

and $\hat{h} = 0$ iff $h = \lambda g$.

Proof. We start by observing that

$$\begin{aligned}
(Lh)(z, w) &= -h(L(z), w) - h(z, L(w)) \\
&= -g(H(L(z)), w) - g(H(w), L(z)) \\
&= -g(L(z), H^*(w)) - g(L(z), H(w)) \\
&= g(L, z \wedge H^*(w)) + g(L, z \wedge H(w)) \\
&= g(L, -H(z) \wedge w + z \wedge H^*(w))
\end{aligned}$$

Note that if $h = \lambda g$ then $H = \lambda I = H^*$, thus $\hat{h} = 0$. Next assume that $\hat{h} = 0$. Then for all z, w we have

$$\begin{aligned}
z \wedge H^*(H(w)) &= H(z) \wedge H(w) \\
&= -H(w) \wedge H(z) \\
&= -w \wedge H^*(H(z)) \\
&= H^*(H(z)) \wedge w
\end{aligned}$$

But that only be true if $H^* H = \lambda^2 I$ and $H = \lambda I$. □

This indicates that we have to control curvatures of the type

$$g\left(\mathfrak{R}\left(-H\left(z\right)\wedge w+z\wedge H^{*}\left(w\right)\right),-H\left(z\right)\wedge w+z\wedge H^{*}\left(w\right)\right).$$

If H is normal, then it can be diagonalized with respect to an orthonormal basis in the complexified tangent bundle. Assuming that $H\left(z\right)=\lambda z$ and $H\left(w\right)=\mu w$ where $z,w\in T_{p}M\otimes\mathbb{C}$ are orthonormal we obtain

$$g\left(\mathfrak{R}\left(-H\left(z\right)\wedge w+z\wedge H^{*}\left(w\right)\right),\overline{-H\left(z\right)\wedge w+z\wedge H^{*}\left(w\right)}\right)=\left|-\lambda+\bar{\mu}\right|^{2}g\left(\mathfrak{R}\left(z\wedge w\right),\overline{z\wedge w}\right).$$

The curvature term $g\left(\mathfrak{R}\left(z\wedge w\right),\overline{z\wedge w}\right)$ looks like a complexified sectional curvature and is in fact called the *complex sectional curvature*. It can be recalculated without references to complexifications. If we let $z=x+\sqrt{-1}y$ and $w=u+\sqrt{-1}v$, $x,y,u,v\in TM$ then

$$\begin{aligned}
g\left(\mathfrak{R}\left(z\wedge w\right),\bar{z}\wedge\bar{w}\right) &= g\left(\mathfrak{R}\left(x\wedge u-y\wedge v\right),x\wedge u-y\wedge v\right)\\
&\quad+g\left(\mathfrak{R}\left(x\wedge v+y\wedge u\right),x\wedge v+y\wedge u\right)\\
&= g\left(\mathfrak{R}\left(x\wedge u\right),x\wedge u\right)+g\left(\mathfrak{R}\left(y\wedge v\right),y\wedge v\right)\\
&\quad+g\left(\mathfrak{R}\left(x\wedge v\right),x\wedge v\right)+g\left(\mathfrak{R}\left(y\wedge u\right),y\wedge u\right)\\
&\quad-2g\left(\mathfrak{R}\left(x\wedge u\right),y\wedge v\right)+2g\left(\mathfrak{R}\left(x\wedge v\right),y\wedge u\right)\\
&= R\left(x,u,u,x\right)+R\left(y,v,v,y\right)+R\left(x,v,v,x\right)+R\left(y,u,u,y\right)\\
&\quad+2R\left(x,u,y,v\right)-2R\left(x,v,y,u\right)\\
&= R\left(x,u,u,x\right)+R\left(y,v,v,y\right)+R\left(x,v,v,x\right)+R\left(y,u,u,y\right)\\
&\quad-2\left(R\left(v,y,x,u\right)+R\left(x,v,y,u\right)\right)\\
&= R\left(x,u,u,x\right)+R\left(y,v,v,y\right)+R\left(x,v,v,x\right)+R\left(y,u,u,y\right)\\
&\quad+2R\left(y,x,v,u\right)\\
&= R\left(x,u,u,x\right)+R\left(y,v,v,y\right)+R\left(x,v,v,x\right)+R\left(y,u,u,y\right)\\
&\quad+2R\left(x,y,u,v\right)
\end{aligned}$$

The first line in this derivation shows that complex sectional curvatures are nonnegative when $\mathfrak{R}\geq0$. Thus we see that it is weaker than working with the curvature operator. On the other hand it is stronger than sectional curvature.

