120A Midterm 2 Instructions

November 7, 2011

The second midterm is on W 11-16-2011.

Here are some previous exam problems. I've generally given four of these types of questions on a midterm. There are also similar questions on the hwk and in the book that I suggest that you look at carefully. We are covering sections 3A,B,C on this midterm. Look at problems:

Chapter 3: 2,3,5,9,11,12,13,14,15.

For the midterm you are allowed to bring ONE sheet of paper with your notes. NO books or electronic devices are allowed.

I'll have extended office hrs next week: Monday 11-Noon and 2-3pm, Tuesday Noon-2:30pm.

Some notation: If we have a curve $c(s) : I \to S \subset \mathbb{R}^3$ on a surface S that is parametrized by arclength, then the normal curvature

$$\kappa_n = \left\langle \frac{d^2c}{ds^2}, \nu \right\rangle$$

and geodesic curvature

$$\kappa_g = \left\langle \frac{d^2c}{ds^2}, \nu \times \frac{dc}{ds} \right\rangle$$

are the normal and tangential components of the acceleration of c as a curve in $\mathbb{R}^3$.

If $f(u, v)$ defines a parametrized surface then we say that the parametrization is equiareal if $g_{uu}g_{vv} - g_{uv}^2 = 1$.

If $f(u, v)$ defines a parametrized surface then we say that the parametrization is conformal or angle preserving if $g_{uu} = g_{vv}$ and $g_{uv} = 0$.

1. Let $\gamma(s) = f(u(s), v(s))$ be a unit speed curve on a surface S. Prove that

$$\frac{d\nu}{ds} = -\mathrm{II}(T, T)\, T - \mathrm{II}(T, C)\, C,$$

where $T = \frac{d\gamma}{ds}$, ν is the normal to S, and $C = \nu \times T$.

2. Let $X, Y \in T_pS$ be an orthonormal basis for the tangent space at p to the surface S. Prove that the mean and Gauss curvatures can be computed as

follows:

$$\begin{aligned} H &= \frac{1}{2}\left(\mathrm{II}\left(X,X\right)+\mathrm{II}\left(Y,Y\right)\right), \\ K &= \mathrm{II}\left(X,X\right)\mathrm{II}\left(Y,Y\right)-\left(\mathrm{II}\left(X,Y\right)\right)^2 \end{aligned}$$

3. Let $\alpha:(a,b)\to\mathbb{R}^3$ be a unit speed curve with $\kappa(s)\neq 0$ for all $s\in(a,b)$. Define
$$f(s,t)=\alpha(s)+t\alpha'(s).$$
Prove that f defines a parametrized surface as long as $t\neq 0$. Compute the first and second fundamental forms and show that the Gauss curvature K vanishes.

4. For a surface of revolution $x(t,\theta)=(r(t)\cos(\theta),r(t)\sin(\theta),z(t))$ compute the first and second fundamental forms and the principal curvatures.

5. Let γ be a curve on the unit sphere S^2. Prove that its normal curvature κ_n is constant.

6. Let $f(u,v)$ be a parametrized surface. Recall that a tangent vector is a principal direction if it is an eigenvector for the Weingarten map. Assume that the principal curvatures are different. Show that $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ are the principal directions if and only if $F=0=M$.

7. Let $\alpha(u)$ be a unit speed curve in the x,y plane $\mathbb{R}^2$. Show that
$$f(u,v)=(\alpha(u),v).$$
yields a parametrized surface. Compute its first and second fundamental forms and principal curvatures. Compute its Gauss curvature.

8. Show that the equation
$$ax+by+cz=d$$
defines a surface if and only if $(a,b,c)\neq(0,0,0)$. Show that this surface has a parametrization that is Cartesian.

9. Let γ be a unit speed curve on a surface S with normal ν. Define $C=\nu\times T$, $T=\dot{\gamma}$ and
$$\kappa_g = \frac{dT}{ds}\cdot C,\ \kappa_n=\frac{dT}{ds}\cdot\nu,\ \tau_g=\frac{dC}{ds}\cdot\nu$$
Prove that
$$\begin{aligned} \frac{dT}{ds} &= \kappa_g C+\kappa_n\nu, \\ \frac{dC}{ds} &= -\kappa_g T+\tau_g\nu, \\ \frac{d\nu}{ds} &= -\kappa_n T-\tau_g C. \end{aligned}$$

10. Let $\gamma(u)$ be a regular curve in the x, y plane $\mathbb{R}^2$. Show that

$$f(u,v) = (\gamma(u), v).$$

yields a parametrized surface. Compute its first fundamental form. Show that it has a Cartesian parametrization.

