DEMYSTIFYING THE CURVATURE TERM IN LICHNEROWICZ LAPLACIANS

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Abstract. The goal is to define two concepts that make computations of Laplacians of Tensors considerably simpler. This allows us in almost no time to prove many major theorems that use the Bochner Technique.

1. The Lichnerowicz Laplacian

The goal is to define the Lichenrowicz Laplacian for a tensor and show that we can deconstruct the curvature term rather easily. One step in our reduction is modelled on W.A. Poor's approach to the Hodge Laplacian, which in turn was inspired by work of Chern. However, we generalize this to all tensors and then take one more step that will further simplify the curvature term in the Lichnerowicz Laplacian.

We define

$$\mathfrak{Ric}(T)(X_1, ..., X_k) = \sum (R(e_j, X_i) T)(X_1, ..., e_j, ..., X_k)$$

and the Lichnerowicz Laplacian

$$\Delta_L T = \nabla^* \nabla T + c\mathfrak{Ric}(T)$$

for a suitable constant $c > 0$. We shall see below that the Hodge Laplacian on forms is of this type and that interesting information can also be extracted from both the Ricci tensor and curvature tensor via this operator when we use $c = \frac{1}{2}$. The goal here is to show that one can perform a type change for T that significantly simplifies $g(\mathfrak{Ric}(T), T)$ and immediately shows that it is nonnegative when the curvature operator is nonnegative. The formula also makes it very easy to show what happens when T is a $(0,1)$ or $(0,2)$ tensor.

1.1. The Bochner Technique.

The Bochner technique generally works in one of two ways. First using the maximum principle

Lemma 1. *Let T be a tensor such that*

$$\begin{aligned} \nabla_U^* \nabla T + c\mathfrak{Ric}(T) &= 0, \\ g(\mathfrak{Ric}(T), T) &\geq 0. \end{aligned}$$

If $|T|$ has a maximum, then T is parallel.

Proof. The first observation we have to make is

$$\frac{1}{2} \nabla^* \nabla |T|^2 = g(\nabla^* \nabla T, T) - |\nabla T|^2$$

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Thus

$$\frac{1}{2}\nabla^{*}\nabla\left|T\right|^{2}=-\left(g\left(c\mathfrak{Ric}\left(T\right),T\right)+\left|\nabla T\right|^{2}\right)\leq 0.$$

The maximum principle, then implies that $\left|T\right|^{2}$ must be constant if it has a maximum. Having shown that it is constant we then obtain

$$g\left(c\mathfrak{Ric}\left(T\right),T\right)+\left|\nabla T\right|^{2}\equiv 0$$

which is only possible if both

$$\begin{aligned} g\left(\mathfrak{Ric}\left(T\right),T\right) &\equiv 0 \\ \left|\nabla T\right|^{2} &\equiv 0. \end{aligned}$$

□

Next by using integration.

Lemma 2. *Let T be a tensor such that*

$$\begin{aligned} \nabla^{*}\nabla T+c\mathfrak{Ric}\left(T\right) &= 0, \\ g\left(\mathfrak{Ric}\left(T\right),T\right) &\geq 0, \end{aligned}$$

then T is parallel.

Proof. We have

$$\begin{aligned} 0 &\leq \int\left|\nabla T\right|^{2}d\mathrm{vol} \\ &= \int g\left(\nabla^{*}\nabla T,T\right)d\mathrm{vol} \\ &= -c\int g\left(\mathfrak{Ric}\left(T\right),T\right)d\mathrm{vol} \\ &\leq 0 \end{aligned}$$

showing that all integrals vanish. As the integrands are all nonnegative they must in addition vanish. □

Both methods work when M is compact, although in the second case we might have to pass to the orientation double cover of M to get integration to work. Both methods also allow for extensions to complete manifolds provided we have a maximum or that we are allowed to use integration by parts:

$$\int\left|\nabla T\right|^{2}d\mathrm{vol}=\int g\left(\nabla^{*}\nabla T,T\right)d\mathrm{vol}$$

The two assumptions we make about T also require some discussion. The first assumption is usually implied by showing that we have a more general Weitzenböck formula

$$\Delta=\nabla^{*}\nabla T+c\mathfrak{Ric}\left(T\right)$$

and then assuming that $\Delta T=0$. The Laplacian Δ might be more naturally defined or have some other interesting properties. So the goal is to first show that it is also a Lichnerowicz Laplacian. That is not always obvious, but the calculations are rarely that bad.

The second assumption

$$g\left(\mathfrak{Ric}\left(T\right),T\right)\geq 0$$

is often difficult to check and in many cases it took decades to figure what curvature assumptions gave the best results. The goal here is to give a simple and natural approach that coveres many known cases including several that have generally not been so easy to understand.

1.2. **Natural Derivations.** The goal to define a natural way for $(1,1)$ tensors to act as derivations on the space of all tensors. For fixed X, Y the curvature $R(X,Y)$ is naturally a $(1,1)$ tensor, but it can also beapplied to tensors. The hope is that this might allow us to gain a better understaning of the more obscure operation $\mathfrak{Ric}(T)$.

For any tensor T we have

$$\begin{aligned} R(X,Y)T &= (\nabla_X(\nabla_Y T)) - (\nabla_Y(\nabla_X T)) - \left(\nabla_{[X,Y]}T\right) \\ &= \nabla^2_{X,Y}T - \nabla^2_{Y,X}T \end{aligned}$$

and

$$\begin{aligned} (R(X,Y)T)(X_1,\ldots,X_k) &= R(X,Y)(T(X_1,\ldots,X_k)) \\ &\quad -T(R(X,Y)X_1,\ldots,X_k) \\ &\quad \vdots \\ &\quad -T(X_1,\ldots,R(X,Y)X_k) \end{aligned}$$

This can be rewritten in a more abstract and algebraic fashion. We have a natural homomorphism

$$Gl(V) \to Gl(T(V))$$

where $T(V)$ is the space of all tensors over the vector space V. We know that (s,t)-tensors are spanned by

$$v_1 \otimes \cdots \otimes v_s \otimes \phi_1 \otimes \cdots \otimes \phi_t$$

where $v_1, ..., v_s \in V$ and $\phi_1, ..., \phi_t : V \to \mathbb{R}$ are linear functions. The natural homomorphism acts on tensors as follows:

$$\begin{aligned} & g \cdot (v_1 \otimes \cdots \otimes v_s \otimes \phi_1 \otimes \cdots \otimes \phi_t) \\ = \; & g(v_1) \otimes \cdots \otimes g(v_s) \otimes \left(\phi_1 \circ g^{-1}\right) \otimes \cdots \otimes \left(\phi_t \circ g^{-1}\right) \end{aligned}$$

The derivative of this action yields a linear derivation

$$\mathrm{End}(V) \to \mathrm{End}(T(V))$$

where now

$$\begin{aligned} & L(v_1 \otimes \cdots \otimes v_s \otimes \phi_1 \otimes \cdots \otimes \phi_t) \\ = \; & L(v_1) \otimes \cdots \otimes v_s \otimes \phi_1 \otimes \cdots \otimes \phi_t \\ & + \cdots \\ & + v_1 \otimes \cdots \otimes L(v_s) \otimes \phi_1 \otimes \cdots \otimes \phi_t \\ & - v_1 \otimes \cdots \otimes v_s \otimes (\phi_1 \circ L) \otimes \cdots \otimes \phi_t \\ & - \cdots \\ & - v_1 \otimes \cdots \otimes v_s \otimes \phi_1 \otimes \cdots \otimes (\phi_t \circ L) \end{aligned}$$

Keep in mind that for $(0,k)$-tensors we have

$$R(X,Y)(T(X_1,\ldots,X_k)) = 0$$

since $T(X_1,\dots,X_k)$ is a function. In particular, we see that $R(X,Y)(T)$ acts on the tensor T in the way the linear map $L=R(X,Y)$ on T_pM extends to a derivation on tensors.