There are three special cases depending on the dimension of span$\left\{x,y,u,v\right\}$. When $y=v=0$ we obtain the standard definition of sectional curvature. When x, y, u, v are orthonormal we obtain the so called *isotropic curvature*, and finally if $u=v$ we get a sum of two sectional curvatures

$$2R\left(x,u,u,x\right)+2R\left(y,u,u,y\right)$$

also called a *second Ricci curvature*, when x,y,u are orthonormal.

The next result is a general version of two separate theorems. Simons and Berger did the case of symmetric tensors and Micallef-Wang the case of 2-forms.

Proposition 3.8. *Let h be a $(0,2)$ tensor such that H is normal. If the complex sectional curvatures are nonnegative, then* $g\left(\mathfrak{R}\left(\hat{h}\right),\hat{h}\right)\geq0$.

Proof. We can use complex orthonormal bases as well as real bases to compute $g\left(\mathfrak{R}\left(\hat{h}\right),\hat{h}\right)$. Using that H is normal we obtain a complex orthonormal basis e_{i}

of eigenvectors $H(e_i) = \lambda_i e_i$ and $H^*(e_i) = \bar{\lambda}_i e_i$. Using that we quickly obtain

$$
\begin{aligned}
g\left(\mathfrak{R}\left(\hat{h}\right), \hat{h}\right) &= \sum g\left(\mathfrak{R}\left(\hat{h}(e_i, e_j)\right), \overline{\hat{h}(e_i, e_j)}\right) \\
&= \sum g\left(\mathfrak{R}\left(-H(e_i) \wedge e_j + e_i \wedge H^*(e_j)\right), \overline{-H(e_i) \wedge e_j + e_i \wedge H^*(e_j)}\right) \\
&= \sum \left|-\lambda_i + \bar{\lambda}_j\right|^2 g\left(\mathfrak{R}(e_i \wedge e_j), \overline{e_i \wedge e_j}\right)
\end{aligned}
$$

□

In the special case where H is self-adjoint the eigenvalues/vectors are real we need only use the real sectional curvatures. When H is skew-adjoint the eigenvectors are purely imaginary unless they correspond to zero eigenvalues. This shows that we must use the isotropic curvatures and also the second Ricci curvatures when M is odd dimensional. However, none of the terms involve real sectional curvatures.

These characterizations can be combined to show

Proposition 3.9. *$g\left(\mathfrak{R}\left(\hat{h}\right), \hat{h}\right) \geq 0$ for all $(0,2)$-tensors on T_pM if and only if all complex sectional curvatures on T_pM are nonnegative.*

Proof. We decompose $h = h_s + h_a$ into symmetric and skew symmetric parts. Then

$$
\begin{aligned}
g\left(\mathfrak{R}\left(\hat{h}\right), \hat{h}\right) &= g\left(\mathfrak{R}\left(\hat{h}_s\right), \hat{h}_s\right) + g\left(\mathfrak{R}\left(\hat{h}_a\right), \hat{h}_a\right) + g\left(\mathfrak{R}\left(\hat{h}_s\right), \hat{h}_a\right) + g\left(\mathfrak{R}\left(\hat{h}_a\right), \hat{h}_s\right) \\
&= g\left(\mathfrak{R}\left(\hat{h}_s\right), \hat{h}_s\right) + g\left(\mathfrak{R}\left(\hat{h}_a\right), \hat{h}_a\right) + 2g\left(\mathfrak{R}\left(\hat{h}_s\right), \hat{h}_a\right)
\end{aligned}
$$

