120A Final Instructions

November 28, 2011

The final is on Tu Dec 6 3-6pm in MS 5127 (classroom.)

Here are some previous exam problems. I've generally given 11-13 of these types of questions on a final. There are also similar questions on the hwk and in the book that I suggest that you look at carefully. We are covering sections 2A-F and 3A-D on this midterm. Look at problems:

Chapter 2: 1,3,4,5,6,9,10,12,13,16,17,18,19,20,21,22

Chapter 3: 2,3,5,9,11,12,13,14,15,17,18.

For the final you are allowed to bring ONE sheet of paper with your notes. NO books or electronic devices are allowed.

I'll have extended office hrs next week: Monday 10-Noon, 1-2pm and Tuesday 10:30-12:30.

1. Let $\gamma(s) = f(u(s), v(s))$ be a unit speed curve on a surface S. Prove that
$$\frac{d\nu}{ds} = -\mathrm{II}(T,T)\,T - \mathrm{II}(T,C)\,C,$$
where $T = \frac{d\gamma}{ds}$, ν is the normal to S, and $C = \nu \times T$.

2. Let $X, Y \in T_pS$ be an orthonormal basis for the tangent space at p to the surface S. Prove that the mean and Gauss curvatures can be computed as follows:
$$\begin{aligned} H &= \frac{1}{2}\left(\mathrm{II}(X,X) + \mathrm{II}(Y,Y)\right), \\ K &= \mathrm{II}(X,X)\,\mathrm{II}(Y,Y) - \left(\mathrm{II}(X,Y)\right)^2 \end{aligned}$$

3. Let $\alpha : (a,b) \to \mathbb{R}^3$ be a unit speed curve with $\kappa(s) \neq 0$ for all $s \in (a,b)$. Define
$$f(s,t) = \alpha(s) + t\alpha'(s).$$
Prove that f defines a parametrized surface as long as $t \neq 0$. Compute the first and second fundamental forms and show that the Gauss curvature K vanishes.

4. For a surface of revolution $x(t,\theta) = (r(t)\cos(\theta), r(t)\sin(\theta), z(t))$ compute the first and second fundamental forms and the principal curvatures.

5. Let γ be a curve on the unit sphere S^2. Prove that its normal curvature κ_n is constant.

6. Let $f(u,v)$ be a parametrized surface. Recall that a tangent vector is a principal direction if it is an eigenvector for the Weingarten map. Assume that the principal curvatures are different. Show that $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ are the principal directions if and only if $F = 0 = M$.

7. Let $\alpha(u)$ be a unit speed curve in the x, y plane $\mathbb{R}^2$. Show that

$$f(u,v) = (\alpha(u), v).$$

yields a parametrized surface. Compute its first and second fundamental forms and principal curvatures. Compute its Gauss curvature.

8. Show that the equation

$$ax + by + cz = d$$

defines a surface if and only if $(a,b,c) \neq (0,0,0)$. Show that this surface has a parametrization that is Cartesian.

9. Let γ be a unit speed curve on a surface S with normal ν. Define $C = \nu \times T$, $T = \dot{\gamma}$ and

$$\kappa_g = \frac{dT}{ds} \cdot C,\ \kappa_n = \frac{dT}{ds} \cdot \nu,\ \tau_g = \frac{dC}{ds} \cdot \nu$$

Prove that

$$\begin{aligned}
\frac{dT}{ds} &= \kappa_g C + \kappa_n \nu, \\
\frac{dC}{ds} &= -\kappa_g T + \tau_g \nu, \\
\frac{d\nu}{ds} &= -\kappa_n T - \tau_g C.
\end{aligned}$$

10. Let $\gamma(u)$ be a regular curve in the x, y plane $\mathbb{R}^2$. Show that

$$f(u,v) = (\gamma(u), v).$$

yields a parametrized surface. Compute its first fundamental form. Show that it has a Cartesian parametrization.

11. For a regular curve $\gamma(u) : I \to \mathbb{R}^3 - \{(0,0,0)\}$ show that $f(u,v) = v\gamma(u)$ defines a surface for $v > 0$ provided γ and $\dot{\gamma}$ are linearly independent. Compute its first fundamental form. Show that it admits Cartesian coordinates by rewriting the surface as $f(r,\theta) = rX(\theta)$ for a suitable unit speed curve $X(\theta)$.