We note further that $T\to L(T)$ preserves (skew)-symmetry of L when an inner product is present since

$$g\left(\bigotimes_{i=1}^{s} v_i\otimes\bigotimes_{i=1}^{t}\phi_j,\bigotimes_{i=1}^{s} w_i\otimes\bigotimes_{i=1}^{t}\psi_j\right)=\sum g(v_i,w_i)\,g\left(\phi_j,\psi_j\right)$$

Finally we need to observe that $L(T)$ has the same symmetry properties that T has. This is a direct consequence of it being the differential of an action that preserves such symmetries, but a direct proof is also very simple. E.g., assume T is a $(0,2)$ tensor, when T is skew symmetric in, then

$$\begin{aligned}(LT)(Z,Z) &= -T(L(Z),Z)-T(Z,L(Z))\\ &= T(Z,L(Z))-T(Z,L(Z))\\ &= 0,\end{aligned}$$

and when T is symmetric, then

$$\begin{aligned}(LT)(X,Y) &= -T(L(X),Y)-T(X,L(Y))\\ &= -T(L(X),Y)-T(L(Y),X)\\ &= -T(L(X+Y),X+Y)+T(L(X),X)+T(L(Y),Y)\end{aligned}$$

which is symmetric in X,Y as well.

1.3. **Redefining $\mathfrak{Ric}(T)$.** Since $R(X,Y):T_pM\to T_pM$ is always skew-symmetric it can be decomposed using an orthonormal basis of skew-symmetric transformations $\Xi_\alpha\in\mathfrak{so}(T_pM)$. A tricky point enters our formulas at this point. It comes from the fact that if v and w are orthonormal, then $v\wedge w\in\Lambda^2T_pM$ is a unit vector, while the corresponding skew symmetric operator, which is a rotation of $\pi/2$ in $\operatorname{span}\{v,w\}$, has Euclidean norm $\sqrt{2}$. To avoid confusion and unnecessary factors we assume that $\mathfrak{so}(T_pM)$ is endowed with the metric that comes from Λ^2T_pM. With that in mind we have

$$\begin{aligned}R(X,Y) &= g(R(X,Y),\Xi_\alpha)\,\Xi_\alpha\\ &= g(\mathfrak{R}(X\wedge Y),\Xi_\alpha)\,\Xi_\alpha\\ &= g(\mathfrak{R}(\Xi_\alpha),X\wedge Y)\,\Xi_\alpha\\ &= -g(R(\Xi_\alpha)X,Y)\,\Xi_\alpha\end{aligned}$$

where the last line is due to the convention

$$g((x\wedge y)(v),w)=-g(x\wedge y,v\wedge w).$$

This allows us to rewrite the curvature term in the Lichnerowicz Laplacian as was done in [] for forms.

Lemma 3. *For any $(0,k)$ tensor T we have*

$$\begin{aligned}\mathfrak{Ric}(T) &= -\sum R(\Xi_\alpha)(\Xi_\alpha T),\\ \Delta_L T &= \nabla^*\nabla T-c\sum R(\Xi_\alpha)(\Xi_\alpha T)\end{aligned}$$

Proof. We only need to establish the first identity. This works as follows.

$$\begin{aligned}\mathfrak{Ric}(T)(X_1,...,X_k) &= \sum (R(e_j,X_i)T)(X_1,...,e_j,...,X_k)\\ &= -\sum g(R(\Xi_\alpha)e_j,X_i)(\Xi_\alpha T)(X_1,...,e_j,...,X_k)\\ &= -\sum (\Xi_\alpha T)(X_1,...,g(R(\Xi_\alpha)e_j,X_i)e_j,...,X_k)\\ &= \sum (\Xi_\alpha T)(X_1,...,R(\Xi_\alpha)X_i,...,X_k)\\ &= -\sum (R(\Xi_\alpha)(\Xi_\alpha T))(X_1,...,X_i,...,X_k)\end{aligned}$$

□

At first sight we have replaced a simple sum over j and i with a possibly more complicated sum. The next result justifies the reformulation.

Corollary 1. *If* $\mathfrak{R}\geq 0$, *then* $g(\mathfrak{Ric}(T),T)\geq 0$.

Proof. Select the orthonormal basis Ξ_α to consist of eigenvectors for $\mathfrak{R}$, i.e., $\mathfrak{R}(\Xi_\alpha)=\lambda_\alpha\Xi_\alpha$. When taking inner products with T and using that $R(\Xi_\alpha)$ is skew symmetric we obtain

$$\begin{aligned}-\sum g(R(\Xi_\alpha)(\Xi_\alpha T),T) &= \sum g(\Xi_\alpha T,R(\Xi_\alpha)T)\\ &= \sum \lambda_\alpha|\Xi_\alpha T|^2\end{aligned}$$

This shows that the curvature term is nonnegative when the curvature operator is nonnegative. □

Having redefined the Ricci curvature of tensors, we can now take it a step further and also get rid of the orthonormal basis Ξ_α. To assist in this we introduce a type change for a $(0,k)$-tensor T to a tensor $\hat{T}$ with values in Λ^2TM as follows

$$g\left(L,\hat{T}(X_1,...,X_k)\right)=(LT)(X_1,...,X_k)\text{ for all }L\in\mathfrak{so}(TM)=\Lambda^2TM.$$

Lemma 4. *For any* $(0,k)$ *tensor* T *we have*

$$g(\mathfrak{Ric}(T),T)=g\left(\mathfrak{R}\left(\hat{T}\right),\hat{T}\right).$$

Proof. This is a straight forward calculation

$$\begin{aligned}g(\mathfrak{Ric}(T),T) &= \sum g(\Xi_\alpha T,R(\Xi_\alpha)T)\\ &= \sum (\Xi_\alpha T)(e_{i_1},...,e_{i_k})(R(\Xi_\alpha)T)(e_{i_1},...,e_{i_k})\\ &= \sum g\left(\Xi_\alpha,\hat{T}(e_{i_1},...,e_{i_k})\right)g\left(R(\Xi_\alpha),\hat{T}(e_{i_1},...,e_{i_k})\right)\\ &= \sum g\left(\mathfrak{R}\left(g\left(\Xi_\alpha,\hat{T}(e_{i_1},...,e_{i_k})\right)\Xi_\alpha\right),\hat{T}(e_{i_1},...,e_{i_k})\right)\\ &= \sum g\left(\mathfrak{R}\left(\hat{T}(e_{i_1},...,e_{i_k})\right),\hat{T}(e_{i_1},...,e_{i_k})\right)\\ &= g\left(\mathfrak{R}\left(\hat{T}\right),\hat{T}\right)\end{aligned}$$

□

This new expression for $g(\mathfrak{Ric}(T),T)$ is clearly nonnegative when the curvature operator is nonnegative. In addition it also occasionally allows us to show that it is nonnegative under less restrictive hypotheses.

