Here is a collection of old exam problems:

1. Let $\beta(t) : I \to \mathbb{R}^3$ be a regular curve with speed $\frac{ds}{dt} = \left|\frac{d\beta}{dt}\right|$, where s is the arclength parameter. Prove that

$$\kappa = \frac{\sqrt{\frac{d^2\beta}{dt^2} \cdot \frac{d^2\beta}{dt^2} - \left(\frac{d^2s}{dt^2}\right)^2}}{\left(\frac{ds}{dt}\right)^2}$$

2. Let $\beta(t) : I \to \mathbb{R}^3$ be a regular curve such that its tangent field $\mathbf{T}(t)$ is also regular. Let s be the arclength parameter for β and θ the arclength parameter for $\mathbf{T}$. Show that

$$\kappa = \frac{d\theta}{ds}$$

and

$$\det\left(\mathbf{T}, \frac{d\mathbf{T}}{d\theta}, \frac{d^2\mathbf{T}}{d\theta^2}\right) = \left[\mathbf{T}, \frac{d\mathbf{T}}{d\theta}, \frac{d^2\mathbf{T}}{d\theta^2}\right] = \frac{\tau}{\kappa}.$$

3. Let $\gamma(\theta)$ be an oval parametrized by θ defined by $\mathbf{T} = (\cos\theta, \sin\theta)$of constant width

$$w = \mathbf{N}(\theta) \cdot (\gamma(\theta + \pi) - \gamma(\theta))$$

Show that:

$$w = \frac{1}{\kappa(\theta)} + \frac{1}{k(\theta + \pi)}.$$

You can use that

$$\begin{aligned} \frac{d\gamma}{d\theta} &= \frac{1}{\kappa}\mathbf{T} \\ \frac{d\mathbf{T}}{d\theta} &= \mathbf{N} \\ \frac{d\mathbf{N}}{d\theta} &= -\mathbf{T}, \\ \mathbf{N} \cdot \mathbf{T} &= 0, |\mathbf{N}| = |\mathbf{T}| = 1 \\ \mathbf{T}(\theta + \pi) &= -\mathbf{T}(\theta) \end{aligned}$$

4. Let $\alpha(s)$ unit speed curve with $\kappa > 0$. Let θ be the arclength parameter for $\mathbf{T} = \frac{d\alpha}{ds}$. Show that the curvature satisfies:

$$\kappa = \frac{d\theta}{ds}$$

5. Prove that if $\alpha(s)$ is an oval (a closed planar curve with positive curvature and no self intersections), then the unit tangent field $\mathbf{T}$ is parallel to $\mathbf{T}''$ at four or more points.

6. Let $\beta(t)$ be a regular curve in $\mathbb{R}^3$ with $\kappa > 0$. Prove that β is planar if and only if the triple product

$$\left[\frac{d\beta}{dt}, \frac{d^2\beta}{dt^2}, \frac{d^3\beta}{dt^3}\right] \equiv 0$$

7. Let $\gamma(t) : I \to \mathbb{R}^3$ be a regular curve with positive curvature. Show that γ lies in a plane if and only if the torsion vanishes.

8. Let $\alpha(s) = (x(s), y(s))$ be a planar unit speed curve. Show that the signed curvature can be computed by

$$\kappa = \det[\alpha', \alpha'']$$

9. Let $\alpha(s)$ be a unit speed curve in $\mathbb{R}^3$ Prove that

$$\det[\alpha', \alpha'', \alpha'''] = \kappa^2\tau.$$

   It is also possible to find formulas for

$$\det[\alpha'', \alpha''', \alpha'''']$$

   etc.

10. Prove that the concept of a vertex for a planar curve does not depend on the parametrization.

11. Let $\gamma(t) : I \to \mathbb{R}^3$ be a regular curve. Prove that

$$\kappa = \frac{\sqrt{\frac{d^2\gamma}{dt^2} \cdot \frac{d^2\gamma}{dt^2} - \left(\frac{d}{dt}\left|\frac{d\gamma}{dt}\right|\right)^2}}{\left|\frac{d\gamma}{dt}\right|^2}$$

12. Let $\gamma(t) : I \to \mathbb{R}^3$ be a regular curve with positive curvature. Show that the unit tangent $\mathbf{T}(t)$ is a regular and that, if θ is an arclength parameter for $\mathbf{T}$, then

$$\begin{aligned}
\frac{d\gamma}{d\theta} &= \frac{1}{\kappa}\mathbf{T} \\
\frac{d\mathbf{T}}{d\theta} &= \mathbf{N} \\
\frac{d\mathbf{N}}{d\theta} &= -\mathbf{T} + \frac{\tau}{\kappa}\mathbf{B} \\
\frac{d\mathbf{B}}{d\theta} &= -\frac{\tau}{\kappa}\mathbf{B}
\end{aligned}$$

13. Let $\gamma(t) : I \to \mathbb{R}^3$ be a regular curve with positive curvature. Show that γ lies in a plane if and only if the torsion vanishes.