However,

$$
\begin{aligned}
g\left(\mathfrak{R}\left(\hat{h}_s\right), \hat{h}_a\right) &= \sum g\left(\mathfrak{R}\left(\hat{h}_s(e_i, e_j)\right), \hat{h}_a(e_i, e_j)\right) \\
&= -\sum g\left(\mathfrak{R}\left(\hat{h}_s(e_j, e_i)\right), \hat{h}_a(e_j, e_i)\right) \\
&= -g\left(\mathfrak{R}\left(\hat{h}_s\right), \hat{h}_a\right)
\end{aligned}
$$

So

$$
g\left(\mathfrak{R}\left(\hat{h}\right), \hat{h}\right) = g\left(\mathfrak{R}\left(\hat{h}_s\right), \hat{h}_s\right) + g\left(\mathfrak{R}\left(\hat{h}_a\right), \hat{h}_a\right)
$$

and the result follows from the previous proposition. □

From this one can now easily recover the results about 2-forms, the Ricci tensor and second fundamental form found in.....

4. Laplacians with Diffusion

The goal here is to add first order terms to the Lichnerowicz Laplacian.

4.1. Generalized Divergence.

When the Riemannian measure is changed from being $d\mathrm{vol}$ to $e^{-f}d\mathrm{vol}$ it is natural to also change the way we compute divergences so as to make sure they are still adjoints to exterior and covariant derivatives. To this end we define

$$
\begin{aligned}
\delta_f &= e^f \delta e^{-f} = \delta + i_{\nabla f} \\
&= \nabla^* + i_{\nabla f} = e^f \nabla^* e^{-f} = \nabla^*_f
\end{aligned}
$$

Proposition 4.1. *∇^*_f is the adjoint to ∇ and d with respect to the measure $e^{-f}d\mathrm{vol}$.*

Proof. We calculate

$$\begin{aligned}\int g(d\omega,\omega')\,e^{-f}d\mathrm{vol} &= \int g\left(d\omega,e^{-f}\omega'\right)d\mathrm{vol}\\ &= \int g\left(\omega,\delta\left(e^{-f}\omega'\right)\right)d\mathrm{vol}\\ &= \int g\left(\omega,e^{f}\delta\left(e^{-f}\omega'\right)\right)e^{-f}d\mathrm{vol}\\ &= \int g\left(\omega,\delta_f\omega'\right)e^{-f}d\mathrm{vol}\end{aligned}$$

Likewise

$$\begin{aligned}\int g(\nabla S,T)\,e^{-f}d\mathrm{vol} &= \int g\left(\nabla S,e^{-f}T\right)d\mathrm{vol}\\ &= \int g\left(S,\nabla^*\left(e^{-f}T\right)\right)d\mathrm{vol}\\ &= \int g\left(S,e^{f}\nabla^*\left(e^{-f}T\right)\right)e^{-f}d\mathrm{vol}\\ &= \int g\left(S,\nabla_f^*T\right)e^{-f}d\mathrm{vol}\end{aligned}$$

□

The previous proposition certainly works for tensors with compact support and thus by extension in $W^{1,2}$, the Hilbert space of tensors in $L^2\left(e^{-f}d\mathrm{vol}\right)$ with weak derivatives also in $L^2\left(e^{-f}d\mathrm{vol}\right)$. This is quite interesting as we can, e.g., use $(M,g)=(\mathbb{R}^n,\mathrm{can})$ with $f=\frac{1}{2}|x|^2$. In this case the measure is proportional to the Gaussian measure and thus has finite volume. This means that bounded tensors with bounded derivatives lie in $W^{1,2}$.

More generally one can consider

$$\begin{aligned}\delta_U &= \delta+i_U\\ \nabla_U^* &= \nabla^*+i_U\end{aligned}$$

for a vector field U, but this divergence operator will not necessarily be the adjoint to d or ∇ for any measure. Nevertheless calculations of Lichnerowicz Laplacians are just as simple using this more general divergence.

4.2. **Bochner with Diffusion.** As long as we use the maximum principle we can easily generalize the Bochner technique to work when we have a diffusion term. The important observation is

Lemma 4.2. *Let T be a tensor such that*

$$g\left(\nabla_U^*\nabla T,T\right)\le 0,$$

If $|T|$ has a maximum, then T is parallel.