2. The Hodge Laplacian

The first obvious case to try this philosophy on is that of the Hodge Laplacian on k-forms as we already know that harmonic forms compute the topology of the underlying manifold. First we show

Theorem 1. (Weitzenbock, 1923) *For any form the Hodge Lapalcian is the Lichnerowicz Laplacian with* $c = 1$. *Specifically*

$$\triangle\omega = \nabla^*\nabla\omega + \mathfrak{Ric}(\omega).$$

Proof. We shall follow the proof discovered by W.A. Poor. To perform the calculations we need

$$\begin{aligned} \delta\omega(X_2,...,X_k) &= -\sum(\nabla_{E_i}\omega)(E_i,X_2,...,X_k), \\ d\omega(X_0,...,X_k) &= \sum(-1)^i(\nabla_{X_i}\omega)\left(X_0,...,\hat{X}_i,...,X_k\right) \end{aligned}$$

and employ the usual assumptions about all covariant derivatives of vector fields vanishing at a fixed point $p \in M$. We this in mind we get

$$\begin{aligned} d\delta\omega(X_1,...,X_k) &= \sum(-1)^{i+1}\nabla_{X_i}\delta\omega\left(X_1,...,\hat{X}_i,...,X_k\right) \\ &= \sum(-1)^i\nabla_{X_i}\nabla_{E_j}\omega\left(E_j,X_1,...,\hat{X}_i,...,X_k\right) \\ &= -\sum\nabla_{X_i}\nabla_{E_j}\omega(X_1,...,E_j,...,X_k) \end{aligned}$$

$$\begin{aligned} \delta d\omega(X_1,...,X_k) &= -\sum\nabla_{E_j}d\omega(E_j,X_1,...,X_k) \\ &= -\sum\nabla_{E_j}\nabla_{E_j}\omega(X_1,...,X_k) \\ &\quad -\sum(-1)^i\nabla_{E_j}\nabla_{X_i}\omega\left(E_j,X_1,...,\hat{X}_i,...,X_k\right) \\ &= (\nabla^*\nabla\omega)(X_1,...,X_k) \\ &\quad +\sum\nabla_{E_j}\nabla_{X_i}\omega(X_1,...,E_j,...,X_k) \end{aligned}$$

Thus

$$\begin{aligned} \triangle\omega &= \nabla^*\nabla\omega + \sum(R(E_j,X_i)\omega)(X_1,...,E_j,...,X_k) \\ &= \nabla^*\nabla\omega - \sum(R(E_j,X_i)\omega)(X_1,...,E_j,...,X_k) \\ &= \nabla^*\nabla\omega + \mathfrak{Ric}(\omega) \end{aligned}$$

□

Given our reworking of the curvature term in this formula the next result is now almost obvious.

Corollary 2. (D. Meyer 1971 and D. Meyer-Gallot 1975) *If the curvature operator is nonnegative, then all harmonic forms are parallel. Moreover, when the curvature operator is positive the only parallel* p*-forms have* $p = 1, n$.

Proof. Assume that M is closed and oriented. If ω is any p-form and $\mathfrak{R}(\Xi_\alpha) = \lambda_\alpha \Xi_\alpha$, then

$$\begin{aligned} 0 &= \int g(\nabla^* \nabla \omega, \omega)\, d\text{vol} + \int g(\mathfrak{Ric}(\omega), \omega)\, d\text{vol} \\ &= \int |\nabla \omega|^2\, d\text{vol} + \sum_\alpha \int \lambda_\alpha |\Xi_\alpha \omega|^2\, d\text{vol} \end{aligned}$$

So if $\lambda_\alpha \geq 0$ then we clearly get that $\nabla \omega = 0$. In case $\lambda_\alpha > 0$ we get that $\Xi_\alpha \omega = 0$ for all α, and hence by linearity that $L\omega = 0$ for all skew-symmetric L. If we assume $k < n$ and select L so that $L(e_i) = 0$ for $i < k$, $L(e_k) = e_{k+1}$, then

$$0 = (L\omega)(e_1, ..., e_k) = -\omega(e_1, ..., e_{k-1}, e_{k+1})$$

Since the basis was arbitrary this shows that $\omega = 0$.

When M is not orientable we can pass to the universal covering. When M is noncompact and complete, we have to invoke the soul theorem. □

It is instructive to try to recover the standard formula for 1-forms using the new type change of ω to $\hat{\omega}$.

Corollary 3. *If ω is a 1-form and X the dual vector field, then*

$$g(\mathfrak{R}(\hat{\omega}), \hat{\omega}) = \text{Ric}(X, X).$$

Proof. In this case we have

$$\begin{aligned} (L\omega)(Z) &= -\omega(L(Z)) \\ &= -g(X, L(Z)) \\ &= g(L, Z \wedge X) \end{aligned}$$

so

$$\hat{\omega}(Z) = Z \wedge X.$$

This shows that the curvature term in the Bochner formula becomes

$$\begin{aligned} -\sum g(R(\Xi_\alpha)(\Xi_\alpha \omega), \omega) &= \sum g(\Xi_\alpha \omega, R(\Xi_\alpha)\omega) \\ &= \sum g(\hat{\omega}(E_i), \mathfrak{R}(\hat{\omega}(E_i))) \\ &= \sum g(\mathfrak{R}(E_i \wedge X), E_i \wedge X) \\ &= \sum R(X, E_i, E_i, X) \\ &= \text{Ric}(X, X) \end{aligned}$$

□

3. (0, 2)-Tensors

The goal is to calculate $\hat{h}$ and completely understand $\mathfrak{Ric}\left(\hat{h}\right)$ for any $(0,2)$-tensor. We have the corresponding $(1,1)$-tensor called H

$$h(z, w) = g(H(z), w)$$

and the dual H^*

$$g(H(z), w) = g(z, H^*(w)).$$

Proposition 1.

$$\hat{h}(z,w) = -H(z)\wedge w + z\wedge H^*(w)$$

Proof.

