120A Lecture Notes

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1 Vector-Matrix Preliminaries

Given a basis e, f for a two dimensional vector space we expand vectors using matrix multiplication

$$v = v^e e + v^f f = \begin{bmatrix} e & f \end{bmatrix} \begin{bmatrix} v^e \\ v^f \end{bmatrix}$$

and the matrix representation $[L]$ for a linear map/transformation L can be found from

$$\begin{aligned} \begin{bmatrix} L(e) & L(f) \end{bmatrix} &= \begin{bmatrix} e & f \end{bmatrix} [L] \\ &= \begin{bmatrix} e & f \end{bmatrix} \begin{bmatrix} L_e^e & L_f^e \\ L_e^f & L_f^f \end{bmatrix} \end{aligned}$$

Next we relate matrix multiplication and the dot product in $\mathbb{R}^3$. We think of vectors as being columns or 3×1 matrices. Keeping that in mind and using transposition of matrices we immediately obtain:

$$\begin{aligned} X^t Y &= X \cdot Y, \\ X^t \begin{bmatrix} X_2 & Y_2 \end{bmatrix} &= \begin{bmatrix} X \cdot X_2 & X \cdot Y_2 \end{bmatrix} \\ \begin{bmatrix} X_1 & Y_1 \end{bmatrix}^t X &= \begin{bmatrix} X_1 \cdot X \\ Y_1 \cdot X \end{bmatrix} \\ \begin{bmatrix} X_1 & Y_1 \end{bmatrix}^t \begin{bmatrix} X_2 & Y_2 \end{bmatrix} &= \begin{bmatrix} X_1 \cdot X_2 & X_1 \cdot Y_2 \\ Y_1 \cdot X_2 & Y_1 \cdot Y_2 \end{bmatrix}, \\ \begin{bmatrix} X_1 & Y_1 & Z_1 \end{bmatrix}^t \begin{bmatrix} X_2 & Y_2 & Z_2 \end{bmatrix} &= \begin{bmatrix} X_1 \cdot X_2 & X_1 \cdot Y_2 & X_1 \cdot Z_2 \\ Y_1 \cdot X_2 & Y_1 \cdot Y_2 & Y_1 \cdot Z_2 \\ Z_1 \cdot X_2 & Z_1 \cdot Y_2 & Z_1 \cdot Z_2 \end{bmatrix} \end{aligned}$$

These formulas can be used to calculate the coefficients of a vector with respect to a general basis. Recall first that if E_1, E_2 is an orthonormal basis for $\mathbb{R}^2$, then

$$\begin{aligned} X &= (X \cdot E_1) E_1 + (X \cdot E_2) E_2 \\ &= \begin{bmatrix} E_1 & E_2 \end{bmatrix} \begin{bmatrix} E_1 & E_2 \end{bmatrix}^t X \end{aligned}$$

So the coefficients for X are simply the dot products with the basis elements. More generally we have

Theorem 1 *Let $U.V$ be a basis for $\mathbb{R}^2$, then*

$$\begin{aligned} X &= \begin{bmatrix} U & V \end{bmatrix} \left(\begin{bmatrix} U & V \end{bmatrix}^t \begin{bmatrix} U & V \end{bmatrix} \right)^{-1} \begin{bmatrix} U & V \end{bmatrix}^t X \\ &= \begin{bmatrix} U & V \end{bmatrix} \left(\begin{bmatrix} U & V \end{bmatrix}^t \begin{bmatrix} U & V \end{bmatrix} \right)^{-1} \begin{bmatrix} U \cdot X \\ V \cdot X \end{bmatrix} \end{aligned}$$

Proof. First write

$$X = \begin{bmatrix} U & V \end{bmatrix} \begin{bmatrix} X^u \\ X^v \end{bmatrix}$$

The goal is to find a formula for the coefficients X^u, X^v in terms of the dot products $X \cdot U, X \cdot V$. To that end we notice

$$\begin{aligned} \begin{bmatrix} U \cdot X \\ V \cdot X \end{bmatrix} &= \begin{bmatrix} U & V \end{bmatrix}^t X \\ &= \begin{bmatrix} U & V \end{bmatrix}^t \begin{bmatrix} U & V \end{bmatrix} \begin{bmatrix} X^u \\ X^v \end{bmatrix} \end{aligned}$$

Showing directly that

$$\begin{bmatrix} X^u \\ X^v \end{bmatrix} = \left(\begin{bmatrix} U & V \end{bmatrix}^t \begin{bmatrix} U & V \end{bmatrix} \right)^{-1} \begin{bmatrix} U \cdot X \\ V \cdot X \end{bmatrix}$$

and conseqeuntly

$$X = \begin{bmatrix} U & V \end{bmatrix} \left(\begin{bmatrix} U & V \end{bmatrix}^t \begin{bmatrix} U & V \end{bmatrix} \right)^{-1} \begin{bmatrix} U \cdot X \\ V \cdot X \end{bmatrix}$$

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There is a similar formula in $\mathbb{R}^3$ which is a bit longer. In pratice we shall only need it in the case where the third basis vector is perpendicular to the first two. Also note that if U, V are orthonormal then

$$\begin{bmatrix} U & V \end{bmatrix}^t \begin{bmatrix} U & V \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and we recover the standard formula for the expansion of a vector in an orthonormal basis.

Finally we mention the triple product formula

$$\begin{aligned} \det \begin{bmatrix} X & Y & Z \end{bmatrix} &= X \cdot (Y \times Z) \\ &= X^t (Y \times Z) \end{aligned}$$

2 Vector Calculus

2.1 Chain Rule

$$\frac{d\left(V\circ c\right)}{dt}=\left[\begin{array}{ccc}\frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} & \frac{\partial V}{\partial z}\end{array}\right]\left[\frac{dc}{dt}\right]$$

2.2 Directional Derivatives

If h is a function on $\mathbb{R}^3$ and $X=(P,Q,R)$ then

$$\begin{aligned}D_X h &= P\frac{\partial h}{\partial x}+Q\frac{\partial h}{\partial y}+R\frac{\partial h}{\partial z}\\ &= (\nabla h)\cdot X\\ &= [\nabla h]^t\,[X]\\ &= \left[\begin{array}{ccc}\frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} & \frac{\partial h}{\partial z}\end{array}\right][X]\end{aligned}$$

and for a vector field V we get

$$D_X V=\left[\begin{array}{ccc}\frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} & \frac{\partial V}{\partial z}\end{array}\right][X].$$

We can also calculate directional derivatives by selecting any curve such that $\dot{c}\left(0\right)=X$. Along the curve the chain rule says:

$$\frac{d\left(V\circ c\right)}{dt}=\left[\begin{array}{ccc}\frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} & \frac{\partial V}{\partial z}\end{array}\right]\left[\frac{dc}{dt}\right]=D_{\dot{c}}V$$

Thus

$$D_X V=\frac{d\left(V\circ c\right)}{dt}\left(0\right)$$

3 Curves

Local theory. arc length. That arclength is a good measure for the length of a curve requires some justification.

Theorem 2 *The straight line is the shortest curve between any two points in Euclidean space.*

Proof. We shall give two almost identical proofs. Without loss of generality we assume that we have a curve $c\left(t\right):[a,b]\to\mathbb{R}^k$ where $c\left(a\right)=0$, and $c\left(b\right)=p$. We wish to show that $L\left(c\right)\geq|p|$. To that end we select a unit vector field X which is also a gradient field $X=\nabla f$. Two natural choices are possible: For the first simply let $f\left(x\right)=x\cdot\frac{p}{|p|}$, for the second $f\left(x\right)=|x|$. In the first case the gradient is simply a parallel field and defined everywhere, in the second case we obtain the radial field which is not defined at the origin. When using the second field we need to restrict the domain of the curve to $[a_0,b]$ such that $c\left(a_0\right)=0$ but $c\left(t\right)\neq0$ for $t>a_0$. This is clearly possible as the set of points where $c\left(t\right)=0$ is a closed subset of $[a,b]$, so a_0 is just the maximum value where c vanishes.

This allows us to perform the following calculation using Cauchy-Schwarz, the chain rule, and the fundamental theorem of calculus. When we are in the second case the intergrals are possibly improper at $t = a_0$, but clearly turn out to be perfectly well defined since the integrand has a continuous limit as t approaches a_0

$$\begin{aligned}
L(c) &= \int_{a_0}^{b} |\dot{c}|\, dt \\
&= \int_{a_0}^{b} |\dot{c}|\, |\nabla f|\, dt \\
&\geq \int_{a_0}^{b} |\dot{c} \cdot \nabla f|\, dt \\
&= \int_{a_0}^{b} \left| \frac{d(f \circ c)}{dt} \right| dt \\
&\geq \left| \int_{a_0}^{b} \frac{d(f \circ c)}{dt} dt \right| \\
&= |f(c(b)) - f(c(a_0))| \\
&= |f(x) - f(0)| \\
&= |f(x)| \\
&= |p|
\end{aligned}$$

We can even go backwards and check what happens when $L(c) = |p|$. It appears that we must have equality in two places where we had inequality. Thus we have $\frac{d(f \circ c)}{dt} \geq 0$ everywhere and $\dot{c}$ is proportional to ∇f everywhere. This implies that c is a possibly singular reparametrization of the straight line from 0 to p. ■

Discuss, parametrized curves, implicitly given curves (level sets), integral curves of a vector field, orthogonal curves, integral curves to second order system.

4 General Frames

We shall now consider the general problem of taking derivatives of a basis $U(t), V(t)$ that depends on t, and veiwed as a choice of basis at $c(t)$. Given $U(t)$, a natural choice for $V(t)$ would be the unit vector orthogonal to $U(t)$. Also we shall usually use $U(t) = \dot{c}(t)$ or $U(t) = T(t)$. The goal is to identify the matrix $[D]$ that appears in

$$\frac{d}{dt} \begin{bmatrix} U & V \end{bmatrix} = \begin{bmatrix} \frac{d}{dt} U & \frac{d}{dt} V \end{bmatrix} = \begin{bmatrix} U & V \end{bmatrix} [D]$$

We know a complicated formula

$$[D] = \left(\begin{bmatrix} U & V \end{bmatrix}^t \begin{bmatrix} U & V \end{bmatrix} \right)^{-1} \begin{bmatrix} U & V \end{bmatrix}^t \begin{bmatrix} \frac{d}{dt} U & \frac{d}{dt} V \end{bmatrix}$$

which simplifies to

Theorem 3 *Let $U(t), V(t)$ be an orthonormal frame that depends on a parameter t, then*

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} U & V \end{bmatrix} &= \begin{bmatrix} U & V \end{bmatrix}\begin{bmatrix} 0 & \lambda \\ -\lambda & 0 \end{bmatrix}, \\ \lambda &= U \cdot \frac{d}{dt}V = -V \cdot \frac{d}{dt}U \end{aligned}$$

or

$$\begin{aligned} \frac{d}{dt}U &= \lambda V, \\ \frac{d}{dt}V &= -\lambda U \end{aligned}$$

Proof. We use that

$$\begin{bmatrix} U & V \end{bmatrix}^t \begin{bmatrix} U & V \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and the derivative of this

$$\begin{aligned} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} &= \begin{bmatrix} \frac{d}{dt}U & \frac{d}{dt}V \end{bmatrix}^t \begin{bmatrix} U & V \end{bmatrix} + \begin{bmatrix} U & V \end{bmatrix}^t \begin{bmatrix} \frac{d}{dt}U & \frac{d}{dt}V \end{bmatrix} \\ &= \begin{bmatrix} \left(\frac{d}{dt}U\right) \cdot U & \left(\frac{d}{dt}U\right) \cdot V \\ \left(\frac{d}{dt}V\right) \cdot U & \left(\frac{d}{dt}V\right) \cdot V \end{bmatrix} + \begin{bmatrix} U \cdot \frac{d}{dt}U & U \cdot \frac{d}{dt}V \\ V \cdot \frac{d}{dt}U & V \cdot \frac{d}{dt}V \end{bmatrix} \end{aligned}$$

Showing that

$$\begin{aligned} \left(\frac{d}{dt}U\right) \cdot U &= 0 = \left(\frac{d}{dt}V\right) \cdot V, \\ \left(\frac{d}{dt}V\right) \cdot U &= -V \cdot \frac{d}{dt}U \end{aligned}$$

Our formula for $[D]$ then becomes

$$\begin{aligned} [D] &= \begin{bmatrix} U & V \end{bmatrix}^t \begin{bmatrix} \frac{d}{dt}U & \frac{d}{dt}V \end{bmatrix} \\ &= \begin{bmatrix} U \cdot \frac{d}{dt}U & U \cdot \frac{d}{dt}V \\ V \cdot \frac{d}{dt}U & V \cdot \frac{d}{dt}V \end{bmatrix} \\ &= \begin{bmatrix} 0 & \lambda \\ -\lambda & 0 \end{bmatrix} \end{aligned}$$

■

Occasionally we need one more derivative

$$\begin{aligned} \frac{d^2}{dt^2}\begin{bmatrix} U & V \end{bmatrix} &= \begin{bmatrix} U & V \end{bmatrix}\begin{bmatrix} -\lambda^2 & \frac{d\lambda}{dt} \\ -\frac{d\lambda}{dt} & -\lambda^2 \end{bmatrix}, \\ \frac{d^2U}{dt^2} &= -\lambda^2 U - \frac{d\lambda}{dt}V, \\ \frac{d^2V}{dt^2} &= \frac{d\lambda}{dt}U - \lambda^2 V. \end{aligned}$$

5 Global stuff

rotation index with tangent/normal circular image. convex curves. ovals. Isoperimetric.

6 Space Curves

Serret-Frenet. Observe that there are no relations between curvature and torsion. Generalized helices and other curiosities.

7 Surfaces

We define a parametrized surface as a function $\boldsymbol{x}(u,v) : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ where $\frac{\partial \boldsymbol{x}}{\partial u}$ and $\frac{\partial \boldsymbol{x}}{\partial v}$ are linearly independent. For parametrized surfaces we generally do not worry about self inetersections or other topological pathologies. This is just as with curves and allows us a great deal of flexibility. When we need to worry about these issues, or rather we wish to avoid them, then we resort to the more restrictive class of surfaces that comes from the next two general constructions.

A special case is the Monge patch

Implicitly given surfaces as level sets.

A surface $M \subset \mathbb{R}^3$ is then a subset which is locally represented as a graph over two coordinates. Note that a parametrized surface might not be a surface in this sense if it intersects itself or otherwise gets arbitrarily close to itself.

The tangent space

$$T_pM = \operatorname{span}\left\{\frac{\partial \boldsymbol{x}}{\partial u}, \frac{\partial \boldsymbol{x}}{\partial v}\right\},$$

and normal space

$$N_pM = (T_pM)^{\perp}$$

Proposition 4 *Both tangent and normal spaces are subspaces that do not depend on a choice of parametrization.*

Proof. This would seem intuitively clear, just as with curves, where the tangent line does not depend on parametrizations. For cuves it boils down to the simple fact that velocities for different parametrizations are proportional and hence define the same tangent lines. With surfaces something similar happens, but it is a bit more involved. Suppose we have two different parametrizations of the same surface:

$$\boldsymbol{x}(s,t) = \boldsymbol{x}(u,v)$$

This tells us that the parameters are functions of each other

$$\begin{aligned} u &= u(s,t), v = v(s,t) \\ s &= s(u,v), t = t(u,v) \end{aligned}$$

The chain rule then gives us

$$\frac{\partial \boldsymbol{x}}{\partial u} = \frac{\partial \boldsymbol{x}}{\partial s}\frac{\partial s}{\partial u} + \frac{\partial \boldsymbol{x}}{\partial t}\frac{\partial t}{\partial u} \in \operatorname{span}\left\{\frac{\partial \boldsymbol{x}}{\partial s}, \frac{\partial \boldsymbol{x}}{\partial t}\right\}$$

similarly

$$\frac{\partial \boldsymbol{x}}{\partial v} \in \operatorname{span}\left\{\frac{\partial \boldsymbol{x}}{\partial s}, \frac{\partial \boldsymbol{x}}{\partial t}\right\}$$

and in the other direction

$$\frac{\partial \boldsymbol{x}}{\partial s}, \frac{\partial \boldsymbol{x}}{\partial t} \in \operatorname{span}\left\{\frac{\partial \boldsymbol{x}}{\partial u}, \frac{\partial \boldsymbol{x}}{\partial v}\right\}$$

This shows that a a fixed point p on a surface the tangent space does not depend on how the surface is parametrized. The normal space is then also well defined. ■

It is often useful to find coordinates suited to a particular situation. Most often this entails finding parameters so that $\frac{\partial \boldsymbol{x}}{\partial u}$ and $\frac{\partial \boldsymbol{x}}{\partial v}$ are proportional to some fixed directions.

Theorem 5 *Assume that we have linearly independent tangent vector fields X, Y defined on a surface M. Then it is possible to find a parametrization $\boldsymbol{x}(u, v)$ in a neighborhood of any point such that $\frac{\partial \boldsymbol{x}}{\partial u}$ is proportional to X and $\frac{\partial \boldsymbol{x}}{\partial v}$ is proportional to Y.*

Proof. The vector fields have integral curves forming a net on the surface. Apparently the goal is to reparametrize the curves in this net in some fashion. The difficulty lies in ensuring that the levels where u is constant correspond to the v-curves, and vice versa. We proceed as with a classical construction of Cartesian coordinates. Select a point p and let the u-axis be the integral curve for X through p, similarily set the v-axis be the integral curve for Y through p. Both of these curves retain the parametrizations that make them integral curves for X and Y. Thus p will naturally correspond to $(u, v) = (0, 0)$. We now wish to assign (u, v) coordinates to a point q near p. There are also unique integral curves for X and Y through q. These will be our way of projecting onto the chosen axes and will in this way yield the desired coordinates. Specifically $u(q)$ is the parameter where the integral curve for Y through q intersecs the u-axis, and similarly with $v(q)$. In general integral curves can intersect axes in several places or might not intersect them at all. However, a continuity argument offers some justification when we consider that the axes themselves are the proper integral curves for the qs that lie on these axes and so q sufficiently close to both axes should have a well defined set of coordinates. We also note that as the projection happens along integral curves we have ensured that coordinate curves are simply reparametrizations of integral curves. To completely justify this construction we need to know quite a bit about the existence, uniqueness and smoothness of solutions to differential equations and the inverse function theorem also comes in handy. ■

Excercise: A *generalized cylinder* is determined by a planar regular curve and a vector not in the same plane. Construct a natural parametrization and show that it gives a parametrized surface. What if the planar curve is given by an equation and you also want the surface to be given by an equation?

Excercise: A *generalized cone* is generated by a planar regular curve and a point not in that plane. Construct a natural parametrization and determine where it yields a parametrized surface. What if the planar curve is given by an equation and you also want the surface to be given by an equation?

Excercise: A *ruled surface* is given by a parametrization of the form

$$\boldsymbol{x}(s,t) = \alpha(s) + t\beta(s)$$

It is evidently a surface that is a union of lines (rulers). Give conditions on α, β and the parameter t that guarantee we get a parametrized surface. A special case occures when α is unit speed and $\beta = \alpha'$. These are also called *tangent developables*.

Excercise: A *surface of revolution* is determined by a planar regular curve and a line that is never perpendicular to the tangent vectors of the curve. The surface is generated by rotating the curve around the line. Construct a natural parametrization and show that it is a parametrized surface. What if the planar curve is given by an equation and you also want the surface to be given by an equation?

Excercise: Many classical surfaces are of the form

$$F(x,y,z) = ax^2 + by^2 + cz^2 + dx + ey + fz + g = 0$$

Give conditions on the coefficients such that it is generates a surface ($g = 0$ takes special care). Under what conditions does it become a surface of revolution around the z-axis? Under what conditions does it become a cylinder or cone? Why are these elliptic when $abc > 0$ and hyperbolic when $abc < 0$? When $abc \neq 0$ rewritte it in the form

$$F(x,y,z) = a(x-x_0)^2 + b(y-y_0)^2 + c(z-z_0)^2 + h = 0$$

8 The Abstract Framework

As with curves, parametrized surfaces can have intersections and other nasty complications that do not come up with the other three cases. Nevertheless it is usually easier to develop formulas for parametrized surfaces.

For a parametrized surface $\boldsymbol{x}(u,v)$ we have the velocities of the coordinate vector fields

$$\frac{\partial \boldsymbol{x}}{\partial u}, \frac{\partial \boldsymbol{x}}{\partial v}$$

While these can be normalized to be unit vectors we can't guarantee that they are orthogonal. Nor can we find parameters that make the coordinate fields orthonormal. We shall see that there are geometric obstructions to finding such parametrizations.

Before discussing general surfaces it might be instructive to see what happens if $\boldsymbol{x}(u,v)$ is simply a reparametrization of the plane. Thus $\frac{\partial \boldsymbol{x}}{\partial u}, \frac{\partial \boldsymbol{x}}{\partial v}$ form a basis at each point $\boldsymbol{x}$. Taking partial derivatives of these fields give us

$$\begin{aligned}
\frac{\partial}{\partial u}\begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{bmatrix} &= \begin{bmatrix} \frac{\partial^2 \boldsymbol{x}}{\partial u^2} & \frac{\partial^2 \boldsymbol{x}}{\partial u \partial v} \end{bmatrix} = \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{bmatrix} [\Gamma_u], \\
\frac{\partial}{\partial v}\begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{bmatrix} &= \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial v \partial u} & \frac{\partial^2 \boldsymbol{x}}{\partial v^2} \end{bmatrix} = \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{bmatrix} [\Gamma_v]
\end{aligned}$$

or in condensed form

$$\frac{\partial}{\partial w}\left[\begin{array}{cc}\frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v}\end{array}\right]=\left[\begin{array}{cc}\frac{\partial^2 \boldsymbol{x}}{\partial w \partial u} & \frac{\partial^2 \boldsymbol{x}}{\partial w \partial v}\end{array}\right]=\left[\begin{array}{cc}\frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v}\end{array}\right][\Gamma_w]\,, w=u,v$$

The matrices $[\Gamma_w]$ tell us how the tangent fields change with respect to themselves. A good example comes from considering polar coordinates $\boldsymbol{x}(r,\theta)=(r\cos\theta, r\sin\theta)$

$$\begin{aligned}
\frac{\partial \boldsymbol{x}}{\partial r} &= \begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}, \frac{\partial \boldsymbol{x}}{\partial \theta}=\begin{bmatrix}-r\sin\theta\\ r\cos\theta\end{bmatrix}\\
\frac{\partial^2 \boldsymbol{x}}{\partial r\partial\theta} &= \frac{\partial^2 \boldsymbol{x}}{\partial\theta\partial r}=\begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}, \frac{\partial^2 \boldsymbol{x}}{\partial r^2}=0, \frac{\partial^2 \boldsymbol{x}}{\partial\theta^2}=\begin{bmatrix}-r\cos\theta\\ -r\sin\theta\end{bmatrix}
\end{aligned}$$

so

$$\begin{aligned}
\frac{\partial}{\partial r}\left[\begin{array}{cc}\frac{\partial \boldsymbol{x}}{\partial r} & \frac{\partial \boldsymbol{x}}{\partial \theta}\end{array}\right] &= \left[\begin{array}{cc}\frac{\partial^2 \boldsymbol{x}}{\partial r\partial r} & \frac{\partial^2 \boldsymbol{x}}{\partial r\partial \theta}\end{array}\right]=\left[\begin{array}{cc}\frac{\partial \boldsymbol{x}}{\partial r} & \frac{\partial \boldsymbol{x}}{\partial \theta}\end{array}\right]\begin{bmatrix}0 & 0\\ 0 & \frac{1}{r}\end{bmatrix}\\
\frac{\partial}{\partial \theta}\left[\begin{array}{cc}\frac{\partial \boldsymbol{x}}{\partial r} & \frac{\partial \boldsymbol{x}}{\partial \theta}\end{array}\right] &= \left[\begin{array}{cc}\frac{\partial^2 \boldsymbol{x}}{\partial \theta\partial r} & \frac{\partial^2 \boldsymbol{x}}{\partial \theta\partial \theta}\end{array}\right]=\left[\begin{array}{cc}\frac{\partial \boldsymbol{x}}{\partial r} & \frac{\partial \boldsymbol{x}}{\partial \theta}\end{array}\right]\begin{bmatrix}0 & -r\\ \frac{1}{r} & 0\end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
[\Gamma_r] &= \begin{bmatrix}0 & 0\\ 0 & \frac{1}{r}\end{bmatrix},\\
[\Gamma_\theta] &= \begin{bmatrix}0 & -r\\ \frac{1}{r} & 0\end{bmatrix}
\end{aligned}$$

The key is that only Cartesian coordinates have the property that its coordinate fields are constant. When using general coordinates we are naturally forced to find these quantities. To see why this is consider a curve $c(t)=(r(t)\cos\theta(t), r(t)\sin\theta(t))$ in the plane. It velocity is the naturally given by

$$\dot{c}=\dot{r}\frac{\partial \boldsymbol{x}}{\partial r}+\dot{\theta}\frac{\partial \boldsymbol{x}}{\partial \theta}$$

If we wish to calculate its acceleration then we must compute the derivatives of the coordinate fields. This involves the chain rule as well as the formulas just developed

$$\begin{aligned}
\ddot{c} &= \ddot{r}\frac{\partial \boldsymbol{x}}{\partial r}+\ddot{\theta}\frac{\partial \boldsymbol{x}}{\partial \theta}+\dot{r}\frac{d}{dt}\frac{\partial \boldsymbol{x}}{\partial r}+\dot{\theta}\frac{d}{dt}\frac{\partial \boldsymbol{x}}{\partial \theta}\\
&= \ddot{r}\frac{\partial \boldsymbol{x}}{\partial r}+\ddot{\theta}\frac{\partial \boldsymbol{x}}{\partial \theta}+\dot{r}\left(\frac{dr}{dt}\frac{\partial}{\partial r}+\frac{d\theta}{dt}\frac{\partial}{\partial \theta}\right)\frac{\partial \boldsymbol{x}}{\partial r}+\dot{\theta}\left(\frac{dr}{dt}\frac{\partial}{\partial r}+\frac{d\theta}{dt}\frac{\partial}{\partial \theta}\right)\frac{\partial \boldsymbol{x}}{\partial \theta}\\
&= \ddot{r}\frac{\partial \boldsymbol{x}}{\partial r}+\ddot{\theta}\frac{\partial \boldsymbol{x}}{\partial \theta}+\dot{r}^2\frac{\partial^2 \boldsymbol{x}}{\partial r^2}+2\dot{r}\dot{\theta}\frac{\partial^2 \boldsymbol{x}}{\partial r\partial\theta}+\dot{\theta}^2\frac{\partial^2 \boldsymbol{x}}{\partial \theta^2}\\
&= \ddot{r}\frac{\partial \boldsymbol{x}}{\partial r}+\ddot{\theta}\frac{\partial \boldsymbol{x}}{\partial \theta}+2\dot{r}\dot{\theta}\frac{1}{r}\frac{\partial \boldsymbol{x}}{\partial \theta}-\dot{\theta}^2 r\frac{\partial \boldsymbol{x}}{\partial r}\\
&= \left(\ddot{r}-r\dot{\theta}^2\right)\frac{\partial \boldsymbol{x}}{\partial r}+\left(\ddot{\theta}+\frac{2\dot{r}\dot{\theta}}{r}\right)\frac{\partial \boldsymbol{x}}{\partial \theta}
\end{aligned}$$

Note that $r\dot{\theta}^2$ corresponds to the centrifugal force that you feel when forced to move in a circle. The term $\ddot{\theta} + \frac{2\dot{r}\dot{\theta}}{r}$ is related to Kepler's second law under a central force field. In this context that simply means that

$$\ddot{\theta} + \frac{2\dot{r}\dot{\theta}}{r} = 0$$

if the force and hence acceleration is radial. This in turn implies that $r^2\dot{\theta}$ is constant as Kepler's law states.

The general goal will be to develop a similar set of ideas for surfaces and in addition to find other ways of calculating $[\Gamma_w]$ that depend on the geometry of the tangent fields.

