

1 The Lichnerowicz Laplacian

The goal is to define the Lichnerowicz Laplacian for a tensor and show that we can deconstruct the curvature term rather easily. This appears to be new, but is modelled on W.A. Poor's approach to the Hodge Laplacian, which in turn was inspired by work of Chern.

We define

$$\mathfrak{Ric}(T)(X_1, \dots, X_k) = \sum (R(e_j, X_i)T)(X_1, \dots, e_j, \dots, X_k)$$

and the Lichnerowicz Laplacian

$$\Delta_L T = \nabla^* \nabla T + c \mathfrak{Ric}(T)$$

for a suitable constant $c > 0$. We shall see below that the Hodge Laplacian on forms is of this type and that interesting information can also be extracted from the Ricci tensor and curvature tensor via this operator when we use $c = \frac{1}{2}$. The goal here is to give a simple formula for $\mathfrak{Ric}(T)$ showing that it is nonnegative when the curvature operator is nonnegative.

For any tensor T we have

$$\begin{aligned} R(X, Y)T &= (\nabla_X(\nabla_Y T)) - (\nabla_Y(\nabla_X T)) - (\nabla_{[X, Y]}T) \\ &= \nabla_{X, Y}^2 T - \nabla_{Y, X}^2 T \end{aligned}$$

and

$$\begin{aligned} (R(X, Y)T)(X_1, \dots, X_k) &= R(X, Y)(T(X_1, \dots, X_k)) \\ &\quad - T(R(X, Y)X_1, \dots, X_k) \\ &\quad \vdots \\ &\quad - T(X_1, \dots, R(X, Y)X_k) \end{aligned}$$

This can be rewritten in a more abstract and algebraic fashion. We have a natural homomorphism

$$Gl(V) \rightarrow Gl(T(V))$$

where $T(V)$ is the space of all tensors over the vector space V . We know that (s, t) -tensors are spanned by

$$v_1 \otimes \dots \otimes v_s \otimes \phi_1 \otimes \dots \otimes \phi_t$$

where $v_1, \dots, v_s \in V$ and $\phi_1, \dots, \phi_t : V \rightarrow \mathbb{R}$ are linear functions. The natural homomorphism acts on tensors as follows:

$$\begin{aligned} &L \cdot (v_1 \otimes \dots \otimes v_s \otimes \phi_1 \otimes \dots \otimes \phi_t) \\ &= L(v_1) \otimes \dots \otimes L(v_s) \otimes (\phi_1 \circ L^{-1}) \otimes \dots \otimes (\phi_t \circ L^{-1}) \end{aligned}$$

The derivative of this action yields a linear derivation

$$\text{End}(V) \rightarrow \text{End}(T(V))$$

where now

$$\begin{aligned} &L(v_1 \otimes \dots \otimes v_s \otimes \phi_1 \otimes \dots \otimes \phi_t) \\ &= L(v_1) \otimes \dots \otimes v_s \otimes \phi_1 \otimes \dots \otimes \phi_t \\ &\quad + \dots \\ &\quad + v_1 \otimes \dots \otimes L(v_s) \otimes \phi_1 \otimes \dots \otimes \phi_t \\ &\quad - v_1 \otimes \dots \otimes v_s \otimes (\phi_1 \circ L) \otimes \dots \otimes \phi_t \\ &\quad - \dots \\ &\quad - v_1 \otimes \dots \otimes v_s \otimes \phi_1 \otimes \dots \otimes (\phi_t \circ L) \end{aligned}$$

In particular, we see that $R(X, Y)(T)$ acts on the tensor T in the way the linear map $R(X, Y)$ on $T_p M$ extends to a derivation on tensors. Keep in mind that for $(0, k)$ -tensors we have

$$R(X, Y)(T(X_1, \dots, X_k)) = 0$$

since $T(X_1, \dots, X_k)$ is a function.

We note further that $T \rightarrow L(T)$ preserves (skew)-symmetry of L when an inner product is present since

$$g \left(\bigotimes_{i=1}^s v_i \otimes \bigotimes_{i=1}^t \phi_j, \bigotimes_{i=1}^s w_i \otimes \bigotimes_{i=1}^t \psi_j \right) = \sum g(v_i, w_i) g(\phi_j, \psi_j)$$

Since $R(X, Y) : T_p M \rightarrow T_p M$ is a skew-symmetric linear transformation we can decompose it using an orthonormal basis of skew-symmetric transformations $\Xi_\alpha \in \mathfrak{so}(T_p M)$. This gives

$$\begin{aligned} R(X, Y) &= g(R(X, Y), \Xi_\alpha) \Xi_\alpha \\ &= g(\mathfrak{R}(X \wedge Y), \Xi_\alpha) \Xi_\alpha \\ &= g(\mathfrak{R}(\Xi_\alpha), X \wedge Y) \Xi_\alpha \\ &= -g(R(\Xi_\alpha) X, Y) \Xi_\alpha \end{aligned}$$

where the last line is due to the convention

$$g((x \wedge y)(v), w) = -g(x \wedge y, v \wedge w).$$

With this in mind we can show

Lemma 1 *For any $(0, k)$ tensor T we have*

$$\begin{aligned} \mathfrak{Ric}(T) &= -\sum (R(\Xi_\alpha)(\Xi_\alpha T)), \\ \Delta_L T &= \nabla^* \nabla T - c \sum R(\Xi_\alpha)(\Xi_\alpha T) \end{aligned}$$

Proof. We only need to establish the first identity. This works as follows.

$$\begin{aligned} \mathfrak{Ric}(T)(X_1, \dots, X_k) &= \sum (R(e_j, X_i) T)(X_1, \dots, e_j, \dots, X_k) \\ &= -\sum g(R(\Xi_\alpha) e_j, X_i) (\Xi_\alpha T)(X_1, \dots, e_j, \dots, X_k) \\ &= -\sum (\Xi_\alpha T)(X_1, \dots, g(R(\Xi_\alpha) e_j, X_i) e_j, \dots, X_k) \\ &= \sum (\Xi_\alpha T)(X_1, \dots, R(\Xi_\alpha) X_i, \dots, X_k) \\ &= -\sum (R(\Xi_\alpha)(\Xi_\alpha T))(X_1, \dots, X_i, \dots, X_k) \end{aligned}$$

■

At first sight we have replaced a simple sum over j and i with a possibly more complicated sum. The next result justifies the reformulation.