$$\begin{aligned}(Lh)(v,y) &= -h(L(z),w) - h(z,L(w)) \\ &= -g(H(L(z)),w) - g(H(w),L(z)) \\ &= -g(L(z),H^*(w)) - g(L(z),H(w)) \\ &= g(L,z\wedge H^*(w)) + g(L,z\wedge H(w)) \\ &= g(L,-H(z)\wedge w + z\wedge H^*(w))\end{aligned}$$

□

This indicates that we have to control curvatures of the type

$$g(\mathfrak{R}(-H(z)\wedge w + z\wedge H^*(w)), -H(z)\wedge w + z\wedge H^*(w)).$$

If H is normal, then it can be diagonalized with respect to an orthonormal basis in the complexified tangent bundle. Assuming that $H(z) = \lambda z$ and $H(w) = \mu w$ where $z,w\in T_pM\otimes\mathbb{C}$ are orthonormal we obtain

$$g\left(\mathfrak{R}(-H(z)\wedge w + z\wedge H^*(w)), \overline{-H(z)\wedge w + z\wedge H^*(w)}\right) = \left|-\lambda+\bar{\mu}\right|^2 g\left(\mathfrak{R}(z\wedge w), \overline{z\wedge w}\right)$$

where g and $\mathfrak{R}$ are the natural complexifications.

The curvature term $g\left(\mathfrak{R}(z\wedge w), \overline{z\wedge w}\right)$ looks like a complexified sectional curvature and is in fact called the *complex sectional curvature*. It can be recalculated without references to complexifications. If we let $z = x+\sqrt{-1}y$ and $w = u+\sqrt{-1}v$, $x,y,u,v\in TM$ then

$$\begin{aligned}g(\mathfrak{R}(z\wedge w),\bar{z}\wedge\bar{w}) &= g(\mathfrak{R}(x\wedge u - y\wedge v), x\wedge u - y\wedge v) \\ &\quad + g(\mathfrak{R}(x\wedge v + y\wedge u), x\wedge v + y\wedge u) \\ &= g(\mathfrak{R}(x\wedge u), x\wedge u) + g(\mathfrak{R}(y\wedge v), y\wedge v) \\ &\quad + g(\mathfrak{R}(x\wedge v), x\wedge v) + g(\mathfrak{R}(y\wedge u), y\wedge u) \\ &\quad - 2g(\mathfrak{R}(x\wedge u), y\wedge v) + 2g(\mathfrak{R}(x\wedge v), y\wedge u) \\ &= \sec(x,u) + \sec(y,v) + \sec(x,v) + \sec(y,u) \\ &\quad + 2R(x,u,y,v) - 2R(x,v,y,u) \\ &= \sec(x,u) + \sec(y,v) + \sec(x,v) + \sec(y,u) \\ &\quad - 2(R(v,y,x,u) + R(x,v,y,u)) \\ &= \sec(x,u) + \sec(y,v) + \sec(x,v) + \sec(y,u) \\ &\quad + 2R(y,x,v,u) \\ &= \sec(x,u) + \sec(y,v) + \sec(x,v) + \sec(y,u) \\ &\quad + 2R(x,y,u,v)\end{aligned}$$

The first line in this derivation shows that complex sectional curvatures are nonnegative when $\mathfrak{R}\geq 0$. Thus we see that it is weaker than working with the curvature operator. On the other hand it is stronger than sectional curvature.

There are three special cases depending on the dimension of span $\{x,y,u,v\}$. When $y = v = 0$ we obtain the standard definition of sectional curvature. When

x, y, u, v are orthonormal we obtain the so called *isotropic curvature*, and finally if $u = v$ we get a sum of two sectional curvatures

$$2\sec(x, u) + 2\sec(y, u)$$

also called a *second Ricci curvature*, when x, y, u are orthonormal.

Proposition 2. *Let h be a $(0,2)$ tensor such that H is normal. If the complex sectional curvatures are nonnegative, then $g\left(\mathfrak{R}\left(\hat{h}\right), \hat{h}\right) \geq 0$.*

Proof. We can use complex orthornormal bases as well as real bases to compute $g\left(\mathfrak{R}\left(\hat{h}\right), \hat{h}\right)$. Using that H is normal we obtain a complex orthonormal basis e_i of eigenvectors $H(e_i) = \lambda_i e_i$ and $H^*(e_i) = \bar{\lambda}_i e_i$. Using that we quickly obtain

$$\begin{aligned} g\left(\mathfrak{R}\left(\hat{h}\right), \hat{h}\right) &= \sum g\left(\mathfrak{R}\left(\hat{h}(e_i, e_j)\right), \overline{\hat{h}(e_i, e_j)}\right) \\ &= \sum g\left(\mathfrak{R}\left(-H(e_i) \wedge e_j + e_i \wedge H^*(e_j)\right), \overline{-H(e_i) \wedge e_j + e_i \wedge H^*(e_j)}\right) \\ &= \sum \left|-\lambda_i + \bar{\lambda}_j\right|^2 g\left(\mathfrak{R}(e_i \wedge e_j), \overline{e_i \wedge e_j}\right) \end{aligned}$$

□

In the special case where H is self-adjoint the eigenvalues/vectors are real and so we need only use the real sectional curvatures. When H is skew-symmetric the eigenvectors are complex unless they correspond to zero eigenvalues. This shows that we must use the isotropic curvatures and also the second Ricci curvatures when M is odd dimensional, however none of the terms involve real sectional curvatures.

These characterizations can be combined to show

Proposition 3. $g\left(\mathfrak{R}\left(\hat{h}\right), \hat{h}\right) \geq 0$ *for all $(0,2)$-tensors on T_pM if and only if all complex sectional curvatures on T_pM are nonnegative.*

Proof. We decompose $h = h_s + h_a$ into symmetric and skew symmetric parts. Then

$$\begin{aligned} g\left(\mathfrak{R}\left(\hat{h}\right), \hat{h}\right) &= g\left(\mathfrak{R}\left(\hat{h}_s\right), \hat{h}_s\right) + g\left(\mathfrak{R}\left(\hat{h}_a\right), \hat{h}_a\right) + g\left(\mathfrak{R}\left(\hat{h}_s\right), \hat{h}_a\right) + g\left(\mathfrak{R}\left(\hat{h}_a\right), \hat{h}_s\right) \\ &= g\left(\mathfrak{R}\left(\hat{h}_s\right), \hat{h}_s\right) + g\left(\mathfrak{R}\left(\hat{h}_a\right), \hat{h}_a\right) + 2g\left(\mathfrak{R}\left(\hat{h}_s\right), \hat{h}_a\right) \end{aligned}$$

However,

$$\begin{aligned} g\left(\mathfrak{R}\left(\hat{h}_s\right), \hat{h}_a\right) &= \sum g\left(\mathfrak{R}\left(\hat{h}_s(e_i, e_j)\right), \hat{h}_a(e_i, e_j)\right) \\ &= -\sum g\left(\mathfrak{R}\left(\hat{h}_s(e_j, e_i)\right), \hat{h}_a(e_j, e_i)\right) \\ &= -g\left(\mathfrak{R}\left(\hat{h}_s\right), \hat{h}_a\right) \end{aligned}$$