14. Let $\gamma(s) = \sigma(u(s), v(s))$ be a unit speed curve on a surface S. Prove that

$$\frac{dn}{ds} = -\mathrm{II}(T,T)\,T - \mathrm{II}(T,C)\,C,$$

where $T = \frac{d\gamma}{ds}$, n is the normal to S, and $C = n \times T$.

15. Let $X, Y \in T_pS$ be an orthonormal basis for the tangent space at p to the surface S. Prove that the mean and Gauss curvatures can be computed as follows:

$$\begin{aligned} H &= \frac{1}{2}\left(\mathrm{II}(X,X) + \mathrm{II}(Y,Y)\right), \\ K &= \mathrm{II}(X,X)\,\mathrm{II}(Y,Y) - \left(\mathrm{II}(X,Y)\right)^2 \end{aligned}$$

16. Let $\alpha : (a,b) \to \mathbb{R}^3$ be a unit speed curve with $\kappa(s) \neq 0$ for all $s \in (a,b)$. Define

$$\sigma(s,t) = \alpha(s) + t\alpha'(s).$$

Prove that σ defines a parametrization surface as long as $t \neq 0$. Compute the first and second fundamental forms and show that the Gauss curvature K vanishes.

17. For a surface of revolution $x(t,\theta) = (r(t)\cos(\theta), r(t)\sin(\theta), z(t))$ compute the first and second fundamental forms and the principal curvatures.

18. Let γ be a curve on the unit sphere S^2. Prove that its normal curvature κ_n is constant.

19. Let $\sigma(u,v)$ be a parametrized surface. Recall that a tangent vector is a principal direction if it is an eigenvector for the Weingarten map. Assume that the principal curvature are different and show that $\frac{\partial\sigma}{\partial u}$ and $\frac{\partial\sigma}{\partial v}$ are the principal directions if and only if $F = 0 = M$.

20. Let $\alpha(u)$ be a unit speed curve in the x, y plane $\mathbb{R}^2$. Show that

$$\sigma(u,v) = (\alpha(u), v).$$

yields a parametrized surface. Compute its first and second fundamental forms and principal curvatures. Compute its Gauss curvature.

21. Show that the equation

$$ax + by + cz = d$$

defines a surface if and only if $(a,b,c) \neq (0,0,0)$. Show that this surface has a parametrization that is Cartesian.

22. Let γ be a unit speed curve on a surface S with normal N. Define $C = N \times T$, $T = \dot{\gamma}$ and

$$\kappa_g = \frac{dT}{ds} \cdot C,\ \kappa_n = \frac{dT}{ds} \cdot N,\ \tau_g = \frac{dC}{ds} \cdot N$$

Prove that

$$\begin{aligned} \frac{dT}{ds} &= \kappa_g C + \kappa_n N, \\ \frac{dC}{ds} &= -\kappa_g T + \tau_g N, \\ \frac{dN}{ds} &= -\kappa_n T - \tau_g C. \end{aligned}$$

23. Let $\gamma(u)$ be a regular curve in the x, y plane $\mathbb{R}^2$. Show that

$$\sigma(u, v) = (\gamma(u), v).$$

yields a parametrized surface. Compute its first fundamental form and construct a local isometry from a subset of the plane to the surface.

24. For a regular curve $\gamma(u) : I \to \mathbb{R}^3 - \{(0,0,0)\}$ show that $\sigma(u, v) = v\gamma(u)$ defines a surface for $v > 0$ provided γ and $\dot{\gamma}$ are linearly independent. Compute its first fundamental form. Show that it admits Cartesian coordinates by rewriting the surface as $\sigma(r, \theta) = r\delta(\theta)$ for a suitable unit speed curve $\delta(\theta)$.

25. Let $\sigma(z, \theta) = \left(\sqrt{1-z^2}\cos\theta, \sqrt{1-z^2}\sin\theta, z\right)$ with $-1 < z < 1$ and $-\pi < \theta < \pi$. Show that σ defines a patch on a surface. What is the surface?

26. Let σ be a coordinate patch such that $E = 1$ and $F = 0$. Prove that the u curves are unit speed with acceleration that is perpendicular to the surface. The u curves are given by $\gamma(u) = \sigma(u, v)$ where v is fixed.