Corollary 2 *If $\mathfrak{R} \geq 0$, then $g(\mathfrak{Ric}(T), T) \geq 0$.*

Proof. Select the orthonormal basis Ξ_α to consist of eigenvectors for \mathfrak{R} , i.e., $\mathfrak{R}(\Xi_\alpha) = \lambda_\alpha \Xi_\alpha$, then

$$R(\Xi_\alpha) = \lambda_\alpha \Xi_\alpha$$

When taking inner products with T and using that $R(\Xi_\alpha)$ is skew symmetric we obtain

$$\begin{aligned} -\sum g(R(\Xi_\alpha)(\Xi_\alpha T), T) &= \sum g(\Xi_\alpha T, R(\Xi_\alpha) T) \\ &= \sum \lambda_\alpha |\Xi_\alpha T|^2 \end{aligned}$$

This shows that the curvature term is nonnegative when the curvature operator is nonnegative. ■

Having redefined the Ricci curvature of tensors, we can now take it a step further and also get rid of the orthonormal basis Ξ_α . To assist in this we introduce a type change for $(0, k)$ -tensors T to a tensor \hat{T} with values in $\Lambda^2 TM = \mathfrak{so}(TM)$ as follows

$$g(L, \hat{T}(X_1, \dots, X_k)) = (LT)(X_1, \dots, X_k) \text{ for all } L \in \mathfrak{so}(TM).$$

The curvature formula then becomes

$$\begin{aligned}
g(\mathfrak{Ric}(T), T) &= \sum g(\Xi_\alpha T, R(\Xi_\alpha)T) \\
&= \sum (\Xi_\alpha T)(e_{i_1}, \dots, e_{i_k})(R(\Xi_\alpha)T)(e_{i_1}, \dots, e_{i_k}) \\
&= \sum g(\Xi_\alpha, \hat{T}(e_{i_1}, \dots, e_{i_k}))g(R(\Xi_\alpha), \hat{T}(e_{i_1}, \dots, e_{i_k})) \\
&= \sum g(R(g(\Xi_\alpha, \hat{T}(e_{i_1}, \dots, e_{i_k}))\Xi_\alpha), \hat{T}(e_{i_1}, \dots, e_{i_k})) \\
&= \sum g(R(\hat{T}(e_{i_1}, \dots, e_{i_k})), \hat{T}(e_{i_1}, \dots, e_{i_k})) \\
&= g(R \circ \hat{T}, \hat{T})
\end{aligned}$$

This is again clearly nonnegative when the curvature operator is nonnegative, but it also occasionally allows us to show that it is nonnegative under less restrictive hypotheses.

2 The Hodge Laplacian

The first obvious case to try this philosophy on is that of the Hodge Laplacian on k -forms as we already know that harmonic forms compute the topology of the underlying manifold. We'll show that

Theorem 3 (D. Meyer, 1971)

$$\Delta\omega = \nabla^*\nabla\omega + \mathfrak{Ric}\omega.$$

In particular harmonic forms are parallel when $\mathfrak{R} \geq 0$ and vanish when $\mathfrak{R} > 0$ for $k = 1, \dots, n-1$.

Proof. We shall follow a proof discovered by W.A. Poor. To perform these calculations we need

$$\begin{aligned}
\delta\omega(X_2, \dots, X_k) &= -\sum (\nabla_{E_i}\omega)(E_i, X_2, \dots, X_k), \\
d\omega(X_0, \dots, X_k) &= \sum (-1)^i (\nabla_{X_i}\omega)(X_0, \dots, \hat{X}_i, \dots, X_k)
\end{aligned}$$

and employ the usual assumptions about all covariant derivatives of vector fields vanishing at a fixed point $p \in M$. We this in mind we get

$$\begin{aligned}
\delta d\omega(X_1, \dots, X_k) &= \sum (-1)^{i+1} \nabla_{X_i}\delta\omega(X_1, \dots, \hat{X}_i, \dots, X_k) \\
&= \sum (-1)^i \nabla_{X_i}\nabla_{E_j}\omega(E_j, X_1, \dots, \hat{X}_i, \dots, X_k) \\
&= -\sum \nabla_{X_i}\nabla_{E_j}\omega(X_1, \dots, E_j, \dots, X_k)
\end{aligned}$$

$$\begin{aligned}
\delta d\omega(X_1, \dots, X_k) &= -\sum \nabla_{E_j}d\omega(E_j, X_1, \dots, X_k) \\
&= -\sum \nabla_{E_j}\nabla_{E_j}\omega(X_1, \dots, X_k) \\
&\quad -\sum (-1)^i \nabla_{E_j}\nabla_{X_i}\omega(E_j, X_1, \dots, \hat{X}_i, \dots, X_k) \\
&= (\nabla^*\nabla\omega)(X_1, \dots, X_k) \\
&\quad + \sum \nabla_{E_j}\nabla_{X_i}\omega(X_1, \dots, E_j, \dots, X_k)
\end{aligned}$$

Thus

$$\begin{aligned}
\Delta\omega &= \nabla^*\nabla\omega + \sum (R(E_j, X_i)\omega)(X_1, \dots, E_j, \dots, X_k) \\
&= \nabla^*\nabla\omega - \sum (R(E_j, X_i)\omega)(X_1, \dots, E_j, \dots, X_k) \\
&= \nabla^*\nabla\omega + \mathfrak{Ric}(\omega)
\end{aligned}$$

In case the curvature operator is positive we get that $\Xi_\alpha \omega = 0$ for all α , and hence by linearity that $L\omega = 0$ for all skew-symmetric L . If we assume $k < n$ and select L so that $L(e_i) = 0$ for $i < k$, $L(e_k) = e_{k+1}$, then

$$0 = (L\omega)(e_1, \dots, e_k) = -\omega(e_1, \dots, e_{k-1}, e_{k+1})$$

Since the basis was arbitrary this shows that $\omega = 0$. ■

It is instructive to try to recover the standard formula for 1-forms using this more abstract approach.

Corollary 4 *If ω is a 1-form and X the dual vector field, then*

$$-\sum g(R(\Xi_\alpha)(\Xi_\alpha \omega), \omega) = \text{Ric}(X, X).$$

Proof. In this case we have

$$\begin{aligned} (L\omega)(Z) &= -\omega(L(Z)) \\ &= -g(X, L(Z)) \\ &= g(L, Z \wedge X) \end{aligned}$$

so

$$\hat{\omega}(Z) = Z \wedge X.$$

This shows that the curvature term in the Bochner formula becomes

$$\begin{aligned} -\sum g(R(\Xi_\alpha)(\Xi_\alpha \omega), \omega) &= \sum g(\Xi_\alpha \omega, R(\Xi_\alpha) \omega) \\ &= \sum g(R(\hat{\omega}(E_i)), \hat{\omega}(E_i)) \\ &= \sum g(\mathfrak{R}(E_i \wedge X), E_i \wedge X) \\ &= \sum R(X, E_i, E_i, X) \\ &= \text{Ric}(X, X) \end{aligned}$$

■

3 (0, 2)-tensors

In this section we calculate \hat{h} for a general (0, 2)-tensor and relate it to curvatures. We have the corresponding (1, 1)-tensor called H

$$h(v, w) = g(H(v), w)$$

and the dual H^*

$$g(H(v), w) = g(v, H^*(w)).$$

Proposition 5

$$\hat{h}(x, y) = -H(x) \wedge y + x \wedge H^*(y)$$

Proof.

$$\begin{aligned} (Lh)(x, y) &= -h(L(x), y) - h(x, L(y)) \\ &= -g(H(L(x)), y) - g(H(x), L(y)) \\ &= -g(L(x), H^*(y)) - g(L(y), H(x)) \\ &= g(L, x \wedge H^*(y)) + g(L, y \wedge H(x)) \\ &= g(L, -H(x) \wedge y + x \wedge H^*(y)) \end{aligned}$$

■

This formula is particularly interesting when H is skew symmetric as

$$\begin{aligned}
\hat{h}(x, y) &= -H(x) \wedge y + x \wedge H^*(y) \\
&= -H(x) \wedge y - x \wedge H(y) \\
&= -H(x \wedge y) \\
&= -2(H \wedge I)(x \wedge y)
\end{aligned}$$

We start by relating symmetric $(0, 2)$ -tensors to sectional curvature:

Proposition 6 *The following are equivalent:*

Theorem 7 1. $\text{sec} \geq 0$ on $T_p M$.