11. For a regular curve $\gamma(u) : I \to \mathbb{R}^3 - \{(0,0,0)\}$ show that $f(u,v) = v\gamma(u)$ defines a surface for $v > 0$ provided γ and $\dot{\gamma}$ are linearly independent. Compute its first fundamental form. Show that it admits Cartesian coordinates by rewriting the surface as $f(r,\theta) = rX(\theta)$ for a suitable unit speed curve $X(\theta)$.

12. Let $f(z,\theta) = \left(\sqrt{1-z^2}\cos\theta, \sqrt{1-z^2}\sin\theta, z\right)$ with $-1 < z < 1$ and $-\pi < \theta < \pi$. Show that f defines a parametrized surface. What is the surface?

13. Let f be a parametrized surface such that $E = 1$ and $F = 0$. Prove that the u curves are unit speed with acceleration that is perpendicular to the surface. The u curves are given by $\gamma(u) = f(u,v)$ where v is fixed.

14. For a surface of revolution $\sigma(t,\theta) = (r(t)\cos(\theta), r(t)\sin(\theta), z(t))$ show that the first fundamental form is given by

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} \dot{r}^2 + \dot{z}^2 & 0 \\ 0 & r^2 \end{bmatrix}$$

and that the longitudes/meridians $\gamma(t) = \sigma((t,\theta)$ have acceleration perpendicular to the surface provided that $(r(t), 0, z(t))$ is unit speed.

15. Reparametrize the curve $(r(u), z(u))$ so that the new parametrization $\sigma(t,\theta) = (r(t)\cos(\theta), r(t)\sin(\theta), z(t))$ is conformal.

16. Reparametrize the curve $(r(u), z(u))$ so that the new parametrization $\sigma(t,\theta) = (r(t)\cos(\theta), r(t)\sin(\theta), z(t))$ is equiareal.

17. Let $f : U \to S^2$ be a parametrization of part of the unit sphere. Show that the normal $\frac{\partial f}{\partial u} \times \frac{\partial f}{\partial v}$ is always proportional to f.

18. Show that a Monge patch $z = f(x,y)$ is equiareal if and only if f is constant.

19. Show that a Monge patch $z = f(x,y)$ is conformal if and only if f is constant.

20. Show that the equation

$$ax + by + cz = d$$

defines a surface if and only if $(a,b,c) \neq (0,0,0)$. Show that this surface has a parametrization that is Cartesian.

21. The conoid is a special type of ruled surface given by

$$\begin{aligned} f(t,\theta) &= (r(t)\cos\theta, r(t)\sin\theta, z(\theta)) \\ &= (0,0,z(\theta)) + r(t)(\cos\theta,\sin\theta,0) \end{aligned}$$

Compute its first fundamental form. Show that if $z(\theta) = a\theta$ for some constant a, then $r(t)$ can be reparametrized in such a way that we get a conformal parametrization.

22. Consider the two parametrized surfaces given by

$$\begin{aligned} f_1(\phi,u) &= (\sinh\phi\cos u, \sinh\phi\sin u, u) \\ &= (0,0,u) + \sinh\phi(\cos u,\sin u,0) \\ f_2(t,\theta) &= (\cosh t\cos\theta, \cosh t\sin\theta, t) \end{aligned}$$

Compute the first fundamental forms for both surfaces and show that they can be reparametrized in such a way that they have the same first fundamental forms. (The first surface is a ruled surface with a one-to-one parametrization called the helicoid, the second surface is a surface of revolution called the catenoid.)

23. Let $S = \left\{x \in \mathbb{R}^3 : |x - m|^2 = R^2\right\}$. Show that S is a surface, and that if I and II denote the first and second fundamental forms, then

$$\mathrm{II} = \pm\frac{1}{R}\mathrm{I}$$

24. The conoid is a special type of ruled surface given by

$$\begin{aligned} f(t,\theta) &= (t\cos\theta, t\sin\theta, z(\theta)) \\ &= (0,0,z(\theta)) + t(\cos\theta,\sin\theta,0) \end{aligned}$$

Compute its first and second fundamental forms as well as the Gauss and mean curvatures.

25. Let $\gamma(t) : I \to S$ be a regular curve on a surface S, with ν being the normal to the surface. Show that

$$\kappa_n = \frac{\mathrm{II}(\dot\gamma,\dot\gamma)}{\mathrm{I}(\dot\gamma,\dot\gamma)}, \ \kappa_g = \frac{\det(\dot\gamma,\ddot\gamma,\nu)}{(\mathrm{I}(\dot\gamma,\dot\gamma))^{3/2}}$$

26. Show that the principal curvatures at a point $p \in S$ are equal if and only if at p the mean and Gauss curvatures are related by $H^2 = K$.

27. Compute the matrix representation of the Weingarten map for a Monge patch $f(x,y) = (x, y, f(x,y))$ with respect to the basis $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$.