27. For a surface of revolution $\sigma(t, \theta) = (r(t)\cos(\theta), r(t)\sin(\theta), z(t))$ show that the first fundamental form is given by

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} \dot{r}^2 + \dot{z}^2 & 0 \\ 0 & r^2 \end{bmatrix}$$

and that the longitudes/meridians $\gamma(t) = \sigma((t, \theta)$ have acceleration perpendicular to the surface provided that $(r(t), 0, z(t))$ is unit speed.

28. Find a conformal map from a surface of revolution $\sigma_1(r, \theta) = (r\cos\theta, r\sin\theta, z_1(r))$ to a circular cylinder $\sigma_2(r, \theta) = (\cos\theta, \sin\theta, z_2(r))$.

29. Reparametrize the curve $(r(u), z(u))$ so that the new parametrization $\sigma(t, \theta) = (r(t)\cos(\theta), r(t)\sin(\theta), z(t))$ is conformal.

30. Find an equiareal map from a surface of revolution $\sigma_1(r, \theta) = (r\cos\theta, r\sin\theta, z_1(r))$ to a circular cylinder $\sigma_2(r, \theta) = (\cos\theta, \sin\theta, z_2(r))$.

31. Reparametrize the curve $(r(u), z(u))$ so that the new parametrization $\sigma(t, \theta) = (r(t)\cos(\theta), r(t)\sin(\theta), z(t))$ is equiareal.

32. Let $\sigma : U \to S^2$ be a parametrization of part of the unit sphere. Show that the normal $\frac{\partial \sigma}{\partial u} \times \frac{\partial \sigma}{\partial v}$ is always proportional to σ.

33. Show that a Monge patch $z = f(x, y)$ is equiareal if and only if f is constant.

34. Show that a Monge patch $z = f(x, y)$ is conformal if and only if f is constant.

35. Show that the equation
$$ax + by + cz = d$$
defines a surface if and only if $(a, b, c) \neq (0, 0, 0)$. Show that this surface has a parametrization that is Cartesian.

36. The conoid is a special type of ruled surface given by
$$\begin{aligned} \sigma(t, \theta) &= (r(t)\cos\theta, r(t)\sin\theta, z(\theta)) \\ &= (0, 0, z(\theta)) + r(t)(\cos\theta, \sin\theta, 0) \end{aligned}$$
Compute its first fundamental form. Show that if $z(\theta) = a\theta$ for some constant a, then $r(t)$ can be reparametrized in such a way that we get a conformal parametrization.

37. Consider the two parametrized surfaces given by
$$\begin{aligned} \sigma_1(\phi, u) &= (\sinh\phi\cos u, \sinh\phi\sin u, u) \\ &= (0, 0, u) + \sinh\phi(\cos u, \sin u, 0) \\ \sigma_2(t, \theta) &= (\cosh t\cos\theta, \cosh t\sin\theta, t) \end{aligned}$$
Compute the first fundamental forms for both surfaces and construct a local isometry from the first surface to the second. (The first surface is a ruled surface with a one-to-one parametrization called the helicoid, the second surface is a surface of revolution called the catenoid.)

38. Let $S = \left\{ x \in \mathbb{R}^3 : |x - m|^2 = R^2 \right\}$. Show that S is a surface, and that if I and II denote the first and second fundamental forms, then
$$\mathrm{II} = \pm\frac{1}{R}\mathrm{I}$$

39. 7. The conoid is a special type of ruled surface given by
$$\begin{aligned} \sigma(t, \theta) &= (t\cos\theta, t\sin\theta, z(\theta)) \\ &= (0, 0, z(\theta)) + t(\cos\theta, \sin\theta, 0) \end{aligned}$$
Compute its first and second fundamental forms as well as the Gauss and mean curvatures.

40. Let $\gamma(t) : I \to S$ be a regular curve on a surface S, with N being the normal to the surface. Show that

$$\kappa_n = \frac{\mathrm{II}(\dot{\gamma}, \dot{\gamma})}{\mathrm{I}(\dot{\gamma}, \dot{\gamma})}, \; \kappa_g = \frac{\det(\dot{\gamma}, \ddot{\gamma}, N)}{(\mathrm{I}(\dot{\gamma}, \dot{\gamma}))^{3/2}}$$

41. Show that the principal curvatures at a point $p \in S$ are equal if and only if at p the mean and Gauss curvatures are related by $H^2 = K$.

42. Compute the matrix representation of the Weingarten map for a Monge patch $\sigma(x, y) = (x, y, f(x, y))$ with respect to the basis $\frac{\partial \sigma}{\partial x}, \frac{\partial \sigma}{\partial y}$.