12. Let $f(z,\theta) = \left(\sqrt{1-z^2}\cos\theta, \sqrt{1-z^2}\sin\theta, z\right)$ with $-1 < z < 1$ and $-\pi < \theta < \pi$. Show that f defines a parametrized surface. What is the surface?

13. Let f be a parametrized surface such that $E = 1$ and $F = 0$. Prove that the u curves are unit speed with acceleration that is perpendicular to the surface. The u curves are given by $\gamma(u) = f(u, v)$ where v is fixed.

14. For a surface of revolution $\sigma(t, \theta) = (r(t)\cos(\theta), r(t)\sin(\theta), z(t))$ show that the first fundamental form is given by

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} \dot{r}^2 + \dot{z}^2 & 0 \\ 0 & r^2 \end{bmatrix}$$

and that the longitudes/meridians $\gamma(t) = \sigma((t, \theta)$ have acceleration perpendicular to the surface provided that $(r(t), 0, z(t))$ is unit speed.

15. Reparametrize the curve $(r(u), z(u))$ so that the new parametrization $\sigma(t, \theta) = (r(t)\cos(\theta), r(t)\sin(\theta), z(t))$ is conformal.

16. Reparametrize the curve $(r(u), z(u))$ so that the new parametrization $\sigma(t, \theta) = (r(t)\cos(\theta), r(t)\sin(\theta), z(t))$ is equiareal.

17. Let $f : U \to S^2$ be a parametrization of part of the unit sphere. Show that the normal $\frac{\partial f}{\partial u} \times \frac{\partial f}{\partial v}$ is always proportional to f.

18. Show that a Monge patch $z = f(x, y)$ is equiareal if and only if f is constant.

19. Show that a Monge patch $z = f(x, y)$ is conformal if and only if f is constant.

20. Show that the equation

$$ax + by + cz = d$$

defines a surface if and only if $(a, b, c) \neq (0, 0, 0)$. Show that this surface has a parametrization that is Cartesian.

21. The conoid is a special type of ruled surface given by

$$\begin{aligned} f(t, \theta) &= (r(t)\cos\theta, r(t)\sin\theta, z(\theta)) \\ &= (0, 0, z(\theta)) + r(t)(\cos\theta, \sin\theta, 0) \end{aligned}$$

Compute its first fundamental form. Show that if $z(\theta) = a\theta$ for some constant a, then $r(t)$ can be reparametrized in such a way that we get a conformal parametrization.

22. Consider the two parametrized surfaces given by

$$\begin{aligned} f_1(\phi, u) &= (\sinh\phi\cos u, \sinh\phi\sin u, u) \\ &= (0, 0, u) + \sinh\phi(\cos u, \sin u, 0) \\ f_2(t, \theta) &= (\cosh t\cos\theta, \cosh t\sin\theta, t) \end{aligned}$$

Compute the first fundamental forms for both surfaces and show that they can be reparametrized in such a way that they have the same first fundamental forms. (The first surface is a ruled surface with a one-to-one parametrization called the helicoid, the second surface is a surface of revolution called the catenoid.)

23. Let $S = \left\{ x \in \mathbb{R}^3 : |x - m|^2 = R^2 \right\}$. Show that S is a surface, and that if I and II denote the first and second fundamental forms, then

$$\mathrm{II} = \pm \frac{1}{R} \mathrm{I}$$

24. The conoid is a special type of ruled surface given by

$$\begin{aligned} f(t, \theta) &= (t \cos\theta, t \sin\theta, z(\theta)) \\ &= (0, 0, z(\theta)) + t(\cos\theta, \sin\theta, 0) \end{aligned}$$

Compute its first and second fundamental forms as well as the Gauss and mean curvatures.

25. Let $\gamma(t) : I \to S$ be a regular curve on a surface S, with ν being the normal to the surface. Show that

$$\kappa_n = \frac{\mathrm{II}(\dot{\gamma}, \dot{\gamma})}{\mathrm{I}(\dot{\gamma}, \dot{\gamma})}, \ \kappa_g = \frac{\det(\dot{\gamma}, \ddot{\gamma}, \nu)}{(\mathrm{I}(\dot{\gamma}, \dot{\gamma}))^{3/2}}$$

26. Show that the principal curvatures at a point $p \in S$ are equal if and only if at p the mean and Gauss curvatures are related by $H^2 = K$.

27. Compute the matrix representation of the Weingarten map for a Monge patch $f(x, y) = (x, y, f(x, y))$ with respect to the basis $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$.