In case $U=\nabla f$, we can also use integration.

Lemma 4.3. *Assume that $\int e^{-f}d\mathrm{vol}<\infty$. Let $T\in L^2\left(e^{-f}d\mathrm{vol}\right)$ be a tensor such that*

$$g\left(\nabla_f^*\nabla T,T\right)\le 0,$$

then T is parallel.

4.3. **Lichnerowicz Laplacians with Diffusion.** We start by checking what happens for Hodge Laplacians using the generalized divergence operator. In this case the natural U-Hodge Laplacian becomes

$$\begin{aligned}\Delta_U &= \delta_U d + d\delta_U \\ &= \delta d + d\delta + i_U d + d i_U \\ &= \Delta + L_U\end{aligned}$$

and is thus the standard Hodge Laplacian with a Lie derivative as diffusion term.

Proposition 4.4. *The U-Hodge Laplacian on forms satisfies the Weitzenböck formula*

$$\Delta_U \omega = \nabla_U^* \nabla \omega + \mathfrak{Ric}(\omega) - (\nabla U)\,\omega.$$

Proof. Lie derivatives and covariant derivatives are related by the derivation coming from the $(1,1)$ tensor ∇U

$$L_U = \nabla_U - \nabla U$$

Since we already know that $\Delta = \nabla^* \nabla + \mathfrak{Ric}$ on forms this will balance the terms in the formula that only depend on U. □

The Lichnerowicz Laplacians also generalize in a natural fashion. In view of this proposition is it natural to define a new Weitzenböck curvature as follows:

$$\mathfrak{Ric}_U = \mathfrak{Ric} - (\nabla U).$$

In the case of 1-forms it was introduced by Lichnerowicz who simply called it the C tensor. Today it is better known as the Bakry-Emery tensor.

This leads to the U-Lichnerowicz Laplacian on tensors

$$\begin{aligned}\Delta_{L,U} &= \nabla_U^* \nabla + c\,\mathfrak{Ric}_U \\ &= \nabla_U^* \nabla + c\,(\mathfrak{Ric} - (\nabla U)),\ c > 0.\end{aligned}$$

In case $U = \nabla f$ we also use the notation

$$\begin{aligned}\mathfrak{Ric}_f &= \mathfrak{Ric} - (\nabla\nabla f) = \mathfrak{Ric} - S_f, \\ \Delta_{L,f} &= \nabla_f^* \nabla + c\,\mathfrak{Ric}_f.\end{aligned}$$

It is useful to have a formula for $g((\nabla U)\,T, T)$ since that term now gets added to the curvature term. Let $\nabla U = S_U + S'_U$ be the decomposition of the operator ∇U into symmetric and skew symmetric parts.

Proposition 4.5. *If T is a $(0,k)$ tensor then*

$$g((\nabla U)\,T, T) = g(S_U T, T) = \frac{1}{2}(L_U g)(T,T)$$

Proof. Since S'_U is skew symmetric we clearly have that

$$g(S'_U T, T) = 0$$

This proves the first equality.

To check the second equality note that:

$$\begin{aligned}g((\nabla U)\,T, T) &= -g(L_U T, T) + g(\nabla_U T, T) \\ &= -g(L_U T, T) + \frac{1}{2} D_U |T|^2 \\ &= \frac{1}{2}(L_U g)(T,T).\end{aligned}$$

□

Next we show that diffusion terms can also be included when considering curvature tensors.

