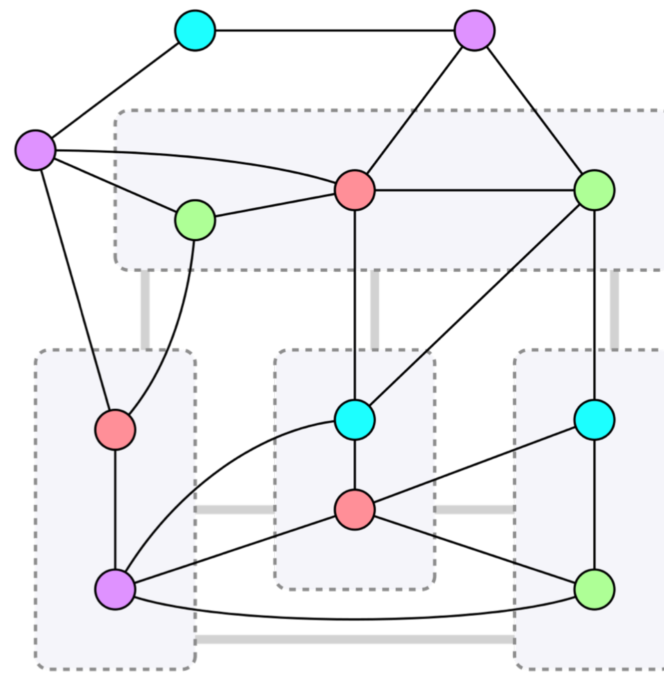
Computation of Hadwiger Number and Related Contraction Problems: Tight Lower Bound



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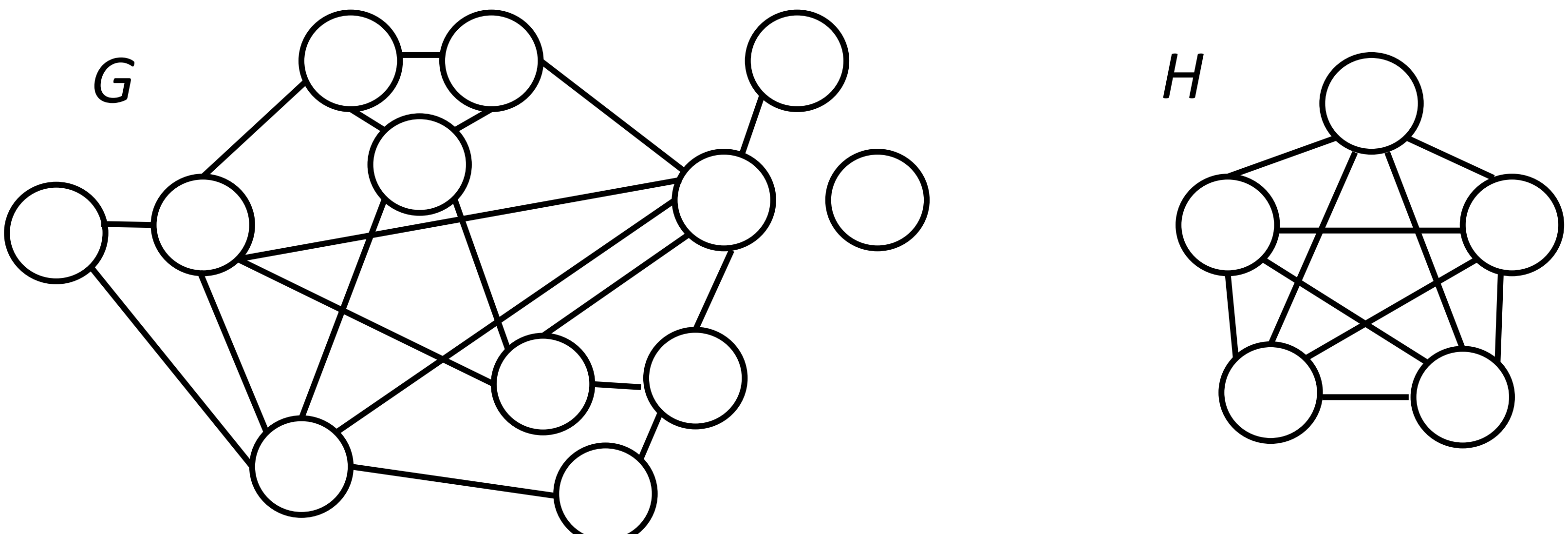
Introduction

Minor. *H* is a **minor** of *G* if there is a series of edge deletions, edge contractions and vertex deletions in *G* that yields *H*.