2. $g(R \circ \hat{h}, \hat{h}) \geq 0$ for all symmetric $(0, 2)$ -tensors h on $T_p M$.

3. $g(R \circ H, H) \geq 0$ for all symmetric endomorphisms H on $T_p M$.

Proof. We calculate the expressions in 2. and 3. by using an orthonormal basis e_i that diagonalizes $H(e_i) = \lambda_i e_i$.

We start with 2.

$$\begin{aligned}
g(R \circ \hat{h}, \hat{h}) &= \sum g(R(\hat{h}(e_i, e_j)), \hat{h}(e_i, e_j)) \\
&= \sum g(R(-H(e_i) \wedge e_j + e_i \wedge H^*(e_j)), -H(e_i) \wedge e_j + e_i \wedge H^*(e_j)) \\
&= \sum g(R((\lambda_j - \lambda_i) e_i \wedge e_j), (\lambda_j - \lambda_i) e_i \wedge e_j) \\
&= \sum (\lambda_j - \lambda_i)^2 g(R(e_i \wedge e_j), e_i \wedge e_j) \\
&= \sum (\lambda_j - \lambda_i)^2 \text{sec}(e_i, e_j) \\
&= 2 \sum_{i < j} (\lambda_j - \lambda_i)^2 \text{sec}(e_i, e_j)
\end{aligned}$$

For 3. we have

$$\begin{aligned}
g(R \circ H, H) &= \sum_{i < j} g(R(H(e_i \wedge e_j)), H(e_i \wedge e_j)) \\
&= \sum_{i < j} g(R((\lambda_i + \lambda_j) e_i \wedge e_j), (\lambda_i + \lambda_j) e_i \wedge e_j) \\
&= \sum_{i < j} (\lambda_i + \lambda_j)^2 g(R(e_i \wedge e_j), e_i \wedge e_j) \\
&= \sum_{i < j} (\lambda_i + \lambda_j)^2 \text{sec}(e_i, e_j)
\end{aligned}$$

In each case it is clear that nonnegative sectional curvature implies that these inner products are nonnegative. The converse statements are also fairly simple. To check that $\text{sec}(e_1, e_2) \geq 0$ simply assume that $\lambda_3 = \dots = \lambda_n = 0$ and, $\lambda_1 = 2, \lambda_2 = 1$. ■

Note that it is possible to restrict attention to positive or nonnegative definite tensors in these characterizations.

These characterizations lead to the following variational conditions:

Proposition 8 *Assume that $\text{sec} \geq 0$ on $T_p M$. If $g(R \circ \hat{h}, \hat{h}) = 0$ (resp. $g(R \circ H, H) = 0$), then $g(R \circ \hat{h}, \hat{l}) = 0$ for all symmetric l ($g(R \circ H, L) = 0$ for all symmetric L).*

Proof. We have

$$\begin{aligned}
0 &\leq g\left(R\circ\left(\widehat{h+tl}\right),\widehat{h+tl}\right) \\
&= g\left(R\circ\left(\widehat{h}\right),\widehat{h}\right)+2tg\left(R\circ\left(\widehat{h}\right),\widehat{l}\right)+O\left(t^2\right) \\
&= 2tg\left(R\circ\left(\widehat{h}\right),\widehat{l}\right)+O\left(t^2\right).
\end{aligned}$$

■

Analyzing when $g\left(R\circ\widehat{h},\widehat{h}\right)=0$ quickly shows that no new info is obtained. On an orthonormal basis that diagonalizes H it simply tells us that either eigenvalues agree or the sectional curvature vanishes.

We now do a similar analysis for skew-symmetric tensors. This time cases 2. and 3. really are the same as we saw that $\widehat{h}(x,y)=-H(x\wedge y)$.

Recall that (M,g) has nonnegative isotropic curvature if for all orthonormal vectors e_1,\dots,e_4 we have

$$\begin{aligned}
&Iso(R)(e_1,e_2,e_3,e_4) \\
&= g\left(R(e_1\wedge e_3+e_2\wedge e_4),e_1\wedge e_3+e_2\wedge e_4\right) \\
&\quad +g\left(R(e_2\wedge e_3-e_1\wedge e_4),e_2\wedge e_3-e_1\wedge e_4\right)\geq 0
\end{aligned}$$

Bianchi's identity

$$\begin{aligned}
&2g\left(R(e_1\wedge e_3),e_2\wedge e_4\right)-2g\left(R(e_2\wedge e_3),e_1\wedge e_4\right) \\
&= -2g\left(R(e_3\wedge e_1),e_2\wedge e_4\right)-2g\left(R(e_2\wedge e_3),e_1\wedge e_4\right) \\
&= 2g\left(R(e_1\wedge e_2),e_3\wedge e_4\right)
\end{aligned}$$

shows that these two curvatures yields the more standard expression for isotropic curvature:

$$\begin{aligned}
&g\left(R(e_1\wedge e_3),e_1\wedge e_3\right)+g\left(R(e_1\wedge e_4),e_1\wedge e_4\right) \\
&\quad +g\left(R(e_2\wedge e_3),e_2\wedge e_3\right)+g\left(R(e_2\wedge e_4),e_2\wedge e_4\right) \\
&\quad +2g\left(R(e_1\wedge e_2),e_3\wedge e_4\right) \\
&= g\left(R(e_1\wedge e_3+e_2\wedge e_4),e_1\wedge e_3+e_2\wedge e_4\right) \\
&\quad +g\left(R(e_2\wedge e_3-e_1\wedge e_4),e_2\wedge e_3-e_1\wedge e_4\right)
\end{aligned}$$

Having nonnegative isotropic curvature can be strengthened a bit by allowing $e_3=e_4$. Thus we sometimes also assume:

$$\begin{aligned}
&Iso(R)(e_1,e_2,e_3,e_3) \\
&= g\left(R(e_1\wedge e_3+e_2\wedge e_3),e_1\wedge e_3+e_2\wedge e_3\right) \\
&\quad +g\left(R(e_2\wedge e_3-e_1\wedge e_3),e_2\wedge e_3-e_1\wedge e_3\right) \\
&= 2\left(g\left(R(e_1\wedge e_3),e_1\wedge e_3\right)+g\left(R(e_2\wedge e_3),e_2\wedge e_3\right)\right) \\
&\geq 0
\end{aligned}$$

Note that this implies that the Ricci curvature is nonnegative. For odd dimensional manifolds we shall always make this additional condition. Note that this also allows us to work with 3-manifolds. We can go further and also let $e_1=e_2$ in which case we end up with sectional curvature. More generally one can consider complex curvatures

$$Iso(R)(x,y,u,v)$$

without further assumptions on how the vectors are related.