Before generalizing we make another rather startling observation. Taking one more derivative we obtain

$$\begin{aligned}
\frac{\partial^2}{\partial w_2 \partial w_1} \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{bmatrix} &= \frac{\partial}{\partial w_2} \left(\begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{bmatrix} [\Gamma_{w_1}] \right) \\
&= \left(\frac{\partial}{\partial w_2} \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{bmatrix} \right) [\Gamma_{w_1}] + \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{bmatrix} \left[\frac{\partial \Gamma_{w_1}}{\partial w_2} \right] \\
&= \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{bmatrix} [\Gamma_{w_2}] [\Gamma_{w_1}] + \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{bmatrix} \left[\frac{\partial \Gamma_{w_1}}{\partial w_2} \right] \\
&= \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{bmatrix} \left([\Gamma_{w_2}] [\Gamma_{w_1}] + \left[\frac{\partial \Gamma_{w_1}}{\partial w_2} \right] \right)
\end{aligned}$$

Switching the order of the derivatives should not change the outcome,

$$\frac{\partial^2}{\partial w_1 \partial w_2} \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{bmatrix} \left([\Gamma_{w_1}] [\Gamma_{w_2}] + \left[\frac{\partial \Gamma_{w_2}}{\partial w_1} \right] \right)$$

but it does look different when we use $w_1 = u$ and $w_2 = v$. Therefore we can conclude that

$$[\Gamma_v] [\Gamma_u] + \left[\frac{\partial \Gamma_u}{\partial v} \right] = [\Gamma_u] [\Gamma_v] + \left[\frac{\partial \Gamma_v}{\partial u} \right]$$

or

$$\left[\frac{\partial \Gamma_v}{\partial u} \right] - \left[\frac{\partial \Gamma_u}{\partial v} \right] + [\Gamma_u] [\Gamma_v] - [\Gamma_v] [\Gamma_u] = 0.$$

For polar coordinates this can be verified directly:

$$\begin{aligned}
\left[\frac{\partial \Gamma_r}{\partial \theta} \right] - \left[\frac{\partial \Gamma_\theta}{\partial r} \right] &= 0 - \begin{bmatrix} 0 & -1 \\ -\frac{1}{r^2} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{1}{r^2} & 0 \end{bmatrix} \\
[\Gamma_r] [\Gamma_\theta] - [\Gamma_\theta] [\Gamma_r] &= \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{r} \end{bmatrix} \begin{bmatrix} 0 & -r \\ \frac{1}{r} & 0 \end{bmatrix} - \begin{bmatrix} 0 & -r \\ \frac{1}{r} & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{r} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 \\ \frac{1}{r^2} & 0 \end{bmatrix}
\end{aligned}$$

This means that the two matrices of functions $[\Gamma_u], [\Gamma_v]$ have some nontrivial relations between them that are not evident from the definition.

For a surface $x(u,v)$ in $\mathbb{R}^3$ we add to the tangent vectors the normal

$$n(u,v) = \frac{\frac{\partial \boldsymbol{x}}{\partial u} \times \frac{\partial \boldsymbol{x}}{\partial v}}{\left| \frac{\partial \boldsymbol{x}}{\partial u} \times \frac{\partial \boldsymbol{x}}{\partial v} \right|}$$

in order to get a basis. While n does depend on the parametrizations we note that as it is normal to a plane in $\mathbb{R}^3$ there are in fact only two choices $\pm n$, just as with planar curves.

This means we shall consider frames $\begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} & n \end{bmatrix}$ and derivatives of such frames

$$\frac{\partial}{\partial w} \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} & n \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 \boldsymbol{x}}{\partial w \partial u} & \frac{\partial^2 \boldsymbol{x}}{\partial w \partial v} & \frac{\partial n}{\partial w} \end{bmatrix} = \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} & n \end{bmatrix} [D_w]$$

where w can be either u or v.

The entries of D_w are divided up into parts. The first depends only on tangential information, the first two rows and columns, and corresponds to the $[\Gamma_w]$ that we defined in the plane using general coordinates. The second depends on normal information, the third row and column. Since n is a unit vector the 33 entry actually vanishes:

$$0 = \frac{\partial |n|^2}{\partial w} = 2n \cdot \frac{\partial n}{\partial w}$$

showing that $\frac{\partial n}{\partial w}$ lies in the tangent space and hence does not have a normal component.

As before we have

$$\frac{\partial^2}{\partial w_1 \partial w_2} \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} & n \end{bmatrix} = \frac{\partial^2}{\partial w_2 \partial w_1} \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} & n \end{bmatrix}$$

In particular,

$$[D_u][D_v] + \left[\frac{\partial D_v}{\partial u}\right] = [D_v][D_u] + \left[\frac{\partial D_u}{\partial v}\right]$$

or

$$\left[\frac{\partial D_v}{\partial u}\right] - \left[\frac{\partial D_u}{\partial v}\right] + [D_u][D_v] - [D_v][D_u] = 0$$

As we shall see, other interesting features emerge when we try to restrict attention to the tangential and normal parts of these matrices.

Elie Cartan developed an approach that uses orthonormal bases, but he clearly had to give up on the idea of using coordinate vector fields. Thus he chose an orthonormal frame E_1, E_2, E_3 along part of the surface with the property that $E_3 = n$ is normal to the surface, and consequently E_1, E_2 form an orthonormal basis for the tangent space. The goal is again to take derivatives. For that purpose we can still use parameters

$$\frac{\partial}{\partial w} \begin{bmatrix} E_1 & E_2 & E_3 \end{bmatrix} = \begin{bmatrix} \frac{\partial E_1}{\partial w} & \frac{\partial E_2}{\partial w} & \frac{\partial E_3}{\partial w} \end{bmatrix} = \begin{bmatrix} E_1 & E_2 & E_3 \end{bmatrix} [D_w]$$

The first observation is that $[D_w]$ is skew-symmetric since we used an orthonormal basis:

$$\begin{bmatrix} E_1 & E_2 & E_3 \end{bmatrix}^t \begin{bmatrix} E_1 & E_2 & E_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so

$$\begin{aligned}
0 &= \frac{\partial}{\partial w}\left(\begin{bmatrix} E_1 & E_2 & E_3 \end{bmatrix}^t \begin{bmatrix} E_1 & E_2 & E_3 \end{bmatrix}\right) \\
&= \left(\frac{\partial}{\partial w}\begin{bmatrix} E_1 & E_2 & E_3 \end{bmatrix}\right)^t \begin{bmatrix} E_1 & E_2 & E_3 \end{bmatrix} \\
&\quad + \begin{bmatrix} E_1 & E_2 & E_3 \end{bmatrix}^t \frac{\partial}{\partial w}\begin{bmatrix} E_1 & E_2 & E_3 \end{bmatrix} \\
&= \left(\begin{bmatrix} E_1 & E_2 & E_3 \end{bmatrix}[D_w]\right)^t \begin{bmatrix} E_1 & E_2 & E_3 \end{bmatrix} \\
&\quad + \begin{bmatrix} E_1 & E_2 & E_3 \end{bmatrix}^t \begin{bmatrix} E_1 & E_2 & E_3 \end{bmatrix}[D_w] \\
&= [D_w]^t + [D_w]
\end{aligned}$$

In particular, there will only be 3 entries to sort out. This is a significant reduction from what we had to deal with above. What is more, the entries can easily be found by computing the dot products

$$E_i \cdot \frac{\partial E_j}{\partial w}$$

This is also in sharp contrast to what happens in the above situation as we shall see. Taking one more derivative will again yield a formula

$$\left[\frac{\partial D_{w_2}}{\partial w_1}\right] - \left[\frac{\partial D_{w_1}}{\partial w_2}\right] = [D_{w_2}]\,[D_{w_1}] - [D_{w_1}]\,[D_{w_2}]$$

where both sides are skew symmetric.

Given the simplicity of using orthonormal frames it is perhaps puzzling why one would bother developing the more cumbersome approach that uses coordinate fields. The answer lies, as with curves, in the unfortunate fact that it is often easier to find coordinate fields than orthonormal bases that are easy to work with. Monge patches are prime examples. For specific examples and many theoretical developments, however, Cartan's approach has some advantages.

9 The First Fundamental Form

Let $\boldsymbol{x}(u,v) : U \to \mathbb{R}^3$ be a parametrized surface. At each point of this surface we get a basis

$$\begin{aligned}
&\frac{\partial \boldsymbol{x}}{\partial u}(u,v), \\
&\frac{\partial \boldsymbol{x}}{\partial v}(u,v), \\
n(u,v) \quad = \quad &\frac{\frac{\partial \boldsymbol{x}}{\partial u} \times \frac{\partial \boldsymbol{x}}{\partial v}}{\left|\frac{\partial \boldsymbol{x}}{\partial u} \times \frac{\partial \boldsymbol{x}}{\partial v}\right|}
\end{aligned}$$

These vectors are again parametrized by u, v. The first two vectors are tangent to the surface and give us an unnormalized version of the tangent vector for a curve, while the third is the normal and is naturally normalized just as the normal vector is for a curve.

One of the issues that make surface theory more difficult than curve theory is that there is no canonical parametrization along the lines of the arclength parametrization for curves.

The *first fundamental form* is the symmetric positive definite form that comes from the matrix

$$\begin{aligned}
[\mathrm{I}] &= \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{bmatrix}^t \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} \cdot \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial u} \cdot \frac{\partial \boldsymbol{x}}{\partial v} \\ \frac{\partial \boldsymbol{x}}{\partial v} \cdot \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \cdot \frac{\partial \boldsymbol{x}}{\partial v} \end{bmatrix} \\
&= \begin{bmatrix} g_{uu} & g_{uv} \\ g_{vu} & g_{vv} \end{bmatrix}
\end{aligned}$$

For a curve the analogous term would simply be the square of the speed

$$\left(\frac{d\gamma}{dt}\right)^t \frac{d\gamma}{dt} = \frac{d\gamma}{dt} \cdot \frac{d\gamma}{dt}.$$

This form dictates how one computes dot products of vectors tangent to the surface assuming they are expanded according to the basis $\frac{\partial \boldsymbol{x}}{\partial u}, \frac{\partial \boldsymbol{x}}{\partial v}$

$$\begin{aligned}
X &= X^u \frac{\partial \boldsymbol{x}}{\partial u} + X^v \frac{\partial \boldsymbol{x}}{\partial v} = \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{bmatrix} \begin{bmatrix} X^u \\ X^v \end{bmatrix} \\
Y &= Y^u \frac{\partial \boldsymbol{x}}{\partial u} + Y^v \frac{\partial \boldsymbol{x}}{\partial v} = \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{bmatrix} \begin{bmatrix} Y^u \\ Y^v \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\mathrm{I}(X,Y) &= \begin{bmatrix} X^u & X^v \end{bmatrix} \begin{bmatrix} g_{uu} & g_{uv} \\ g_{vu} & g_{vv} \end{bmatrix} \begin{bmatrix} Y^u \\ Y^v \end{bmatrix} \\
&= \begin{bmatrix} X^u & X^v \end{bmatrix} \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{bmatrix}^t \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{bmatrix} \begin{bmatrix} Y^u \\ Y^v \end{bmatrix} \\
&= \left(\begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{bmatrix} \begin{bmatrix} X^u \\ X^v \end{bmatrix} \right)^t \left(\begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{bmatrix} \begin{bmatrix} Y^u \\ Y^v \end{bmatrix} \right) \\
&= X^t Y \\
&= X \cdot Y
\end{aligned}$$

In particular, we see that while the *metric coefficients* $g_{w_1 w_2}$ depend on our parametrization. The dot product $\mathrm{I}(X,Y)$ of two tangent vectors remains the same if we change parameters. Note that I stands for the bilinear form $\mathrm{I}(X,Y)$ which does not depend on parametrizations, while [I] is the matrix representation for a fixed parametrization.

Our first surprising observation is that the normalization factor $\left|\frac{\partial \boldsymbol{x}}{\partial u} \times \frac{\partial \boldsymbol{x}}{\partial v}\right|$ can be computed from $[\mathrm{I}]$.

Lemma 6

$$\left|\frac{\partial \boldsymbol{x}}{\partial u} \times \frac{\partial \boldsymbol{x}}{\partial v}\right|^2 = \det[\mathrm{I}] = g_{uu} g_{vv} - (g_{uv})^2$$

Proof. The proof is a bit more general. Fix two vectors $m, n \in \mathbb{R}^3$. The quantity $|m \times n|$ represents the area of the parallelogram with sides m and n. This area can also be calculated by the height×base formula. If m is the base then we have to find $h\,|m|$. The height can be calculated by the Pythagorean theorem if we know the projection onto m. The projection of n onto m is

$$\frac{(n \cdot m)\, m}{|m|^2}$$

So we have

$$h^2 + \left| \frac{(n \cdot m)\, m}{|m|^2} \right|^2 = |n|^2$$

Isolating h^2 and multiplying my $|m|^2$ yields

$$\begin{aligned} |m \times n|^2 &= h^2\, |m|^2 \\ &= |m|^2 \left(|n|^2 - \left| \frac{(n \cdot m)\, m}{|m|^2} \right|^2 \right) \\ &= |m|^2\, |n|^2 - |m|^2\, \frac{|(n \cdot m)|^2\, |m|^2}{|m|^4} \\ &= (m \cdot m)\,(n \cdot n) - (m \cdot n)^2 \end{aligned}$$

This is what we wanted to prove. ■

The inverse

$$[\mathrm{I}]^{-1} = \begin{bmatrix} g_{uu} & g_{uv} \\ g_{vu} & g_{vv} \end{bmatrix}^{-1} = \begin{bmatrix} g^{uu} & g^{uv} \\ g^{vu} & g^{vv} \end{bmatrix}$$

can be used to find the expansion of a tangent vector by computing its dots products with the basis:

Proposition 7 *If $X \in T_pM$, then*

$$\begin{aligned} X &= \left(g^{uu} \left(X \cdot \frac{\partial \boldsymbol{x}}{\partial u} \right) + g^{uv} \left(X \cdot \frac{\partial \boldsymbol{x}}{\partial v} \right) \right) \frac{\partial \boldsymbol{x}}{\partial u} + \left(g^{vu} \left(X \cdot \frac{\partial \boldsymbol{x}}{\partial u} \right) + g^{vv} \left(X \cdot \frac{\partial \boldsymbol{x}}{\partial v} \right) \right) \frac{\partial \boldsymbol{x}}{\partial v} \\ &= \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{bmatrix} [\mathrm{I}]^{-1} \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{bmatrix}^t X \end{aligned}$$

and more generally for any $Z \in \mathbb{R}^3$

$$\begin{aligned} Z &= \left(g^{uu} \left(Z \cdot \frac{\partial \boldsymbol{x}}{\partial u} \right) + g^{uv} \left(Z \cdot \frac{\partial \boldsymbol{x}}{\partial v} \right) \right) \frac{\partial \boldsymbol{x}}{\partial u} + \left(g^{vu} \left(Z \cdot \frac{\partial \boldsymbol{x}}{\partial u} \right) + g^{vv} \left(Z \cdot \frac{\partial \boldsymbol{x}}{\partial v} \right) \right) \frac{\partial \boldsymbol{x}}{\partial v} + (Z \cdot n)\, n \\ &= \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{bmatrix} [\mathrm{I}]^{-1} \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{bmatrix}^t Z + (Z \cdot n)\, n \end{aligned}$$

Proof. We already suspect that this formula works for $X \in T_pM$ as we worked with it in $\mathbb{R}^2$. Clearly a similar formula holds in $\mathbb{R}^3$ as well. Note that the operation

$$\begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{bmatrix} [\mathrm{I}]^{-1} \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{bmatrix}^t$$

can be applied to any vector in $\mathbb{R}^3$. It simply projects the vector to a vector in the tangent space. For a general vector $Z \in \mathbb{R}^3$ we therefore have to split it up in the tangential component and normal component

$$\begin{aligned} Z &= X + (Z \cdot n)\, n, \\ X &= Z - (Z \cdot n)\, n \end{aligned}$$

and then apply our result to X. ■

Defining the gradient of a function is another important use of the first fundamental form as well as its inverse. Let $f(u, v)$ be viewed as a function on the surface $\boldsymbol{x}(u, v)$. Our definition of the gradient should definitly be so that it conforms with the chain rule for a curve $c(t) = \boldsymbol{x}(u(t), v(t))$. Thus on one hand we want

$$\begin{aligned} \frac{d(f \circ c)}{dt} &= \nabla f \cdot \dot{c} \\ &= \begin{bmatrix} (\nabla f)^u & (\nabla f)^v \end{bmatrix} [\mathrm{I}] \begin{bmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{bmatrix} \end{aligned}$$

while the chain rule also dictates

$$\frac{d(f \circ c)}{dt} = \begin{bmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \end{bmatrix} \begin{bmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{bmatrix}$$

Thus

$$\begin{bmatrix} (\nabla f)^u & (\nabla f)^v \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \end{bmatrix} [\mathrm{I}]^{-1}$$

or

$$\begin{aligned} \nabla f &= \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{bmatrix} \begin{bmatrix} (\nabla f)^u \\ (\nabla f)^v \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{bmatrix} \left(\begin{bmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \end{bmatrix} [\mathrm{I}]^{-1} \right)^t \\ &= \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{bmatrix} [\mathrm{I}]^{-1} \begin{bmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \end{bmatrix}^t \\ &= \left(g^{uu} \frac{\partial f}{\partial u} + g^{uv} \frac{\partial f}{\partial v} \right) \frac{\partial \boldsymbol{x}}{\partial u} + \left(g^{vu} \frac{\partial f}{\partial u} + g^{vv} \frac{\partial f}{\partial v} \right) \frac{\partial \boldsymbol{x}}{\partial v} \end{aligned}$$

In particular, we see that changing coordinates changes the gradiant in such a way that it isn't simply the vector corresponding to the partial derivatives! The other nice feature is that we now have a concept of the gradient that gives a vector field independently of parametrizations. The defining equation

$$\frac{d(f \circ c)}{dt} = \nabla f \cdot \dot{c} = \mathrm{I}(\nabla f, \dot{c})$$

gives an implicit definition of ∇f that makes sense without reference to parametrizations of the surface.

Exercise: If we have a parametrization where

$$[\mathrm{I}] = \begin{bmatrix} 1 & 0 \\ 0 & g_{vv} \end{bmatrix}$$

then the coordinate function $f(u,v) = u$ has

$$\nabla u = \frac{\partial \boldsymbol{x}}{\partial u}.$$

Exercise: Show that it is always possible to find an orthogonal parametrization, i.e., g_{uv} vanishes.

Exercise: Show that if

$$\frac{\partial g_{uu}}{\partial v} = \frac{\partial g_{vv}}{\partial u} = g_{uv} = 0$$

then we can reparametrize u and v separately, i.e., $u = u(s)$ and $v = v(t)$, in such a way that we have *Cartesian coordinates*:

$$\begin{aligned} g_{ss} &= g_{tt} = 1, \\ g_{st} &= 0 \end{aligned}$$

Exercise: Show that if

$$\frac{\partial^2 \boldsymbol{x}}{\partial u \partial v} = 0$$

then

$$\boldsymbol{x}(u,v) = F(u) + G(v)$$

and conclude that we we are in the situation of the previous exercise.

10 The Gauss Formulas

With all of this in mind we are now going to compute the partial derivatives of our basis in both the u and v directions. Since these derivatives might not be tangential we get a formula that looks like

$$\frac{\partial^2 \boldsymbol{x}}{\partial w_1 \partial w_2} = \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{bmatrix} [\mathrm{I}]^{-1} \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{bmatrix}^t \frac{\partial^2 \boldsymbol{x}}{\partial w_1 \partial w_2} + \left(\frac{\partial^2 \boldsymbol{x}}{\partial w_1 \partial w_2} \cdot n \right) n$$

The goal here and in the next section is to show that the tangential part of this formula

$$\Gamma_{w_1 w_2} = \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{bmatrix} [\mathrm{I}]^{-1} \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{bmatrix}^t \frac{\partial^2 \boldsymbol{x}}{\partial w_1 \partial w_2}$$

can be computed directly form the first fundamental form and without knowledge of the second derivatives $\frac{\partial^2 \boldsymbol{x}}{\partial w_1 \partial w_2}$. Note that this is similar to what we did for a reparametrization of the plane.

To accomplish this we need some more notation:

$$\begin{aligned} \Gamma_{w_1 w_2 w} &= \frac{\partial^2 \boldsymbol{x}}{\partial w_1 \partial w_2} \cdot \frac{\partial \boldsymbol{x}}{\partial w} \\ L_{w_1 w_2} &= \frac{\partial^2 \boldsymbol{x}}{\partial w_1 \partial w_2} \cdot n \end{aligned}$$

The first line defines the *Christoffel symbols of the first kind*. The second line the *second fundamental form*

$$\begin{aligned}
\mathrm{II}^{n}(X,Y) &= \begin{bmatrix} X^u & X^v \end{bmatrix} [\mathrm{II}^n] \begin{bmatrix} Y^u \\ Y^v \end{bmatrix} \\
&= \begin{bmatrix} X^u & X^v \end{bmatrix} \begin{bmatrix} L_{uu} & L_{uv} \\ L_{vu} & L_{vv} \end{bmatrix} \begin{bmatrix} Y^u \\ Y^v \end{bmatrix}
\end{aligned}$$

The superscript n refers to the choice of normal and is usually supressed since there are only two choices for the normal $\pm n$. This also tells us that $n\mathrm{II}^n$ is independent of the normal.

To further simplify expressions we also need to do the appropriate multiplication with $g^{w_4w_5}$ to find the coefficients also called the *Christoffel symbols of the second kind*:

$$\begin{aligned}
\Gamma^{w}_{w_1w_2} &= g^{wu}\Gamma_{w_1w_2u} + g^{wv}\Gamma_{w_1w_2v}, \\
\begin{bmatrix} \Gamma^{u}_{w_1w_2} \\ \Gamma^{v}_{w_1w_2} \end{bmatrix} &= \begin{bmatrix} g^{uu} & g^{uv} \\ g^{vu} & g^{vv} \end{bmatrix} \begin{bmatrix} \Gamma_{w_1w_2u} \\ \Gamma_{w_1w_2v} \end{bmatrix} \\
&= [\mathrm{I}]^{-1} \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{bmatrix}^t \frac{\partial^2 \boldsymbol{x}}{\partial w_1 \partial w_2}
\end{aligned}$$

This now gives us the tangential component as

$$\begin{aligned}
\Gamma_{w_1w_2} &= \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{bmatrix} [\mathrm{I}]^{-1} \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{bmatrix}^t \frac{\partial^2 \boldsymbol{x}}{\partial w_1 \partial w_2} \\
&= \Gamma^{u}_{w_1w_2} \frac{\partial \boldsymbol{x}}{\partial u} + \Gamma^{v}_{w_1w_2} \frac{\partial \boldsymbol{x}}{\partial v}
\end{aligned}$$

The second derivatives of $\boldsymbol{x}(u,v)$ can now be expressed as follows in terms of the Christoffel symbols of the second kind and the second fundamental form. These are often called the *Gauss formulas*:

$$\begin{aligned}
\frac{\partial^2 \boldsymbol{x}}{\partial u^2} &= \Gamma^{u}_{uu} \frac{\partial \boldsymbol{x}}{\partial u} + \Gamma^{v}_{uu} \frac{\partial \boldsymbol{x}}{\partial v} + L_{uu} n \\
\frac{\partial^2 \boldsymbol{x}}{\partial u \partial v} &= \Gamma^{u}_{uv} \frac{\partial \boldsymbol{x}}{\partial u} + \Gamma^{v}_{uv} \frac{\partial \boldsymbol{x}}{\partial v} + L_{uv} n = \frac{\partial^2 \boldsymbol{x}}{\partial v \partial u} \\
\frac{\partial^2 \boldsymbol{x}}{\partial v^2} &= \Gamma^{u}_{vv} \frac{\partial \boldsymbol{x}}{\partial u} + \Gamma^{v}_{vv} \frac{\partial \boldsymbol{x}}{\partial v} + L_{vv} n
\end{aligned}$$

or

$$\frac{\partial^2 \boldsymbol{x}}{\partial w_1 \partial w_2} = \Gamma^{u}_{w_1w_2} \frac{\partial \boldsymbol{x}}{\partial u} + \Gamma^{v}_{w_1w_2} \frac{\partial \boldsymbol{x}}{\partial v} + L_{w_1w_2} n$$

or

$$\frac{\partial}{\partial w} \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} & n \end{bmatrix} \begin{bmatrix} \Gamma^{u}_{wu} & \Gamma^{u}_{wv} \\ \Gamma^{v}_{wu} & \Gamma^{v}_{wv} \\ L_{wu} & L_{wv} \end{bmatrix}$$

This means that we have introduced notation for the first two columns in $[D_w]$. We shall wait a bit to deal with the last column.

As we shall see, and indeed already saw when considering polar coordinates in the plane, these formulas are important for defining accelerations of curves. They are however also important for giving a proper definition of the Hessian or second derivative matrix of a function on a surface. This will be explored in an exercise later.

11 Calculating Christoffel Symbols

Next we seek formulas for the Christoffel symbols that involve only the first fundamental form. This shows that they can be computed knowing only the first derivatives of $\boldsymbol{x}(u,v)$ despite the fact that they are defined using the second derivatives!

Proposition 8

$$\begin{aligned}
\Gamma_{uuu} &= \frac{1}{2}\frac{\partial g_{uu}}{\partial u} \\
\Gamma_{uvu} &= \frac{1}{2}\frac{\partial g_{uu}}{\partial v} = \Gamma_{vuu} \\
\Gamma_{vvv} &= \frac{1}{2}\frac{\partial g_{vv}}{\partial v} \\
\Gamma_{uvv} &= \frac{1}{2}\frac{\partial g_{vv}}{\partial u} = \Gamma_{vuv} \\
\Gamma_{uuv} &= \frac{\partial g_{uv}}{\partial u} - \frac{1}{2}\frac{\partial g_{uu}}{\partial v} \\
\Gamma_{vvu} &= \frac{\partial g_{uv}}{\partial v} - \frac{1}{2}\frac{\partial g_{vv}}{\partial u}
\end{aligned}$$

Proof. We select to prove only two of these as the proofs are all similar. We use the product rule just as we did when computing derivatives of dot products for the Frenet-Serret formulas. Note that we use the first calculation to finish off the second calculation.

$$\Gamma_{uvu} = \frac{\partial^2 \boldsymbol{x}}{\partial u \partial v}\cdot\frac{\partial \boldsymbol{x}}{\partial u} = \left(\frac{\partial}{\partial v}\left(\frac{\partial \boldsymbol{x}}{\partial u}\right)\right)\cdot\frac{\partial \boldsymbol{x}}{\partial u} = \frac{1}{2}\frac{\partial}{\partial v}\left(\frac{\partial \boldsymbol{x}}{\partial u}\cdot\frac{\partial \boldsymbol{x}}{\partial u}\right) = \frac{1}{2}\frac{\partial g_{uu}}{\partial v}$$

$$\begin{aligned}
\Gamma_{uuv} &= \frac{\partial^2 \boldsymbol{x}}{\partial u \partial u}\cdot\frac{\partial \boldsymbol{x}}{\partial v} = \left(\frac{\partial}{\partial u}\left(\frac{\partial \boldsymbol{x}}{\partial u}\right)\right)\cdot\frac{\partial \boldsymbol{x}}{\partial v} \\
&= \frac{\partial}{\partial u}\left(\frac{\partial \boldsymbol{x}}{\partial u}\cdot\frac{\partial \boldsymbol{x}}{\partial v}\right) - \left(\frac{\partial \boldsymbol{x}}{\partial u}\cdot\frac{\partial}{\partial u}\frac{\partial \boldsymbol{x}}{\partial v}\right) \\
&= \frac{\partial g_{uv}}{\partial u} - \frac{\partial \boldsymbol{x}}{\partial u}\cdot\frac{\partial^2 \boldsymbol{x}}{\partial u \partial v} \\
&= \frac{\partial g_{uv}}{\partial u} - \frac{1}{2}\frac{\partial g_{uu}}{\partial v}
\end{aligned}$$

■

There is a unified formula for all of these equations. While it unifies it also complicates and is less useful for actual calculations:

$$\Gamma_{w_1 w_2 w} = \frac{1}{2}\left(\frac{\partial g_{w_2 w}}{\partial w_1} + \frac{\partial g_{w_1 w}}{\partial w_2} - \frac{\partial g_{w_1 w_2}}{\partial w}\right)$$

The product rule for derivatives also tells us that

$$\frac{\partial g_{w_1 w_2}}{\partial w} = \Gamma_{w w_2 w_1} + \Gamma_{w w_1 w_2}$$

Note that this formula is now also a direct consequence of our new formulas for the Christoffel symbols in terms of the derivatives of the metric coeffients.