28. Prove that if $\alpha(s)$ is an oval (a closed planar curve with positive curvature and no self intersections), then the unit tangent field e_1 is parallel to e_1'' at four or more points.

29. Prove that the concept of a vertex for a planar curve does not depend on the parametrization.

30. Let $c(t)$ be a closed Frenet curve in $\mathbb{R}^3$. Show that if its curvature is $\leq R^{-1}$, then its length is $\geq 2\pi R$.

31. Let $c(t) : I \to \mathbb{R}^3$ be a regular curve with speed $\frac{ds}{dt} = \left|\frac{dc}{dt}\right|$, where s is the arclength parameter. Prove that

$$\kappa = \frac{\sqrt{\frac{d^2c}{dt^2} \cdot \frac{d^2c}{dt^2} - \left(\frac{d^2s}{dt^2}\right)^2}}{\left(\frac{ds}{dt}\right)^2}$$

32. Let $c(t) : I \to \mathbb{R}^3$ be a regular curve such that its unit tangent field $e_1(t)$ is also regular. Let s be the arclength parameter for c and θ the arclength parameter for e_1. Show that

$$\kappa = \frac{d\theta}{ds}$$

and

$$\det\left(e_1, \frac{de_1}{d\theta}, \frac{d^2 e_1}{d\theta^2}\right) = \frac{\tau}{\kappa}.$$

33. Let $c(t)$ be a regular curve in $\mathbb{R}^3$ with $\kappa > 0$. Prove that c is planar if and only if the triple product

$$\left[\frac{dc}{dt}, \frac{d^2c}{dt^2}, \frac{d^3c}{dt^3}\right] \equiv 0$$

34. Let $c(s) = (x(s), y(s))$ be a planar unit speed curve. Show that the signed curvature can be computed by

$$\kappa = \det[c', c'']$$

35. Let $c(s)$ be a unit speed curve in $\mathbb{R}^3$ Prove that

$$\det[c', c'', c'''] = \kappa^2 \tau.$$

It is also possible to find formulas for

$$\det[c'', c''', c'''']$$

etc.

36. Let $c(t) : I \to \mathbb{R}^3$ be a regular curve with positive curvature. Show that the unit tangent $e_1(t)$ is a regular and that, if θ is an arclength parameter for e_1, then

$$\begin{aligned}
\frac{dc}{d\theta} &= \frac{1}{\kappa} e_1 \\
\frac{de_1}{d\theta} &= e_2 \\
\frac{de_2}{d\theta} &= -e_1 + \frac{\tau}{\kappa} e_3 \\
\frac{de_3}{d\theta} &= -\frac{\tau}{\kappa} e_2
\end{aligned}$$

37. Let $c(t) : I \to \mathbb{R}^3$ be a regular curve with positive curvature. Show that c lies in a plane if and only if the torsion vanishes.

38. Let $\gamma(\theta)$ be a simple closed planar curve with $\kappa > 0$ parametrized by θ, where θ is defined as the arclength parameter of the unit tangent field e_1. Further assume that the width

$$w = \langle e_2(\theta), (\gamma(\theta + \pi) - \gamma(\theta)) \rangle$$

is constant. Show that:

$$w = \frac{1}{\kappa(\theta)} + \frac{1}{\kappa(\theta + \pi)}.$$

Start by establishing the facts:

$$\begin{aligned} \frac{d\gamma}{d\theta} &= \frac{1}{\kappa} e_1 \\ \frac{de_1}{d\theta} &= e_2 \\ \frac{de_2}{d\theta} &= -e_1 \\ e_1(\theta + \pi) &= -e_1(\theta) \end{aligned}$$

39. Let $\alpha : (a, b) \to \mathbb{R}^3$ be a unit speed curve with $\kappa(s) \neq 0$ for all $s \in (a, b)$. Define

$$f(s, t) = \alpha(s) + t\alpha'(s).$$

Prove that f defines a parametrized surface as long as $t \neq 0$. Compute the first and second fundamental forms and show that the Gauss curvature K vanishes.

40. For a surface of revolution $f(t, \theta) = (r(t)\cos(\theta), r(t)\sin(\theta), z(t))$ compute the first and second fundamental forms and the principal curvatures.