So

$$g\left(\mathfrak{R}\left(\hat{h}\right), \hat{h}\right) = g\left(\mathfrak{R}\left(\hat{h}_s\right), \hat{h}_s\right) + g\left(\mathfrak{R}\left(\hat{h}_a\right), \hat{h}_a\right)$$

and the result follows from the previous proposition. □

4. The Curvature Tensor

The Bochner technique can also be applied to the curvature tensor. It is by no means clear that this will yield anything. It seems both miraculous and profound that it works. The goal is to show that compact Riemannian manifolds with $\mathrm{div}R = 0$ and nonnegative sectional curvature, respectively nonnegative curvature operator, have parallel Ricci tensor, respectively parallel curvature tensor. The key is to show that when the curvature tensor is divergence free then

$$\begin{aligned} 0 &= \Delta_L \mathrm{Ric} = \nabla^* \nabla \mathrm{Ric} + \frac{1}{2} \mathfrak{Ric} \left(\mathrm{Ric} \right), \\ 0 &= \Delta_L R = \nabla^* \nabla R + \frac{1}{2} \mathfrak{Ric} \left(R \right) \end{aligned}$$

We start with the Ricci tensor calculations. They are a bit easier and give us an idea of what to expect for the curvature tensor. Recall that the divergence of the curvature tensor can be defined as the $(1,3)$-tensor

$$-\nabla^* R = \mathrm{div} R \left(\cdot, \cdot, \cdot \right) = \sum \left(\nabla_{E_i} R \right) \left(E_i, \cdot, \cdot, \cdot \right)$$

Theorem 2. (Berger) *Suppose that a Riemannian manifold (M,g) satisfies $\mathrm{div}R = 0$, then*

$$0 = \nabla^* \nabla \mathrm{Ric} - \frac{1}{2} \sum R \left(\Xi_\alpha \right) \left(\Xi_\alpha \mathrm{Ric} \right).$$

Moreover, if (M,g) is closed, oriented and $\sec \geq 0$, then $\nabla \mathrm{Ric} = 0$.

Proof. We start by proving that

$$\mathrm{div} R \left(X, Y, Z \right) = \left(\nabla_Z \mathrm{Ric} \right) \left(Y, X \right) - \left(\nabla_Y \mathrm{Ric} \right) \left(Z, X \right).$$

The left hand side is defined by

$$\begin{aligned} \mathrm{div} R \left(X, Y, Z \right) &= \sum \left(\nabla_{E_i} R \right) \left(E_i, X, Y, Z \right) \\ &= \sum \left(\nabla_{E_i} R \right) \left(Y, Z, E_i, X \right) \\ &= \sum \left(\nabla_Z R \right) \left(Y, E_i, E_i, X \right) - \sum \left(\nabla_Y R \right) \left(Z, E_i, E_i, X \right) \\ &= \left(\nabla_Z \mathrm{Ric} \right) \left(Y, X \right) - \left(\nabla_Y \mathrm{Ric} \right) \left(Z, X \right) \end{aligned}$$

With this in mind we can then calculate

$$\begin{aligned} -\left(\nabla^* \nabla \mathrm{Ric} \right) \left(X, X \right) &= \sum \left(\nabla^2_{E_i, E_i} \mathrm{Ric} \right) \left(X, X \right) \\ &= \sum \left(\nabla^2_{E_i, X} \mathrm{Ric} \right) \left(E_i, X \right) + \sum \left(\nabla_{E_i} \mathrm{div} R \right) \left(X, X, E_i \right) \end{aligned}$$

If we assume $\mathrm{div}R = 0$, then we get

$$\begin{aligned} -\left(\nabla^* \nabla \mathrm{Ric} \right) \left(X, X \right) &= \sum \left(\nabla^2_{E_i, X} \mathrm{Ric} \right) \left(E_i, X \right) \\ &= \sum R \left(E_i, X \right) \left(\mathrm{Ric} \right) \left(E_i, X \right) + \sum \left(\nabla^2_{X, E_i} \mathrm{Ric} \right) \left(E_i, X \right) \end{aligned}$$

Here the last term vanishes since

$$\begin{aligned} \sum \left(\nabla^2_{X, E_i} \mathrm{Ric} \right) \left(E_i, X \right) &= \sum \left(\nabla^2_{X, E_i} R \right) \left(E_i, E_j, E_j, X \right) \\ &= \sum \left(\nabla_X \mathrm{div} R \right) \left(E_j, E_j, X \right) \\ &= 0 \end{aligned}$$

Thus we obtain

$$\begin{aligned}
-\left(\nabla^*\nabla \mathrm{Ric}\right)(X,X) &= \sum R\left(E_i,X\right)\left(\mathrm{Ric}\right)\left(E_i,X\right)\\
&= \frac{1}{2}\sum R\left(E_i,X\right)\left(\mathrm{Ric}\right)\left(E_i,X\right)+R\left(E_i,X\right)\left(\mathrm{Ric}\right)\left(X,E_i\right)\\
&= \frac{1}{2}\mathfrak{Ric}\left(\mathrm{Ric}\right)(X,X)
\end{aligned}$$

where in the second line we used that Ric is symmetric.

Assuming that $\mathrm{Ric}\left(E_i\right)=\rho_i E_i$ the curvature term then looks like

$$\begin{aligned}
g\left(\mathfrak{Ric}\left(\mathrm{Ric}\right),\mathrm{Ric}\right) &= g\left(\mathfrak{R}\left(\widehat{\mathrm{Ric}}\right),\widehat{\mathrm{Ric}}\right)\\
&\ \sum\left(\rho_i-\rho_j\right)^2\sec\left(E_i,E_j\right)
\end{aligned}$$

Since this is nonnegative when $\sec\geq 0$ we get the desired conclusion in the usual fashion by integrating over M. Note in addition that if the sectional curvatures are positive at a point, the the eigenvalues of Ric are all the same. Hence the metric becomes an Einstein metric. □

Note that the Berger spheres have divergence-free Ricci tensor, but only the standard sphere has parallel Ricci tensor. Thus it is not clear that one can get results without assuming divergence free curvature tensor.

We can now address what happens for the full curvature tensor.