The proposition can also be used to find the Christoffel symbols of the second kind. For example

$$\begin{aligned} \Gamma^u_{uv} &= g^{uu}\Gamma_{uvu} + g^{uv}\Gamma_{uvv} \\ &= \frac{1}{2}\left(g^{uu}\frac{\partial g_{uu}}{\partial v} + g^{uv}\frac{\partial g_{vv}}{\partial u}\right) \end{aligned}$$

While this can't be made simpler as such, it is possible to be a bit more efficient when calculations are done. Specifically we often do calculations in orthogonal coordinates, i.e., $g_{uv} \equiv 0$. In such coordinates

$$\begin{aligned} g^{uv} &= 0 \\ g^{uu} &= (g_{uu})^{-1} \\ g^{vv} &= (g_{vv})^{-1} \end{aligned}$$

$$\begin{aligned} \Gamma_{uuu} &= \frac{1}{2}\frac{\partial g_{uu}}{\partial u} \\ \Gamma_{uvu} &= \frac{1}{2}\frac{\partial g_{uu}}{\partial v} = \Gamma_{vuu} \\ \Gamma_{vvv} &= \frac{1}{2}\frac{\partial g_{vv}}{\partial v} \\ \Gamma_{uvv} &= \frac{1}{2}\frac{\partial g_{vv}}{\partial u} = \Gamma_{vuv} \\ \Gamma_{uuv} &= -\frac{1}{2}\frac{\partial g_{uu}}{\partial v} \\ \Gamma_{vvu} &= -\frac{1}{2}\frac{\partial g_{vv}}{\partial u} \end{aligned}$$

and

$$
\begin{aligned}
\Gamma_{uu}^{u} &= \frac{1}{2}g^{uu}\frac{\partial g_{uu}}{\partial u} = \frac{1}{2}\frac{\partial \ln g_{uu}}{\partial u} \\
\Gamma_{uu}^{v} &= -\frac{1}{2}g^{vv}\frac{\partial g_{uu}}{\partial v} \\
\Gamma_{vv}^{v} &= \frac{1}{2}g^{vv}\frac{\partial g_{vv}}{\partial v} = \frac{1}{2}\frac{\partial \ln g_{vv}}{\partial v} \\
\Gamma_{vv}^{u} &= -\frac{1}{2}g^{uu}\frac{\partial g_{vv}}{\partial u} \\
\Gamma_{uv}^{u} &= \frac{1}{2}g^{uu}\frac{\partial g_{uu}}{\partial v} = \frac{1}{2}\frac{\partial \ln g_{uu}}{\partial v} \\
\Gamma_{uv}^{v} &= \frac{1}{2}g^{vv}\frac{\partial g_{vv}}{\partial u} = \frac{1}{2}\frac{\partial \ln g_{vv}}{\partial u}
\end{aligned}
$$

We often have more specific information. This could be that the metric coefficients only depend on one of the parameters, or that $g_{uu} = 1$. In such circumstances it is quite managable to calculate the Christoffel symbols. What is more, it is always possible to find parametrizations where $g_{uu} \equiv 1$ and $g_{uv} \equiv 0$ as we shall see.

12 Generalized and Abstract Surfaces

It is possible to work with *generalized surfaces* into Euclidean spaces of arbitrary dimension: $\boldsymbol{x}(u,v) : U \to \mathbb{R}^k$ for any $k \geq 2$. What changes is that we no longer have a normal vector n. In fact for $k = 2$ we could just let n be $(0,0,1)$ after letting $\mathbb{R}^2$ be the (x,y) coordinates in space. While for $k \geq 4$ we get a whole family of normal vectors, not unlike what happened for space curves. What all of these surfaces do have in common is that we can define the first fundamental form and with it also the Christoffel symbols of the first and second kind using the formulas in terms of derivatives of g. This leads us to the possibility of an abstract definition of a surface that is independent of a particular map into some coordinate space $\mathbb{R}^k$.

One of the simplest examples of a generalized surface is the flat torus in $\mathbb{R}^4$. It is parametrized by

$$\boldsymbol{x}(u,v) = (\cos u, \sin u, \cos v, \sin v)$$

and its first fundamental form is

$$\mathrm{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

just as we have for Cartesian coordinates in the plane. This is why it is called the flat torus. It is in fact not possible to have a flat torus in $\mathbb{R}^3$.

An *abstract parametrized surface* consists of a domain $U \subset \mathbb{R}^2$ and a first fundamental form

$$\mathrm{I} = \begin{bmatrix} g_{uu} & g_{uv} \\ g_{uv} & g_{vv} \end{bmatrix}$$

that defines inner products of vectors X, Y with the same base point $p \in U$

$$\mathrm{I}(X,Y) = \begin{bmatrix} X^u & X^v \end{bmatrix} \begin{bmatrix} g_{uu} & g_{uv} \\ g_{vu} & g_{vv} \end{bmatrix} \begin{bmatrix} Y^u \\ Y^v \end{bmatrix}$$

where

$$X = (X^u, X^v), Y = (Y^u, Y^v)$$

are the representations of the vectors using the standard (u, v) coordinates on U. Note the first fundamental form consists of three functions and so gives an inner products that varies from point to point. For this to give us an inner product we also have to make sure that it is positive definite:

$$\begin{aligned} 0 &< \mathrm{I}(X, X) \\ &= \begin{bmatrix} X^u & X^v \end{bmatrix} \begin{bmatrix} g_{uu} & g_{uv} \\ g_{uv} & g_{vv} \end{bmatrix} \begin{bmatrix} X^u \\ X^v \end{bmatrix} \\ &= X^u X^u g_{uu} + 2X^u X^v g_{uv} + X^v X^v g_{vv} \end{aligned}$$

Proposition 9 I *is positive definite if and only if* $g_{uu} + g_{vv}$, *and* $g_{uu}g_{vv} - (g_{uv})^2$ *are positive.*

Proof. If I is positive definite, then we see that g_{uu} and g_{vv} are positive by letting $X = (1, 0)$ and $(0, 1)$ resp. Next let $X = \left(\sqrt{g_{vv}}, \pm\sqrt{g_{uu}}\right)$ to get

$$0 < \mathrm{I}(X, X) = 2g_{uu}g_{vv} \pm 2\sqrt{g_{uu}}\sqrt{g_{vv}}g_{uv}$$

Thus we have

$$\pm g_{uv} < \sqrt{g_{uu}}\sqrt{g_{vv}}$$

showing that

$$g_{uu}g_{vv} > (g_{uv})^2 .$$

To check that I is positive definite we have to use that it is symmetric. The characteristic polynomial is

$$\lambda^2 - (g_{uu} + g_{vv})\lambda + g_{uu}g_{vv} - (g_{uv})^2$$

The minimum of this upward pointing parabola is obtained at

$$\lambda = \frac{1}{2}(g_{uu} + g_{vv})$$

and has the value

$$g_{uu}g_{vv} - (g_{uv})^2 - \frac{1}{4}(g_{uu} + g_{vv})^2 = -\frac{1}{4}(g_{uu} - g_{vv})^2 - (g_{uv})^2 < 0$$

Thus there are two real roots. The spectral theorem for symmetric matrices could also have been invoked at this point to establish that the eigenvalues are both real. It is now easy to see that [I] is positive definite if its eigenvalues are positive. Two real numbers have to be positive if their sum and product are both positive. In this case the sum of the eigenvalues is the trace $g_{uu} + g_{vv}$ while the product is the determinant $g_{uu}g_{vv} - (g_{uv})^2$ so our assumptions guarantee that the eigenvalues are positive. ■

There is an interesting example of an abstract surface on the upper half plane defined by $H = \{(u, v) : v > 0\}$, where the metric coefficients are

$$[\mathrm{I}] = \begin{bmatrix} g_{uu} & g_{uv} \\ g_{vu} & g_{vv} \end{bmatrix} = \frac{1}{v^2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

One can show that for each point $p \in H$, there is a small neighborhood U containing p and $\boldsymbol{x}(u,v) : U \to \mathbb{R}^3$ such that

$$\begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{bmatrix}^t \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} \cdot \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial u} \cdot \frac{\partial \boldsymbol{x}}{\partial v} \\ \frac{\partial \boldsymbol{x}}{\partial v} \cdot \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \cdot \frac{\partial \boldsymbol{x}}{\partial v} \end{bmatrix} = \frac{1}{v^2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

In other words we can locally represent the abstract surface as a surface in space. However, a very difficult theorem of Hilbert shows that one cannot represent the entire surface in space, i.e., there is no function $\boldsymbol{x}(u,v) : H \to \mathbb{R}^3$ defined on the entire domain such that

$$\begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{bmatrix}^t \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} \cdot \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial u} \cdot \frac{\partial \boldsymbol{x}}{\partial v} \\ \frac{\partial \boldsymbol{x}}{\partial v} \cdot \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \cdot \frac{\partial \boldsymbol{x}}{\partial v} \end{bmatrix} = \frac{1}{v^2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Janet-Burstin-Cartan showed that if the metric coefficients of an abstract surface are analytic, then one can always locally represent the abstract surface in $\mathbb{R}^3$. Nash showed that any abstract surface can be represented by a map $\boldsymbol{x}(u,v) : U \to \mathbb{R}^k$ on the entire domain, but only at the expense of making k very large. Based in part on Nash's work Greene and Gromov independently showed that one can always locally represent an abstract surface in $\mathbb{R}^5$.

13 Acceleration and Geodesics

We'll now consider curves on a parametrized surface $\boldsymbol{x}(u,v) : U \to \mathbb{R}^3$. The curve is parametrized in U as $(u(t), v(t))$ and becomes a space curve $c(t) = \boldsymbol{x}(u(t), v(t))$ that lies on our parametrized surface.

The velocity is

$$\dot{c} = \frac{dc}{dt} = \frac{d\boldsymbol{x}}{dt} = \frac{\partial \boldsymbol{x}}{\partial u}\frac{du}{dt} + \frac{\partial \boldsymbol{x}}{\partial v}\frac{dv}{dt} = \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{bmatrix} \begin{bmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{bmatrix}$$

Next we calculate the acceleration as if it were a space curve, but using the velocity representation we just gave. Recall that we can decompose any vector into normal and tangential components. For the acceleration this is

$$\ddot{c} = \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{bmatrix} [\mathrm{I}] \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{bmatrix}^t \ddot{c} + (\ddot{c} \cdot n)\, n$$

The goal is to calculate each of these components in terms of $\frac{du}{dt}, \frac{dv}{dt}$ and $\frac{d^2u}{dt^2}, \frac{d^2v}{dt^2}$. This will lead us to another surprising result.

Theorem 10 *The acceleration can be calculated as*

$$\begin{aligned} \ddot{c} &= \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} & n \end{bmatrix} \begin{bmatrix} \frac{d^2u}{dt^2} + \Gamma^u(\dot{c},\dot{c}) \\ \frac{d^2v}{dt^2} + \Gamma^v(\dot{c},\dot{c}) \\ \mathrm{II}(\dot{c},\dot{c}) \end{bmatrix} \\ &= \left(\frac{d^2u}{dt^2} + \Gamma^u(\dot{c},\dot{c})\right)\frac{\partial \boldsymbol{x}}{\partial u} + \left(\frac{d^2v}{dt^2} + \Gamma^v(\dot{c},\dot{c})\right)\frac{\partial \boldsymbol{x}}{\partial v} + n\mathrm{II}(\dot{c},\dot{c}), \end{aligned}$$

where

$$\Gamma^{w}\left(\dot{c},\dot{c}\right)=\sum_{w_1,w_2=u,v}\Gamma^{w}_{w_1w_2}\frac{dw_1}{dt}\frac{dw_2}{dt}=\left[\begin{array}{cc}\frac{du}{dt} & \frac{dv}{dt}\end{array}\right]\left[\begin{array}{cc}\Gamma^{w}_{uu} & \Gamma^{w}_{uv}\\ \Gamma^{w}_{vu} & \Gamma^{w}_{vv}\end{array}\right]\left[\begin{array}{c}\frac{du}{dt}\\ \frac{dv}{dt}\end{array}\right]$$

Proof. We start from the formula for the velocity and take derivatives. This clearly requires us to be able to calculate derivatives of the tangent fields $\frac{\partial \boldsymbol{x}}{\partial u}, \frac{\partial \boldsymbol{x}}{\partial v}$. Fortunately the Gauss formulas tell us how that is done. This leads us to the acceleration as follows

$$\begin{aligned}
\ddot{c} &= \frac{d}{dt}\left(\left[\begin{array}{cc}\frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v}\end{array}\right]\left[\begin{array}{c}\frac{du}{dt}\\ \frac{dv}{dt}\end{array}\right]\right)\\
&= \left[\begin{array}{cc}\frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v}\end{array}\right]\left[\begin{array}{c}\frac{d^2u}{dt^2}\\ \frac{d^2v}{dt^2}\end{array}\right]+\left(\frac{d}{dt}\left[\begin{array}{cc}\frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v}\end{array}\right]\right)\left[\begin{array}{c}\frac{du}{dt}\\ \frac{dv}{dt}\end{array}\right]
\end{aligned}$$

which after the chain rule

$$\frac{d}{dt}=\frac{du}{dt}\frac{\partial}{\partial u}+\frac{dv}{dt}\frac{\partial}{\partial v}$$

becomes

$$\begin{aligned}
\ddot{c} &= \left[\begin{array}{cc}\frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v}\end{array}\right]\left[\begin{array}{c}\frac{d^2u}{dt^2}\\ \frac{d^2v}{dt^2}\end{array}\right]\\
&\quad+\frac{du}{dt}\left(\frac{\partial}{\partial u}\left[\begin{array}{cc}\frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v}\end{array}\right]\right)\left[\begin{array}{c}\frac{du}{dt}\\ \frac{dv}{dt}\end{array}\right]\\
&\quad+\frac{dv}{dt}\left(\frac{\partial}{\partial v}\left[\begin{array}{cc}\frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v}\end{array}\right]\right)\left[\begin{array}{c}\frac{du}{dt}\\ \frac{dv}{dt}\end{array}\right]
\end{aligned}$$

The Gauss formulas help us with the last two terms

$$\begin{aligned}
\left(\frac{\partial}{\partial w}\left[\begin{array}{cc}\frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v}\end{array}\right]\right)\left[\begin{array}{c}\frac{du}{dt}\\ \frac{dv}{dt}\end{array}\right] &= \left[\begin{array}{ccc}\frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} & n\end{array}\right]\left[\begin{array}{cc}\Gamma^{u}_{wu} & \Gamma^{u}_{wv}\\ \Gamma^{v}_{wu} & \Gamma^{v}_{wv}\\ L_{wu} & L_{wv}\end{array}\right]\left[\begin{array}{c}\frac{du}{dt}\\ \frac{dv}{dt}\end{array}\right]\\
&= \frac{\partial \boldsymbol{x}}{\partial u}\left[\begin{array}{cc}\Gamma^{u}_{wu} & \Gamma^{u}_{wv}\end{array}\right]\left[\begin{array}{c}\frac{du}{dt}\\ \frac{dv}{dt}\end{array}\right]\\
&\quad+\frac{\partial \boldsymbol{x}}{\partial v}\left[\begin{array}{cc}\Gamma^{v}_{wu} & \Gamma^{v}_{wv}\end{array}\right]\left[\begin{array}{c}\frac{du}{dt}\\ \frac{dv}{dt}\end{array}\right]\\
&\quad+n\left[\begin{array}{cc}L_{wu} & L_{wv}\end{array}\right]\left[\begin{array}{c}\frac{du}{dt}\\ \frac{dv}{dt}\end{array}\right]
\end{aligned}$$

which after further rearranging allows us to conclude

$$\begin{aligned}
\ddot{c} &= \left[\begin{array}{cc} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{array}\right] \left[\begin{array}{c} \frac{d^2u}{dt^2} \\ \frac{d^2v}{dt^2} \end{array}\right] \\
&\quad + \frac{\partial \boldsymbol{x}}{\partial u} \left[\begin{array}{cc} \frac{du}{dt} & \frac{dv}{dt} \end{array}\right] \left[\begin{array}{cc} \Gamma^u_{uu} & \Gamma^u_{uv} \\ \Gamma^u_{vu} & \Gamma^u_{vv} \end{array}\right] \left[\begin{array}{c} \frac{du}{dt} \\ \frac{dv}{dt} \end{array}\right] \\
&\quad + \frac{\partial \boldsymbol{x}}{\partial v} \left[\begin{array}{cc} \frac{du}{dt} & \frac{dv}{dt} \end{array}\right] \left[\begin{array}{cc} \Gamma^v_{uu} & \Gamma^v_{vu} \\ \Gamma^v_{vu} & \Gamma^v_{vv} \end{array}\right] \left[\begin{array}{c} \frac{du}{dt} \\ \frac{dv}{dt} \end{array}\right] \\
&\quad + n \left[\begin{array}{cc} \frac{du}{dt} & \frac{dv}{dt} \end{array}\right] \left[\begin{array}{cc} L_{uu} & L_{uv} \\ L_{vu} & L_{vv} \end{array}\right] \left[\begin{array}{c} \frac{du}{dt} \\ \frac{dv}{dt} \end{array}\right] \\
&= \left[\begin{array}{ccc} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} & n \end{array}\right] \left[\begin{array}{c} \frac{d^2u}{dt^2} + \Gamma^u(\dot{c},\dot{c}) \\ \frac{d^2v}{dt^2} + \Gamma^v(\dot{c},\dot{c}) \\ \mathrm{II}(\dot{c},\dot{c}) \end{array}\right]
\end{aligned}$$

Alternately the whole calculation could have been done using summations

$$\begin{aligned}
\ddot{c} &= \frac{d^2c}{dt^2} \\
&= \frac{\partial \boldsymbol{x}}{\partial u}\frac{d^2u}{dt^2} + \frac{\partial \boldsymbol{x}}{\partial v}\frac{d^2v}{dt^2} \\
&\quad + \left(\frac{\partial^2 \boldsymbol{x}}{\partial u^2}\frac{du}{dt} + \frac{\partial^2 \boldsymbol{x}}{\partial u \partial v}\frac{dv}{dt}\right)\frac{du}{dt} + \left(\frac{\partial^2 \boldsymbol{x}}{\partial u \partial v}\frac{du}{dt} + \frac{\partial^2 \boldsymbol{x}}{\partial v^2}\frac{dv}{dt}\right)\frac{dv}{dt} \\
&= \frac{\partial \boldsymbol{x}}{\partial u}\frac{d^2u}{dt^2} + \frac{\partial \boldsymbol{x}}{\partial v}\frac{d^2v}{dt^2} + \sum_{w_1,w_2=u,v} \frac{\partial^2 \boldsymbol{x}}{\partial w_1 \partial w_2}\frac{dw_1}{dt}\frac{dw_2}{dt} \\
&= \frac{\partial \boldsymbol{x}}{\partial u}\left(\frac{d^2u}{dt^2} + \sum_{w_1,w_2=u,v} \Gamma^u_{w_1 w_2}\frac{dw_1}{dt}\frac{dw_2}{dt}\right) \\
&\quad + \frac{\partial \boldsymbol{x}}{\partial v}\left(\frac{d^2v}{dt^2} + \sum_{w_1,w_2=u,v} \Gamma^v_{w_1 w_2}\frac{dw_1}{dt}\frac{dw_2}{dt}\right) \\
&\quad + n\left(\sum_{w_1,w_2=u,v} L_{w_1 w_2}\frac{dw_1}{dt}\frac{dw_2}{dt}\right) \\
&= \frac{\partial \boldsymbol{x}}{\partial u}\left(\frac{d^2u}{dt^2} + \Gamma^u(\dot{c},\dot{c})\right) + \frac{\partial \boldsymbol{x}}{\partial v}\left(\frac{d^2v}{dt^2} + \Gamma^v(\dot{c},\dot{c})\right) + n\mathrm{II}(\dot{c},\dot{c})
\end{aligned}$$

■

Note that we have shown

Theorem 11 (Meusnier) *The normal component of the acceleration satisfies*

$$(\ddot{c} \cdot n)\, n = \ddot{c}^{\mathrm{II}} = n\mathrm{II}(\dot{c},\dot{c})$$

In particular two curves with the same velocity at a point have the same normal acceleration components.

The tangential component is more complicated

$$\begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{bmatrix} [\mathrm{I}] \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{bmatrix}^t \ddot{c} = \ddot{c}^{\mathrm{I}} = \frac{\partial \boldsymbol{x}}{\partial u}\left(\frac{d^2u}{dt^2} + \Gamma^u(\dot{c},\dot{c})\right) + \frac{\partial \boldsymbol{x}}{\partial v}\left(\frac{d^2v}{dt^2} + \Gamma^v(\dot{c},\dot{c})\right)$$

But it seems to be a more genuine acceleration as it inlcudes second derivatives. It actually tells us what acceleration we feel on the surface. Note that the tangential acceleration only depends on the first fundamental form.

We say that c is a *geodesic* on the surface if the tangential part of the acceleration vanishes $\ddot{c}^{\mathrm{I}} = 0$, or specifically

$$\begin{aligned} \frac{d^2u}{dt^2} + \Gamma^u(\dot{c},\dot{c}) &= 0, \\ \frac{d^2v}{dt^2} + \Gamma^v(\dot{c},\dot{c}) &= 0. \end{aligned}$$

This is equivalent to saying that $\ddot{c}$ is normal to the surface or that $\ddot{c} = n\mathrm{II}(\dot{c},\dot{c})$.

Proposition 12 *A geodesic has constant speed.*

Proof. Let $c(t)$ be a geodesic. We compute the derivative of the square of the speed:

$$\frac{d}{dt}\mathrm{I}(\dot{c},\dot{c}) = \frac{d}{dt}(\dot{c}\cdot\dot{c}) = 2\ddot{c}\cdot\dot{c} = 2\mathrm{II}(\dot{c},\dot{c})\,n\cdot\dot{c} = 0$$

since n and $\dot{c}$ are perpendicular. Thus c has constant speed. ■

Note that we used the second fundamental form to give a simple proof of this result. It is desirable and indeed possible to give a proof that only refers to the first fundamental form. The key lies in showing that we have a product rule for $\mathrm{I}(\dot{c},\dot{c})$ that works just inside the surface. Since

$$\mathrm{I}\left(\ddot{c}^{\mathrm{I}},\dot{c}\right) = \ddot{c}\cdot\dot{c}$$

this is fairly obvious. The goal would be to do the calculation using only the first fundamental form and that takes quite a bit more work.

Next we address existence of geodesics.

Theorem 13 *Given a point $p = \boldsymbol{x}(u_0,v_0)$ and a tangent vector $V = V^u\frac{\partial \boldsymbol{x}}{\partial u}(u_0,v_0) + V^v\frac{\partial \boldsymbol{x}}{\partial v}(u_0,v_0) \in T_pM$ there is a unique geodesic $c(t) = \boldsymbol{x}(u(t),v(t))$ defined on some small interval $t \in (-\varepsilon,\varepsilon)$ with the inital values*

$$\begin{aligned} c(0) &= p, \\ \dot{c}(0) &= V. \end{aligned}$$

Proof. The existence and uniqueness part is a very general statement about solutions to differential equations. In this case we note that in the (u,v) parameters we must solve a system of second order equations

$$\begin{aligned}
\frac{d^2u}{dt^2} &= -\begin{bmatrix} \frac{du}{dt} & \frac{dv}{dt} \end{bmatrix} \begin{bmatrix} \Gamma^u_{uu} & \Gamma^u_{uv} \\ \Gamma^u_{vu} & \Gamma^u_{vv} \end{bmatrix} \begin{bmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{bmatrix} \\
\frac{d^2v}{dt^2} &= -\begin{bmatrix} \frac{du}{dt} & \frac{dv}{dt} \end{bmatrix} \begin{bmatrix} \Gamma^v_{uu} & \Gamma^v_{uv} \\ \Gamma^v_{vu} & \Gamma^v_{vv} \end{bmatrix} \begin{bmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{bmatrix}
\end{aligned}$$

with the initial values

$$\begin{aligned}
(u(0), v(0)) &= (u_0, v_0), \\
(\dot{u}(0), \dot{v}(0)) &= (V^u, V^v).
\end{aligned}$$

As long as Γ is sufficiently smooth there is a unique solution to such a system of equations given the initial values. The domain $(-\varepsilon, \varepsilon)$ on which such a solution exists is quite hard to determine. It'll depend on the domain of parameters U, the initial values, and finally on Γ. ■

This theorem allows us to find all geodesics on spheres and in the plane without calculation.

In the plane straight lines $c(t) = p + vt$ are clearly geodesics. And since these solve all possible intial problems there are no other geodesics.

On S^2 we claim that the great circles

$$\begin{aligned}
c(t) &= p\cos(|v|\,t) + \frac{v}{|v|}\sin(|v|\,t) \\
p &\in S^2, \\
p \cdot v &= 0
\end{aligned}$$

are geodesics. Note that this is a curve on S^2, and that $c(0) = p$, $\dot{c}(0) = v$. Next we see that the acceleration

$$\ddot{c}(t) = -p\,|v|^2\cos(|v|\,t) - v\,|v|\sin(|v|\,t) = -|v|^2\,c(t)$$

computed in $\mathbb{R}^3$ is normal to the sphere. Thus $\ddot{c}^{\mathrm{I}} = 0$. This means that we have also solved all initial value problems on the sphere.

Exercise: Let $c(s)$ be a unit speed curve on a surface with normal n. Show that it is a geodesic if and only if

$$[c', c'', n] = 0$$

Exercise: Let $c(s)$ be a unit speed curve on a surface with normal n. Define T as the usual tangent to the curve and

$$S = n \times T$$

as the normal to the curve in the surface. Show that

$$\frac{d}{ds}\begin{bmatrix} T & S & n \end{bmatrix} = \begin{bmatrix} T & S & n \end{bmatrix} \begin{bmatrix} 0 & -\kappa_g & -\kappa_n \\ \kappa_g & 0 & -\tau_g \\ \kappa_n & \tau_g & 0 \end{bmatrix}$$

for functions $\kappa_g, \kappa_n, \tau_g$. They are called *geodesic curvature, normal curvature,* and *geodesic torsion* respectively. Further show that S and $\ddot{c}^{\mathrm{I}}$ are proportional and that

$$\begin{aligned}
\kappa_g &= \mathrm{I}\left(S, \ddot{c}^{\mathrm{I}}\right) = S \cdot \frac{dT}{ds}, \\
\kappa_n &= \mathrm{II}\left(\dot{c}, \dot{c}\right) = n \cdot \frac{dT}{ds}, \\
\tau_g &= \mathrm{II}\left(S, \dot{c}\right) = n \cdot \frac{dS}{ds}.
\end{aligned}$$

Exercise: Show that the geodesic curvature can be computed as

$$\kappa_g = \frac{\frac{\partial}{\partial u}\left(T \cdot \frac{\partial \boldsymbol{x}}{\partial v}\right) - \frac{\partial}{\partial v}\left(T \cdot \frac{\partial \boldsymbol{x}}{\partial u}\right)}{\sqrt{\det\left[\mathrm{I}\right]}}$$

Exercise: Define the Hessian of a function on a surface abstractly by

$$\mathrm{Hess} f\left(X, Y\right) = \mathrm{I}\left(D_X \nabla f, Y\right)$$

Show that the entries in the matrix $[\mathrm{Hess} f]$ defined by

$$\mathrm{Hess} f\left(X, Y\right) = \begin{bmatrix} X^u & X^v \end{bmatrix} [\mathrm{Hess} f] \begin{bmatrix} Y^u \\ Y^v \end{bmatrix}$$

are given as

$$\frac{\partial^2 f}{\partial w_1 \partial w_2} + \begin{bmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \end{bmatrix} \begin{bmatrix} \Gamma^u_{w_1 w_2} \\ \Gamma^v_{w_1 w_2} \end{bmatrix}$$

Further relate these entries to the dot products

$$\frac{\partial \nabla f}{\partial w_1} \cdot \frac{\partial \boldsymbol{x}}{\partial w_2}$$

14 Unparametrized Geodesics

It is often simpler to find the unparametrized form of the geodesics, i.e., in a given parametrization they are easier to find as functions $u\left(v\right)$ or $v\left(u\right)$. We start with a tricky characterization showing that one can characterize geodesics without referring to the arclength parameter. The idea is that a regular curve can be reparametrized to be a geodesic if and only if its tangential acceleration $\ddot{c}^{\mathrm{I}}$ is tangent to the curve.