Proposition 4.6. *The curvature tensor satisfies*

$$\begin{aligned}\nabla_U^*\nabla R+\frac{1}{2}\mathfrak{Ric}_U(R) &= \Delta_{L,U}R\\ &= \frac{1}{2}(\nabla_X\nabla_U^*R)(Y,Z,W)-\frac{1}{2}(\nabla_Y\nabla_U^*R)(X,Z,W)\\ &\quad \frac{1}{2}(\nabla_Z\nabla_U^*R)(W,X,Y)-\frac{1}{2}(\nabla_W\nabla_U^*R)(Z,X,Y)\end{aligned}$$

Proof. We start with the formula

$$\begin{aligned}\nabla^*\nabla R+\frac{1}{2}\mathfrak{Ric}(R) &= \Delta_L R\\ &= \frac{1}{2}(\nabla_X\nabla^*R)(Y,Z,W)-\frac{1}{2}(\nabla_Y\nabla^*R)(X,Z,W)\\ &\quad \frac{1}{2}(\nabla_Z\nabla^*R)(W,X,Y)-\frac{1}{2}(\nabla_W\nabla^*R)(Z,X,Y)\end{aligned}$$

To verify the proposition we need the extra terms that involve U to cancel out. This relies on the second Bianchi identity. Assume as usual that X,Y,Z,W are parallel at a some fixed point. On the left hand side we have

$$\nabla_U R-\frac{1}{2}(\nabla U)(R)$$

To understand the right hand side we first need to observe that

$$\begin{aligned}(\nabla_X\nabla_U^*R)(Y,Z,W) &= (\nabla_X(\nabla^*R+i_UR))(Y,Z,W)\\ &= (\nabla_X\nabla^*R)(Y,Z,W)+\nabla_X(R(U,Y,Z,W))\\ &= (\nabla_X\nabla^*R)(Y,Z,W)+(\nabla_XR)(U,Y,Z,W)+R(\nabla_XU,Y,Z,W)\end{aligned}$$

This allows us to simplify the U terms on the right hand side:

$$\begin{aligned}
& +\frac{1}{2}\left(\nabla_X R\right)(U,Y,Z,W)-\frac{1}{2}\left(\nabla_Y R\right)(U,X,Z,W) \\
& +\frac{1}{2}\left(\nabla_Z R\right)(U,W,X,Y)-\frac{1}{2}\left(\nabla_W R\right)(U,Z,X,Y) \\
& +\frac{1}{2}R\left(\nabla_X U,Y,Z,W\right)-\frac{1}{2}R\left(\nabla_Y U,X,Z,W\right) \\
& +\frac{1}{2}R\left(\nabla_Z U,W,X,Y\right)-\frac{1}{2}R\left(\nabla_W U,Z,X,Y\right) \\
= & -\frac{1}{2}\left(\nabla_X R\right)(Y,U,Z,W)-\frac{1}{2}\left(\nabla_Y R\right)(U,X,Z,W) \\
& -\frac{1}{2}\left(\nabla_Z R\right)(W,U,X,Y)-\frac{1}{2}\left(\nabla_W R\right)(U,Z,X,Y) \\
& +\frac{1}{2}R\left(\nabla_X U,Y,Z,W\right)+\frac{1}{2}R\left(X,\nabla_Y U,Z,W\right) \\
& +\frac{1}{2}R\left(\nabla_Z U,W,X,Y\right)+\frac{1}{2}R\left(Z,\nabla_W U,X,Y\right) \\
= & \frac{1}{2}\left(\nabla_U R\right)(X,Y,Z,W)+\frac{1}{2}\left(\nabla_U R\right)(X,Y,Z,W) \\
& +\frac{1}{2}R\left(\nabla_X U,Y,Z,W\right)+\frac{1}{2}R\left(X,\nabla_Y U,Z,W\right) \\
& +\frac{1}{2}R\left(X,Y,\nabla_Z U,W\right)+\frac{1}{2}R\left(X,Y,Z,\nabla_W U\right) \\
= & \left(\nabla_U R\right)(X,Y,Z,W)-\frac{1}{2}\left(\nabla U\right)R(X,Y,Z,W)
\end{aligned}$$

□

Finally for $(0,2)$-tensors we obtain.