Theorem 3. *The curvature tensor R on a Riemannian manifold satisfies*

$$\begin{aligned}
&\left(\nabla^*\nabla R\right)(X,Y,Z,W)+\frac{1}{2}\mathfrak{Ric}\left(R\right)(X,Y,Z,W)\\
=\ &\frac{1}{2}\left(\nabla_X\nabla^*R\right)(Y,Z,W)-\frac{1}{2}\left(\nabla_Y\nabla^*R\right)(X,Z,W)\\
&\frac{1}{2}\left(\nabla_Z\nabla^*R\right)(W,X,Y)-\frac{1}{2}\left(\nabla_W\nabla^*R\right)(Z,X,Y)
\end{aligned}$$

Proof. By far the most important ingredient in the proof is that we have the second Bianchi identity at our disposal. We will begin the calculation by considering the (0,4)-curvature tensor R. Fix a point p, let X,Y,Z,W be vector fields with $\nabla X=\nabla Y=\nabla Z=\nabla W=0$ at p and let E_i be normal coordinates at p. Then

$$\begin{aligned}
\left(\nabla^*\nabla R\right)(X,Y,Z,W) &= -\sum_{i=1}^{n}\left(\nabla^2_{E_i,E_i}R\right)(X,Y,Z,W)\\
&= \sum_{i=1}^{n}\left(\nabla^2_{E_i,X}R\right)\left(Y,E_i,Z,W\right)+\left(\nabla^2_{E_i,Y}R\right)\left(E_i,X,Z,W\right)\\
&= \sum_{i=1}^{n}\left(\nabla^2_{X,E_i}R\right)\left(Y,E_i,Z,W\right)+\left(\nabla^2_{Y,E_i}R\right)\left(E_i,X,Z,W\right)\\
&\quad+\sum_{i=1}^{n}\left(R\left(E_i,X\right)(R)\right)\left(Y,E_i,Z,W\right)+\left(R\left(E_i,Y\right)(R)\right)\left(E_i,X,Z,W\right)\\
&= \left(\nabla_X\nabla^*R\right)(Y,Z,W)-\left(\nabla_Y\nabla^*R\right)(X,Z,W)\\
&\quad-\sum_{i=1}^{n}\left(R\left(E_i,X\right)(R)\right)\left(E_i,Y,Z,W\right)+\left(R\left(E_i,Y\right)(R)\right)\left(X,E_i,Z,W\right)
\end{aligned}$$

where we note that the last two terms are half of the expected terms in $-\mathfrak{Ric}(R)(X,Y,Z,W)$.

Using that R is symmetric in the pairs X,Y and Z,W we then obtain

$$
\begin{aligned}
(\nabla^*\nabla R)(X,Y,Z,W) &= \frac{1}{2}(\nabla^*\nabla R)(X,Y,Z,W)-\frac{1}{2}(\nabla^*\nabla R)(Z,W,X,Y)\\
&= \frac{1}{2}((\nabla_X\nabla^* R)(Y,Z,W)-(\nabla_Y\nabla^* R)(X,Z,W))\\
&\quad +\frac{1}{2}((\nabla_Z\nabla^* R)(W,X,Y)-(\nabla_W\nabla^* R)(Z,X,Y))\\
&\quad -\frac{1}{2}\sum_{i=1}^{n}(R(E_i,X)(R))(E_i,Y,Z,W)+(R(E_i,Y)(R))(X,E_i,Z,W)\\
&\quad -\frac{1}{2}\sum_{i=1}^{n}(R(E_i,Z)(R))(E_i,W,X,Y)+(R(E_i,W)(R))(Z,E_i,X,Y)\\
&= \frac{1}{2}((\nabla_X\nabla^* R)(Y,Z,W)-(\nabla_Y\nabla^* R)(X,Z,W))\\
&\quad +\frac{1}{2}((\nabla_Z\nabla^* R)(W,X,Y)-(\nabla_W\nabla^* R)(Z,X,Y))\\
&\quad -\frac{1}{2}\mathfrak{Ric}(R)(X,Y,Z,W)
\end{aligned}
$$

□

We can now go over to the more complicated result we are interested in. It was first established in [?], and then with a modified proof in [?]. We shall present a very simple proof based on the above formula relating the divergence and connection Laplacians of the curvature tensor.

Theorem 4. (Tachibana, 1974) *If (M,g) is a compact oriented Riemannian manifold with* $\operatorname{div}R=0$ *and* $\mathfrak{R}\geq 0$, *then* $\nabla R=0$. *If in addition,* $\mathfrak{R}>0$, *then* (M,g) *has constant curvature.*

Proof. We know from above that

$$\nabla^*\nabla R+\frac{1}{2}\mathfrak{Ric}(R)=0$$

Taking inner products with R and then integrating yields

$$
\begin{aligned}
\int_M|\nabla R|^2\,d\mathrm{vol} &= -\frac{1}{2}\sum\int_M g(R(\Xi_\alpha)(\Xi_\alpha R),R)\,d\mathrm{vol}\\
&= \frac{1}{2}\sum\int_M g(\Xi_\alpha R,R(\Xi_\alpha)R)\,d\mathrm{vol}\\
&= \frac{1}{2}\sum\int_M \lambda_\alpha|\Xi_\alpha R|^2\,d\mathrm{vol}
\end{aligned}
$$

using a basis Ξ_α that diagonalizes $\mathfrak{R}$. So if all eigenvalues are nonnegative we obviously get that $\nabla R=0$. Moreover, should all eigenvalues be positive then $LR=0$ for all $L\in\mathfrak{so}(T_pM)$. This condition will imply that $R(x,y,y,z)=0$ and $R(x,y,v,w)=0$ when the vectors are perpendicular. This in turn shows that any bi-vector $X\wedge Y$ is an eigenvector for $\mathfrak{R}$, but this can only happen if $\mathfrak{R}=kI$ for some constant k.

To show that the mixed curvatures vanish first select L so that $L(y)=0$ and $L(x)=z$, then

$$\begin{aligned} 0 &= LR(x,y,y,x) = -R(L(x),y,y,x) - R(x,y,y,L(x)) \\ &= -2R(x,y,y,z). \end{aligned}$$

Polarizing in $y=v+w$, then shows that

$$R(x,v,w,z) = -R(x,w,v,z)$$

The Bianchi identity implies

$$\begin{aligned} R(x,v,w,z) &= R(w,v,x,z) - R(w,x,v,z) \\ &= -2R(w,x,v,z) \\ &= 2R(x,w,v,z) \\ &= -2R(x,v,w,z) \end{aligned}$$

showing that $R(x,v,w,z)=0$. □

Finally we recover Tachibana's original formula by calculating $\hat{S}$ for a symmetric tensor $S:\Lambda^2TM\times\Lambda^2TM\to\mathbb{R}$. This includes 4-forms.

$$\begin{aligned} & -(LS)(X,Y,V,W) \\ =\ & S(L(X),Y,V,W) + S(X,L(Y),V,W) + S(X,Y,L(V),W) + S(X,Y,V,L(W)) \\ =\ & -S(V,W,Y,L(X)) + S(V,W,X,L(Y)) - S(X,Y,W,L(V)) + S(X,Y,V,L(W)) \\ =\ & -g(S(V,W)Y,L(X)) + g(S(V,W)X,L(Y)) - g(S(X,Y)W,L(V)) + g(S(X,Y)V,L(W)) \\ =\ & g(L,X\wedge S(V,W)Y) - g(L,Y\wedge S(V,W)X) + g(L,V\wedge S(X,Y)W) - g(L,W\wedge S(X,Y)V) \\ =\ & g(L,S(V,W)X\wedge Y + X\wedge S(V,W)Y + S(X,Y)V\wedge W + V\wedge S(X,Y)W) \\ =\ & g(L,S(V,W)(X\wedge Y) + S(X,Y)(V\wedge W)) \end{aligned}$$