Lemma 14 *A regular curve $c\left(t\right) = \boldsymbol{x}\left(u\left(t\right), v\left(t\right)\right)$ can be reparametrized as a geodesic if and only if*

$$\frac{dv}{dt}\left(\frac{d^2 u}{dt^2} + \Gamma^u\left(\dot{c}, \dot{c}\right)\right) = \frac{du}{dt}\left(\frac{d^2 v}{dt^2} + \Gamma^v\left(\dot{c}, \dot{c}\right)\right).$$

Proof. Let s correspond to a reparametrization of the curve. When switching from t to s we note that the left hand side becomes

$$\begin{aligned}
\frac{dv}{dt}\left(\frac{d^2u}{dt^2}+\Gamma^u(\dot{c},\dot{c})\right) &= \frac{dv}{dt}\left(\frac{d^2u}{dt^2}+\Gamma^u\left(\frac{dc}{dt},\frac{dc}{dt}\right)\right)\\
&= \frac{ds}{dt}\frac{dv}{ds}\left(\frac{d^2s}{dt^2}\frac{du}{ds}+\left(\frac{ds}{dt}\right)^2\frac{d^2u}{ds^2}+\Gamma^u\left(\frac{ds}{dt}\frac{dc}{ds},\frac{ds}{dt}\frac{dc}{ds}\right)\right)\\
&= \frac{ds}{dt}\frac{dv}{ds}\left(\frac{d^2s}{dt^2}\frac{du}{ds}+\left(\frac{ds}{dt}\right)^2\frac{d^2u}{ds^2}+\left(\frac{ds}{dt}\right)^2\Gamma^u\left(\frac{dc}{ds},\frac{dc}{ds}\right)\right)\\
&= \frac{ds}{dt}\frac{d^2s}{dt^2}\frac{dv}{ds}\frac{du}{ds}+\left(\frac{ds}{dt}\right)^3\frac{dv}{ds}\left(\frac{d^2u}{ds^2}+\Gamma^u\left(\frac{dc}{ds},\frac{dc}{ds}\right)\right)
\end{aligned}$$

with a similar formula for the right hand side. Here the first term

$$\frac{ds}{dt}\frac{d^2s}{dt^2}\frac{dv}{ds}\frac{du}{ds}$$

is the same on both sides, so we have shown that the equation is actually independent of parametrizations. In other words if it holds for one parametrization it holds for all reparametrizations.

If c is a geodesic then the formula clearly holds for the arclength parameter.

Conversely if the equation holds for some parameter then it also holds for the arclength parameter. Being parametrized by arclength gives us the equation

$$\mathrm{I}\left(\dot{c},\ddot{c}^{\mathrm{I}}\right)=\begin{bmatrix}\frac{du}{dt} & \frac{dv}{dt}\end{bmatrix}\begin{bmatrix}g_{uu} & g_{uv}\\ g_{vu} & g_{vv}\end{bmatrix}\begin{bmatrix}\frac{d^2u}{dt^2}+\Gamma^u(\dot{c},\dot{c})\\ \frac{d^2v}{dt^2}+\Gamma^v(\dot{c},\dot{c})\end{bmatrix}=0$$

Thus we have two equations

$$\begin{aligned}
\frac{dv}{dt}\left(\frac{d^2u}{dt^2}+\Gamma^u(\dot{c},\dot{c})\right)-\frac{du}{dt}\left(\frac{d^2v}{dt^2}+\Gamma^v(\dot{c},\dot{c})\right) &= 0,\\
\left(g_{uu}\frac{du}{dt}+g_{vu}\frac{dv}{dt}\right)\left(\frac{d^2u}{dt^2}+\Gamma^u(\dot{c},\dot{c})\right)+\left(g_{uv}\frac{du}{dt}+g_{vv}\frac{dv}{dt}\right)\left(\frac{d^2v}{dt^2}+\Gamma^v(\dot{c},\dot{c})\right) &= 0
\end{aligned}$$

Since

$$\begin{aligned}
&\det\begin{bmatrix}\frac{dv}{dt} & -\frac{du}{dt}\\ g_{uu}\frac{du}{dt}+g_{uv}\frac{dv}{dt} & g_{vu}\frac{du}{dt}+g_{vv}\frac{dv}{dt}\end{bmatrix}\\
&= \frac{dv}{dt}\left(g_{vu}\frac{du}{dt}+g_{vv}\frac{dv}{dt}\right)+\frac{du}{dt}\left(g_{uu}\frac{du}{dt}+g_{uv}\frac{dv}{dt}\right)\\
&= |\dot{c}|^2=1
\end{aligned}$$

the only possible solution is

$$\frac{d^2u}{dt^2}+\Gamma^u(\dot{c},\dot{c})=0=\frac{d^2v}{dt^2}+\Gamma^v(\dot{c},\dot{c}),$$

showing that c is a geodesic. ■

Depending on our parametrization (u,v) geodesics can be pictured in many ways. We'll study a few cases where geodesics take on some familiar shapes.

Consider the sphere where great circles are described by

$$\begin{aligned} ax+by+cz &= 0, \\ x^2+y^2+z^2 &= 1 \end{aligned}$$

If we use the parametrization $\frac{1}{\sqrt{1+s^2+t^2}}(s,t,1)$, or in other words $\frac{x}{z}=s$, $\frac{y}{z}=t$ then these equations simply become straight lines in (s,t) coordinates:

$$as+bt+c=0$$

Or we could use $\left(u,v,\sqrt{1-u^2-v^2}\right)$ and note that the equations become

$$\left(a^2+c^2\right)u^2+2abuv+\left(b^2+c^2\right)v^2=c^2$$

which are the equations of ellipses whose axes go through the origin and are inscribed as well as tangent to the unit circle. This is how you draw great circles on the sphere! The first fundamental form is given by

$$[\mathrm{I}]=\begin{bmatrix} 1+\frac{u^2}{1-u^2-v^2} & \frac{uv}{1-u^2-v^2} \\ \frac{uv}{1-u^2-v^2} & 1+\frac{v^2}{1-u^2-v^2} \end{bmatrix}$$

Here is an intrinsic metric on the (u,v) plane where we have simply switched signs from above

$$[\mathrm{I}]=\begin{bmatrix} 1-\frac{u^2}{1+u^2+v^2} & -\frac{uv}{1+u^2+v^2} \\ -\frac{uv}{1+u^2+v^2} & 1-\frac{v^2}{1+u^2+v^2} \end{bmatrix}$$

Using the parameter independent approach to geodesics one can show that they turn out to be hyperbolas whose axes go through the origin

$$\left(a^2-c^2\right)u^2+2abuv+\left(b^2-c^2\right)v^2=c^2$$

This metric can also be reparametrized to have its geodesics be straight lines. The later reparametrization is:

$$\begin{aligned} s &= \frac{u}{\sqrt{1+u^2+v^2}} \\ t &= \frac{v}{\sqrt{1+u^2+v^2}} \end{aligned}$$

and the geodesics given by

$$as+bt+c=0.$$

We shall later explicitly find the geodesics on the upper half plane using the equations developed here.

Exercise: Show that geodesics satisfy a second order equation of the type

$$\frac{d^2v}{du^2}=A\left(\frac{dv}{du}\right)^3+B\left(\frac{dv}{du}\right)^2+C\frac{dv}{du}+D$$

and identify the functions A,B,C,D with the appropriate Christoffel symbols.

15 Shortest Curves

The goal is to show that the shortest curves are geodesics, and conversely that sufficiently short geodesics are minimal in length. For the latter using geodesic coordinates makes an argument that is similar to the Euclidean version using a unit gradient field.

16 Invariance Issues

We offer a geometric approach to show that the second fundamental form is, like the first fundamental form, defined in such a way that selecting a different parametrization will not affect it.

The key observation is that if we have a surface M and a point $p \in M$, then the tangent space T_pM is defined independently of our parametrizations. Correspondingly the *normal space* $N_pM = (T_pM)^{\perp}$ of vectors in $\mathbb{R}^3$ perpendicular to the tangent space are also defined independently of parametrizations. Therefore, if we have a vector Z in Euclidean space then its projection onto both the tangent space and the normal space are also independently defined.

Consider a curve $c(t)$ on the surface. We know that the velocity $\dot{c}$ and acceleration $\ddot{c}$ can be calculated without reference to parametrizations. This means that the projections of $\ddot{c}$ onto the tanget space, $\ddot{c}^{\mathrm{I}}$, and onto the normal space, $n\mathrm{II}^n(\dot{c}, \dot{c})$, can also be computed without reference to parametrizations. This shows that tangential and normal accelerations are well defined.

This also takes care of $n\mathrm{II}^n(X, Y)$ if we use two important observations. The first is called *polarization*, the idea is that symmetric bilinear forms have the property:

$$n\mathrm{II}^n(X, Y) = \frac{1}{2}\left(n\mathrm{II}^n(X+Y, X+Y) - n\mathrm{II}^n(X, X) - n\mathrm{II}^n(Y, Y)\right)$$

Thus it suffices to show that $n\mathrm{II}^n(Z, Z)$ is well defined. But this follows from knowing that $n\mathrm{II}^n(\dot{c}, \dot{c})$ is invariant and that any tangent vector is the velocity of some curve.

17 The Weingarten Map and Equations

There is a similar set of equations for the entries in the second fundamental form that also lead us to the partial derivatives of $n(u, v)$. Together these are also known as the *Weingarten equations*. But first we need to introduce the *Weingarten map*. It is related to the second fundamental form in the same way the Christoffel symbols of the second kind are related to the symbols of the first kind. Its matrix or the entries of its matrix are

$$\begin{aligned} L_{w_1}^{w_2} &= g^{w_2 u} L_{u w_1} + g^{w_2 v} L_{v w_1}, \\ [L] &= [\mathrm{I}]^{-1}[\mathrm{II}], \\ \begin{bmatrix} L_u^u & L_v^u \\ L_u^v & L_v^v \end{bmatrix} &= \begin{bmatrix} g^{uu} & g^{uv} \\ g^{vu} & g^{vv} \end{bmatrix} \begin{bmatrix} L_{uu} & L_{uv} \\ L_{vu} & L_{vv} \end{bmatrix} \end{aligned}$$

When using matrix language we must be careful in our definitions as $[\mathrm{I}]^{-1}[\mathrm{II}]$ and $[\mathrm{II}][\mathrm{I}]^{-1}$ are generally not the same. The abstract Weingarten map L will be a self-adjoint map with respect to the first fundamental form

$$\mathrm{I}(L(X), Y) = \mathrm{I}(X, L(Y))$$

but this does not guarantee that its matrix representation $[L]$ is symmetric. This will only be the case if we are lucky enough to have used an orthonormal basis.

Proposition 15

$$\mathrm{I}(L(X), Y) = \mathrm{II}(X, Y)$$

In particular L is self-adjoint as II is symmetric.

Proof. We have

$$\begin{aligned} \mathrm{II}(X, Y) &= \begin{bmatrix} X^u & X^v \end{bmatrix} [\mathrm{II}] \begin{bmatrix} Y^u \\ Y^v \end{bmatrix}, \\ \mathrm{I}(L(X), Y) &= \begin{bmatrix} X^u & X^v \end{bmatrix} [L]^t [\mathrm{I}] \begin{bmatrix} Y^u \\ Y^v \end{bmatrix} \end{aligned}$$

and by definition

$$[L] = [\mathrm{I}]^{-1}[\mathrm{II}]$$

so

$$[L]^t = [\mathrm{II}][\mathrm{I}]^{-1}$$

as $[\mathrm{I}]$ and $[\mathrm{II}]$ are symmetric. This shows that $\mathrm{I}(L(X), Y) = \mathrm{II}(X, Y)$. ■

Next we find a new formula for L.

Proposition 16 (Weingarten Equations)

$$\begin{aligned} L_{w_1 w_2} &= -\frac{\partial \boldsymbol{x}}{\partial w_2} \cdot \frac{\partial n}{\partial w_1}, \\ [\mathrm{II}] &= -\begin{bmatrix} \frac{\partial n}{\partial u} & \frac{\partial n}{\partial v} \end{bmatrix}^t \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{bmatrix} \\ &= -\begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{bmatrix}^t \begin{bmatrix} \frac{\partial n}{\partial u} & \frac{\partial n}{\partial v} \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \frac{\partial n}{\partial u} &= -L_u^u \frac{\partial \boldsymbol{x}}{\partial u} - L_u^v \frac{\partial \boldsymbol{x}}{\partial v} = -L\left(\frac{\partial \boldsymbol{x}}{\partial u}\right) \\ \frac{\partial n}{\partial v} &= -L_v^u \frac{\partial \boldsymbol{x}}{\partial u} - L_v^v \frac{\partial \boldsymbol{x}}{\partial v} = -L\left(\frac{\partial \boldsymbol{x}}{\partial v}\right) \end{aligned}$$

Proof. The strategy is just as with Christoffel symbols, but works out a bit more easily

$$\begin{aligned} L_{w_1 w_2} &= \frac{\partial^2 \boldsymbol{x}}{\partial w_1 \partial w_2} \cdot n \\ &= \left(\frac{\partial}{\partial w_1} \left(\frac{\partial \boldsymbol{x}}{\partial w_2} \right) \right) \cdot n \\ &= \frac{\partial}{\partial w_1} \left(\frac{\partial \boldsymbol{x}}{\partial w_2} \cdot n \right) - \frac{\partial \boldsymbol{x}}{\partial w_2} \cdot \frac{\partial n}{\partial w_1} \\ &= -\frac{\partial \boldsymbol{x}}{\partial w_2} \cdot \frac{\partial n}{\partial w_1} \end{aligned}$$

were we used that n is perpendicular to $\frac{\partial \boldsymbol{x}}{\partial w_2}$.

For the second set of equations we note

$$\begin{aligned} \left[\begin{array}{cc} L\left(\frac{\partial \boldsymbol{x}}{\partial u}\right) & L\left(\frac{\partial \boldsymbol{x}}{\partial v}\right) \end{array} \right] &= \left[\begin{array}{cc} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{array} \right] [L] \\ &= \left[\begin{array}{cc} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{array} \right] [\mathrm{I}]^{-1} [\mathrm{II}] \\ &= -\left[\begin{array}{cc} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{array} \right] [\mathrm{I}]^{-1} \left[\begin{array}{cc} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{array} \right]^t \left[\begin{array}{cc} \frac{\partial n}{\partial u} & \frac{\partial n}{\partial v} \end{array} \right] \end{aligned}$$

But n is a unit vector field so

$$n \cdot \frac{\partial n}{\partial w} = \frac{1}{2} \frac{\partial}{\partial w} |n|^2 = 0$$

showing that $\frac{\partial n}{\partial w}$ is a tanget vector. In particular

$$\left[\begin{array}{cc} L\left(\frac{\partial \boldsymbol{x}}{\partial u}\right) & L\left(\frac{\partial \boldsymbol{x}}{\partial v}\right) \end{array} \right] = -\left[\begin{array}{cc} \frac{\partial n}{\partial u} & \frac{\partial n}{\partial v} \end{array} \right]$$

■

The Weingarten equations can also be combined into one equation

$$\frac{\partial n}{\partial w} = -L_w^u \frac{\partial \boldsymbol{x}}{\partial u} - L_w^v \frac{\partial \boldsymbol{x}}{\partial v} = -L\left(\frac{\partial \boldsymbol{x}}{\partial w}\right).$$

The Gauss formulas and Weingarten equations together tell us how the derivatives of our basis $\frac{\partial \boldsymbol{x}}{\partial u}, \frac{\partial \boldsymbol{x}}{\partial v}, n$ relate back to the basis. They can be collected as follows:

Corollary 17 (Gauss and Weingarten Formulas)

$$\begin{aligned} \frac{\partial}{\partial w} \left[\begin{array}{ccc} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} & n \end{array} \right] &= \left[\begin{array}{ccc} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} & n \end{array} \right] [D_w] \\ &= \left[\begin{array}{ccc} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} & n \end{array} \right] \left[\begin{array}{ccc} \Gamma_{wu}^u & \Gamma_{wv}^u & -L_w^u \\ \Gamma_{wu}^v & \Gamma_{wv}^v & -L_w^v \\ L_{wu} & L_{wv} & 0 \end{array} \right] \end{aligned}$$

18 The Gauss Curvature and Map

One of the interesting features of the Weingarten map is that its trace $2H = L_u^u + L_v^v$ and determinant $K = L_u^u L_v^v - L_u^v L_v^u$ yield functions on the surface that are independent of the chosen parametrization. Clearly the entries themselves do depend on parametrizations. This means that if we have two parametrizations around a point $p \in M$, then the calculation of H and K at p will not depend on what parametrization we use! H is called the *mean curvature* and K the *Gauss curvature*. We saw that for a fixed choice of normal the second fundamental form is defined independently of the parameters. This will clearly also be true of the Weingarten map. The next theorem is therefore obvious. Nevertheless it is instructive to offer a less inspired proof.

Theorem 18 *The mean and Gauss curvatures do not depend on the parametrizations.*

Proof. The key to proving this is to first realize that we know that the trace and determinant of a matrix do not depend on the basis that is used to represent the matrix, second we need to see that the Weingarten map changes according to the change of basis rules when we change parametrizations. In these calculations we assume that the normal vector field is fixed rather than given as a formula that depnds on the tangent fields. There are only two choices for the normal field $\pm n$, and II as well as L will also change sign if we change sign for n. Note that this sign change does affect H, but not K!

The Weingarten map is calculated by

$$\begin{bmatrix} \frac{\partial n}{\partial u} & \frac{\partial n}{\partial v} \end{bmatrix} = -\begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{bmatrix} \begin{bmatrix} L_u^u & L_v^u \\ L_u^v & L_v^v \end{bmatrix}$$

and similarly in (s,t) coordinates

$$\begin{bmatrix} \frac{\partial n}{\partial s} & \frac{\partial n}{\partial t} \end{bmatrix} = -\begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial s} & \frac{\partial \boldsymbol{x}}{\partial t} \end{bmatrix} \begin{bmatrix} L_s^s & L_t^s \\ L_s^t & L_t^t \end{bmatrix}$$

Changing parametrizations is done using the chain rule which in matrix form looks like

$$\begin{aligned} \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial s} & \frac{\partial \boldsymbol{x}}{\partial t} \end{bmatrix} &= \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial s} & \frac{\partial v}{\partial t} \end{bmatrix} \\ \begin{bmatrix} \frac{\partial n}{\partial s} & \frac{\partial n}{\partial t} \end{bmatrix} &= \begin{bmatrix} \frac{\partial n}{\partial u} & \frac{\partial n}{\partial v} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial s} & \frac{\partial v}{\partial t} \end{bmatrix} \end{aligned}$$

Thus

$$\begin{aligned} \begin{bmatrix} \frac{\partial n}{\partial u} & \frac{\partial n}{\partial v} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial s} & \frac{\partial v}{\partial t} \end{bmatrix} &= \begin{bmatrix} \frac{\partial n}{\partial s} & \frac{\partial n}{\partial t} \end{bmatrix} \\ &= -\begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial s} & \frac{\partial \boldsymbol{x}}{\partial t} \end{bmatrix} \begin{bmatrix} L_s^s & L_t^s \\ L_s^t & L_t^t \end{bmatrix} \\ &= -\begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial s} & \frac{\partial v}{\partial t} \end{bmatrix} \begin{bmatrix} L_s^s & L_t^s \\ L_s^t & L_t^t \end{bmatrix} \end{aligned}$$

showing that

$$\begin{bmatrix} \frac{\partial n}{\partial u} & \frac{\partial n}{\partial v} \end{bmatrix} = -\begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial s} & \frac{\partial v}{\partial t} \end{bmatrix} \begin{bmatrix} L_s^s & L_t^s \\ L_s^t & L_t^t \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial s} & \frac{\partial v}{\partial t} \end{bmatrix}^{-1}$$

or

$$\begin{bmatrix} L_u^u & L_v^u \\ L_u^v & L_v^v \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial s} & \frac{\partial v}{\partial t} \end{bmatrix} \begin{bmatrix} L_s^s & L_t^s \\ L_s^t & L_t^t \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial s} & \frac{\partial v}{\partial t} \end{bmatrix}^{-1}$$

This in turn gives us

$$\begin{aligned} \det \begin{bmatrix} L_u^u & L_v^u \\ L_u^v & L_v^v \end{bmatrix} &= \det\left(\begin{bmatrix} \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial s} & \frac{\partial v}{\partial t} \end{bmatrix} \begin{bmatrix} L_s^s & L_t^s \\ L_s^t & L_t^t \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial s} & \frac{\partial v}{\partial t} \end{bmatrix}^{-1} \right) \\ &= \det \begin{bmatrix} \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial s} & \frac{\partial v}{\partial t} \end{bmatrix} \det \begin{bmatrix} L_s^s & L_t^s \\ L_s^t & L_t^t \end{bmatrix} \det \begin{bmatrix} \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial s} & \frac{\partial v}{\partial t} \end{bmatrix}^{-1} \\ &= \det \begin{bmatrix} L_s^s & L_t^s \\ L_s^t & L_t^t \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} \operatorname{tr} \begin{bmatrix} L_u^u & L_v^u \\ L_u^v & L_v^v \end{bmatrix} &= \operatorname{tr}\left(\begin{bmatrix} \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial s} & \frac{\partial v}{\partial t} \end{bmatrix} \begin{bmatrix} L_s^s & L_t^s \\ L_s^t & L_t^t \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial s} & \frac{\partial v}{\partial t} \end{bmatrix}^{-1} \right) \\ &= \operatorname{tr}\left(\begin{bmatrix} \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial s} & \frac{\partial v}{\partial t} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial s} & \frac{\partial v}{\partial t} \end{bmatrix} \begin{bmatrix} L_s^s & L_t^s \\ L_s^t & L_t^t \end{bmatrix} \right) \\ &= \operatorname{tr} \begin{bmatrix} L_s^s & L_t^s \\ L_s^t & L_t^t \end{bmatrix} \end{aligned}$$

■

Note that we have in fact shown that the linear map $L : T_pM \to T_pM$ does not depend on the parametrizations we use.

We can further find a very interesting formula for the Gauss curvature

Proposition 19 (Gauss)

$$K = \frac{\left(\frac{\partial n}{\partial u} \times \frac{\partial n}{\partial v}\right) \cdot n}{\left|\frac{\partial \boldsymbol{x}}{\partial u} \times \frac{\partial \boldsymbol{x}}{\partial v}\right|}$$

Proof. Simply use the Weingarten equations to calculate

$$\begin{aligned} \frac{\partial n}{\partial u} \times \frac{\partial n}{\partial v} &= \left(-L_u^u \frac{\partial \boldsymbol{x}}{\partial u} - L_u^v \frac{\partial \boldsymbol{x}}{\partial v}\right) \times \left(-L_v^u \frac{\partial \boldsymbol{x}}{\partial u} - L_v^v \frac{\partial \boldsymbol{x}}{\partial v}\right) \\ &= L_u^u L_v^v \frac{\partial \boldsymbol{x}}{\partial u} \times \frac{\partial \boldsymbol{x}}{\partial v} + L_u^v L_v^u \frac{\partial \boldsymbol{x}}{\partial v} \times \frac{\partial \boldsymbol{x}}{\partial u} \\ &= \left(L_u^u L_v^v - L_u^v L_v^u\right) \frac{\partial \boldsymbol{x}}{\partial u} \times \frac{\partial \boldsymbol{x}}{\partial v} \\ &= K \left|\frac{\partial \boldsymbol{x}}{\partial u} \times \frac{\partial \boldsymbol{x}}{\partial v}\right| n \end{aligned}$$

■

Note that the denominator is already computed in terms of the first fundamental form

$$\left|\frac{\partial \boldsymbol{x}}{\partial u} \times \frac{\partial \boldsymbol{x}}{\partial v}\right|^2 = g_{uu}g_{vv} - (g_{uv})^2$$

The numerator is actually very similar in nature as it is simply the corresponding expression for the so called *Gauss map* $n(u,v) : U \to S^2(1) \subset \mathbb{R}^2$ for the surface, i.e., computed from the first fundamental form of $n(u,v)$. This map is our analog of the tangent spherical image. Note that for the unit sphere the unit normal at n is $\pm n$ depending on parametrizations. Thus $\left(\frac{\partial n}{\partial u} \times \frac{\partial n}{\partial v}\right) \cdot n$ represents the oriented area or the parallelogram whose sides are $\frac{\partial n}{\partial u}, \frac{\partial n}{\partial v}$. Recall from curve theory that the tangent spherical image was also related to curvature in a similar way. Here the formulas are a bit more complicated as we use arbitrary parameters.

One classically defines the *third fundamental form* III as the first fundamental form for n

$$\begin{aligned}[\mathrm{III}] &= \begin{bmatrix} \frac{\partial n}{\partial u} & \frac{\partial n}{\partial v} \end{bmatrix}^t \begin{bmatrix} \frac{\partial n}{\partial u} & \frac{\partial n}{\partial v} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial n}{\partial u} \cdot \frac{\partial n}{\partial u} & \frac{\partial n}{\partial u} \cdot \frac{\partial n}{\partial v} \\ \frac{\partial n}{\partial v} \cdot \frac{\partial n}{\partial u} & \frac{\partial n}{\partial v} \cdot \frac{\partial n}{\partial v} \end{bmatrix}\end{aligned}$$

This certainly makes sense, but n might not be a genuine parametrization if the Gauss curvature vanishes. Note however that n is not just the normal to the surface, but also to the unit sphere at n

$$\frac{\partial n}{\partial u} \times \frac{\partial n}{\partial v} = \left|\frac{\partial n}{\partial u} \times \frac{\partial n}{\partial v}\right| n$$

This is part of what we just established.

The three fundamental forms and two curvatures are related by a very interesting formula which also shows that the third fundamental form is almost redundant.

Theorem 20

$$\mathrm{III} - 2H\mathrm{II} + K\mathrm{I} = 0$$

Proof. We first reduce this statement to the Cayley-Hamilton theorem for the linear map L. This relies on showing

$$\begin{aligned}\mathrm{I}(L(X), Y) &= \mathrm{II}(X, Y), \\ \mathrm{I}\left(L^2(X), Y\right) &= \mathrm{III}(X, Y)\end{aligned}$$

and then proving that any 2×2 matrix satisfies:

$$L^2 - (\mathrm{tr}(L))L + \det(L)I = 0$$

where I is the identity matrix. The last step can be done by a straightforward calculation.

We already saw that $\mathrm{I}(L(X), Y) = \mathrm{II}(X, Y)$ as that followed directly from $[\mathrm{II}] = [\mathrm{I}][L]$. We similarly have

$$\begin{aligned}
[\mathrm{III}] &= \begin{bmatrix} L\left(\frac{\partial \boldsymbol{x}}{\partial u}\right) & L\left(\frac{\partial \boldsymbol{x}}{\partial v}\right) \end{bmatrix}^t \begin{bmatrix} L\left(\frac{\partial \boldsymbol{x}}{\partial u}\right) & L\left(\frac{\partial \boldsymbol{x}}{\partial v}\right) \end{bmatrix} \\
&= [L]^t \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{bmatrix}^t \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{bmatrix} [L] \\
&= [L]^t [\mathrm{I}] [L] \\
&= [\mathrm{II}] [L] \\
&= [\mathrm{I}] [L] [L] \\
&= [\mathrm{I}] [L]^2
\end{aligned}$$

showing that $\mathrm{I}\left(L^2(X), Y\right) = \mathrm{III}(X, Y)$. ■

In a related vein we mention Gauss' amazing discovery that the Gauss curvature can be computed knowing only the first fundamental form. Given the definition of K this is certainly a big surprise. A different proof that uses our abstract framework will be given in a later section. Here we use a more direct approach.