41. Let $f(u, v)$ be a parametrized surface. A tangent vector is a principal direction if it is an eigenvector for the Weingarten map. Show that $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ are the principal directions if $g_{uv} = 0 = h_{uv}$, and that the principal curvatures are given by

$$\frac{h_{uu}}{g_{uu}}, \frac{h_{vv}}{g_{vv}}.$$

42. Let $\alpha(u)$ be a unit speed curve in the x, y plane $\mathbb{R}^2$. Show that

$$\sigma(u, v) = (\alpha(u), v).$$

yields a parametrized surface. Compute its first and second fundamental forms and principal curvatures. Compute its Gauss curvature.

43. Consider a surface given by $F(x, y) = C$, i.e., it is given by a function that doesn't depend on the third coordinate z. Compute the normal to this surface and show that its Gauss curvature vanishes.

44. For a regular curve $\gamma(u) : I \to \mathbb{R}^3 - \{(0,0,0)\}$ show that $f(u,v) = v\gamma(u)$ defines a surface for $v > 0$ provided γ and $\dot{\gamma}$ are linearly independent (this a generalized cone.) Compute its first fundamental form. Show that it admits Cartesian coordinates by rewriting the surface as $f(r,\theta) = r\delta(\theta)$ for a suitable unit speed curve $\delta(\theta)$. Hint: δ is the curve gotten by intersecting the generalized cone with the unit sphere.

45. Let f be a parametrization such that $g_{uu} = 1$ and $g_{uv} = 0$. Prove that the u curves are unit speed with acceleration that is perpendicular to the surface. The u curves are given by $\gamma(u) = f(u,v)$ where v is fixed.

46. Given a surface of revolution $\sigma_1(r,\theta) = (r\cos\theta, r\sin\theta, z_1(r))$ show that there is a function $z_2(r)$ so that σ_1 becomes conformal to the cylinder $\sigma_2(r,\theta) = (\cos\theta, \sin\theta, z_2(r))$.

47. Given a surface of revolution $\sigma_1(r,\theta) = (r\cos\theta, r\sin\theta, z_1(r))$ show that there a function $z_2(r)$ so that σ_1 becomes equiareal to the cylinder $\sigma_2(r,\theta) = (\cos\theta, \sin\theta, z_2(r))$.

48. Compute the mean curvature of the Enneper surface:

$$f(u,v) = \left(u - \frac{1}{3}u^3 + uv^2, -v + \frac{1}{3}v^3 - vu^2, u^2 - v^2\right)$$

Scherk minimal surface:

$$f(u,v) = \left(u, v, \log\frac{\cos v}{\cos u}\right)$$

and Catalan surface:

$$f(u,v) = \left(u - \sin u\cosh v, 1 - \cos u\cosh v, 4\sin\frac{u}{2}\sinh\frac{v}{2}\right)$$

49. Show that a generalized cylinder is minimal if and only if it is planar.

50. Show that a generalized cone is minimal if and only if it is planar.

51. Show that a surface given by $z = f(x) + g(y)$ is a minimal if and only if

$$\frac{\frac{d^2 f}{dx^2}}{1 + \left(\frac{df}{dx}\right)^2} + \frac{\frac{d^2 g}{dy^2}}{1 + \left(\frac{dg}{dy}\right)^2} = 0$$

52. Show that the surface given by $\sin z = \sinh x \sinh y$ is a minimal surface.

53. Show that the surface defined by

$$x\sin\frac{z}{a} = y\cos\frac{z}{a}$$

is a minimal surface.

54. Let $f(u,v) : U \to \mathbb{R}^3$ be a parametrized surface and $f^\epsilon(u,v) = f(u,v) + \epsilon\nu(u,v)$ the parallel surface. If f is minimal, then the principal curvatures for f^ϵ satisfy that
$$\frac{1}{\kappa_1^\epsilon} + \frac{1}{\kappa_2^\epsilon} = \text{constant}.$$

55. Let $f(u,v) : U \to \mathbb{R}^3$ be a parametrized surface such that f and its Gauss map $\nu(u,v) : U \to S^2$ are conformally equivalent. Show that either f is minimal or parametrizing part of a sphere. Hint: Show that the surface is umbilic at any point where the mean curvature is not zero. Then use our characterization of surfaces which are totally umbilic.

56. Let $f(u,v) : U \to \mathbb{R}^3$ be a parametrized surface. Show that
$$\frac{\partial \nu}{\partial u} \times \frac{\partial \nu}{\partial v} = K \frac{\partial f}{\partial u} \times \frac{\partial f}{\partial v}$$