Proposition 9 *A Riemannian manifold satisfies the condition $g(R\circ A,A)\geq 0$ for all skew symmetric $A\in\mathfrak{so}(T_pM)$ if and only if the isotropic curvatures on T_pM are nonnegative. Moreover, if the isotropic curvature is nonnegative and $g(R\circ A,A)=0$, then $g(R\circ A,L)=0$ for all $L\in\mathfrak{so}(T_pM)$.*

Proof. The first part of this proof is a bit more involved, the second statement is obvious. We need to select an orthonormal basis that puts A in canonical form. This means that

$$\begin{aligned} A(e_i) &= \lambda_i e_{i+1}, i = 1, 3, \dots, 2k-1, \\ A(e_{i+1}) &= -\lambda_i e_i, i = 1, 3, \dots, 2k-1, \\ A(e_n) &= 0, \text{ if } n \text{ is odd.} \\ 0 &\leq \lambda_1, \lambda_3, \dots, \lambda_{2k-1} \end{aligned}$$

The inner product can be calculated as

$$g(R \circ A, A) = \sum_{i < j} g(R(A(e_i \wedge e_j)), A(e_i \wedge e_j))$$

But it turns out to be more convenient to select a new basis for $\Lambda^2 T_p M$ that also puts $A : \Lambda^2 T_p M \rightarrow \Lambda^2 T_p M$ into canonical form. When $i < j$ are both odd and $j < n$ we note that

$$\begin{aligned} A(e_i \wedge e_j + e_{i+1} \wedge e_{j+1}) &= (\lambda_i - \lambda_j)(e_{i+1} \wedge e_j - e_i \wedge e_{j+1}), \\ A(e_{i+1} \wedge e_j - e_i \wedge e_{j+1}) &= -(\lambda_i - \lambda_j)(e_i \wedge e_j + e_{i+1} \wedge e_{j+1}), \\ A(e_i \wedge e_j - e_{i+1} \wedge e_{j+1}) &= (\lambda_i + \lambda_j)(e_{i+1} \wedge e_j + e_i \wedge e_{j+1}), \\ A(e_{i+1} \wedge e_j + e_i \wedge e_{j+1}) &= -(\lambda_i + \lambda_j)(e_i \wedge e_j - e_{i+1} \wedge e_{j+1}). \end{aligned}$$

While if $i < j = n$ are both odd

$$\begin{aligned} A(e_i \wedge e_n + e_{i+1} \wedge e_n) &= \lambda_i(e_{i+1} \wedge e_n - e_i \wedge e_n), \\ A(e_{i+1} \wedge e_n - e_i \wedge e_n) &= -\lambda_i(e_i \wedge e_n + e_{i+1} \wedge e_n) \end{aligned}$$

Finally we note that for all odd $i < n$

$$A(e_i \wedge e_{i+1}) = 0$$

Normalizing these vectors suitably we get an orthonormal basis that puts $A : \Lambda^2 T_p M \rightarrow \Lambda^2 T_p M$ into canonical form. Using this basis the inner product becomes

$$\begin{aligned} g(R \circ A, A) &= \frac{1}{4} \sum_{i < j < n, i, j \text{ odd}} g(R(A(e_i \wedge e_j + e_{i+1} \wedge e_{j+1})), A(e_i \wedge e_j + e_{i+1} \wedge e_{j+1})) \\ &+ \frac{1}{4} \sum_{i < j < n, i, j \text{ odd}} g(R(A(e_{i+1} \wedge e_j - e_i \wedge e_{j+1})), A(e_{i+1} \wedge e_j - e_i \wedge e_{j+1})) \\ &+ \frac{1}{4} \sum_{i < j < n, i, j \text{ odd}} g(R(A(e_i \wedge e_j - e_{i+1} \wedge e_{j+1})), A(e_i \wedge e_j - e_{i+1} \wedge e_{j+1})) \\ &+ \frac{1}{4} \sum_{i < j < n, i, j \text{ odd}} g(R(A(e_{i+1} \wedge e_j + e_i \wedge e_{j+1})), A(e_{i+1} \wedge e_j + e_i \wedge e_{j+1})) \\ &+ \frac{1}{2} \sum_{i < n, i, n \text{ odd}} g(R(A(e_i \wedge e_n + e_{i+1} \wedge e_n)), A(e_i \wedge e_n + e_{i+1} \wedge e_n)) \\ &+ \frac{1}{2} \sum_{i < n, i, n \text{ odd}} g(R(A(e_{i+1} \wedge e_n - e_i \wedge e_n)), A(e_{i+1} \wedge e_n - e_i \wedge e_n)) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \sum_{i < j < n, i, j \text{ odd}} (\lambda_i - \lambda_j)^2 g(R(e_i \wedge e_j + e_{i+1} \wedge e_{j+1}), e_i \wedge e_j + e_{i+1} \wedge e_{j+1}) \\
&+ \frac{1}{2} \sum_{i < j < n, i, j \text{ odd}} (\lambda_i - \lambda_j)^2 g(R(e_{i+1} \wedge e_j - e_i \wedge e_{j+1}), e_{i+1} \wedge e_j - e_i \wedge e_{j+1}) \\
&+ \frac{1}{2} \sum_{i < j < n, i, j \text{ odd}} (\lambda_i + \lambda_j)^2 g(R(e_i \wedge e_j - e_{i+1} \wedge e_{j+1}), e_i \wedge e_j - e_{i+1} \wedge e_{j+1}) \\
&+ \frac{1}{2} \sum_{i < j < n, i, j \text{ odd}} (\lambda_i + \lambda_j)^2 g(R(e_{i+1} \wedge e_j + e_i \wedge e_{j+1}), e_{i+1} \wedge e_j + e_i \wedge e_{j+1}) \\
&+ \frac{1}{2} \sum_{i < n, i, n \text{ odd}} \lambda_i^2 g(R(e_i \wedge e_n + e_{i+1} \wedge e_n), e_i \wedge e_n + e_{i+1} \wedge e_n) \\
&+ \frac{1}{2} \sum_{i < n, i, n \text{ odd}} \lambda_i^2 g(R(e_{i+1} \wedge e_n - e_i \wedge e_n), e_{i+1} \wedge e_n - e_i \wedge e_n) \\
&= \frac{1}{2} \sum_{i < j < n, i, j \text{ odd}} (\lambda_i - \lambda_j)^2 Iso(R)(e_i, e_j, e_{i+1}, e_{j+1}) \\
&+ \frac{1}{2} \sum_{i < j < n, i, j \text{ odd}} (\lambda_i + \lambda_j)^2 Iso(R)(e_i, e_j, e_{i+1}, -e_{j+1}) \\
&+ \frac{1}{2} \sum_{i < n, i, n \text{ odd}} \lambda_i^2 Iso(R)(e_i, e_{i+1}, e_n, e_n)
\end{aligned}$$

■

These characterizations can be combined to show

Proposition 10 $g(R \circ \hat{h}, \hat{h}) \geq 0$ for all $(0, 2)$ -tensors on $T_p M$ if and only if all sectional and isotropic curvatures on $T_p M$ are nonnegative.