Theorem 21 (Theorema Egregium) *The Gauss curvature can be computed knowing only the first fundamental form.*

Proof. We start with the observation that

$$\begin{aligned}
K &= \det L = \det [\mathrm{I}]^{-1} \det [\mathrm{II}], \\
\det [\mathrm{I}] &= g_{uu} g_{vv} - (g_{uv})^2
\end{aligned}$$

So we concentrate on

$$\begin{aligned}
\det [\mathrm{II}] &= \det \begin{bmatrix} L_{uu} & L_{uv} \\ L_{vu} & L_{vv} \end{bmatrix} \\
&= \det \begin{bmatrix} \frac{\partial^2 \boldsymbol{x}}{\partial u^2} \cdot n & \frac{\partial^2 \boldsymbol{x}}{\partial u \partial v} \cdot n \\ \frac{\partial^2 \boldsymbol{x}}{\partial v \partial u} \cdot n & \frac{\partial^2 \boldsymbol{x}}{\partial v^2} \cdot n \end{bmatrix} \\
&= \frac{1}{g_{uu} g_{vv} - (g_{uv})^2} \det \begin{bmatrix} \frac{\partial^2 \boldsymbol{x}}{\partial u^2} \cdot \left(\frac{\partial \boldsymbol{x}}{\partial u} \times \frac{\partial \boldsymbol{x}}{\partial v}\right) & \frac{\partial^2 \boldsymbol{x}}{\partial u \partial v} \cdot \left(\frac{\partial \boldsymbol{x}}{\partial u} \times \frac{\partial \boldsymbol{x}}{\partial v}\right) \\ \frac{\partial^2 \boldsymbol{x}}{\partial v \partial u} \cdot \left(\frac{\partial \boldsymbol{x}}{\partial u} \times \frac{\partial \boldsymbol{x}}{\partial v}\right) & \frac{\partial^2 \boldsymbol{x}}{\partial v^2} \cdot \left(\frac{\partial \boldsymbol{x}}{\partial u} \times \frac{\partial \boldsymbol{x}}{\partial v}\right) \end{bmatrix}
\end{aligned}$$

Which then reduces us to consider

$$\det \begin{bmatrix} \frac{\partial^2 \boldsymbol{x}}{\partial u^2} \cdot \left(\frac{\partial \boldsymbol{x}}{\partial u} \times \frac{\partial \boldsymbol{x}}{\partial v}\right) & \frac{\partial^2 \boldsymbol{x}}{\partial u \partial v} \cdot \left(\frac{\partial \boldsymbol{x}}{\partial u} \times \frac{\partial \boldsymbol{x}}{\partial v}\right) \\ \frac{\partial^2 \boldsymbol{x}}{\partial v \partial u} \cdot \left(\frac{\partial \boldsymbol{x}}{\partial u} \times \frac{\partial \boldsymbol{x}}{\partial v}\right) & \frac{\partial^2 \boldsymbol{x}}{\partial v^2} \cdot \left(\frac{\partial \boldsymbol{x}}{\partial u} \times \frac{\partial \boldsymbol{x}}{\partial v}\right) \end{bmatrix}$$

Here each entry in the matrix is a tripel product and hence a determinant of a 3×3 matrix

$$\det \begin{bmatrix} \frac{\partial^2 \boldsymbol{x}}{\partial w_1 \partial w_2} & \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{bmatrix} = \frac{\partial^2 \boldsymbol{x}}{\partial w_1 \partial w_2} \cdot \left(\frac{\partial \boldsymbol{x}}{\partial u} \times \frac{\partial \boldsymbol{x}}{\partial v}\right)$$

With that observation and recalling that a matrix and its transpose have the same determinant we can calculate the products that appear in our 2×2 determinant

$$\begin{aligned}
& \left(\frac{\partial^2 \boldsymbol{x}}{\partial u^2} \cdot \left(\frac{\partial \boldsymbol{x}}{\partial u} \times \frac{\partial \boldsymbol{x}}{\partial v}\right)\right)\left(\frac{\partial^2 \boldsymbol{x}}{\partial v^2} \cdot \left(\frac{\partial \boldsymbol{x}}{\partial u} \times \frac{\partial \boldsymbol{x}}{\partial v}\right)\right) \\
= \; & \det \left[\begin{array}{ccc} \frac{\partial^2 \boldsymbol{x}}{\partial u^2} & \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{array}\right] \det \left[\begin{array}{ccc} \frac{\partial^2 \boldsymbol{x}}{\partial v^2} & \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{array}\right] \\
= \; & \det \left[\begin{array}{ccc} \frac{\partial^2 \boldsymbol{x}}{\partial u^2} & \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{array}\right]^t \det \left[\begin{array}{ccc} \frac{\partial^2 \boldsymbol{x}}{\partial v^2} & \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{array}\right] \\
= \; & \det \left(\left[\begin{array}{ccc} \frac{\partial^2 \boldsymbol{x}}{\partial u^2} & \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{array}\right]^t \left[\begin{array}{ccc} \frac{\partial^2 \boldsymbol{x}}{\partial v^2} & \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{array}\right]\right) \\
= \; & \det \left[\begin{array}{ccc} \frac{\partial^2 \boldsymbol{x}}{\partial u^2} \cdot \frac{\partial^2 \boldsymbol{x}}{\partial v^2} & \frac{\partial \boldsymbol{x}}{\partial u} \cdot \frac{\partial^2 \boldsymbol{x}}{\partial v^2} & \frac{\partial \boldsymbol{x}}{\partial v} \cdot \frac{\partial^2 \boldsymbol{x}}{\partial v^2} \\ \frac{\partial^2 \boldsymbol{x}}{\partial u^2} \cdot \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial u} \cdot \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \cdot \frac{\partial \boldsymbol{x}}{\partial v} \\ \frac{\partial^2 \boldsymbol{x}}{\partial u^2} \cdot \frac{\partial \boldsymbol{x}}{\partial v} & \frac{\partial \boldsymbol{x}}{\partial u} \cdot \frac{\partial \boldsymbol{x}}{\partial v} & \frac{\partial \boldsymbol{x}}{\partial v} \cdot \frac{\partial \boldsymbol{x}}{\partial v} \end{array}\right] \\
= \; & \det \left[\begin{array}{ccc} \frac{\partial^2 \boldsymbol{x}}{\partial u^2} \cdot \frac{\partial^2 \boldsymbol{x}}{\partial v^2} & \Gamma_{vvu} & \Gamma_{vvv} \\ \Gamma_{uuu} & g_{uu} & g_{uv} \\ \Gamma_{uuv} & g_{vu} & g_{vv} \end{array}\right] \\
= \; & \frac{\partial^2 \boldsymbol{x}}{\partial u^2} \cdot \frac{\partial^2 \boldsymbol{x}}{\partial v^2} \det [\mathrm{I}] + \det \left[\begin{array}{ccc} 0 & \Gamma_{vvu} & \Gamma_{vvv} \\ \Gamma_{uuu} & g_{uu} & g_{uv} \\ \Gamma_{uuv} & g_{vu} & g_{vv} \end{array}\right]
\end{aligned}$$

and similarly

$$\begin{aligned}
& \left(\frac{\partial^2 \boldsymbol{x}}{\partial u \partial v} \cdot \left(\frac{\partial \boldsymbol{x}}{\partial u} \times \frac{\partial \boldsymbol{x}}{\partial v}\right)\right)\left(\frac{\partial^2 \boldsymbol{x}}{\partial v \partial u} \cdot \left(\frac{\partial \boldsymbol{x}}{\partial u} \times \frac{\partial \boldsymbol{x}}{\partial v}\right)\right) \\
= \; & \det \left[\begin{array}{ccc} \frac{\partial^2 \boldsymbol{x}}{\partial u \partial v} \cdot \frac{\partial^2 \boldsymbol{x}}{\partial u \partial v} & \frac{\partial \boldsymbol{x}}{\partial u} \cdot \frac{\partial^2 \boldsymbol{x}}{\partial u \partial v} & \frac{\partial \boldsymbol{x}}{\partial v} \cdot \frac{\partial^2 \boldsymbol{x}}{\partial u \partial v} \\ \frac{\partial^2 \boldsymbol{x}}{\partial u \partial v} \cdot \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial u} \cdot \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \cdot \frac{\partial \boldsymbol{x}}{\partial v} \\ \frac{\partial^2 \boldsymbol{x}}{\partial u \partial v} \cdot \frac{\partial \boldsymbol{x}}{\partial v} & \frac{\partial \boldsymbol{x}}{\partial u} \cdot \frac{\partial \boldsymbol{x}}{\partial v} & \frac{\partial \boldsymbol{x}}{\partial v} \cdot \frac{\partial \boldsymbol{x}}{\partial v} \end{array}\right] \\
= \; & \det \left[\begin{array}{ccc} \frac{\partial^2 \boldsymbol{x}}{\partial u \partial v} \cdot \frac{\partial^2 \boldsymbol{x}}{\partial u \partial v} & \Gamma_{uvu} & \Gamma_{uvv} \\ \Gamma_{uvu} & g_{uu} & g_{uv} \\ \Gamma_{uvv} & g_{vu} & g_{vv} \end{array}\right] \\
= \; & \frac{\partial^2 \boldsymbol{x}}{\partial u \partial v} \cdot \frac{\partial^2 \boldsymbol{x}}{\partial u \partial v} \det [\mathrm{I}] + \det \left[\begin{array}{ccc} 0 & \Gamma_{uvu} & \Gamma_{uvv} \\ \Gamma_{uvu} & g_{uu} & g_{uv} \\ \Gamma_{uvv} & g_{vu} & g_{vv} \end{array}\right]
\end{aligned}$$

Since we need to subtract these quantities we are finally reduced to check the difference

$$\begin{aligned}
& \frac{\partial^2 \boldsymbol{x}}{\partial u^2} \cdot \frac{\partial^2 \boldsymbol{x}}{\partial v^2} - \frac{\partial^2 \boldsymbol{x}}{\partial u \partial v} \cdot \frac{\partial^2 \boldsymbol{x}}{\partial u \partial v} \\
= \; & \frac{\partial}{\partial v}\left(\frac{\partial^2 \boldsymbol{x}}{\partial u^2} \cdot \frac{\partial \boldsymbol{x}}{\partial v}\right) - \frac{\partial^3 \boldsymbol{x}}{\partial v \partial u^2} \cdot \frac{\partial \boldsymbol{x}}{\partial v} \\
& -\frac{\partial}{\partial u}\left(\frac{\partial^2 \boldsymbol{x}}{\partial u \partial v} \cdot \frac{\partial \boldsymbol{x}}{\partial v}\right) + \frac{\partial^3 \boldsymbol{x}}{\partial^2 u \partial v} \cdot \frac{\partial \boldsymbol{x}}{\partial v} \\
= \; & \frac{\partial}{\partial v}\Gamma_{uuv} - \frac{\partial}{\partial u}\Gamma_{uvv}
\end{aligned}$$

Thus

$$
\begin{aligned}
K &= \frac{\det\begin{bmatrix} \frac{\partial^2 \boldsymbol{x}}{\partial u^2}\cdot\left(\frac{\partial \boldsymbol{x}}{\partial u}\times\frac{\partial \boldsymbol{x}}{\partial v}\right) & \frac{\partial^2 \boldsymbol{x}}{\partial u \partial v}\cdot\left(\frac{\partial \boldsymbol{x}}{\partial u}\times\frac{\partial \boldsymbol{x}}{\partial v}\right) \\ \frac{\partial^2 \boldsymbol{x}}{\partial v \partial u}\cdot\left(\frac{\partial \boldsymbol{x}}{\partial u}\times\frac{\partial \boldsymbol{x}}{\partial v}\right) & \frac{\partial^2 \boldsymbol{x}}{\partial v^2}\cdot\left(\frac{\partial \boldsymbol{x}}{\partial u}\times\frac{\partial \boldsymbol{x}}{\partial v}\right)\end{bmatrix}}{(\det [\mathrm{I}])^2} \\
&= \frac{\left(\frac{\partial}{\partial v}\Gamma_{uuv} - \frac{\partial}{\partial u}\Gamma_{uvv}\right)}{\det [\mathrm{I}]} \\
&\quad + \frac{\det\begin{bmatrix} 0 & \Gamma_{vvu} & \Gamma_{vvv} \\ \Gamma_{uuu} & g_{uu} & g_{uv} \\ \Gamma_{uuv} & g_{vu} & g_{vv}\end{bmatrix} - \det\begin{bmatrix} 0 & \Gamma_{uvu} & \Gamma_{uvv} \\ \Gamma_{uvu} & g_{uu} & g_{uv} \\ \Gamma_{uvv} & g_{vu} & g_{vv}\end{bmatrix}}{(\det [\mathrm{I}])^2}
\end{aligned}
$$

■

Exercise: Compute the mean and Gauss curvatures of the generalized cones, cylinders, and tangent developables. It turns out that these are essentially the only surfaces in space with vanishing Gauss curvature.

19 Principal Curvatures

The principal curvatures at a point p on a surface are the eigenvalues of the Weingraten map associated to that point, and the principal directions are the corresponding eigenvectors. The fact that L is self-adjoint with respect to the first fundamental form guarantees that we can always find an orthonormal set of principal directions, and that the principal curvatures are real. This is a nice and general theorem from linear algebra, variously called diagonalization of symmetric matrices or the spectral theorem. Since the Weingraten map is a linear map on a two dimensional vector space we can give a direct proof.

Theorem 22 *For a fixed point $p \in M$, we can find orthonormal principal directions $E_1, E_2 \in T_pM$*

$$
\begin{aligned}
L(E_1) &= \kappa_1 E_1, \\
L(E_2) &= \kappa_2 E_2.
\end{aligned}
$$

Moreover κ_1, κ_2 are both real.

Proof. The characteristic polynomial for L looks like

$$\lambda^2 - 2H\lambda + K = 0.$$

The roots of this polynomial are real if and only if the discriminant is nonnegative:

$$
\begin{aligned}
4H^2 - 4K &\geq 0, \text{ or} \\
H^2 &\geq K.
\end{aligned}
$$

If we select an orthonormal basis for T_pM (it doesn't have to be related to a parametrization), the the matrix representation for L is symmetric

$$[L] = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$$

and so

$$\begin{aligned} H &= \frac{a+d}{2}, \\ K &= ad - b^2. \end{aligned}$$

This means we need to show that

$$\begin{aligned} ad - b^2 &\leq \left(\frac{a+d}{2}\right)^2, \text{ or} \\ -b^2 &\leq \frac{a^2+d^2}{4} \end{aligned}$$

So the principal curvatures really are real. If they are also equal, then all vectors are eigenvectors and so we can certainly find an orthonormal basis that diagonalizes L. If the principal curvatures are not eual, then the corresponding principal directions are forced to be orthogonal:

$$\begin{aligned} \kappa_1 \mathrm{I}(E_1, E_2) &= \mathrm{I}(L(E_1), E_2) = \mathrm{I}(E_1, L(E_2)) = \kappa_2 \mathrm{I}(E_1, E_2), \text{ or} \\ (\kappa_1 - \kappa_2)\,\mathrm{I}(E_1, E_2) &= 0. \end{aligned}$$

■

This also makes it possible to calculate the second fundamental form in general directions.

Theorem 23 (Euler) *If $X \in T_pM$, and the principal curvatures are κ_1, κ_2, then*

$$\mathrm{II}(X, X) = \left(\kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta\right)|X|^2$$

where θ is the angle between X and the principal direction corresponding to κ_1.

Proof. Simply selct an orthonormal basis E_1, E_2 or principal directions and use that

$$\begin{aligned} X &= |X|(\cos\theta E_1 + \sin\theta E_2), \\ \mathrm{II}(E_1, E_1) &= \kappa_1, \\ \mathrm{II}(E_2, E_2) &= \kappa_2, \\ \mathrm{II}(E_1, E_2) &= 0 = \mathrm{II}(E_2, E_1). \end{aligned}$$

■

As an important corollary we get a nice characterization of the principal curvatures.

Corollary 24 *Assume that the principal curvatures are ordered $\kappa_1 \geq \kappa_2$, then*

$$\begin{aligned}\max_{|X|=1} \mathrm{II}(X,X) &= \kappa_1,\\ \min_{|X|=1} \mathrm{II}(X,X) &= \kappa_2.\end{aligned}$$

....

Surfaces with constant L are parts of planes or spheres.

The height function that measures the distance from a point on the surface to the tangent space T_pM is given by

$$f(\boldsymbol{x}) = (\boldsymbol{x} - p) \cdot n(p)$$

its partial derivatives in some parametrization are

$$\begin{aligned}\frac{\partial f}{\partial w} &= \frac{\partial \boldsymbol{x}}{\partial w} \cdot n(p),\\ \frac{\partial^2 f}{\partial w_1 \partial w_2} &= \frac{\partial^2 \boldsymbol{x}}{\partial w_1 \partial w_2} \cdot n(p)\end{aligned}$$

So f has a critical point at p, and the second derivatives matrix there is simply $[\mathrm{II}]$. The second derivative test then tells us something about how the surface is placed in relation to T_pM. Specifically we see that if both principal curvatures have the same sign, or $K > 0$, then the surface must locally be on one side of the tangent plane, while if the principal curvatures have opposite signs, or $K < 0$, then the surface lies on both sides.

Theorem 25 *A parametrized surface all of whose principal curvatures are $\geq \varepsilon > 0$ is convex on regions of a fixed size depending on ε and the domain.*

Proof. The important observation is that critical points for the height function are isolated, and, unlike the curve situation, the complement is connected! Also critical points are max or min by second derivative test. Probably need a domain $B(0,R) \subset \mathbb{R}^2$ where the metric in polar coordinates has the form

$$[\mathrm{I}] = \begin{bmatrix} 1 & 0 \\ 0 & \rho^2(r,\theta) \end{bmatrix}$$

and then restrict to $B(0,\varepsilon R)$. ■

Exercise: Show that the principal curvatures are constant if and only if the Gauss and mean curvatures are constant.

Exercise: A surface is called a *ruled surface* if it is a union of lines. Specifically given curves $\alpha(u)$ and $\beta(u)$

$$\boldsymbol{x}(u,v) = \alpha(u) + v\beta(u)$$

Show that $x^2 + y^2 - z^2 = 1$ is a surface of revolution that is also a ruled surface. Compute its Gauss and mean curvatures.

Exercise: Show that if a surface has the property that it has a straight line passing through every point, then it has $K \leq 0$.

Exercise: Show that a surface where $\kappa_1 > \kappa_2 = 0$ everywhere, must be a ruled surface. Hint: Construct an orthogonal parametrization where $L\left(\frac{\partial \boldsymbol{x}}{\partial v}\right) = 0$, $\left|\frac{\partial \boldsymbol{x}}{\partial v}\right| = 1$. Then show that

$$\boldsymbol{x}(u,v) = \alpha(u) + v\beta(u)$$

Exercise: Show that ruled surfaces with vanishing Gauss curvature are, cones, cylinders, or have $\beta(u)$ proportional to $\frac{d\alpha}{du}$.

20 Special Coordinates

The simplest types of coordinates one can expect to obtain on a general surface have first fundamental forms where only one entry is a general function. We'll mention some examples at the end of the section. We start by giving some special examples.

20.1 Cartesian and Oblique Coordinates

Cartesian coordinates on a surface is a parametrization where

$$[\mathrm{I}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Oblique coordinates more generally come from a parametrization where

$$[\mathrm{I}] = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$$

for constants a, b, c with $a, c > 0$ and $ad - b^2 > 0$.

Note that the Christoffel symbols all vanish if we have a parametrization where the metric coefficients are constant. In particular, the rather nasty formula we developed in the proof of Theorema Egregium shows that the Gauss curvature vanishes. This immediately tells us that Cartesian or oblique coordinates cannot exist if the Gauss curvature doesn't vanish. When we have defined geodesic coordinates below we'll also be able to show that surfaces with zero Gauss curvature admit Cartesian coordinates.

20.2 Surfaces of Revolution

Many features of surfaces show themselves for surfaces of revolution. While this is certainly a special class of surfaces it is broad enough to give a rich family examples.

We consider

$$\boldsymbol{x}(t,\theta) = (r(t)\cos\theta, r(t)\sin\theta, z(t)).$$

It is often convenient to select or reparametrized (r, z) so that it is a unit speed curve. In this case we use the parametrization

$$\begin{aligned} \boldsymbol{x}(s,\theta) &= (r(s)\cos\theta, r(s)\sin\theta, h(s)), \\ (r')^2 + (h')^2 &= 1 \end{aligned}$$

We get the unit sphere by using $r = \sin s$ and $h = \cos s$.

We get a cone, cylinder or plane, by considering $r = (\alpha t + \beta)$ and $h = \gamma t$. When $\gamma = 0$ this is simply polar coordinates in the $\boldsymbol{x}, y$ plane. When $\alpha = 0$ we get a cylinder, while if both α and γ are nontrivial we get a cone. When $\alpha^2 + \gamma^2 = 1$ we have a parametrization by arclength.

The basis is given by

$$\begin{aligned}
\frac{\partial \boldsymbol{x}}{\partial t} &= \left(\dot{r}\cos\theta, \dot{r}\sin\theta, \dot{h}\right), \\
\frac{\partial \boldsymbol{x}}{\partial \theta} &= \left(-r\sin\theta, r\cos\theta, 0\right), \\
n &= \frac{\left(-\dot{h}\cos\theta, -\dot{h}\sin\theta, \dot{r}\right)}{\sqrt{\dot{h}^2 + \dot{r}^2}}
\end{aligned}$$

and first fundamental form by

$$\begin{aligned}
g_{tt} &= \dot{h}^2 + \dot{r}^2, \\
g_{\theta\theta} &= r^2 \\
g_{t\theta} &= 0
\end{aligned}$$

Note that the cylinder has the same first fundamental form as the plane if we use Cartesian coordinates in the plane. The cone also allows for Cartesian coordinates, but they are less easy to construct directly. This is not so surprising as we just saw that it took different types of coordinates for the cylinder and the plane to recognize that they admitted Cartesian coordinates. Pictorially one can put Cartesian coordinates on the cone by slicing it open along a meridian and the unfolding it to be flat. Think of unfolding a lamp shade.

Taking a surface of revolution using the arclength parameter s, we see that

$$\begin{aligned}
\frac{\partial n}{\partial s} &= \frac{\partial}{\partial s}\left(-h'\cos\theta, -h'\sin\theta, r'\right) \\
&= \left(-h''\cos\theta, -h''\sin\theta, r''\right) \\
\frac{\partial n}{\partial \theta} &= \frac{\partial}{\partial \theta}\left(-h'\cos\theta, -h'\sin\theta, r'\right) \\
&= \left(h'\sin\theta, -h'\cos\theta, 0\right)
\end{aligned}$$

The Weingarten map is now found by expanding these two vectors. For the last equation this is simply

$$\begin{aligned}
\frac{\partial n}{\partial \theta} &= \left(h'\sin\theta, -h'\cos\theta, 0\right) \\
&= -\frac{h'}{r}\left(-r\sin\theta, r\cos\theta, 0\right) \\
&= -\frac{h'}{r}\frac{\partial \boldsymbol{x}}{\partial \theta}
\end{aligned}$$

Thus we have

$$
\begin{aligned}
L_\theta^s &= L_s^\theta = 0, \\
L_\theta^\theta &= \frac{h'}{r}
\end{aligned}
$$

This leaves us with finding L_s^s. Since $\frac{\partial \boldsymbol{x}}{\partial s}$ is a unit vector this is simply

$$
\begin{aligned}
L_s^s &= -\frac{\partial n}{\partial s} \cdot \frac{\partial \boldsymbol{x}}{\partial s} \\
&= (h'' \cos\theta, h'' \sin\theta, -r'') \cdot (r' \cos\theta, r' \sin\theta, h') \\
&= h'' r' - r'' h'
\end{aligned}
$$

Thus

$$
\begin{aligned}
K &= (h'' r' - r'' h') \frac{h'}{r} \\
H &= \frac{h'}{r} + h'' r' - r'' h'
\end{aligned}
$$

In the case of cylinder, plane, and cone we note that K vanishes, but H only vanishes when it is a plane. This means that we have a selection of surfaces all with Cartesian coordinates with different H.

We can in general simplify the Gauss curvature by noting that

$$
\begin{aligned}
1 &= (r')^2 + (h')^2 \\
0 &= \left((r')^2 + (h')^2 \right)' = 2r'r'' + 2h'h''
\end{aligned}
$$

Thus yielding

$$
\begin{aligned}
K &= \left(r'' \frac{(r')^2}{h'} - r'' h' \right) \frac{h'}{r} \\
&= \frac{r''}{r} \left(-(r')^2 - (h')^2 \right) \\
&= -\frac{r''}{r} \\
&= -\frac{\frac{\partial^2}{\partial s^2} \left(\sqrt{g_{rr}} \right)}{\sqrt{g_{rr}}}
\end{aligned}
$$

This makes it particularly easy to calculate the Gauss curvature and also to construct examples with a given curvature function. It also shows that the Gauss curvature can be computed directly from the first fundamental form! For instance if we want $K = -1$, then we can just use $r(s) = \exp(-s)$ for $s > 0$ and then adjust $h(s)$ for $s \in (0, \infty)$ such that

$$1 = (r')^2 + (h')^2$$

If we introduce a new parameter $t = \exp(s) > 1$, then we obtain a new parametrization of the same surface

$$
\begin{aligned}
\boldsymbol{x}(t,\theta) &= \boldsymbol{x}(\ln(t),\theta) \\
&= (\exp(-\ln t)\cos\theta, \exp(-\ln t)\sin\theta, h(\ln t)) \\
&= \left(\frac{1}{t}\cos\theta, \frac{1}{t}\sin\theta, h(\ln t)\right)
\end{aligned}
$$

To find the first fundamental form of this surface we have to calculate

$$
\begin{aligned}
\frac{d}{dt}h(\ln t) &= \frac{dh}{ds}\frac{1}{t} \\
&= \sqrt{1-(r')^2}\frac{1}{t} \\
&= \sqrt{1-(-\exp(-s))^2}\frac{1}{t} \\
&= \sqrt{1-\exp(-2\ln t)}\frac{1}{t} \\
&= \sqrt{1-\frac{1}{t^2}}\frac{1}{t}
\end{aligned}
$$

Thus

$$
\mathrm{I} = \begin{bmatrix} \frac{1}{t^4} + \left(1-\frac{1}{t^2}\right)\frac{1}{t^2} & 0 \\ 0 & \frac{1}{t^2} \end{bmatrix} = \begin{bmatrix} \frac{1}{t^2} & 0 \\ 0 & \frac{1}{t^2} \end{bmatrix}
$$

This is exactly what the first fundamental form for the upper half plane looked like. But the domians for the two are quite different. What we have achieved is a local representation of part of the upper half plane.

Exercise: Show that geodesics on a surface of revolution satisfy Clairaut's condition: $r\sin\omega$ is constant, where ω is the angle the geodesic forms with the meridians.