Corollary 4.7. *If h is a symmetric $(0,2)$ tensor, then*

$$\left(\nabla_X \nabla_U^* h\right)(X)+\left(\nabla_U^* d^{\nabla} h\right)(X,X)=\left(\nabla_U^* \nabla h\right)(X,X)+\frac{1}{2}\left(\mathfrak{Ric}_U(h)\right)(X,X)$$

Proof. Since we know that

$$\left(\nabla_X \nabla^* h\right)(X)+\left(\nabla^* d^{\nabla} h\right)(X,X)=\left(\nabla^* \nabla h\right)(X,X)+\frac{1}{2}\left(\mathfrak{Ric}(h)\right)(X,X)$$

we can isolate the terms that depend on U. Thus we must show

$$\left(\nabla_X i_U h\right)(X)+\left(i_U d^{\nabla} h\right)(X,X)=\left(\nabla_U h\right)(X,X)-\frac{1}{2}\left(\left(\nabla U\right)h\right)(X,X)$$

This follows if we start on the left hand side

$$\begin{aligned}
\left(\nabla_X h\right)(U,X)+\left(\nabla_U h\right)(X,X)-\left(\nabla_X h\right)(U,X)+h\left(\nabla_X U,X\right) & = \left(\nabla_U h\right)(X,X)+h\left(\nabla_X U,X\right) \\
& = \left(\nabla_U h\right)(X,X)-\frac{1}{2}\left(\left(\nabla U\right)h\right)(X,X)
\end{aligned}$$

□

Add something about what a natural assumption on h might be. Having it be Codazzi would seem natural but we know that that is not what happens to the Ricci tensor for solitons. In fact the natural condition for the curvature tensor $\nabla_U^* R = 0$ translates into

$$\nabla_U^* \mathrm{Ric} = 0$$

and

$$\left(d^{\nabla} \mathrm{Ric}\right)(Z, W, Y) = (\nabla^* R)(Y, Z, W) = -R(U, Y, Z, W) = R(Z, W, Y, U).$$

4.4. **Ricci Solitons.** A Ricci soliton is a metric that satisfies

$$\mathrm{Ric} + \frac{1}{2} L_U g = \lambda g,$$

As endormorphisms

$$\mathrm{Ric} + S_U = \lambda I$$

on all tensors

References

[1] A.L. Besse, *Einstein Manifolds*, Berlin-Heidelberg: Springer-Verlag, 1978.

[2] S. Gallot and D. Meyer, *Opérateur de courbure et laplacien des formes différentielles d'une variété riemannianne*, J. Math. Pures. Appl. 54 (1975), 259-284.

[3] M.L. Labbi, *On Weitzenböck Curvature Operators*, arXiv:math/0607521v2

[4] H.B. Lawson Jr. and M.-L. Michelsohn, *Spin Geometry*, Princeton: Princeton University Press, 1989.

[5] A. Lichnerowicz, *Géométrie des groupes de transformations*, Paris: Dunod, 1958.

[6] A. Lichnerowicz, *Propagateurs et Commutateurs en relativité générale*, Publ. Math. IHES 10 (1961) 293-343.

[7] M.J. Micallef and J.D. Moore, *Minimal 2-spheres and the topology of manifolds with positive curvature on totally isotropic 2-planes*, Ann. of Math. 127 (1988), 199-227.

[8] M.J.Micallef and M.Y. Wang, *Metrics with nonnegative isotropic curvature*. Duke Math. J. 72 (1993), no. 3, 649–672.

[9] N. Mok, *The uniformization theorem for compact Kähler manifolds of non-negative holomorphic bi-sectional curvature*, J. Diff. Geo. 27 (1988), 179-214.

[10] J.W. Morgan, *The Seiberg-Witten equations and applications to the topology of smooth four-manifolds*, Princeton: Princeton Univ. Press, 1996.

[11] J. Simons, *Minimal varieties in riemannian manifolds*. Ann. of Math. (2) 88 1968 62–105.

[12] S. Tachibana, *A theorem on Riemannian manifolds of positive curvature operator*, Proc. Japan Acad. 50 (1974), 301-302.

[13] H.H. Wu, *The Bochner Technique in Differential Geometry*, Mathematical Reports vol. 3, part 2, London: Harwood Academic Publishers, 1988.

Dept. of Mathematics, UCLA, 520 Portola Pl, Los Angeles, CA, 90095
E-mail address: petersen@math.ucla.edu
URL: http://www.math.ucla.edu/~petersen