Proof. We decompose $h = h_s + h_a$ into symmetric and skew symmetric parts. Then

$$\begin{aligned}
g(R \circ \hat{h}, \hat{h}) &= g(R \circ \hat{h}_s, \hat{h}_s) + g(R \circ \hat{h}_a, \hat{h}_a) + g(R \circ \hat{h}_s, \hat{h}_a) + g(R \circ \hat{h}_a, \hat{h}_s) \\
&= g(R \circ \hat{h}_s, \hat{h}_s) + g(R \circ \hat{h}_a, \hat{h}_a) + 2g(R \circ \hat{h}_s, \hat{h}_a)
\end{aligned}$$

However,

$$\begin{aligned}
g(R \circ \hat{h}_s, \hat{h}_a) &= \sum g(R \circ \hat{h}_s(e_i, e_j), \hat{h}_a(e_i, e_j)) \\
&= -\sum g(R \circ \hat{h}_s(e_j, e_i), \hat{h}_a(e_j, e_i)) \\
&= -g(R \circ \hat{h}_s, \hat{h}_a)
\end{aligned}$$

So

$$g(R \circ \hat{h}, \hat{h}) = g(R \circ \hat{h}_s, \hat{h}_s) + g(R \circ \hat{h}_a, \hat{h}_a)$$

and we get the result by combining the above results. ■

It also seems clear that we can bound b_2 in terms of lower bounds on isotropic curvature and upper diameter bounds. This requires lower Ricci bounds for the Sobolev constant, but that's fine if we assume, say, that $M \times \mathbb{R}$ has a lower bound on the isotropic curvature.

I'd also really like to show that all Betti numbers can be controlled by a combination of isotropic and sectional curvature. I think that this might be weaker than curvature operator in high dimensions, but I'm not sure.

4 The Curvature Tensor

The Bochner technique can also be applied to the curvature tensor. It is by no means clear that this will yield anything. It seems both miraculous and profound that it works. The goal is to show that a compact Riemannian manifolds with $\operatorname{div}R = 0$ and nonnegative sectional curvature, respectively nonnegative curvature operator, has parallel Ricci tensor, respectively parallel curvature tensor. The key is to show that when the curvature tensor is divergence free then we have

$$\begin{aligned} 0 &= \Delta_L \operatorname{Ric} = \nabla^* \nabla \operatorname{Ric} - \frac{1}{2} \sum R(\Xi_\alpha)(\Xi_\alpha \operatorname{Ric}), \\ 0 &= \Delta_L R = \nabla^* \nabla R - \frac{1}{2} \sum R(\Xi_\alpha)(\Xi_\alpha R) \end{aligned}$$

We start with the Ricci tensor calculations. They are a bit easier and give us an idea of what to expect for the curvature tensor. Recall that the divergence of the curvature tensor can be defined as the $(1, 3)$ -tensor

$$\operatorname{div}R(\cdot, \cdot, \cdot) = \sum (\nabla_{E_i} R)(E_i, \cdot, \cdot, \cdot)$$

Theorem 11 (Berger) *Suppose that a Riemannian manifold (M, g) satisfies $\operatorname{div}R = 0$, then*

$$0 = \nabla^* \nabla \operatorname{Ric} - \frac{1}{2} \sum R(\Xi_\alpha)(\Xi_\alpha \operatorname{Ric}).$$

Moreover, if (M, g) is closed, oriented and $\sec \geq 0$, then $\nabla \operatorname{Ric} = 0$.

Proof. We start by proving that

$$\operatorname{div}R(X, Y, Z) = (\nabla_Z \operatorname{Ric})(Y, X) - (\nabla_Y \operatorname{Ric})(Z, X).$$

The left hand side is defined by

$$\begin{aligned} \operatorname{div}R(X, Y, Z) &= \sum (\nabla_{E_i} R)(E_i, X, Y, Z) \\ &= \sum (\nabla_{E_i} R)(Y, Z, E_i, X) \\ &= \sum (\nabla_Z R)(Y, E_i, E_i, X) - \sum (\nabla_Y R)(Z, E_i, E_i, X) \\ &= (\nabla_Z \operatorname{Ric})(Y, X) - (\nabla_Y \operatorname{Ric})(Z, X) \end{aligned}$$

With this in mind we can then calculate

$$\begin{aligned} -(\nabla^* \nabla \operatorname{Ric})(X, X) &= \sum (\nabla_{E_i, E_i}^2 \operatorname{Ric})(X, X) \\ &= \sum (\nabla_{E_i, X}^2 \operatorname{Ric})(E_i, X) + \sum (\nabla_{E_i} \operatorname{div}R)(X, X, E_i) \end{aligned}$$

If we assume $\operatorname{div}R = 0$, then we get

$$\begin{aligned} -(\nabla^* \nabla \operatorname{Ric})(X, X) &= \sum (\nabla_{E_i, X}^2 \operatorname{Ric})(E_i, X) \\ &= \sum R(E_i, X)(\operatorname{Ric})(E_i, X) + \sum (\nabla_{X, E_i}^2 \operatorname{Ric})(E_i, X) \end{aligned}$$

Here the last term vanishes since

$$\begin{aligned} \sum (\nabla_{X, E_i}^2 \operatorname{Ric})(E_i, X) &= \sum (\nabla_{X, E_i}^2 R)(E_i, E_j, E_j, X) \\ &= \sum (\nabla_X \operatorname{div}R)(E_j, E_j, X) \\ &= 0 \end{aligned}$$

Thus we obtain

$$\begin{aligned} -(\nabla^* \nabla \operatorname{Ric})(X, X) &= \sum R(E_i, X)(\operatorname{Ric})(E_i, X) \\ &= \sum R(E_i, X, \Xi_\alpha)(\Xi_\alpha \operatorname{Ric})(E_i, X) \\ &= \sum (\Xi_\alpha \operatorname{Ric})(R(\Xi_\alpha, X, E_i)E_i, X) \\ &= \sum (\Xi_\alpha \operatorname{Ric})(R(\Xi_\alpha)X, X) \\ &= -\frac{1}{2} \sum R(\Xi_\alpha)(\Xi_\alpha \operatorname{Ric})(X, X) \end{aligned}$$

where in the last line we used that Ric is symmetric.

Assuming that $\text{Ric}(E_i) = \rho_i E_i$ the curvature term then looks like

$$-\sum g\left(\frac{1}{2}R(\Xi_\alpha)(\Xi_\alpha \text{Ric}), \text{Ric}\right) = \sum \frac{1}{2}(\rho_i - \rho_j)^2 \text{sec}(E_i, E_j)$$

Since this is nonnegative when $\text{sec} \geq 0$ we get the desired conclusion in the usual fashion by integrating over M . Note in addition that if the sectional curvatures are positive at a point, the the eigenvalues of Ric are all the same. Hence the metric becomes an Einstein metric. ■

Note that the Berger spheres have divergence-free Ricci tensor, but only the standard sphere has parallel Ricci tensor. Thus it is not clear that one can get results without assuming divergence free curvature tensor.