20.3 Monge Patches

This is more complicated than the previous case, but that is only to be expected as all surfaces admit Monge patches. We consider $\boldsymbol{x}(u,v) = (u, v, f(u,v))$. Thus

$$
\begin{aligned}
\frac{\partial \boldsymbol{x}}{\partial u} &= \left(1, 0, \frac{\partial f}{\partial u}\right), \\
\frac{\partial \boldsymbol{x}}{\partial v} &= \left(0, 1, \frac{\partial f}{\partial v}\right)
\end{aligned}
$$

$$
n = -\frac{\left(\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}, -1\right)}{\sqrt{1+\left(\frac{\partial f}{\partial u}\right)^2+\left(\frac{\partial f}{\partial v}\right)^2}}
$$

$$
\begin{aligned}
g_{uu} &= 1+\left(\frac{\partial f}{\partial u}\right)^2, \\
g_{vv} &= 1+\left(\frac{\partial f}{\partial v}\right)^2, \\
g_{uv} &= \frac{\partial f}{\partial u}\frac{\partial f}{\partial v}, \\
[\mathrm{I}] &= \left[\begin{array}{cc} 1+\left(\frac{\partial f}{\partial u}\right)^2 & \frac{\partial f}{\partial u}\frac{\partial f}{\partial v} \\ \frac{\partial f}{\partial u}\frac{\partial f}{\partial v} & 1+\left(\frac{\partial f}{\partial v}\right)^2 \end{array}\right] \\
\det[\mathrm{I}] &= 1+\left(\frac{\partial f}{\partial u}\right)^2+\left(\frac{\partial f}{\partial v}\right)^2
\end{aligned}
$$

$$
\frac{\partial^2 \boldsymbol{x}}{\partial w_1 \partial w_2} = \left(0, 0, \frac{\partial^2 f}{\partial w_1 \partial w_2}\right)
$$

So we immediately get

$$
\Gamma_{w_1 w_2 w_3} = \frac{\partial^2 f}{\partial w_1 \partial w_2}\frac{\partial f}{\partial w_3}
$$

$$
L_{w_1 w_2} = \frac{\frac{\partial^2 f}{\partial w_1 \partial w_2}}{\sqrt{1+\left(\frac{\partial f}{\partial u}\right)^2+\left(\frac{\partial f}{\partial v}\right)^2}}
$$

The Gauss curvature is then the determinant of

$$
L = \left[\begin{array}{cc} L^u_u & L^u_v \\ L^v_u & L^v_v \end{array}\right] = \left[\begin{array}{cc} g^{uu} & g^{uv} \\ g^{vu} & g^{vv} \end{array}\right]\left[\begin{array}{cc} L_{uu} & L_{uv} \\ L_{vu} & L_{vv} \end{array}\right]
$$

$$
\begin{aligned}
K &= \frac{1}{\det[\mathrm{I}]}\det\left[\begin{array}{cc} L_{uu} & L_{uv} \\ L_{vu} & L_{vv} \end{array}\right] \\
&= \frac{\frac{\partial^2 f}{\partial u^2}\frac{\partial^2 f}{\partial v^2}-\left(\frac{\partial^2 f}{\partial u \partial v}\right)^2}{\det[\mathrm{I}]^2}
\end{aligned}
$$

We note that

$$
\begin{aligned}
[\mathrm{I}]^{-1} &= \frac{1}{\det[\mathrm{I}]}\left[\begin{array}{cc} 1+\left(\frac{\partial f}{\partial v}\right)^2 & -\frac{\partial f}{\partial u}\frac{\partial f}{\partial v} \\ -\frac{\partial f}{\partial u}\frac{\partial f}{\partial v} & 1+\left(\frac{\partial f}{\partial u}\right)^2 \end{array}\right], \\
[\mathrm{II}] &= \frac{1}{\sqrt{\det[\mathrm{I}]}}\left[\begin{array}{cc} \frac{\partial^2 f}{\partial u^2} & \frac{\partial^2 f}{\partial u \partial v} \\ \frac{\partial^2 f}{\partial u \partial v} & \frac{\partial^2 f}{\partial v^2} \end{array}\right]
\end{aligned}
$$

and the Weingarten map

$$\begin{aligned}
[L] &= [\mathrm{I}]^{-1}[\mathrm{II}] \\
&= \frac{1}{(\det[\mathrm{I}])^{\frac{3}{2}}}\begin{bmatrix} 1+\left(\frac{\partial f}{\partial v}\right)^2 & -\frac{\partial f}{\partial u}\frac{\partial f}{\partial v} \\ -\frac{\partial f}{\partial u}\frac{\partial f}{\partial v} & 1+\left(\frac{\partial f}{\partial u}\right)^2 \end{bmatrix}\begin{bmatrix} \frac{\partial^2 f}{\partial u^2} & \frac{\partial^2 f}{\partial u \partial v} \\ \frac{\partial^2 f}{\partial u \partial v} & \frac{\partial^2 f}{\partial v^2} \end{bmatrix} \\
&= \frac{1}{(\det[\mathrm{I}])^{\frac{3}{2}}}\begin{bmatrix} \left(1+\left(\frac{\partial f}{\partial v}\right)^2\right)\frac{\partial^2 f}{\partial u^2}-\frac{\partial f}{\partial u}\frac{\partial f}{\partial v}\frac{\partial^2 f}{\partial u \partial v} & \left(1+\left(\frac{\partial f}{\partial v}\right)^2\right)\frac{\partial^2 f}{\partial u \partial v}-\frac{\partial f}{\partial u}\frac{\partial f}{\partial v}\frac{\partial^2 f}{\partial v^2} \\ \left(1+\left(\frac{\partial f}{\partial u}\right)^2\right)\frac{\partial^2 f}{\partial u \partial v}-\frac{\partial f}{\partial u}\frac{\partial f}{\partial v}\frac{\partial^2 f}{\partial u^2} & \left(1+\left(\frac{\partial f}{\partial u}\right)^2\right)\frac{\partial^2 f}{\partial v^2}-\frac{\partial f}{\partial u}\frac{\partial f}{\partial v}\frac{\partial^2 f}{\partial u \partial v} \end{bmatrix}
\end{aligned}$$

This gives us a general example where the Weingarten map might not be a symmetric matrix.

20.4 Surfaces Given by an Equation

This is again very general. Note that any Monge patch $(u, v, f(u, v))$ also yields a function $F(x, y, z) = z - f(x, y)$ such that the zero level of F is precisely the Monge patch. This case is also complicated by the fact that while the normal is easy to find, it is proportional to the gradient of F, we don't have a basis for the tangent space without resorting to a Monge patch. This is troublesome, but not insurmountable as we can solve for the derivatives of F. Assume that near some point p we know $\frac{\partial F}{\partial z} \neq 0$, then we can use x, y as coordinates. Our coordinates vector fields look like

$$\begin{aligned}
\frac{\partial \boldsymbol{x}}{\partial u} &= \left(1, 0, \frac{\partial f}{\partial u}\right), \\
\frac{\partial \boldsymbol{x}}{\partial v} &= \left(0, 1, \frac{\partial f}{\partial v}\right)
\end{aligned}$$

where

$$\frac{\partial f}{\partial w} = -\frac{\frac{\partial F}{\partial \boldsymbol{x}}}{\frac{\partial F}{\partial z}}$$

Thus we actually get some explicit formulas

$$\begin{aligned}
\frac{\partial \boldsymbol{x}}{\partial u} &= \left(1, 0, -\frac{\frac{\partial F}{\partial \boldsymbol{x}}}{\frac{\partial F}{\partial z}}\right), \\
\frac{\partial \boldsymbol{x}}{\partial v} &= \left(0, 1, -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}\right).
\end{aligned}$$

We can however describe the second fundamental form without resorting to coordinates. We consider a surface given by an equation

$$F(x, y, z,) = C$$

The normal can be calculated directly as

$$n = \frac{\nabla F}{|\nabla F|}$$

This shows first of all that we have a simple equation defining the tangent space at each point p

$$T_pM = \left\{Y \in \mathbb{R}^3 : Y \cdot \nabla F(p) = 0\right\}$$

Next we make the claim that

$$\begin{aligned}\mathrm{II}(X,Y) &= -\frac{1}{|\nabla F|}\mathrm{I}(D_X \nabla F, Y) \\ &= -\frac{1}{|\nabla F|} Y \cdot D_X \nabla F\end{aligned}$$

where D_X is the directional derivative. We can only evaluate II on tangent vectors, but $Y \cdot D_X \nabla F$ clearly makes sense for all vectors. This has the advantage that we can even use Cartesian coordinates in $\mathbb{R}^3$ for our tangent vectors. First we show that

$$L(X) = -D_X n$$

Select a parametrization $\boldsymbol{x}(u,v)$ such that

$$\frac{\frac{\partial \boldsymbol{x}}{\partial u} \times \frac{\partial \boldsymbol{x}}{\partial v}}{\left|\frac{\partial \boldsymbol{x}}{\partial u} \times \frac{\partial \boldsymbol{x}}{\partial v}\right|} = \frac{\nabla F}{|\nabla F|}$$

The Weingarten equations then tell us that

$$L\left(\frac{\partial \boldsymbol{x}}{\partial w}\right) = -\frac{\partial n}{\partial w} = -D_{\frac{\partial \boldsymbol{x}}{\partial w}} n$$

We can now return to the second fundamental form. Let Y be another tangent vector then, $Y \cdot \nabla F = 0$ so

$$\begin{aligned}-\mathrm{II}(X,Y) &= -\mathrm{I}(L(X), Y) \\ &= Y \cdot D_X n \\ &= Y \cdot \left(D_X \frac{1}{|\nabla F|}\right) \nabla F + Y \cdot \frac{1}{|\nabla F|} D_X \nabla F \\ &= Y \cdot \frac{1}{|\nabla F|} D_X \nabla F\end{aligned}$$

Note that even when X is tangent it does not necessarily follow that $D_X \nabla F$ is also tangent to the surface.

To perform a calculation it is useful to know that

$$\begin{bmatrix} \frac{\partial \nabla F}{\partial \boldsymbol{x}} & \frac{\partial \nabla F}{\partial y} & \frac{\partial \nabla F}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 F}{\partial \boldsymbol{x}^2} & \frac{\partial^2 F}{\partial \boldsymbol{x} \partial y} & \frac{\partial^2 F}{\partial \boldsymbol{x} \partial z} \\ \frac{\partial^2 F}{\partial y \partial \boldsymbol{x}} & \frac{\partial^2 F}{\partial y^2} & \frac{\partial^2 F}{\partial y \partial z} \\ \frac{\partial^2 F}{\partial z \partial \boldsymbol{x}} & \frac{\partial^2 F}{\partial z \partial y} & \frac{\partial^2 F}{\partial z^2} \end{bmatrix}$$

is the second derivative matrix of f.

Exercise: If c is a curve, then it is a curve on $F = C$ if $c(0)$ lies on the surface and $\dot{c} \cdot \nabla F$ vanishes. If c is regular and a curve on $F = C$, then it can be reparametrized to be a geodesic if and only if the tripel product $[\nabla F, \dot{c}, \ddot{c}]$ vanishes.

20.5 Geodesic Coordinates

This is a parametrization having a first fundamental form that looks like:

$$\mathrm{I} = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix}$$

This is as with surfaces of revolution, but now r can depend on both u and v. Using a central v curve, we let the u curves be unit speed geodesics orthogonal to the fixed v curve. They are also often call Fermi coordinates after the famous physicist and seem to have been used in his thesis on general relativity. They were however also used by Gauss. These coordinates will be used time and again to simplify calculations in the proofs of several theorems. The v-curves are well defined as the curves that appear when u is constant. At $u = 0$ the u and v curves are perpendicular by constaruction, so by continuity they can't be tangent as long as u is sufficiently small. This shows that we can always find such parametrizations.

Exercies: Show that

$$\begin{aligned}
\Gamma_{uuu} &= 0 \\
\Gamma_{uvu} &= 0 = \Gamma_{vuu} \\
\Gamma_{vvv} &= r\frac{\partial r}{\partial v} \\
\Gamma_{uvv} &= r\frac{\partial r}{\partial u} = \Gamma_{vuv} \\
\Gamma_{uuv} &= 0 \\
\Gamma_{vvu} &= -r\frac{\partial r}{\partial u}, \\
\Gamma^{u}_{w_1w_2} &= \Gamma_{w_1w_2u} \\
\Gamma^{v}_{w_1w_2} &= \frac{1}{r^2}\Gamma_{w_1w_2v}
\end{aligned}$$

and

$$K = -\frac{\frac{\partial^2 r}{\partial u^2}}{r}$$

20.6 Chebyshev Nets

These correspond to a parametrization where the first fundamental form looks like:

$$\begin{aligned}
\mathrm{I} &= \begin{bmatrix} 1 & c \\ c & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & \cos\theta \\ \cos\theta & 1 \end{bmatrix}, \\
\theta &\in (0,\pi)
\end{aligned}$$

These coordinates can be shown to exist even after describing the parameter curves through a fixed point p. Real life interpretations that are generally brought up are fishnet stockings or nonstretchable cloth tailored to the contours of the body. The idea is to have a material where the fibers are not changed in length or stretched, but are allowed to change their mutual angles.

Exercise: Chebyshev notes have the property that

$$\frac{\partial^2 \boldsymbol{x}}{\partial u \partial v} = 0$$

$$\begin{aligned}
\Gamma_{uvw} &= \Gamma_{uuu} = \Gamma_{vvv} = 0, \\
\Gamma_{uuv} &= -\frac{\partial \theta}{\partial v} \sin \theta \\
\Gamma_{vvu} &= -\frac{\partial \theta}{\partial u} \sin \theta \\
\frac{\partial^2 \theta}{\partial u \partial v} &= -K \sin \theta
\end{aligned}$$

Exercise: Show that the geodesic curvatures κ_u and κ_v of the coordinates curves in a Chebyshev net satisfy

$$\begin{aligned}
\kappa_u &= \frac{\partial \theta}{\partial v}, \\
\kappa_v &= \frac{\partial \theta}{\partial u}.
\end{aligned}$$

Exercise: (Hazzidakis) Show that $\sqrt{\det [\mathrm{I}]} = \sin \theta$, and integrating the Gauss curvature over a coordinate rectangle yields:

$$-\int_{[a,b]\times[c,d]} K \sin \theta du dv = 2\pi - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4$$

where the angles α_i are the interior angles.

20.7 Isothermal Coordinates

These are also more generally known as *conformally flat coordinates* and have a first fundamental form that looks like:

$$\mathrm{I} = \begin{bmatrix} \lambda^2 & 0 \\ 0 & \lambda^2 \end{bmatrix}$$

The proof that these always exist is called the *local uniformization theorem*. It is not a simple result, but the importance of these types of coordinates in the development of both classical and modern surface theory cannot be understated. There is also a global result which we will mention at a later point. Gauss was the first to work with such coordinates, and Riemann also heavily depended on their use. They have the properties that

$$\begin{aligned}
\Gamma_{uuu} &= \frac{\partial \ln \lambda}{\partial u} \\
\Gamma_{uvu} &= \frac{\partial \ln \lambda}{\partial v} = \Gamma_{vuu} \\
\Gamma_{vvv} &= \frac{\partial \ln \lambda}{\partial v} \\
\Gamma_{uvv} &= \frac{\partial \ln \lambda}{\partial u} = \Gamma_{vuv} \\
\Gamma_{uuv} &= -\frac{\partial \ln \lambda}{\partial v} \\
\Gamma_{vvu} &= -\frac{\partial \ln \lambda}{\partial u}, \\
\Gamma_{w_1 w_2}^{w_3} &= \frac{1}{\lambda^2} \Gamma_{w_1 w_2 w_3},
\end{aligned}$$

$$K = -\frac{1}{\lambda^2}\left(\frac{\partial^2 \ln \lambda}{\partial u^2} + \frac{\partial^2 \ln \lambda}{\partial v^2}\right)$$

Exercise: A particularly nice special case occurs when

$$\lambda^2 (u, v) = U^2 (u) + V^2 (v)$$

These types of metrics are called *Liouville metrics*. Compute their Christoffel symbols, Gauss curvature, and show that when geodesics are written as $v (u)$ or $u (v)$ they they solve a separable differential equation. Show also that the geodesic have the property that

$$U^2 \sin^2 \omega - V^2 \cos^2 \omega$$

is constant, where ω is the angle the geodesic forms with the u curves.

21 Constant Gauss Curvature

The goal will be to give a canonical local structure for surfaces with constant Gauss curvature. Given the plethora of surfaces with constant Gauss curvature we seek for the moment only canonical coordinates. Minding was the first person to give the classification of the first fundamental form we obtain below. Riemann extended this result to higher dimensions. We start by studying the case of vanishing Gauss curvature.

Theorem 26 *If a surface has zero Gauss curvature, then it admits Cartesian coordinates.*

Proof. We shall assume that we have geodesic coordinates along a unit speed geodesic. Thus the v-curve described by $u = 0$ is a geodesic, and by definition of geodesic coordinates all of the u-curves are unit speed geodesics.

Assuming $K = 0$, we immediately obtain

$$r (u, v) = r (0, v) + u \frac{\partial r}{\partial u} (0, v)$$

But

$$r(0,v)=\left|\frac{\partial \boldsymbol{x}}{\partial v}\right|=1$$

since the v-curve with $u=0$ is unit speed. Next use that this curve is also a geodesic. The explicit form in u,v parameters for the curve is simply $c(v)=(0,v)$ so all second derivatives vanish and the velocity is pointing in the v direction. Thus the geodesic equations in particular tell us

$$\begin{aligned}
0 &= 0+\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} \Gamma^u_{uu} & \Gamma^u_{uv} \\ \Gamma^u_{vu} & \Gamma^u_{vv} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \Gamma^u_{vv}(0,v) \\
&= \Gamma_{vvu}(0,v) \\
&= \frac{\partial r}{\partial u}(0,v)
\end{aligned}$$

This shows that $r=1$, and hence that we have Cartesian coordinates in a neighborhood of a geodesic. ■

There are similar characterizations for spaces with constant positive or negative curvature. These spaces don't have Cartesian coordinates, but geodesic coordinates near a geodesic are obviously completely determined by the curvature regardless of how the metric might otherwise be viewed.

To be more specific

$$\begin{aligned}
r(0,v) &= \left|\frac{\partial \boldsymbol{x}}{\partial v}\right|=1, \\
\frac{\partial r}{\partial u}(0,v) &= 0
\end{aligned}$$

as we just saw. Given these inital conditions the equation

$$K=-\frac{\frac{\partial^2 r}{\partial u^2}}{r}$$

dictates how r changes as a function of u

$$r(u,v)=\begin{cases} \cos\left(\sqrt{K}u\right) & K>0 \\ \cosh\left(\sqrt{-K}u\right) & K<0 \end{cases}$$

22 The Gauss and Codazzi Equations

Recall the Gauss formulas and Weingarten equations in combined form:

$$\frac{\partial}{\partial w_2}\begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} & n \end{bmatrix}=\begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} & n \end{bmatrix}\left[D_{w_2}\right]$$

Taking one more derivative on both sides yields

$$\begin{aligned}\frac{\partial^2}{\partial w_1 \partial w_2}\begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} & n \end{bmatrix} &= \left(\frac{\partial}{\partial w_1}\begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} & n \end{bmatrix}\right)[D_{w_2}] \\ &\quad + \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} & n \end{bmatrix}\left[\frac{\partial}{\partial w_1} D_{w_2}\right] \\ &= \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} & n \end{bmatrix}[D_{w_1}][D_{w_2}] \\ &\quad + \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} & n \end{bmatrix}\left[\frac{\partial}{\partial w_1} D_{w_2}\right]\end{aligned}$$

Now using that

$$\frac{\partial^2}{\partial w_1 \partial w_2}\begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} & n \end{bmatrix} = \frac{\partial^2}{\partial w_2 \partial w_1}\begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} & n \end{bmatrix}$$

we obtain after writing out the entries in the matrices

$$\begin{aligned}&\begin{bmatrix} \frac{\partial \Gamma^u_{w_2 u}}{\partial w_1} & \frac{\partial \Gamma^u_{w_2 v}}{\partial w_1} & -\frac{\partial L^u_{w_2}}{\partial w_1} \\ \frac{\partial \Gamma^v_{w_2 u}}{\partial w_1} & \frac{\partial \Gamma^v_{w_2 v}}{\partial w_1} & -\frac{\partial L^v_{w_2}}{\partial w_1} \\ \frac{\partial L_{w_2 u}}{\partial w_1} & \frac{\partial L_{w_2 v}}{\partial w_1} & 0 \end{bmatrix} + \begin{bmatrix} \Gamma^u_{w_1 u} & \Gamma^u_{w_1 v} & -L^u_{w_1} \\ \Gamma^v_{w_1 u} & \Gamma^v_{w_1 v} & -L^v_{w_1} \\ L_{w_1 u} & L_{w_1 v} & 0 \end{bmatrix}\begin{bmatrix} \Gamma^u_{w_2 u} & \Gamma^u_{w_2 v} & -L^u_{w_2} \\ \Gamma^v_{w_2 u} & \Gamma^v_{w_2 v} & -L^v_{w_2} \\ L_{w_2 u} & L_{w_2 v} & 0 \end{bmatrix} \\ = &\begin{bmatrix} \frac{\partial \Gamma^u_{w_1 u}}{\partial w_2} & \frac{\partial \Gamma^u_{w_1 v}}{\partial w_2} & -\frac{\partial L^u_{w_1}}{\partial w_2} \\ \frac{\partial \Gamma^v_{w_1 u}}{\partial w_2} & \frac{\partial \Gamma^v_{w_1 v}}{\partial w_2} & -\frac{\partial L^v_{w_1}}{\partial w_2} \\ \frac{\partial L_{w_1 u}}{\partial w_2} & \frac{\partial L_{w_1 v}}{\partial w_2} & 0 \end{bmatrix} + \begin{bmatrix} \Gamma^u_{w_2 u} & \Gamma^u_{w_2 v} & -L^u_{w_2} \\ \Gamma^v_{w_2 u} & \Gamma^v_{w_2 v} & -L^v_{w_2} \\ L_{w_2 u} & L_{w_2 v} & 0 \end{bmatrix}\begin{bmatrix} \Gamma^u_{w_1 u} & \Gamma^u_{w_1 v} & -L^u_{w_1} \\ \Gamma^v_{w_1 u} & \Gamma^v_{w_1 v} & -L^v_{w_1} \\ L_{w_1 u} & L_{w_1 v} & 0 \end{bmatrix}\end{aligned}$$

If we restrict attention to the the general terms of the entries in the first two columns and rows using w_3, w_4 as indices instead of u, v we end up with

$$\frac{\partial \Gamma^{w_4}_{w_2 w_3}}{\partial w_1} + \begin{bmatrix} \Gamma^{w_4}_{w_1 u} & \Gamma^{w_4}_{w_1 v} & -L^{w_4}_{w_1} \end{bmatrix}\begin{bmatrix} \Gamma^u_{w_2 w_3} \\ \Gamma^v_{w_2 w_3} \\ L_{w_2 w_3} \end{bmatrix} = \frac{\partial \Gamma^{w_4}_{w_1 w_3}}{\partial w_2} + \begin{bmatrix} \Gamma^{w_4}_{w_2 u} & \Gamma^{w_4}_{w_2 v} & -L^{w_4}_{w_2} \end{bmatrix}\begin{bmatrix} \Gamma^u_{w_1 w_3} \\ \Gamma^v_{w_1 w_3} \\ L_{w_1 w_3} \end{bmatrix}$$

which can further be rearranged by isolating Γs on one side:

$$\frac{\partial \Gamma^{w_4}_{w_2 w_3}}{\partial w_1} - \frac{\partial \Gamma^{w_4}_{w_1 w_3}}{\partial w_2} + \begin{bmatrix} \Gamma^{w_4}_{w_1 u} & \Gamma^{w_4}_{w_1 v} \end{bmatrix}\begin{bmatrix} \Gamma^u_{w_2 w_3} \\ \Gamma^v_{w_2 w_3} \end{bmatrix} - \begin{bmatrix} \Gamma^{w_4}_{w_2 u} & \Gamma^{w_4}_{w_2 v} \end{bmatrix}\begin{bmatrix} \Gamma^u_{w_1 w_3} \\ \Gamma^v_{w_1 w_3} \end{bmatrix} = L^{w_4}_{w_1} L_{w_2 w_3} - L^{w_4}_{w_2} L_{w_1 w_3},$$

These are called the *Gauss Equations*.

The *Riemann curvature tensor* is defined as the left hand side of the Gauss equations

$$R^{w_4}_{w_1 w_2 w_3} = \frac{\partial \Gamma^{w_4}_{w_2 w_3}}{\partial w_1} - \frac{\partial \Gamma^{w_4}_{w_1 w_3}}{\partial w_2} + \begin{bmatrix} \Gamma^{w_4}_{w_1 u} & \Gamma^{w_4}_{w_1 v} \end{bmatrix}\begin{bmatrix} \Gamma^u_{w_2 w_3} \\ \Gamma^v_{w_2 w_3} \end{bmatrix} - \begin{bmatrix} \Gamma^{w_4}_{w_2 u} & \Gamma^{w_4}_{w_2 v} \end{bmatrix}\begin{bmatrix} \Gamma^u_{w_1 w_3} \\ \Gamma^v_{w_1 w_3} \end{bmatrix}$$

It is clearly an object that can be calculated directly from the first fundamental form, although it is certainly not always easy to do so. But there are some symmetries among the indices that show that there is essentially only one nontrivial curvarure on a surface. On the face of it each index has two possibilities so there are potentially 16 different quantities! Here are some fairly obvious symmetries

$$R^{w_4}_{w_1 w_2 w_3} = -R^{w_4}_{w_2 w_1 w_3},$$

In particular there are at least 8 curvatures that vanish

$$R^{w_4}_{www_3} = 0$$

and up to a sign only 4 left to calculate

$$\begin{aligned} R^u_{uvu} &= -R^u_{vuu}, \\ R^v_{uvu} &= -R^v_{vuu}, \\ R^u_{uvv} &= -R^u_{vuv}, \\ R^v_{uvv} &= -R^v_{vuv} \end{aligned}$$

A slightly less obvious formula is the *Bianchi identity*

$$R^{w_4}_{w_1w_2w_3} + R^{w_4}_{w_3w_1w_2} + R^{w_4}_{w_2w_3w_1} = 0$$

It too follows from the above definition, but with more calculations. Unfortunately it doesn't reduce our job of computing curvatures. The final reduction comes about by constructing

$$R_{w_1w_2w_3w_4} = R^u_{w_1w_2w_3} g_{uw_4} + R^v_{w_1w_2w_3} g_{vw_4}$$

and showing that

$$R_{w_1w_2w_3w_4} = -R_{w_1w_2w_4w_3}.$$

This means that the only possibilities for nontrivial curvatures are

$$R_{uvvu} = R_{vuuv} = -R_{uvuv} = -R_{vuvu}.$$

All of the curvatures of both types turn out to be related to an old friend

Theorem 27 (Theorema Egregium) *The Gauss curvature can be computed knowing only the first fundamental form*

$$\begin{aligned} K &= \frac{R^u_{uvv}}{g_{vv}} = \frac{R^v_{vuu}}{g_{uu}} \\ &= -\frac{R^v_{uvv}}{g_{vu}} = -\frac{R^u_{vuu}}{g_{vu}} \\ &= \frac{R_{uvvu}}{\det[\mathrm{I}]} \end{aligned}$$

Proof. We know that

$$K = L^u_u L^v_v - L^u_v L^v_u$$

and

$$\begin{aligned} \begin{bmatrix} L^u_u & L^u_v \\ L^v_u & L^v_v \end{bmatrix} &= \begin{bmatrix} g^{uu} & g^{uv} \\ g^{vu} & g^{vv} \end{bmatrix} \begin{bmatrix} L_{uu} & L_{uv} \\ L_{vu} & L_{vv} \end{bmatrix} \\ [L] &= [\mathrm{I}]^{-1} [\mathrm{II}] \end{aligned}$$

Now let $u = w_1 = w_4$ and $v = w_2 = w_3$ in the Gauss equation. We take the strange route of calculating so that we end up with second fundamental form terms. This is because [II] is always symmetric, while $[L]$ might not be symmetric. Thus several steps are somewhat simplified.

$$\begin{aligned}
R^u_{uvv} &= L^u_u L_{vv} - L^u_v L_{uv} \\
&= \left(g^{uu} L_{uu} + g^{uv} L_{vu}\right) L_{vv} - \left(g^{uu} L_{uv} + g^{uv} L_{vv}\right) L_{uv} \\
&= g^{uu} \left(L_{uu} L_{vv} - L_{uv} L_{uv}\right) \\
&= g^{uu} \det [\mathrm{II}] \\
&= g^{uu} \det [\mathrm{I}] \det L \\
&= g_{vv} \det L \\
&= g_{vv} K
\end{aligned}$$

The second equality follows by a similar calculation. For the third (and in a similar way fourth) the Gauss equations can again be used to calculate

$$\begin{aligned}
R^v_{uvv} &= L^v_u L_{vv} - L^v_v L_{uv} \\
&= \left(g^{vu} L_{uu} + g^{vv} L_{vu}\right) L_{vv} - \left(g^{vu} L_{uv} + g^{vv} L_{vv}\right) L_{uv} \\
&= g^{vu} \left(L_{uu} L_{vv} - L_{uv} L_{uv}\right) \\
&= -g_{vu} K
\end{aligned}$$