If we also think of $S:\Lambda^2TM\to\Lambda^2TM$, then (modulo signs?)

$$\begin{aligned} -(LS)(X,Y,V,W) &= -g(S(L(X\wedge Y)),V\wedge W) - g(S(X\wedge Y),L(V\wedge W)) \\ &= -g(S(L(X\wedge Y)),V\wedge W) - g(L^*S(X\wedge Y),V\wedge W) \end{aligned}$$

so if L is skew symmetric

$$LS = [S,L]$$

Thus L acts trivally on the identity operator, and one can easily check that if T is another (1,1) tensor then $[L,T]$ on Λ^2TM is the same as the derivation coming from $[L,T]$ on T_pM. Finally $L\omega$ is a form if ω is a form, so L preserves the natural orthogonal decomposition

$$S^2\Lambda^2T_pM = I\oplus\mathrm{Ric}_0\oplus W\oplus\Lambda^4T_pM$$

We know that 4-forms are spanned by $\omega_1\wedge\omega_2$ where ω_i are 2-forms. Curvature tensors are similarily spanned by $S_1\wedge S_2$ where $S_i:T_pM\to T_pM$ are symmetric, this is equivalent to the Kulkarni-Nomizu product $s_1\circ s_2$ under type change. Conformally flat curvature tensors correspond to the case where $S_2=I$ or $s_2=g$.

5. Laplacians with Diffusion Terms

When the measure is changed from being $d\mathrm{vol}$ to $e^{-f}d\mathrm{vol}$ then we also need to change the way we compute divergences so as to make sure they are still adjoints to exterior and covariant derivatives. To this end we define

$$\begin{aligned}\delta_f &= e^f\delta e^{-f} = \delta + i_{\nabla f} \\ &= \nabla^* + i_{\nabla f} = e^f\nabla^* e^{-f} = \nabla_f^*\end{aligned}$$

Proposition 4. *∇_f^* is the adjoint to ∇ and d with respect to the measure $e^{-f}d\mathrm{vol}$.*

Proof.

$$\begin{aligned}\int g\left(d\omega,\omega'\right)e^{-f}d\mathrm{vol} &= \int g\left(d\omega, e^{-f}\omega'\right)d\mathrm{vol} \\ &= \int g\left(\omega, \delta\left(e^{-f}\omega'\right)\right)d\mathrm{vol} \\ &= \int g\left(\omega, e^f\delta\left(e^{-f}\omega'\right)\right)e^{-f}d\mathrm{vol} \\ &= \int g\left(\omega, \delta_f\omega'\right)e^{-f}d\mathrm{vol}\end{aligned}$$

Likewise

$$\begin{aligned}\int g\left(\nabla S,T\right)e^{-f}d\mathrm{vol} &= \int g\left(\nabla S, e^{-f}T\right)d\mathrm{vol} \\ &= \int g\left(S, \nabla^*\left(e^{-f}T\right)\right)d\mathrm{vol} \\ &= \int g\left(S, e^f\nabla^*\left(e^{-f}T\right)\right)e^{-f}d\mathrm{vol} \\ &= \int g\left(S, \nabla_f^*T\right)e^{-f}d\mathrm{vol}\end{aligned}$$

□

The previous proposition certainly works for tensors with compact support and thus by extension in $W^{1,2}$ the Hilbert space of tensors in $L^2\left(e^{-f}d\mathrm{vol}\right)$ with weak derivatives also in $L^2\left(e^{-f}d\mathrm{vol}\right)$. This is more inteteresting than in the case where f is not present as we can, e.g., use $(M,g) = (\mathbb{R}^n, \mathrm{can})$ with $f = \frac{1}{2}|x|^2$. In this case the measure is propoertional to the Gaussian measure and thus has finite volume. This means that bounded tensors with bounded derivatives lie in $W^{1,2}$.

More generally one can consider

$$\begin{aligned}\delta_U &= \delta + i_U \\ \nabla_U^* &= \nabla^* + i_U\end{aligned}$$

for a vector field U, but this divergence operator will not necessarily be the adjoint to d or ∇ for any measure.

The U-Hodge Laplacian becomes

$$\begin{aligned}\Delta_U &= \delta_U d + d\delta_U \\ &= \delta d + d\delta + i_U d + d i_U \\ &= \Delta + L_U\end{aligned}$$

and is thus the stardard Hodge Laplacian with a Lie derivative as diffusion term.

Proposition 5. *The U Hodge Laplacian on forms satisfies the Weitzenböck formula*

$$\Delta_U \omega = \nabla_U^* \nabla \omega + \mathfrak{Ric}(\omega) - (\nabla U)\,\omega.$$

Proof. Lie derivatives and covariant derivatives are related by the derivation coming from the $(1,1)$ tensor ∇U

$$L_U = \nabla_U - \nabla U$$

or

$$L_U T = \nabla_U T - (\nabla U)\,T$$

for $(0,p)$ tensors T.

Since we already know that $\Delta = \nabla^* \nabla + \mathfrak{Ric}$ on forms this will balance the terms in the formula that only depend on U. □

This leads us to the new Ricci tensor also called the C tensor by Lichnerowicz

$$\mathfrak{Ric}_U = \mathfrak{Ric} - (\nabla U)$$

and the U-Lichnerowicz Laplacian on tensors

$$\begin{aligned} \Delta_{L,U} &= \nabla_U^* \nabla + c\mathfrak{Ric}_U \\ &= \nabla_U^* \nabla + c\,(\mathfrak{Ric} - (\nabla U))\,,\ c > 0. \end{aligned}$$

In case $U = \nabla f$ we also use the notation

$$\mathfrak{Ric}_f = \mathfrak{Ric} - (\nabla \nabla f) = \mathfrak{Ric} - S_f.$$

Proposition 6. *The curvature tensor satisfies*

$$\begin{aligned} \nabla_U^* \nabla R + \frac{1}{2}\mathfrak{Ric}(R) - \frac{1}{2}(\nabla U)(R) &= \Delta_{L,U} R \\ &= \frac{1}{2}\left(\nabla_X \nabla_U^* R\right)(Y,Z,W) - \frac{1}{2}\left(\nabla_Y \nabla_U^* R\right)(X,Z,W) \\ &\quad \frac{1}{2}\left(\nabla_Z \nabla_U^* R\right)(W,X,Y) - \frac{1}{2}\left(\nabla_W \nabla_U^* R\right)(Z,X,Y) \end{aligned}$$