We can now address what happens for the full curvature tensor.

Theorem 12 *The curvature tensor R on a Riemannian manifold satisfies*

$$\begin{aligned} & (\nabla^* \nabla R)(X, Y, Z, W) - \frac{1}{2} \sum R(\Xi_\alpha)(\Xi_\alpha R)(X, Y, Z, W) \\ &= -\frac{1}{2}(\nabla_X \text{div} R)(Y, Z, W) + \frac{1}{2}(\nabla_Y \text{div} R)(X, Z, W) \\ & \quad -\frac{1}{2}(\nabla_Z \text{div} R)(W, X, Y) + \frac{1}{2}(\nabla_W \text{div} R)(Z, X, Y) \end{aligned}$$

Proof. By far the most important ingredient in the proof is that we have the second Bianchi identity at our disposal. We will begin the calculation by considering the (0,4)-curvature tensor R . Fix a point p , let X, Y, Z, W be vector fields with $\nabla X = \nabla Y = \nabla Z = \nabla W = 0$ at p and let E_i be normal coordinates at p . Then

$$\begin{aligned} -(\nabla^* \nabla R)(X, Y, Z, W) &= \sum_{i=1}^n (\nabla_{E_i, E_i}^2 R)(X, Y, Z, W) \\ &= \sum_{i=1}^n (\nabla_{E_i, X}^2 R)(E_i, Y, Z, W) - (\nabla_{E_i, Y}^2 R)(E_i, X, Z, W) \\ &= \sum_{i=1}^n (\nabla_{X, E_i}^2 R)(E_i, Y, Z, W) - (\nabla_{Y, E_i}^2 R)(E_i, X, Z, W) \\ & \quad + (R(E_i, X)(R))(E_i, Y, Z, W) - (R(E_i, Y)(R))(E_i, X, Z, W) \end{aligned}$$

The terms

$$\sum_{i=1}^n (\nabla_{X, E_i}^2 R)(E_i, Y, Z, W) - (\nabla_{Y, E_i}^2 R)(E_i, X, Z, W)$$

vanish if $\text{div} R = 0$. The second two terms can be interpreted as

$$\begin{aligned} \sum_{i=1}^n (R(E_i, X)(R))(E_i, Y, Z, W) &= \sum_{i=1}^n R(E_i, X)(R)(E_i, Y, Z, W) \\ &= \sum g(R(E_i, X), \Xi_\alpha)(\Xi_\alpha R)(E_i, Y, Z, W) \\ &= \sum g(R(\Xi_\alpha)X, E_i)(\Xi_\alpha R)(E_i, Y, Z, W) \\ &= \sum \Xi_\alpha(R)(g(R(\Xi_\alpha)X, E_i)E_i, Y, Z, W) \\ &= \sum \Xi_\alpha(R)(R(\Xi_\alpha)X, Y, Z, W) \end{aligned}$$

Similarly

$$\begin{aligned} -\sum_{i=1}^n (R(E_i, Y)(R))(E_i, X, Z, W) &= \sum \Xi_\alpha(R)(R(\Xi_\alpha)Y, X, Z, W) \\ &= \sum \Xi_\alpha(R)(X, R(\Xi_\alpha)Y, Z, W) \end{aligned}$$

Using that R is symmetric in the pairs X, Y and Z, W we then obtain

$$\begin{aligned}
-(\nabla^* \nabla R)(X, Y, Z, W) &= -\frac{1}{2}(\nabla^* \nabla R)(X, Y, Z, W) - \frac{1}{2}(\nabla^* \nabla R)(Z, W, X, Y) \\
&= \frac{1}{2} \sum_{i=1}^n (\nabla_{X, E_i}^2 R)(E_i, Y, Z, W) - (\nabla_{Y, E_i}^2 R)(E_i, X, Z, W) \\
&\quad + \frac{1}{2} \sum_{i=1}^n (\nabla_{Z, E_i}^2 R)(E_i, W, X, Y) - (\nabla_{W, E_i}^2 R)(E_i, Z, X, Y) \\
&\quad + \frac{1}{2} \sum (\Xi_\alpha R)(R(\Xi_\alpha) X, Y, Z, W) + (\Xi_\alpha R)(X, R(\Xi_\alpha) Y, Z, W) \\
&\quad + \frac{1}{2} \sum (\Xi_\alpha R)(R(\Xi_\alpha) Z, W, X, Y) + (\Xi_\alpha R)(Z, R(\Xi_\alpha) W, X, Y) \\
&= \frac{1}{2} \sum_{i=1}^n (\nabla_{X, E_i}^2 R)(E_i, Y, Z, W) - (\nabla_{Y, E_i}^2 R)(E_i, X, Z, W) \\
&\quad + \frac{1}{2} \sum_{i=1}^n (\nabla_{Z, E_i}^2 R)(E_i, W, X, Y) - (\nabla_{W, E_i}^2 R)(E_i, Z, X, Y) \\
&\quad + \frac{1}{2} \sum (\Xi_\alpha R)(R(\Xi_\alpha) X, Y, Z, W) + (\Xi_\alpha R)(X, R(\Xi_\alpha) Y, Z, W) \\
&\quad + \frac{1}{2} \sum (\Xi_\alpha R)(X, Y, R(\Xi_\alpha) Z, W) + (\Xi_\alpha R)(X, Y, Z, R(\Xi_\alpha) W) \\
&= \frac{1}{2} \sum_{i=1}^n (\nabla_{X, E_i}^2 R)(E_i, Y, Z, W) - (\nabla_{Y, E_i}^2 R)(E_i, X, Z, W) \\
&\quad + \frac{1}{2} \sum_{i=1}^n (\nabla_{Z, E_i}^2 R)(E_i, W, X, Y) - (\nabla_{W, E_i}^2 R)(E_i, Z, X, Y) \\
&\quad - \frac{1}{2} \sum R(\Xi_\alpha) (\Xi_\alpha R)(X, Y, Z, W)
\end{aligned}$$

■

We can now go over to the more complicated result we are interested in. It was first established in [?], and then with a modified proof in [?]. We shall present a very simple proof based on the above formula relating the divergence and connection Laplacians of the curvature tensor.

Theorem 13 (Tachibana, 1974) *If (M, g) is a compact oriented Riemannian manifold with $\operatorname{div} R = 0$ and $\mathfrak{R} \geq 0$, then $\nabla R = 0$. If in addition, $\mathfrak{R} > 0$, then (M, g) has constant curvature.*

Proof. We know from above that

$$\nabla^* \nabla R - \frac{1}{2} \sum R(\Xi_\alpha) (\Xi_\alpha R) = 0$$

Taking inner products with R and then integrating yields

$$\begin{aligned}
\int_M |\nabla R|^2 d\operatorname{vol} &= -\frac{1}{2} \sum \int_M g(R(\Xi_\alpha) (\Xi_\alpha R), R) d\operatorname{vol} \\
&= \frac{1}{2} \sum \int_M g(\Xi_\alpha R, R(\Xi_\alpha) R) d\operatorname{vol} \\
&= \frac{1}{2} \sum \int_M \lambda_\alpha |\Xi_\alpha R|^2 d\operatorname{vol}
\end{aligned}$$

using a basis Ξ_α that diagonalizes \mathfrak{R} . So if all eigenvalues are nonnegative we obviously get that $\nabla R = 0$. Moreover, should all eigenvalues be positive then $LR = 0$ for all $L \in \mathfrak{so}(T_p M)$. This condition will imply that $R(x, y, y, z) = 0$ and $R(x, y, v, w) = 0$ when the vectors are perpendicular. This in turn shows that any bi-vector $X \wedge Y$ is an eigenvector for \mathfrak{R} , but this can only happen if $\mathfrak{R} = kI$ for some constant k .