Finally note that

$$\begin{aligned}
R_{uvvu} &= R^u_{uvv} g_{uu} + R^v_{uvv} g_{vu} \\
&= K g_{vv} g_{uu} + R^v_{uvv} g_{vu} \\
&= K \left(g_{vv} g_{uu} - g_{uv} g_{vu}\right) \\
&= K \det [\mathrm{I}]
\end{aligned}$$

■

The other entries in the matrices above reduce to the *Codazzi Equations*

$$\frac{\partial L_{w_2 w_3}}{\partial w_1} + \left[\begin{array}{ccc} L_{w_1 u} & L_{w_1 v} & 0 \end{array}\right] \left[\begin{array}{c} \Gamma^u_{w_2 w_3} \\ \Gamma^v_{w_2 w_3} \\ L_{w_2 w_3} \end{array}\right] = \frac{\partial L_{w_1 w_3}}{\partial w_2} + \left[\begin{array}{ccc} L_{w_2 u} & L_{w_2 v} & 0 \end{array}\right] \left[\begin{array}{c} \Gamma^u_{w_1 w_3} \\ \Gamma^v_{w_1 w_3} \\ L_{w_1 w_3} \end{array}\right]$$

or rearranged

$$\frac{\partial L_{w_2 w_3}}{\partial w_1} - \frac{\partial L_{w_1 w_3}}{\partial w_2} = \left[\begin{array}{cc} L_{w_2 u} & L_{w_2 v} \end{array}\right] \left[\begin{array}{c} \Gamma^u_{w_1 w_3} \\ \Gamma^v_{w_1 w_3} \end{array}\right] - \left[\begin{array}{cc} L_{w_1 u} & L_{w_1 v} \end{array}\right] \left[\begin{array}{c} \Gamma^u_{w_2 w_3} \\ \Gamma^v_{w_2 w_3} \end{array}\right]$$

Exercise: Show that all of the possibilities for the Gauss-Codazzi equations can be reduced to the equations that result from:

$$\frac{\partial^2}{\partial u \partial v} \left[\begin{array}{ccc} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} & n \end{array}\right] = \frac{\partial^2}{\partial v \partial u} \left[\begin{array}{ccc} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} & n \end{array}\right]$$

Seven of these nine equations definitely have the potential to be different. Show further that these equations follow from the three equations:

$$\begin{aligned}
\frac{\partial \Gamma_{vu}^u}{\partial u}-\frac{\partial \Gamma_{uu}^u}{\partial v}+\left[\begin{array}{cc}\Gamma_{uu}^u & \Gamma_{uv}^u\end{array}\right]\left[\begin{array}{c}\Gamma_{vu}^u \\ \Gamma_{vu}^v\end{array}\right]-\left[\begin{array}{cc}\Gamma_{vu}^u & \Gamma_{vv}^u\end{array}\right]\left[\begin{array}{c}\Gamma_{uu}^u \\ \Gamma_{uu}^v\end{array}\right] & = L_u^u L_{vu}-L_v^u L_{uu}, \\
\frac{\partial \Gamma_{vu}^v}{\partial u}-\frac{\partial \Gamma_{uu}^v}{\partial v}+\left[\begin{array}{cc}\Gamma_{uu}^v & \Gamma_{uv}^v\end{array}\right]\left[\begin{array}{c}\Gamma_{vu}^u \\ \Gamma_{vu}^v\end{array}\right]-\left[\begin{array}{cc}\Gamma_{vu}^v & \Gamma_{vv}^v\end{array}\right]\left[\begin{array}{c}\Gamma_{uu}^u \\ \Gamma_{uu}^v\end{array}\right] & = L_u^v L_{vu}-L_v^v L_{uu} \\
\frac{\partial L_{vu}}{\partial u}-\frac{\partial L_{uu}}{\partial v}+\left[\begin{array}{cc}L_{uu} & L_{uv}\end{array}\right]\left[\begin{array}{c}\Gamma_{vu}^u \\ \Gamma_{vu}^v\end{array}\right]-\left[\begin{array}{cc}L_{vu} & L_{vv}\end{array}\right]\left[\begin{array}{c}\Gamma_{uu}^u \\ \Gamma_{uu}^v\end{array}\right] & = 0
\end{aligned}$$

Exercise: Use the Codazzi equations to show that if the principal curvatures $\kappa_1 = \kappa_2$ are equal on some connected domain, then they are constant.

Exercise: If the principal curvatures κ_1 and κ_2 are not equal on some part of the surface then we can construct an orthogonal parametrization where the tangent fields are principal directions or said differently the coordinate curves are lines of curvature:

$$\begin{aligned}
L\left(\frac{\partial \boldsymbol{x}}{\partial u}\right) & = \kappa_1 \frac{\partial \boldsymbol{x}}{\partial u}, \\
L\left(\frac{\partial \boldsymbol{x}}{\partial v}\right) & = \kappa_2 \frac{\partial \boldsymbol{x}}{\partial v}.
\end{aligned}$$

Show that the Codazzi Equations can be written as

$$\begin{aligned}
\frac{\partial \kappa_1}{\partial v} & = \frac{1}{2}\left(\kappa_2-\kappa_1\right) \frac{\partial \ln g_{uu}}{\partial v}, \\
\frac{\partial \kappa_2}{\partial u} & = \frac{1}{2}\left(\kappa_1-\kappa_2\right) \frac{\partial \ln g_{vv}}{\partial u}.
\end{aligned}$$

Exercise: (Hilbert) The goal is to show that if there is a point p on a surface with positive Gauss curvature, where κ_1 has a maximum and κ_2 a minimum, then the surface has constant principal curvatures. We assume otherwise, in particular $\kappa_1(p) > \kappa_2(p)$, and construct a coordinate system where the coordinate curves are lines of curvature. At p we have

$$\begin{aligned}
\frac{\partial \kappa_1}{\partial u} & = \frac{\partial \kappa_1}{\partial v}=0, \frac{\partial^2 \kappa_1}{\partial v^2} \leq 0, \\
\frac{\partial \kappa_2}{\partial u} & = \frac{\partial \kappa_2}{\partial v}=0, \frac{\partial^2 \kappa_2}{\partial u^2} \geq 0.
\end{aligned}$$

Using the Codazzi equations from the previous exercise show that at p

$$\frac{\partial \ln g_{uu}}{\partial v}=0=\frac{\partial \ln g_{vv}}{\partial u}$$

and after differentiation also at p that

$$\frac{\partial^2 \ln g_{uu}}{\partial v^2} \geq 0, \frac{\partial^2 \ln g_{vv}}{\partial u^2} \geq 0$$

Next show that at p

$$K=-\frac{1}{2}\left(\frac{1}{g_{vv}}\frac{\partial^2\ln g_{uu}}{\partial v^2}+\frac{1}{g_{uu}}\frac{\partial^2\ln g_{vv}}{\partial u^2}\right)\leq 0$$

Exercise: Using the developments in the previous exercise show that a surface with constant principal curvatures must be part of a plane, sphere, or right circular cylinder. Note that the two former cases happen when the principal curvatures are equal.

Exercise: Show that if we have a parametrization $\boldsymbol{x}(u,v)$ where all geodesics are straight lines

$$au+bv+c=0$$

then

$$\begin{aligned}\Gamma_{uu}^{v} &= \Gamma_{vv}^{u}=0,\\ \Gamma_{uu}^{u} &= 2\Gamma_{uv}^{v},\\ \Gamma_{vv}^{v} &= 2\Gamma_{uv}^{u}\end{aligned}$$

Use the Gauss equations

$$\begin{aligned}g_{vv}K &= R_{uvv}^{u}\\ g_{uu}K &= R_{vuu}^{v}\\ g_{vu}K &= -R_{uvv}^{v}\\ g_{vu}K &= -R_{vuu}^{u}\end{aligned}$$

together with the definitions of $R_{w_1w_2w_3}^{w_4}$ to show that

$$0=\begin{bmatrix}\frac{\partial K}{\partial v} & -\frac{\partial K}{\partial u}\end{bmatrix}\begin{bmatrix}g_{uu} & g_{uv}\\ g_{vu} & g_{vv}\end{bmatrix}$$

and conclude that the Gauss curvature is constant.

23 Local Gauss-Bonnet

Inspired by the idea that the integral of the curvature of a planar curve is related to how the tangent moves we try to prove a similar result on surfaces. First we point out that we cannot expect the same theorem to hold. Consider the equator on a sphere. This curve is a geodesic and so has no geodesic curvature, on the other hand the tangent field clearly turns around 360 degrees. Another similar example comes from a right circular cylinder where meridians are all geodesics and also have tangents that turn 360 degress.

Throughout this section we assume that a parametrized surface is given:

$$\boldsymbol{x}(u,v):(a_u,b_u)\times(a_v,b_v)\to\mathbb{R}^3$$

where the domain is a rectangle. The key is that the domain should not have any holes in it. We further assume that we have a smaller domain

$$R\subset(a_u,b_u)\times(a_v,b_v)$$

that is bounded by a piecewise smooth curve

$$(u(s), v(s)) : [0, L] \to (a_u, b_u) \times (a_v, b_v)$$

running counter clockwise in the plane and such that $c(s) == \boldsymbol{x}(u(s), v(s))$ is a unit speed.

Integration of functions on the surface is done by defining a suitable integral using the parametrization. To make this invariant under parametrizations we define

$$\int_{\boldsymbol{x}(R)} f dA = \int_R f(u, v) \sqrt{\det[\mathrm{I}]} dudv = \int_R f(u, v) \left| \frac{\partial \boldsymbol{x}}{\partial u} \times \frac{\partial \boldsymbol{x}}{\partial v} \right| dudv$$

This ensures that if we use a different parametrization (s, t) where $\boldsymbol{x}(Q) = \boldsymbol{x}(R)$, then

$$\int_R f(u, v) \left| \frac{\partial \boldsymbol{x}}{\partial u} \times \frac{\partial \boldsymbol{x}}{\partial v} \right| dudv = \int_Q f(s, t) \left| \frac{\partial \boldsymbol{x}}{\partial s} \times \frac{\partial \boldsymbol{x}}{\partial t} \right| dsdt.$$

We start by calculating the geodesic curvature of c assming further that the parametrization gives a geodesic coodinate system

$$\mathrm{I} = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix}$$

Lemma 28 *Let θ be the angle between* c *and the* u *curves, then*

$$\kappa_g = \frac{d\theta}{ds} + \frac{\partial r}{\partial u} \frac{1}{r} \sin\theta.$$

Proof. We start by pointing out that the velocity is

$$\begin{aligned} \frac{dc}{ds} &= \frac{du}{ds} \frac{\partial \boldsymbol{x}}{\partial u} + \frac{dv}{ds} \frac{\partial \boldsymbol{x}}{\partial v} \\ &= \cos\theta \frac{\partial \boldsymbol{x}}{\partial u} + \frac{1}{r} \sin\theta \frac{\partial \boldsymbol{x}}{\partial v} \end{aligned}$$

The natural unit normal field to c in the surface is then given by

$$S = -\sin\theta \frac{\partial \boldsymbol{x}}{\partial u} + \frac{1}{r} \cos\theta \frac{\partial \boldsymbol{x}}{\partial v}$$

Our geodesic curvature

$$\begin{aligned} \kappa_g &= \mathrm{I}\left(S, \ddot{c}^{\mathrm{I}}\right) \\ &= S \cdot \left(\left(\frac{d^2 u}{ds^2} + \Gamma^u \left(\frac{dc}{ds}, \frac{dc}{ds} \right) \right) \frac{\partial \boldsymbol{x}}{\partial u} + \left(\frac{d^2 v}{ds^2} + \Gamma^v \left(\frac{dc}{ds}, \frac{dc}{ds} \right) \right) \frac{\partial \boldsymbol{x}}{\partial v} \right) \\ &= -\sin\theta \left(\frac{d^2 u}{ds^2} + \Gamma^u \left(\frac{dc}{ds}, \frac{dc}{ds} \right) \right) + r^2 \frac{1}{r} \cos\theta \left(\frac{d^2 v}{ds^2} + \Gamma^v \left(\frac{dc}{ds}, \frac{dc}{ds} \right) \right) \\ &= -\sin\theta \left(\frac{d^2 u}{ds^2} + \Gamma^u \left(\frac{dc}{ds}, \frac{dc}{ds} \right) \right) + r \cos\theta \left(\frac{d^2 v}{ds^2} + \Gamma^v \left(\frac{dc}{ds}, \frac{dc}{ds} \right) \right) \end{aligned}$$

We further have

$$
\begin{aligned}
\frac{d^2u}{ds^2} &= \frac{d\cos\theta}{ds} = -\sin\theta\frac{d\theta}{ds}, \\
\frac{d^2v}{ds^2} &= \frac{d\frac{1}{r}\sin\theta}{ds} \\
&= \frac{-1}{r^2}\frac{dr}{ds}\sin\theta + \frac{1}{r}\cos\theta\frac{d\theta}{ds} \\
&= \frac{-1}{r^2}\left(\frac{\partial r}{\partial u}\frac{du}{ds} + \frac{\partial r}{\partial v}\frac{dv}{ds}\right)\sin\theta + \frac{1}{r}\cos\theta\frac{d\theta}{ds} \\
&= \frac{-1}{r^2}\frac{\partial r}{\partial u}\cos\theta\sin\theta + \frac{-1}{r^3}\frac{\partial r}{\partial v}\sin^2\theta + \frac{1}{r}\cos\theta\frac{d\theta}{ds}
\end{aligned}
$$

And the Christoffel symbols are not hard to compute

$$
\begin{aligned}
\Gamma^u\left(\frac{dc}{ds},\frac{dc}{ds}\right) &= \Gamma^u_{vv}\left(\frac{dv}{ds}\right)^2 \\
&= -r\frac{\partial r}{\partial u}\frac{1}{r^2}\sin^2\theta \\
&= \frac{-1}{r}\frac{\partial r}{\partial u}\sin^2\theta
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^v\left(\frac{dc}{ds},\frac{dc}{ds}\right) &= 2\Gamma^v_{uv}\frac{du}{ds}\frac{dv}{ds} + \Gamma^v_{uv}\left(\frac{dv}{ds}\right)^2 \\
&= \frac{2}{r}\frac{\partial r}{\partial u}\frac{du}{ds}\frac{dv}{ds} + \frac{1}{r}\frac{\partial r}{\partial v}\left(\frac{dv}{ds}\right)^2 \\
&= \frac{2}{r^2}\frac{\partial r}{\partial u}\sin\theta\cos\theta + \frac{1}{r^3}\frac{\partial r}{\partial v}\sin^2\theta
\end{aligned}
$$

Thus

$$
\begin{aligned}
\kappa_g &= -\sin\theta\left(-\sin\theta\frac{d\theta}{ds} - \frac{1}{r}\frac{\partial r}{\partial u}\sin^2\theta\right) + r\cos\theta\left(\frac{1}{r}\cos\theta\frac{d\theta}{ds} + \frac{1}{r^2}\frac{\partial r}{\partial u}\sin\theta\cos\theta\right) \\
&= \frac{d\theta}{ds} + \frac{1}{r}\frac{\partial r}{\partial u}\sin^3\theta + \frac{1}{r}\frac{\partial r}{\partial u}\sin\theta\cos^2\theta \\
&= \frac{d\theta}{ds} + \frac{\partial r}{\partial u}\frac{1}{r}\sin\theta
\end{aligned}
$$

■

We can now prove the local Gauss-Bonnet theorem. It is stated in the way that Gauss and Bonnet proved it. Gauss considered regions bounded by geodesics thus eliminating the geodesic curvature, while Bonnet presented the version given below.

Theorem 29 (Gauss-Bonnet) *Assume as in the above Lemma that the parametrization gives a geodesic coordinate system. Let θ_i be the exterior angles at the points* *where c has vertices, then*

$$\int_{\boldsymbol{x}(R)} KdA + \int_0^L \kappa_g ds = 2\pi - \sum \theta_i$$

Proof.

$$\begin{aligned}
\int_{\boldsymbol{x}(R)} KdA &= \int_R K\sqrt{\det[\mathrm{I}]}dudv \\
&= -\int_R \frac{\frac{\partial^2 r}{\partial u^2}}{r} rdudv \\
&= -\int_R \frac{\partial^2 r}{\partial u^2} dudv
\end{aligned}$$

The last integral can be turned into a line integral if we use Green's theorem

$$\int_R \frac{\partial^2 r}{\partial u^2} dudv = \int_{\partial R} \frac{\partial r}{\partial u} dv$$

This line integral can now be recognized as one of the terms in the formula for the geodesic curvature

$$\begin{aligned}
\int_{\partial R} \frac{\partial r}{\partial u} dv &= \int_0^L \frac{\partial r}{\partial u} \frac{dv}{ds} ds \\
&= \int_0^L \frac{\partial r}{\partial u} \frac{1}{r} \sin\theta ds \\
&= \int_0^L \left(k_g - \frac{d\theta}{ds} \right) ds \\
&= \int_0^L \kappa_g ds - \int_0^L \frac{d\theta}{ds} ds
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
\int_{\boldsymbol{x}(R)} KdA + \int_0^L \kappa_g ds &= -\int_R \frac{\partial^2 r}{\partial u^2} dudv + \int_{\partial R} \frac{\partial r}{\partial u} dv + \int_0^L \frac{d\theta}{ds} ds \\
&= \int_0^L \frac{d\theta}{ds} ds \\
&= 2\pi - \sum \theta_i
\end{aligned}$$

■

Clearly there are subtle things about the regions R we are allowed to use. Aside from the topological restriction on R there is also an orientation choice (counter clockwise) for ∂R in Green's theorem. If we reverse that orientation there is a sign change, and the geodesic curvature also changes sign when we run backwards.

We used rather special coordinates as well, but it is possible to extend the proof to work for all coordinate systems. The same strategy even works, but is complicated by the nasty formula we have for the Gauss curvature in general coordinates. Using Cartan's approach with selecting orthonormal frames rather than special coordinates makes for a fairly simple proof that works within all coordinate systems. This is exploited in an exercise below, but to keep things in line with what we have already covered we still retsrict attention to how this works in relation to a parametrization.

Let us now return to our examples from above. Without geodesic curvature and exterior angles we expect to end up with the formula

$$\int_{\boldsymbol{x}(R)} KdA = 2\pi$$

But there has to be a region R bounding the closed geodesic. On the sphere we can clearly use the upper hemisphere. As $K = 1$ we end up with the well known fact that the upper hemisphere has area 2π. On the cylinder, however, there is no reasonable region bounding the meridian despite the fact that we have a valid geodesic coordinate system. The issue is that the bounding curve cannot be set up to be a closed curve in a parametrization where there is a rectangle containing the curve.

It is possible to modify the Gauss-Bonnet formula so that more general regions can be used in the statement, but it requires topological information about the region R. This will be studied in detail later and also in some interesting cases in the exercises below.

Another very important observation about our proof is that it only referred to quantities related to the first fundamental form. In fact, the result holds without further ado for generalized surfaces and abstract surfaces as well, again with the proviso of working within coordinates and regions without holes.

It is, however, possible to also get the second fundamental form into the picture if we recall that

$$K\left|\frac{\partial \boldsymbol{x}}{\partial u}\times\frac{\partial \boldsymbol{x}}{\partial v}\right| = \left(\frac{\partial n}{\partial u}\times\frac{\partial n}{\partial v}\right)\cdot n = \pm\left|\frac{\partial n}{\partial u}\times\frac{\partial n}{\partial v}\right|$$

then we see that $\int_R KdA$ also measures the signed area of the spherical image traced by the normal vector, or the image of the Gauss map.

Exercise: Consider a surface of revolution and two meridians c_1 and c_2 on it. These meridians bound a band or annular region $\boldsymbol{x}(R)$. By subdividing the region and using proper orientations and parametrizations on the curves show that

$$\int_{\boldsymbol{x}(R)} KdA = \int_{c_1}\kappa_g ds_1 - \int_{c_2}\kappa_g ds_2.$$

Exercise: Generalize the previous exercise to regions that are bounded both on the inside and outside by smooth (or even piecewise smooth) curves.

Exercise: Assume now that the parametrization is not geodesic. Create tangent vector fields E_1 and E_2 forming an orthonormal basis for the tanget space everywhere with the further property that E_1 is proportional to the first tangent field $\frac{\partial \boldsymbol{x}}{\partial u}$ and

$$E_1\times E_2 = n = \frac{\frac{\partial \boldsymbol{x}}{\partial u}\times\frac{\partial \boldsymbol{x}}{\partial v}}{\left|\frac{\partial \boldsymbol{x}}{\partial u}\times\frac{\partial \boldsymbol{x}}{\partial v}\right|}.$$

First show that

$$\begin{aligned}
\frac{\partial}{\partial w}\left[\begin{array}{ccc} E_1 & E_2 & n \end{array}\right] &= \left[\begin{array}{ccc} E_1 & E_2 & n \end{array}\right][D_w], \\
[D_w] &= \left[\begin{array}{ccc} 0 & -\phi_w & -\phi_{w1} \\ \phi_w & 0 & -\phi_{w2} \\ \phi_{w1} & \phi_{w2} & 0 \end{array}\right],
\end{aligned}$$

and identify the entries with dots products $X \cdot \frac{\partial Y}{\partial w}$ where X, Y are elements of the frame. Next, show that

$$\frac{\partial}{\partial u}[D_v] - \frac{\partial}{\partial v}[D_u] + [D_u][D_v] - [D_v][D_u] = 0,$$

Separating out the middle entry in the first row of that equation we get

$$\frac{\partial \phi_v}{\partial u} - \frac{\partial \phi_u}{\partial v} = \phi_{u2}\phi_{v1} - \phi_{v2}\phi_{u1}$$

Using the Weingarten equations and letting $[L]$ be the matrix of the Weingarten map with respect to E_1, E_2 show that

$$[L]\left[\begin{array}{cc} E_1 & E_2 \end{array}\right]^t \left[\begin{array}{cc} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{array}\right] = \left[\begin{array}{cc} \phi_{u1} & \phi_{v1} \\ \phi_{u2} & \phi_{v2} \end{array}\right]$$

and

$$K\sqrt{\det[\mathrm{I}]} = \phi_{u1}\phi_{v2} - \phi_{u2}\phi_{v1}$$

Thus

$$\begin{aligned}
\int_{\boldsymbol{x}(R)} K dA &= -\int_R \left(\frac{\partial \phi_v}{\partial u} - \frac{\partial \phi_u}{\partial v}\right) du dv \\
&= -\int_c \phi_u du + \phi_v dv
\end{aligned}$$

Finally prove the Gauss-Bonnet theorem by establishing

$$\int_c \phi_u du + \phi_v dv = \int \left(k_g - \frac{d\theta}{ds}\right) ds$$

where θ is the angle with E_1 or $\frac{\partial \boldsymbol{x}}{\partial u}$. To aid the last calculation show that

$$\begin{aligned}
\frac{dc}{ds} &= \cos\theta E_1 + \sin\theta E_2, \\
S &= -\sin\theta E_1 + \cos\theta E_2, \\
\frac{d^2c}{ds^2} &= S\frac{d\theta}{ds} - \sin\theta\left(\cos\theta\phi_u + \sin\theta\phi_v\right)E_1 \\
&\quad + \cos\theta\left(\cos\theta\phi_u + \sin\theta\phi_v\right)E_2 + ()\, n
\end{aligned}$$

where the coefficient in front of n is irrelevant for computing the inner product with S and hnce the geodesic curvature.

24 Symmetries

So far we've discussed how quatities remain invariant if we change parameters at a given point. A *symmetry* is a transformation of a surface that shows that our geometrically defined quantities are the same at different points p and q if the symmetry moves p to q. Basic examples of symmetries are rotations around the z axis for surfaces of revolution around the z axis, or mirror symmetries in meridians on a surface of revolution. The sphere has an even larger number of symmetries as it is a surface of revolution around any line through the origin. The plane also has rotational and mirror symmetries, but in addition translations.

Some other examples we have seen that are less obvious came from geodesic coordinates. There we saw that if we select geodesic coodinates around a geodesic in a space of constant Gauss curvature K, then we always get the same answer. This means that locally any space of constant Gauss curvature must look the same everywhere, and even that any two spaces of the same constant Gauss curvature are locally the same.

Symmetries are usually called *isometries* as they are defined as those maps that preserve metric quatities, i.e., the first fundamental form. An alternate definition more in spirit with the definition of a linear map is to see what it should do to a geodesic. Linear maps preserve lines but not necessarily speed. Symmetries preserve geodesics as well as their speeds. In other words if $F : M \to M$ is a map and $c(t) : I \to M$ is a geodesic, then $(F \circ c)(t) : I \to M$ should also be a geodesic with the same speed as c. This condition is clearly not desirable as we could never check whether a transformation is an isometry without first finding the geodesics. Let us check what it means for F to preserve the speed:

$$\begin{aligned} \left|\frac{dc}{dt}\right| &= \left|\frac{d(F \circ c)}{dt}\right| \\ &= \left|DF\left(\frac{dc}{dt}\right)\right| \end{aligned}$$

The second line is the chain rule for derivatives. The first reduction we can make is to substitute $\dot{c} = V$ with any tangent vector V. After squaring the norms we must check that for all tangent vectors:

$$\mathrm{I}(V, V) = \mathrm{I}(DF(V), DF(V)).$$

To be specific we have to pass to a parametrization $\boldsymbol{x}(u,v)$ and then figure out how F maps the parameters $F(u,v) = (F^u(u,v), F^v(u,v))$. DF is then the matrix of first derivatives

$$[DF] = \begin{bmatrix} \frac{\partial F^u}{\partial u} & \frac{\partial F^u}{\partial v} \\ \frac{\partial F^v}{\partial u} & \frac{\partial F^v}{\partial v} \end{bmatrix}$$

and we have to check that if $V = V^u \frac{\partial \boldsymbol{x}}{\partial u} + V^v \frac{\partial \boldsymbol{x}}{\partial v}$ and $F(p) = q$, then

$$\begin{aligned}
\mathrm{I}(V,V) &= \begin{bmatrix} V^u & V^v \end{bmatrix} \begin{bmatrix} g_{uu}(p) & g_{uv}(p) \\ g_{vu}(p) & g_{vv}(p) \end{bmatrix} \begin{bmatrix} V^u \\ V^v \end{bmatrix} \\
&= \begin{bmatrix} V^u \\ V^v \end{bmatrix}^t \begin{bmatrix} g_{uu}(p) & g_{uv}(p) \\ g_{vu}(p) & g_{vv}(p) \end{bmatrix} \begin{bmatrix} V^u \\ V^v \end{bmatrix} \\
&= \left(\begin{bmatrix} \frac{\partial F^u}{\partial u} & \frac{\partial F^u}{\partial v} \\ \frac{\partial F^v}{\partial u} & \frac{\partial F^v}{\partial v} \end{bmatrix} \begin{bmatrix} V^u \\ V^v \end{bmatrix} \right)^t \begin{bmatrix} g_{uu}(q) & g_{uv}(q) \\ g_{vu}(q) & g_{vv}(q) \end{bmatrix} \begin{bmatrix} \frac{\partial F^u}{\partial u} & \frac{\partial F^u}{\partial v} \\ \frac{\partial F^v}{\partial u} & \frac{\partial F^v}{\partial v} \end{bmatrix} \begin{bmatrix} V^u \\ V^v \end{bmatrix} \\
&= \begin{bmatrix} V^u & V^v \end{bmatrix} \begin{bmatrix} \frac{\partial F^u}{\partial u} & \frac{\partial F^v}{\partial u} \\ \frac{\partial F^u}{\partial v} & \frac{\partial F^v}{\partial v} \end{bmatrix} \begin{bmatrix} g_{uu}(q) & g_{uv}(q) \\ g_{vu}(q) & g_{vv}(q) \end{bmatrix} \begin{bmatrix} \frac{\partial F^u}{\partial u} & \frac{\partial F^u}{\partial v} \\ \frac{\partial F^v}{\partial u} & \frac{\partial F^v}{\partial v} \end{bmatrix} \begin{bmatrix} V^u \\ V^v \end{bmatrix} \\
&= \mathrm{I}(DF(V), DF(V)).
\end{aligned}$$

This comes down to checking that

$$\begin{bmatrix} g_{uu}(p) & g_{uv}(p) \\ g_{vu}(p) & g_{vv}(p) \end{bmatrix} = \begin{bmatrix} \frac{\partial F^u}{\partial u} & \frac{\partial F^v}{\partial u} \\ \frac{\partial F^u}{\partial v} & \frac{\partial F^v}{\partial v} \end{bmatrix} \begin{bmatrix} g_{uu}(q) & g_{uv}(q) \\ g_{vu}(q) & g_{vv}(q) \end{bmatrix} \begin{bmatrix} \frac{\partial F^u}{\partial u} & \frac{\partial F^u}{\partial v} \\ \frac{\partial F^v}{\partial u} & \frac{\partial F^v}{\partial v} \end{bmatrix}$$

or in other words that DF preserves the first fundamental form when mapping from p to q.