Proof. We start with the formula

$$\begin{aligned} \nabla^* \nabla R + \frac{1}{2}\mathfrak{Ric}(R) &= \Delta_L R \\ &= \frac{1}{2}\left(\nabla_X \nabla^* R\right)(Y,Z,W) - \frac{1}{2}\left(\nabla_Y \nabla^* R\right)(X,Z,W) \\ &\quad \frac{1}{2}\left(\nabla_Z \nabla^* R\right)(W,X,Y) - \frac{1}{2}\left(\nabla_W \nabla^* R\right)(Z,X,Y) \end{aligned}$$

To verify the proposition we need the extra terms that involve U to cancel out. This relies on the second Bianchi identity. Assume as usual that X, Y, Z, W are parallel at a some fixed point. On the left hand side we have

$$\nabla_U R - \frac{1}{2}(\nabla U)(R)$$

To understand the right hand side we first need to observe that

$$\begin{aligned} \left(\nabla_X \nabla_U^* R\right)(Y,Z,W) &= \left(\nabla_X \left(\nabla^* R + i_U R\right)\right)(Y,Z,W) \\ &= \left(\nabla_X \nabla^* R\right)(Y,Z,W) + \nabla_X \left(R(U,Y,Z,W)\right) \\ &= \left(\nabla_X \nabla^* R\right)(Y,Z,W) + \left(\nabla_X R\right)(U,Y,Z,W) + R\left(\nabla_X U, Y, Z, W\right) \end{aligned}$$

This allows us to simplify the U terms on the right hand side:

$$\begin{aligned}
&+\frac{1}{2}\left(\nabla_X R\right)(U,Y,Z,W)-\frac{1}{2}\left(\nabla_Y R\right)(U,X,Z,W)\\
&+\frac{1}{2}\left(\nabla_Z R\right)(U,W,X,Y)-\frac{1}{2}\left(\nabla_W R\right)(U,Z,X,Y)\\
&+\frac{1}{2}R\left(\nabla_X U,Y,Z,W\right)-\frac{1}{2}R\left(\nabla_Y U,X,Z,W\right)\\
&+\frac{1}{2}R\left(\nabla_Z U,W,X,Y\right)-\frac{1}{2}R\left(\nabla_W U,Z,X,Y\right)\\
= & -\frac{1}{2}\left(\nabla_X R\right)(Y,U,Z,W)-\frac{1}{2}\left(\nabla_Y R\right)(U,X,Z,W)\\
&-\frac{1}{2}\left(\nabla_Z R\right)(W,U,X,Y)-\frac{1}{2}\left(\nabla_W R\right)(U,Z,X,Y)\\
&+\frac{1}{2}R\left(\nabla_X U,Y,Z,W\right)+\frac{1}{2}R\left(X,\nabla_Y U,Z,W\right)\\
&+\frac{1}{2}R\left(\nabla_Z U,W,X,Y\right)+\frac{1}{2}R\left(Z,\nabla_W U,X,Y\right)\\
= & \ \frac{1}{2}\left(\nabla_U R\right)(X,Y,Z,W)+\frac{1}{2}\left(\nabla_U R\right)(X,Y,Z,W)\\
&+\frac{1}{2}R\left(\nabla_X U,Y,Z,W\right)+\frac{1}{2}R\left(X,\nabla_Y U,Z,W\right)\\
&+\frac{1}{2}R\left(X,Y,\nabla_Z U,W\right)+\frac{1}{2}R\left(X,Y,Z,\nabla_W U\right)\\
= & \left(\nabla_U R\right)(X,Y,Z,W)-\frac{1}{2}\left(\nabla U\right)R(X,Y,Z,W)
\end{aligned}$$

□

As long as we use the maximum principle we can easily generalize the Bochner technique to work when we have a diffusion term. The important observation is

Lemma 5. *Let T be a tensor such that*

$$\begin{aligned}
\nabla_U^*\nabla T+c\mathfrak{Ric}_U(T) &= 0,\\
g\left(\mathfrak{Ric}_U(T),T\right) &\geq 0.
\end{aligned}$$

If $|T|$ has a maximum, then T is parallel.

Proof. The first observation we have to make is

$$\frac{1}{2}\nabla_U^*\nabla|T|^2=g\left(\nabla_U^*\nabla T,T\right)-|\nabla T|^2$$

Thus we have

$$\frac{1}{2}\nabla_U^*\nabla|T|^2=-\left(g\left(c\mathfrak{Ric}_U(T),T\right)+|\nabla T|^2\right)\leq 0.$$

The maximum principle, then implies that $|T|^2$ must be constant if it has a maximum. Having shown that it is constant we then obtain

$$g\left(c\mathfrak{Ric}_U(T),T\right)+|\nabla T|^2\equiv 0$$

which is only possible if both

$$\begin{aligned}
g\left(\mathfrak{Ric}_U(T),T\right) &\equiv 0\\
|\nabla T|^2 &\equiv 0.
\end{aligned}$$

□

In case $U = \nabla f$, we can instead use integration.

Lemma 6. *Assume that* $\int e^{-f} d\text{vol} < \infty$. *Let* T *be a tensor such that*

$$\begin{aligned} \nabla_f^* \nabla T + c\mathfrak{Ric}_f(T) &= 0, \\ g(\mathfrak{Ric}_f(T), T) &\geq 0, \end{aligned}$$

then T *is parallel.*

Proof. We have

$$\begin{aligned} 0 &\leq \int |\nabla T|^2 e^{-f} d\text{vol} \\ &= \int g\left(\nabla_f^* \nabla T, T\right) e^{-f} d\text{vol} \\ &= -c \int g(\mathfrak{Ric}_f(T), T) e^{-f} d\text{vol} \\ &\leq 0 \end{aligned}$$

showing that all integrals vanish. As the integrands are all nonnegative they must in addition vanish. □

6. Ricci Solitons

In case we have a soliton metric

$$\text{Ric} + S_f = \lambda I$$

this means that

$$\begin{aligned} 0 &= \Delta_{L,f} R \\ &= \nabla_f^* \nabla R + \frac{1}{2}\mathfrak{Ric}(R) - \frac{1}{2}(S_f R) \\ &= \nabla_f^* \nabla R + \frac{1}{2}\mathfrak{Ric}(R) - \frac{1}{2}((\lambda I - \text{Ric}) R) \\ &= \nabla_f^* \nabla R + \frac{1}{2}\mathfrak{Ric}(R) + \frac{1}{2}(\text{Ric}R) + 2\lambda R \\ &= \nabla_f^* \nabla R - \frac{1}{2}\sum R(\Xi_\alpha)\Xi_\alpha(R) + \frac{1}{2}(\text{Ric}R) + 2\lambda R. \end{aligned}$$

If we also think of $R : \Lambda^2 TM \to \Lambda^2 TM$, then

$$\begin{aligned} (\text{Ric}R)(X, Y, V, W) &= -g(R(\text{Ric}(X \wedge Y)), V \wedge W) - g(R(X \wedge Y), \text{Ric}(V \wedge W)) \\ &= -g(R(\text{Ric}(X \wedge Y)), V \wedge W) - g(\text{Ric}R(X \wedge Y), V \wedge W) \end{aligned}$$

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