To show that the mixed curvatures vanish first select L so that $L(y) = 0$ and $L(x) = z$, then

$$\begin{aligned} 0 &= LR(x, y, y, x) = -R(L(x), y, y, x) - R(x, y, y, L(x)) \\ &= -2R(x, y, y, z). \end{aligned}$$

Polarizing in $y = v + w$, then shows that

$$R(x, v, w, z) = -R(x, w, v, z)$$

The Bianchi identity implies

$$\begin{aligned} R(x, v, w, z) &= R(w, v, x, z) - R(w, x, v, z) \\ &= -2R(w, x, v, z) \\ &= 2R(x, w, v, z) \\ &= -2R(x, v, w, z) \end{aligned}$$

showing that $R(x, v, w, z) = 0$. ■

Finally we recover Tachibana's original formula by calculating \hat{S} for a symmetric tensor $S : \Lambda^2 TM \times \Lambda^2 TM \rightarrow \mathbb{R}$. This includes 4-forms.

$$\begin{aligned} & - (LS)(X, Y, V, W) \\ &= S(L(X), Y, V, W) + S(X, L(Y), V, W) + S(X, Y, L(V), W) + S(X, Y, V, L(W)) \\ &= -S(V, W, Y, L(X)) + S(V, W, X, L(Y)) - S(X, Y, W, L(V)) + S(X, Y, V, L(W)) \\ &= -g(S(V, W)Y, L(X)) + g(S(V, W)X, L(Y)) - g(S(X, Y)W, L(V)) + g(S(X, Y)V, L(W)) \\ &= g(L, X \wedge S(V, W)Y) - g(L, Y \wedge S(V, W)X) + g(L, V \wedge S(X, Y)W) - g(L, W \wedge S(X, Y)V) \\ &= g(L, S(V, W)X \wedge Y + X \wedge S(V, W)Y + S(X, Y)V \wedge W + V \wedge S(X, Y)W) \\ &= g(L, S(V, W)(X \wedge Y) + S(X, Y)(V \wedge W)) \end{aligned}$$

If we also think of $S : \Lambda^2 TM \rightarrow \Lambda^2 TM$, then (modulo signs?)

$$\begin{aligned} - (LS)(X, Y, V, W) &= -g(S(L(X \wedge Y)), V \wedge W) - g(S(X \wedge Y), L(V \wedge W)) \\ &= -g(S(L(X \wedge Y)), V \wedge W) - g(L^*S(X \wedge Y), V \wedge W) \end{aligned}$$

so if L is skew symmetric

$$LS = [S, L]$$

Thus L acts trivially on the identity operator, and one can easily check that if T is another (1,1) tensor then $[L, T]$ on $\Lambda^2 TM$ is the same as the derivation coming from $[L, T]$ on $T_p M$. Finally $L\omega$ is a form if ω is a form, so L preserves the natural orthogonal decomposition

$$S^2 \Lambda^2 T_p M = I \oplus \text{Ric}_0 \oplus W \oplus \Lambda^4 T_p M$$

We know that 4-forms are spanned by $\omega_1 \wedge \omega_2$ where ω_i are 2-forms. Curvature tensors are similarly spanned by $S_1 \wedge S_2$ where $S_i : T_p M \rightarrow T_p M$ are symmetric, this is equivalent to the Kulkarni-Nomizu product $s_1 \circ s_2$ under type change. Conformally flat curvature tensors correspond to the case where $S_2 = I$ or $s_2 = g$.

5 f -Laplacians

When the measure is changed from being $d\text{vol}$ to $e^{-f}d\text{vol}$ then we also need to change the way we compute divergences so as to make sure they are still adjoints to exterior and covariant derivatives. To this end we define

$$\begin{aligned} \delta_f &= e^f \delta e^{-f} = \delta + i_{\nabla f} \\ \nabla_f^* &= \nabla^* + i_{\nabla f} \end{aligned}$$

We quickly check that this works as follows:

$$\begin{aligned}
\int g(d\omega, \omega') e^{-f} d\text{vol} &= \int g(d\omega, e^{-f}\omega') d\text{vol} \\
&= \int g(\omega, \delta(e^{-f}\omega')) d\text{vol} \\
&= \int g(\omega, e^f \delta(e^{-f}\omega')) e^{-f} d\text{vol} \\
&= \int g(\omega, \delta_f \omega') e^{-f} d\text{vol}
\end{aligned}$$

The f -Hodge Laplacian becomes

$$\begin{aligned}
\Delta_f &= \delta_f d + d\delta_f \\
&= \delta d + d\delta + i_{\nabla f} d + di_{\nabla f}
\end{aligned}$$

where the last term can be rewritten as follows. To simplify calculations we assume that X_i are parallel at the point where calculations occur:

$$\begin{aligned}
(i_{\nabla f} d\omega)(X_1, \dots, X_p) &= d\omega(\nabla f, X_1, \dots, X_p) \\
&= (\nabla_{\nabla f} \omega)(X_1, \dots, X_p) + \sum (-1)^j (\nabla_{X_j} \omega)(\nabla f, X_1, \dots, \hat{X}_j, \dots, X_p) \\
&= (\nabla_{\nabla f} \omega)(X_1, \dots, X_p) \\
&\quad + \sum (-1)^j \nabla_{X_j} \left(\omega(\nabla f, X_1, \dots, \hat{X}_j, \dots, X_p) \right) - \sum (-1)^j \omega(\nabla_{X_j} \nabla f, X_1, \dots, \hat{X}_j, \dots, X_p) \\
&= (\nabla_{\nabla f} \omega)(X_1, \dots, X_p) \\
&\quad + \sum (-1)^j \nabla_{X_j} \left(\omega(\nabla f, X_1, \dots, \hat{X}_j, \dots, X_p) \right) + \sum \omega(X_1, \dots, \nabla_{X_j} \nabla f, \dots, X_p)
\end{aligned}$$

and

$$\begin{aligned}
(di_{\nabla f} \omega)(X_1, \dots, X_p) &= \sum (-1)^{j+1} (\nabla_{X_j} i_{\nabla f} \omega)(X_1, \dots, \hat{X}_j, \dots, X_p) \\
&= \sum (-1)^{j+1} \nabla_{X_j} \left(i_{\nabla f} \omega(X_1, \dots, \hat{X}_j, \dots, X_p) \right) \\
&= \sum (-1)^{j+1} \nabla_{X_j} \left(\omega(\nabla f, X_1, \dots, \hat{X}_j, \dots, X_p) \right)
\end{aligned}$$

so

$$\begin{aligned}
i_{\nabla f} d\omega + di_{\nabla f} \omega &= \nabla_{\nabla f} \omega + \sum \omega(X_1, \dots, \nabla_{X_j} \nabla f, \dots, X_p) \\
&= \nabla_{\nabla f} \omega - S_f \omega
\end{aligned}$$

where

$$S_f(X) = \nabla_X \nabla f$$

acts by derivations on forms as before.