This is still a bit of a mouthful, but no further reductions are possible. The nice result is that any transformation that preserves the first fundamental form as just described will also preserve geodesics. Thus preserving speeds of curves is enough to tell us that geodesics are also preserved. Moreover, checking that speeds are preserved comes down to checking a matrix identity.

Theorem 30 *A symmetry maps geodesics to geodesics and preserves Gauss curvature.*

Proof. Let $c(t)$ be a geodesic and F a symmetry. The geodesic equation depends only on the first fundamental form. By definition symmetries preserve the first fundamental form, thus $F(c(t))$ must also be a geodesic.

Next assume that F is a symmetry such that $F(p) = q$. Again F preserves the first fundamental form so the Gauss curvatures must again be the same. ■

It is possible to construct symmetries that do not preserve the second fundamental form. The simplest example is to image a flat tarp or blanket, here all points have vanishing second fundamental form and also there are symmeteris between all points. Now lift one side of the tarp. Part of it will still be flat on the ground, while the part that's lifted off the ground is curved. The first fundamental form has not changed but the curved part will now have nonzero entries in the second fundamental form.

In order to actually find the set of all symmetries we'd have to somehow solve the equation above. This is not always possible. But as with geodesics there are some uniqueness results that will help.

Theorem 31 *If F and G satisfy $F(p) = G(p)$ and $DF(p) = DG(p)$ then $F = G$ in a neighborhood of p.*

Proof. We just saw that symmetries preserve geodesics. So if $c(t)$ is a geodesic with $c(0) = p$, then $F(c(t))$ and $G(c(t))$ are both geodesics. Moreover they have the same inital values

$$\begin{aligned} F(c(0)) &= F(p), \\ G(c(0)) &= G(p), \\ \frac{d}{dt}F(c(t))|_{t=0} &= DF(\dot{c}(0)), \\ \frac{d}{dt}G(c(t))|_{t=0} &= DG(\dot{c}(0)). \end{aligned}$$

This means that $F(c(t)) = G(c(t))$. By varying the initial velocity of $\dot{c}(0)$ we can reach all points in a neighborhood of p. ■

Often the best method for finding symmetries is to make educated guesses based on what the metric looks like. One general guide line for creating symmetries is the observation that if the first fundamental form doesn't depend on a specific variable such as v, then translations in that variable will be symmetries. This is exemplified by surfaces of revolution where the metric doesn't depend on θ. Translations in θ are the same as rotations by a fixed angle and we know that such transformations are symmetries. Note that reflections in such a parameter where v is mapped to $v_0 - v$ will also be symmetries in such a case.

Here is a slightly more surprising relationship between geodesics and symmetries.

Theorem 32 *Let F be a nontrivial symmetry and $c(t)$ a unit speed curve such that $F(c(t)) = c(t)$ for all t, then $c(t)$ is a geodesic.*

Proof. Since F is a symmetry and it preserves c we must also have that it preserves its velocity and tangential acceleration

$$\begin{aligned} DF(\dot{c}(t)) &= \dot{c}(t), \\ DF\left(\ddot{c}^{\mathrm{I}}(t)\right) &= \ddot{c}^{\mathrm{I}}(t). \end{aligned}$$

As c is unit speed we have $\dot{c} \cdot \ddot{c}^{\mathrm{I}} = 0$. If $\ddot{c}^{\mathrm{I}}(t) \neq 0$, then DF preserves $c(t)$ as well as the basis $\dot{c}(t), \ddot{c}^{\mathrm{I}}(t)$ for the tangent space at $c(t)$. By the uniqueness result above this shows that F is the identity map as that map is always a symmetry that fixes any point and basis. But this contradicts that F is nontrivial. ■

Note that circles in the plane are preserved by rotations, but they are not fixed, nor are they geodesics. The picture we should have in mind for such symmetries and geodesics is a mirror symmetry in a line in the plane, or a mirror symmetry in a great circle on the sphere.

There are some further surprises along these lines.

Theorem 33 *If all geodesics are preserved by nontrivial symmetries, then the space has constant Gauss curvature. Conversely, if the space has constant Gauss curvature, then all geodesics are fixed by symmetries.*

Below we shall construct constant Gauss curvature spaces, and show that the symmetries and geodesics have these properties. More generally one will have to show that locally all constant Gauss curvature spaces can be accounted ofr knowing only these examples.

For now lets us discuss the symmetries of the plane and sphere.....

25 The Upper Half Plane

A particularly interesting case to study is the upper half plane where we don't have much intuition about what might happen. This section is devoted to calculating the symmetries, geodesics, and curvature of this space. Recall that this is an assignment of a first fundamental form

$$\mathrm{I} = \begin{bmatrix} \frac{1}{v^2} & 0 \\ 0 & \frac{1}{v^2} \end{bmatrix}$$

to the tangent space at each point $p = (u, v) \in H = \{(u, v) : v > 0\}$. We saw that it was possible to construct a surface of revolution

$$\begin{aligned} \boldsymbol{x}(t, \theta) &= \left(\frac{1}{t}\cos(\theta), \frac{1}{t}\sin(\theta), h(t)\right), \\ \dot{h} &= \sqrt{1 - \frac{1}{t^2}}\frac{1}{t} \end{aligned}$$

whose first fundamental form is

$$\mathrm{I} = \begin{bmatrix} \frac{1}{t^2} & 0 \\ 0 & \frac{1}{t^2} \end{bmatrix}.$$

This might give us a local picture of the upper half plane but it doesn't really help that much.

Below we shall find the symmetries and geodesics by solving the equations we have for these objects. As we shall see, even in a case where the metric is relatively simple, this is a very difficult task.

25.1 The Symmetries of H

Our first observation is that the first fundamental form doesn't depend on u, so the transformations

$$\begin{aligned} F &: H \to H \\ F(u, v) &= (u + u_0, v) \end{aligned}$$

must be symmetries. Let us check what it means for a general transformation $F(u, v) =$

$(F^u(u,v), F^v(u,v))$ to be a symmetry. Let $p = (u,v)$ and $q = F(u,v)$

$$\begin{aligned}
\begin{bmatrix} \frac{1}{v^2} & 0 \\ 0 & \frac{1}{v^2} \end{bmatrix} &= \begin{bmatrix} \frac{\partial F^u}{\partial u} & \frac{\partial F^v}{\partial u} \\ \frac{\partial F^u}{\partial v} & \frac{\partial F^v}{\partial v} \end{bmatrix} \begin{bmatrix} \frac{1}{(F^v(u,v))^2} & 0 \\ 0 & \frac{1}{(F^v(u,v))^2} \end{bmatrix} \begin{bmatrix} \frac{\partial F^u}{\partial u} & \frac{\partial F^u}{\partial v} \\ \frac{\partial F^v}{\partial u} & \frac{\partial F^v}{\partial v} \end{bmatrix} \\
&= \frac{1}{(F^v(u,v))^2} \begin{bmatrix} \frac{\partial F^u}{\partial u} & \frac{\partial F^v}{\partial u} \\ \frac{\partial F^u}{\partial v} & \frac{\partial F^v}{\partial v} \end{bmatrix} \begin{bmatrix} \frac{\partial F^u}{\partial u} & \frac{\partial F^u}{\partial v} \\ \frac{\partial F^v}{\partial u} & \frac{\partial F^v}{\partial v} \end{bmatrix} \\
&= \frac{1}{(F^v(u,v))^2} \begin{bmatrix} \left(\frac{\partial F^u}{\partial u}\right)^2 + \left(\frac{\partial F^v}{\partial u}\right)^2 & \frac{\partial F^u}{\partial v}\frac{\partial F^u}{\partial u} + \frac{\partial F^v}{\partial u}\frac{\partial F^v}{\partial v} \\ \frac{\partial F^u}{\partial v}\frac{\partial F^u}{\partial u} + \frac{\partial F^v}{\partial u}\frac{\partial F^v}{\partial v} & \left(\frac{\partial F^u}{\partial v}\right)^2 + \left(\frac{\partial F^v}{\partial v}\right)^2 \end{bmatrix}
\end{aligned}$$

This tells us

$$\begin{aligned}
\left(\frac{\partial F^u}{\partial v}\right)^2 + \left(\frac{\partial F^v}{\partial v}\right)^2 &= \frac{(F^v(u,v))^2}{v^2} = \left(\frac{\partial F^u}{\partial u}\right)^2 + \left(\frac{\partial F^v}{\partial u}\right)^2, \\
\frac{\partial F^u}{\partial v}\frac{\partial F^u}{\partial u} + \frac{\partial F^v}{\partial u}\frac{\partial F^v}{\partial v} &= 0
\end{aligned}$$

In particular, we see that the translations $F(u,v) = (u + u_0, v)$ really are symmetries. Could there be symmetries where F only depends on what happens to v? We can check an even more general situation: $F(u,v) = (u, f(u,v))$ where the equations reduce to

$$\begin{aligned}
\left(\frac{\partial f}{\partial v}\right)^2 &= \left(\frac{f(u,v)}{v}\right)^2 = 1 + \left(\frac{\partial f}{\partial u}\right)^2, \\
\frac{\partial f}{\partial u}\frac{\partial f}{\partial v} &= 0
\end{aligned}$$

So first we note that $\frac{\partial f}{\partial v} \pm 0$ from the first equation, the second then implies that $\frac{\partial f}{\partial u} = 0$, which then from the first equation gives us that $f(u,v) = v$ is the only possibility. Next let's try $F(u,v) = (g(u), f(v))$. This reduces to

$$\left(\frac{df}{dv}\right)^2 = \left(\frac{f(v)}{v}\right)^2 = \left(\frac{dg}{du}\right)^2$$

So all transformations of the form $F(u,v) = c(u,v)$ where $c > 0$ is constant are also symmetries. Note that these maps are only similarities in the Euclidean metric, but have now become genuine symmetries. This might give the idea to check maps of the form $F(u,v) = h(u,v)(u,v)$. This is still a bit general so we make the reasonable assumption that h doesn't depend on the direction of (u,v), i.e., $F(u,v) = h(u^2 + v^2)(u,v)$. Then

$$\begin{bmatrix} \frac{\partial F^u}{\partial u} & \frac{\partial F^u}{\partial v} \\ \frac{\partial F^v}{\partial u} & \frac{\partial F^v}{\partial v} \end{bmatrix} = \begin{bmatrix} h + 2u^2h' & 2uvh' \\ 2uvh' & h + 2v^2h' \end{bmatrix}$$

and the equations become

$$\begin{aligned}
(2uvh')^2 + \left(h + 2u^2h'\right)^2 &= h^2 = (2uvh')^2 + \left(h + 2v^2h'\right)^2, \\
\left(h + 2u^2h'\right)2uvh' + \left(h + 2v^2h'\right)2uvh' &= 0.
\end{aligned}$$

Since we just studied the case where $h' = 0$, we can assume that $h' \neq 0$, the last equation then reduces to

$$h = \left(u^2 + v^2\right) h'$$

showing that

$$h = \frac{r}{u^2 + v^2}$$

for some constant $r > 0$. It is then an easy matter to check that the equations in the first line also hold. Note that this map preserves the circle of radius r centered at $(0, 0)$ and switches points inside the circle with points outside the circle. It is called an *inversion* and is a type of mirror symmetry on the upper half plane. Note that the regular mirror symmetries in vertical lines are also symmetries of H. Note that both mirror symmetries and inversions are their own inverses:

$$\begin{aligned}
F \circ F(u, v) &= F\left(\frac{ru}{u^2+v^2}, \frac{rv}{u^2+v^2}\right) \\
&= \frac{r}{\left(\frac{ru}{u^2+v^2}\right)^2 + \left(\frac{rv}{u^2+v^2}\right)^2}\left(\frac{ru}{u^2+v^2}, \frac{rv}{u^2+v^2}\right) \\
&= \frac{1}{\frac{u^2}{u^2+v^2} + \frac{v^2}{u^2+v^2}}(u, v) \\
&= (u, v).
\end{aligned}$$

Between these three types of symmetries we can find all of the symmetries of the half plane. There are two key observations to be made. First, for any pair $p, q \in H$ we have to find a symmetry that takes p to q. This can be done using translations and scalings. Second, for any $p \in H$ and direction $v \in T_p H$ we have to find a symmetry that fixes p and whose differential is a reflections in v. This can be done with inversions or mirror symmetries in vertical lines should v be vertical.

25.2 The Geodesics of H

The fact that the metric is relatively simple allows us to compute the Christoffel symbols without much trouble

$$\begin{aligned}
\Gamma^u_{uu} &= \frac{1}{2} g^{uu} \frac{\partial g_{uu}}{\partial u} = 0 \\
\Gamma^v_{uu} &= -\frac{1}{2} g^{vv} \frac{\partial g_{uu}}{\partial v} = \frac{1}{v} \\
\Gamma^v_{vv} &= \frac{1}{2} g^{vv} \frac{\partial g_{vv}}{\partial v} = -\frac{1}{v} \\
\Gamma^u_{vv} &= -\frac{1}{2} g^{uu} \frac{\partial g_{vv}}{\partial u} = 0 \\
\Gamma^u_{uv} &= \frac{1}{2} g^{uu} \frac{\partial g_{uu}}{\partial v} = -\frac{1}{v} \\
\Gamma^v_{uv} &= \frac{1}{2} g^{vv} \frac{\partial g_{vv}}{\partial u} = 0
\end{aligned}$$

The geodesic equations then become

$$\begin{aligned}
\frac{d^2u}{dt^2} &= -\left[\begin{array}{cc}\frac{du}{dt} & \frac{dv}{dt}\end{array}\right]\left[\begin{array}{cc}\Gamma^u_{uu} & \Gamma^u_{uv}\\ \Gamma^u_{vu} & \Gamma^u_{vv}\end{array}\right]\left[\begin{array}{c}\frac{du}{dt}\\ \frac{dv}{dt}\end{array}\right]\\
&= -\left[\begin{array}{cc}\frac{du}{dt} & \frac{dv}{dt}\end{array}\right]\left[\begin{array}{cc}0 & -\frac{1}{v}\\ -\frac{1}{v} & 0\end{array}\right]\left[\begin{array}{c}\frac{du}{dt}\\ \frac{dv}{dt}\end{array}\right]\\
&= \frac{2}{v}\frac{du}{dt}\frac{dv}{dt}
\end{aligned}$$

$$\begin{aligned}
\frac{d^2v}{dt^2} &= -\left[\begin{array}{cc}\frac{du}{dt} & \frac{dv}{dt}\end{array}\right]\left[\begin{array}{cc}\Gamma^v_{uu} & \Gamma^v_{uv}\\ \Gamma^v_{vu} & \Gamma^v_{vv}\end{array}\right]\left[\begin{array}{c}\frac{du}{dt}\\ \frac{dv}{dt}\end{array}\right]\\
&= -\left[\begin{array}{cc}\frac{du}{dt} & \frac{dv}{dt}\end{array}\right]\left[\begin{array}{cc}\frac{1}{v} & 0\\ 0 & -\frac{1}{v}\end{array}\right]\left[\begin{array}{c}\frac{du}{dt}\\ \frac{dv}{dt}\end{array}\right]\\
&= \frac{1}{v}\left(\frac{dv}{dt}\right)^2-\frac{1}{v}\left(\frac{du}{dt}\right)^2
\end{aligned}$$

We'll try to find these geodesics as graphs over the u axis. Thus we should first address what geodesics might not be such graphs. This corresponds to having points where $\frac{du}{dt}=0$. In fact if we assume that $\frac{du}{dt}\equiv 0$, then the first equation is definitely solved while the second equation becomes

$$\frac{d^2v}{dt^2}=\frac{1}{v}\left(\frac{dv}{dt}\right)^2$$

This shows that vertical lines, if parametrized appropriately will become geodesics. This also means that no other geodesics can have vertical tangents. In particular, we should be able to graph them as functions: $v(u)$. The geodesic equation simplifies to

$$\frac{dv}{du}\left(0-\frac{2}{v}\frac{dv}{du}\right)=\left(\frac{d^2v}{du^2}-\frac{1}{v}\left(\frac{dv}{du}\right)^2+\frac{1}{v}\right)$$

or

$$\begin{aligned}
\frac{d^2v}{du^2} &= -\frac{1}{v}\left(\frac{dv}{du}\right)^2-\frac{1}{v}\\
&= -\frac{1}{v}\left(\left(\frac{dv}{du}\right)^2+1\right)
\end{aligned}$$

As this equation does not depend explicitly on u we are allowed to assume that

$$\begin{aligned}
\frac{dv}{du} &= h(v)\\
\frac{d^2v}{du^2} &= \frac{dh(v)}{du}=\frac{dh(v)}{dv}\frac{dv}{du}=\frac{dh(v)}{dv}h(v)
\end{aligned}$$

Thus

$$\frac{dh}{dv}h=-\frac{1}{v}\left(h^2+1\right),$$

or

$$\frac{h}{h^2+1}dh = -\frac{1}{v}dv$$

so

$$\frac{1}{2}\ln\left(h^2+1\right) = -\ln v + C$$

so

$$h^2+1 = \frac{r^2}{v^2}$$

so

$$\left(\frac{dv}{du}\right)^2 + 1 = \frac{r^2}{v^2}$$

so

$$\begin{aligned}\frac{dv}{du} &= \pm\sqrt{\frac{r^2}{v^2}-1}\\ &= \pm\frac{\sqrt{r^2-v^2}}{v^2}\end{aligned}$$

Even though this is a separable equation and the integerals involved can be computed it is still a mess to sort out. The answer however is fairly simple:

$$v = \sqrt{r^2-(u-u_0)^2}.$$

In other words the geodesics are either vertical lines or semicircles whose center is on the u axis. As these are precisely the curves that are fixed by mirror symmetries in vertical lines or inversions this should not be a big surprise.

25.3 Curvature of H

Having just computed the Christoffel symbols

$$\begin{aligned}\Gamma^u_{uu} &= \frac{1}{2}g^{uu}\frac{\partial g_{uu}}{\partial u} = 0\\ \Gamma^v_{uu} &= -\frac{1}{2}g^{vv}\frac{\partial g_{uu}}{\partial v} = \frac{1}{v}\\ \Gamma^v_{vv} &= \frac{1}{2}g^{vv}\frac{\partial g_{vv}}{\partial v} = -\frac{1}{v}\\ \Gamma^u_{vv} &= -\frac{1}{2}g^{uu}\frac{\partial g_{vv}}{\partial u} = 0\\ \Gamma^u_{uv} &= \frac{1}{2}g^{uu}\frac{\partial g_{uu}}{\partial v} = -\frac{1}{v}\\ \Gamma^v_{uv} &= \frac{1}{2}g^{vv}\frac{\partial g_{vv}}{\partial u} = 0\end{aligned}$$

it is now also possible to calculate the Riemannian curvature tensor

$$\begin{aligned}
R^u_{vvu} &= \frac{\partial \Gamma^u_{vv}}{\partial u} - \frac{\partial \Gamma^u_{vu}}{\partial v} + \Gamma^u_{vv}\Gamma^u_{uu} + \Gamma^v_{vv}\Gamma^u_{uv} - (\Gamma^u_{vu}\Gamma^u_{vu} + \Gamma^v_{vu}\Gamma^u_{vv}) \\
&= 0 - \frac{\partial - \frac{1}{v}}{\partial v} + 0 + \left(-\frac{1}{v}\right)\left(-\frac{1}{v}\right) - \left(\left(-\frac{1}{v}\right)^2 + 0\right) \\
&= -\frac{1}{v^2}
\end{aligned}$$

and the Gauss curvature

$$K = \frac{R^u_{vvu}}{g_{vv}} = -1$$

25.4 Conformal Picture

Triangles and angle sum. Parallel lines.

26 Global Stuff

Closed surfaces must have positive curvature somewhere. Convex surfaces. Constant mean curvature and/or Gauss curvature. Gauss-Bonnet. Hilbert.

27 Riemannian Geometry

As with abstract surfaces we simply define what the dot products of the tangent fields should be:

$$[\mathrm{I}] = \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u^1} & \cdots & \frac{\partial \boldsymbol{x}}{\partial u^n} \end{bmatrix}^t \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u^1} & \cdots & \frac{\partial \boldsymbol{x}}{\partial u^n} \end{bmatrix} = \begin{bmatrix} g_{11} & \cdots & g_{1n} \\ \vdots & \ddots & \vdots \\ g_{n1} & \cdots & g_{nn} \end{bmatrix}$$

The notation $\frac{\partial \boldsymbol{x}}{\partial u^i}$ for the tangent field that corrsponds to the velocity of the u^i curves is borrowed from our view of what happens on a surface.

We have the very general formula for how vectors are expanded

$$\begin{aligned}
V &= \begin{bmatrix} E_1 & \cdots & E_n \end{bmatrix} \left(\begin{bmatrix} E_1 & \cdots & E_n \end{bmatrix}^t \begin{bmatrix} E_1 & \cdots & E_n \end{bmatrix} \right)^{-1} \begin{bmatrix} E_1 & \cdots & E_n \end{bmatrix}^t V \\
&= \begin{bmatrix} E_1 & \cdots & E_n \end{bmatrix} \begin{bmatrix} E_1 \cdot E_1 & \cdots & E_1 \cdot E_n \\ \vdots & \ddots & \vdots \\ E_n \cdot E_1 & \cdots & E_n \cdot E_n \end{bmatrix}^{-1} \begin{bmatrix} E_1 \cdot V \\ \vdots \\ E_n \cdot V \end{bmatrix}
\end{aligned}$$

provided we know how to compute dot products of the basis vectors and dots products of V with the basis vectors.

The key now is to note that we have a way of defining Christoffel symbols in relation to the tangent fields when we know the dot products of those tangent fields:

$$\begin{aligned}\Gamma_{ijk} &= \frac{1}{2}\left(\frac{\partial g_{ki}}{\partial u^j}+\frac{\partial g_{kj}}{\partial u^i}-\frac{\partial g_{ij}}{\partial u^k}\right),\\ \frac{\partial g_{ij}}{\partial u^k} &= \Gamma_{kij}+\Gamma_{kji},\end{aligned}$$

So if we wish to define second partials, i.e., partials of the tangent fields we start by declaring

$$\frac{\partial^2 \boldsymbol{x}}{\partial u^i \partial u^j}\cdot\frac{\partial \boldsymbol{x}}{\partial u^k}=\Gamma_{ijk}$$

and then use

$$\begin{aligned}\frac{\partial^2 \boldsymbol{x}}{\partial u^i \partial u^j} &= \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u^1} & \cdots & \frac{\partial \boldsymbol{x}}{\partial u^1}\end{bmatrix}[\mathrm{I}]^{-1}\begin{bmatrix}\Gamma_{ij1} & \cdots & \Gamma_{ijn}\end{bmatrix}^t\\ &= \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u^1} & \cdots & \frac{\partial \boldsymbol{x}}{\partial u^1}\end{bmatrix}\begin{bmatrix}\Gamma_{ij}^1\\ \vdots \\ \Gamma_{ij}^n\end{bmatrix}\end{aligned}$$

Note that we still have

$$\frac{\partial^2 \boldsymbol{x}}{\partial u^i \partial u^j}=\frac{\partial^2 \boldsymbol{x}}{\partial u^j \partial u^i}$$

since the Christoffel symbols are symmetric in these indices.

This will allow us to define acceleration and hence geodesics. It'll also allow us to show that curves that minimize are geodesics, as well as showing that short geodesics must be minimal.

To define curvature we collect the Gauss formulas

$$\begin{aligned}\frac{\partial}{\partial u^i}\begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u^1} & \cdots & \frac{\partial \boldsymbol{x}}{\partial u^1}\end{bmatrix} &= \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u^1} & \cdots & \frac{\partial \boldsymbol{x}}{\partial u^1}\end{bmatrix}\begin{bmatrix}\Gamma_{i1}^1 & \cdots & \Gamma_{in}^1\\ \vdots & \ddots & \vdots\\ \Gamma_{i1}^n & \cdots & \Gamma_{in}^n\end{bmatrix}\\ &= \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial u^1} & \cdots & \frac{\partial \boldsymbol{x}}{\partial u^1}\end{bmatrix}[\Gamma_i]\end{aligned}$$

and form the expression

$$\frac{\partial}{\partial u^i}[\Gamma_j]-\frac{\partial}{\partial u^j}[\Gamma_i]+[\Gamma_i][\Gamma_j]-[\Gamma_j][\Gamma_i]$$

When we used a frame in $\mathbb{R}^3$ we got this to vanish, but that was due to the inclusion of the second fundamental form terms. Recall that when we restricted attention to terms that only involved Γs then we got something that was related to the Gauss curvature. This time we don't have a Gauss curvature, but we can define the Riemann curvature as the k, l entry in this expression:

$$\begin{aligned}[R_{ij}] &= \frac{\partial}{\partial u^i}[\Gamma_j]-\frac{\partial}{\partial u^j}[\Gamma_i]+[\Gamma_i][\Gamma_j]-[\Gamma_j][\Gamma_i],\\ R_{ijk}^l &= \frac{\partial \Gamma_{jk}^l}{\partial u^i}-\frac{\partial \Gamma_{ik}^l}{\partial u^j}+\begin{bmatrix}\Gamma_{i1}^l & \cdots & \Gamma_{in}^l\end{bmatrix}\begin{bmatrix}\Gamma_{jk}^1\\ \vdots\\ \Gamma_{jk}^n\end{bmatrix}-\begin{bmatrix}\Gamma_{j1}^l & \cdots & \Gamma_{jn}^l\end{bmatrix}\begin{bmatrix}\Gamma_{ik}^1\\ \vdots\\ \Gamma_{ik}^n\end{bmatrix}\end{aligned}$$

This expression shows how certain third order partials might not commute since we have

$$\frac{\partial^3 \boldsymbol{x}}{\partial u^i \partial u^j \partial u^k} - \frac{\partial^3 \boldsymbol{x}}{\partial u^j \partial u^i \partial u^k} = \left[\begin{array}{ccc} \frac{\partial \boldsymbol{x}}{\partial u^1} & \cdots & \frac{\partial \boldsymbol{x}}{\partial u^1} \end{array} \right] \left[\begin{array}{c} R^1_{ijk} \\ \vdots \\ R^n_{ijk} \end{array} \right]$$

But recall that since second order partials do commute we have

$$\frac{\partial^3 \boldsymbol{x}}{\partial u^i \partial u^j \partial u^k} = \frac{\partial^3 \boldsymbol{x}}{\partial u^i \partial u^k \partial u^j}$$

So we see that third order partials commute if and only if the Riemann curvature vanishes. One can in turn show that

Theorem 34 (Riemann) *The Riemann curvature vanishes if and only if there are Cartesian coordinates around any point.*

Proof. The easy direction is to assume that Cartesian coodinates exist. Certainly this shows that the curvatures vanish when we use Cartesian coordinates, but this does not guarantee that they also vanish in some arbitrary coordinate system. For that we need to figure out how the curvature terms change when we change coordinates. A long tedious calculation shows that if the new coordinates are called v^i and the curvature in these coordinates $\tilde{R}^l_{ijk}$, then

$$\tilde{R}^l_{ijk} = \frac{\partial u^\alpha}{\partial v^i} \frac{\partial u^\beta}{\partial v^j} \frac{\partial u^\gamma}{\partial v^k} \frac{\partial v^l}{\partial u^\delta} R^\delta_{\alpha\beta\gamma}.$$

Thus we see that if the all curvatures vanish in one coordinate system, then they vanish in all coordinate systems. ■