This means that

$$\begin{aligned}
\Delta_f \omega &= \Delta \omega + \nabla_{\nabla f} \omega - S_f \omega \\
&= \nabla_f^* \nabla \omega + \mathfrak{Ric}(\omega) - S_f \omega
\end{aligned}$$

which leads us to the new Lichnerowicz Ricci tensor also called C tensor by Lichnerowicz

$$\mathfrak{Ric}_f = \mathfrak{Ric} - S_f$$

and the f -Lichnerowicz Laplacian on tensors

$$\begin{aligned}
\Delta_{L,f} &= \nabla_f^* \nabla + c \mathfrak{Ric}_f \\
&= \nabla_f^* \nabla + c(\mathfrak{Ric} - S_f), \quad c > 0.
\end{aligned}$$

Having f -harmonic curvature tensor becomes:

$$\operatorname{div}_f R = \operatorname{div} R - i_{\nabla f} R = 0$$

or

$$(\nabla_{E_i} R)(E_i, X, Y, Z) = R(\nabla f, X, Y, Z)$$

which is also a consequence of being a Ricci soliton. With this in mind the formula

$$\begin{aligned} \nabla^* \nabla R + \frac{1}{2} \mathfrak{Ric}(R) &= \Delta_L R \\ &= \frac{1}{2} (\nabla_X \operatorname{div} R)(Y, Z, W) - \frac{1}{2} (\nabla_Y \operatorname{div} R)(X, Z, W) \\ &\quad + \frac{1}{2} (\nabla_Z \operatorname{div} R)(W, X, Y) - \frac{1}{2} (\nabla_W \operatorname{div} R)(Z, X, Y) \end{aligned}$$

becomes

$$\begin{aligned} \nabla_f^* \nabla R + \frac{1}{2} \mathfrak{Ric}(R) - \frac{1}{2} S_f(R) &= \Delta_{L,f} R \\ &= -\frac{1}{2} (\nabla_X \operatorname{div}_f R)(Y, Z, W) + \frac{1}{2} (\nabla_Y \operatorname{div}_f R)(X, Z, W) \\ &\quad - \frac{1}{2} (\nabla_Z \operatorname{div}_f R)(W, X, Y) + \frac{1}{2} (\nabla_W \operatorname{div}_f R)(Z, X, Y) \end{aligned}$$

To verify this we need the extra terms that involve f to cancel out. This relies on the second Bianchi identity. Assume as usual that X, Y, Z, W are parallel at a some fixed point. On the left hand side we have

$$\nabla_{\nabla f} R - \frac{1}{2} S_f(R)$$

while on the right hand side we first need to observe that

$$\begin{aligned} (\nabla_X \operatorname{div}_f R)(Y, Z, W) &= (\nabla_X (\operatorname{div} R - i_{\nabla f} R))(Y, Z, W) \\ &= (\nabla_X \operatorname{div} R)(Y, Z, W) - \nabla_X (R(\nabla f, Y, Z, W)) \\ &= (\nabla_X \operatorname{div} R)(Y, Z, W) - (\nabla_X R)(\nabla f, Y, Z, W) + R(\nabla_X \nabla f, Y, Z, W) \end{aligned}$$

Thus we need to simplify

$$\begin{aligned} &+ \frac{1}{2} (\nabla_X R)(\nabla f, Y, Z, W) - \frac{1}{2} (\nabla_Y R)(\nabla f, X, Z, W) \\ &+ \frac{1}{2} (\nabla_Z R)(\nabla f, W, X, Y) - \frac{1}{2} (\nabla_W R)(\nabla f, Z, X, Y) \\ &+ \frac{1}{2} R(\nabla_X \nabla f, Y, Z, W) - \frac{1}{2} R(\nabla_Y \nabla f, X, Z, W) \\ &+ \frac{1}{2} R(\nabla_Z \nabla f, W, X, Y) - \frac{1}{2} R(\nabla_W \nabla f, Z, X, Y) \\ &= -\frac{1}{2} (\nabla_X R)(Y, \nabla f, Z, W) - \frac{1}{2} (\nabla_Y R)(\nabla f, X, Z, W) \\ &\quad - \frac{1}{2} (\nabla_Z R)(W, \nabla f, X, Y) - \frac{1}{2} (\nabla_W R)(\nabla f, Z, X, Y) \\ &\quad + \frac{1}{2} R(\nabla_X \nabla f, Y, Z, W) + \frac{1}{2} R(X, \nabla_Y \nabla f, Z, W) \\ &\quad + \frac{1}{2} R(\nabla_Z \nabla f, W, X, Y) + \frac{1}{2} R(Z, \nabla_W \nabla f, X, Y) \\ &= \frac{1}{2} (\nabla_{\nabla f} R)(X, Y, Z, W) + \frac{1}{2} (\nabla_{\nabla f} R)(X, Y, Z, W) \\ &\quad + \frac{1}{2} R(\nabla_X \nabla f, Y, Z, W) + \frac{1}{2} R(X, \nabla_Y \nabla f, Z, W) \\ &\quad + \frac{1}{2} R(X, Y, \nabla_Z \nabla f, W) + \frac{1}{2} R(X, Y, Z, \nabla_W \nabla f) \\ &= (\nabla_{\nabla f} R)(X, Y, Z, W) - \frac{1}{2} (S_f R)(X, Y, Z, W) \end{aligned}$$

In case we have a soliton metric

$$\text{Ric} + S_f = \lambda I$$

this means that

$$\begin{aligned} 0 &= \Delta_{L,f} R \\ &= \nabla_f^* \nabla R + \frac{1}{2} \mathfrak{Ric}(R) - \frac{1}{2} (S_f R) \\ &= \nabla_f^* \nabla R + \frac{1}{2} \mathfrak{Ric}(R) - \frac{1}{2} ((\lambda I - \text{Ric}) R) \\ &= \nabla_f^* \nabla R + \frac{1}{2} \mathfrak{Ric}(R) + \frac{1}{2} (\text{Ric} R) + 2\lambda R \\ &= \nabla_f^* \nabla R - \frac{1}{2} \sum R(\Xi_\alpha) \Xi_\alpha(R) + \frac{1}{2} (\text{Ric} R) + 2\lambda R. \end{aligned}$$

If we also think of $R : \Lambda^2 TM \rightarrow \Lambda^2 TM$, then

$$\begin{aligned} (\text{Ric} R)(X, Y, V, W) &= -g(R(\text{Ric}(X \wedge Y)), V \wedge W) - g(R(X \wedge Y), \text{Ric}(V \wedge W)) \\ &= -g(R(\text{Ric}(X \wedge Y)), V \wedge W) - g(\text{Ric} R(X \wedge Y), V \wedge W) \end{aligned}$$