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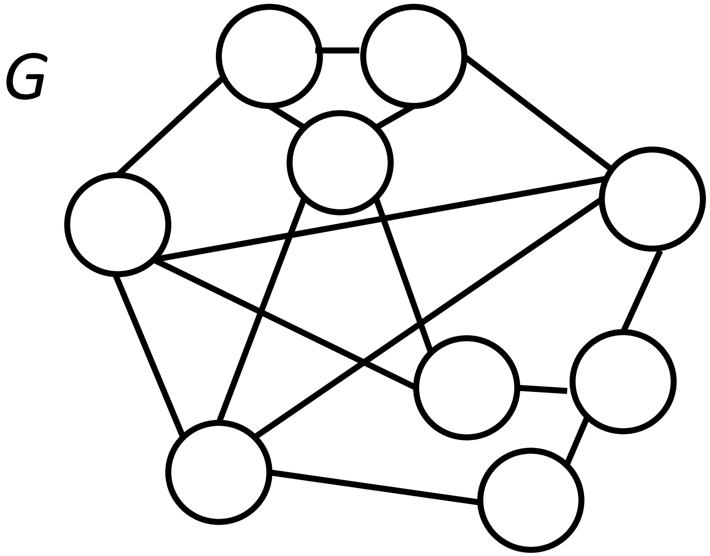

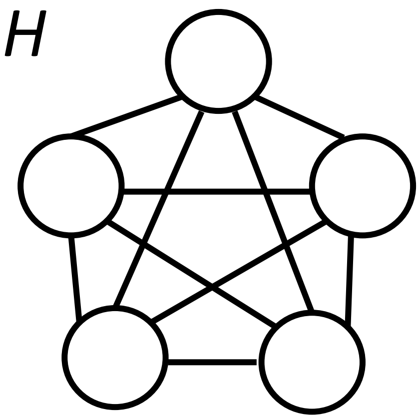

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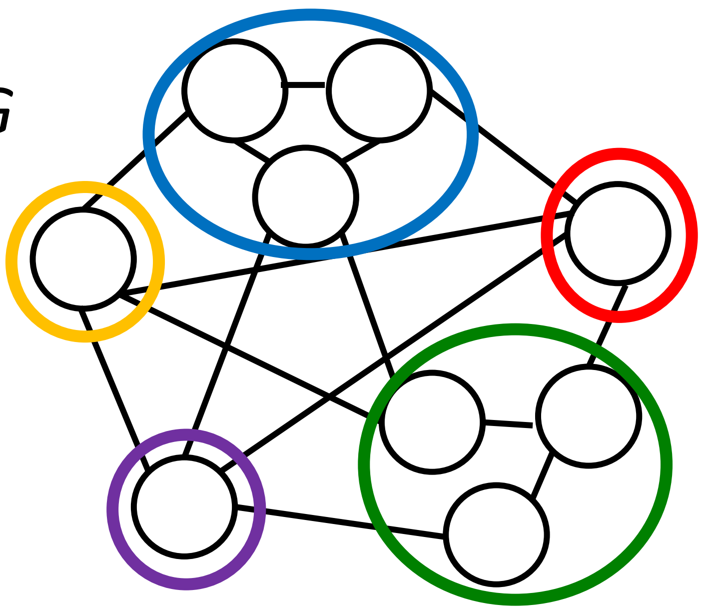

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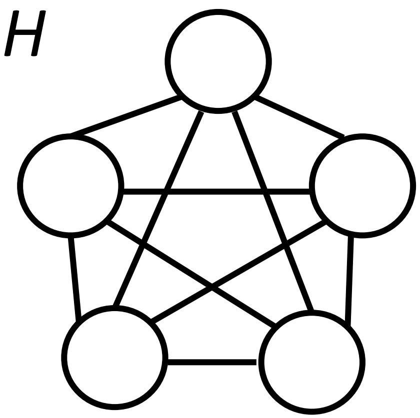

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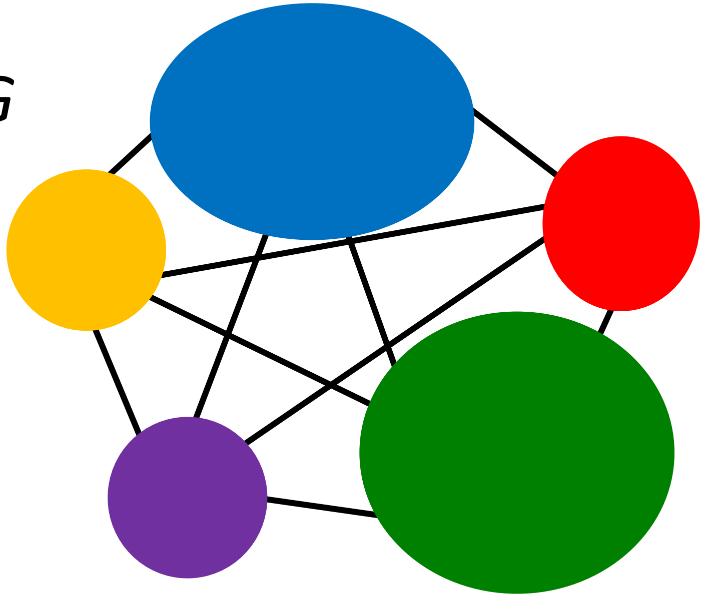

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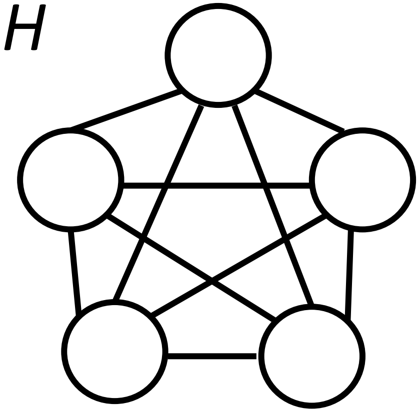
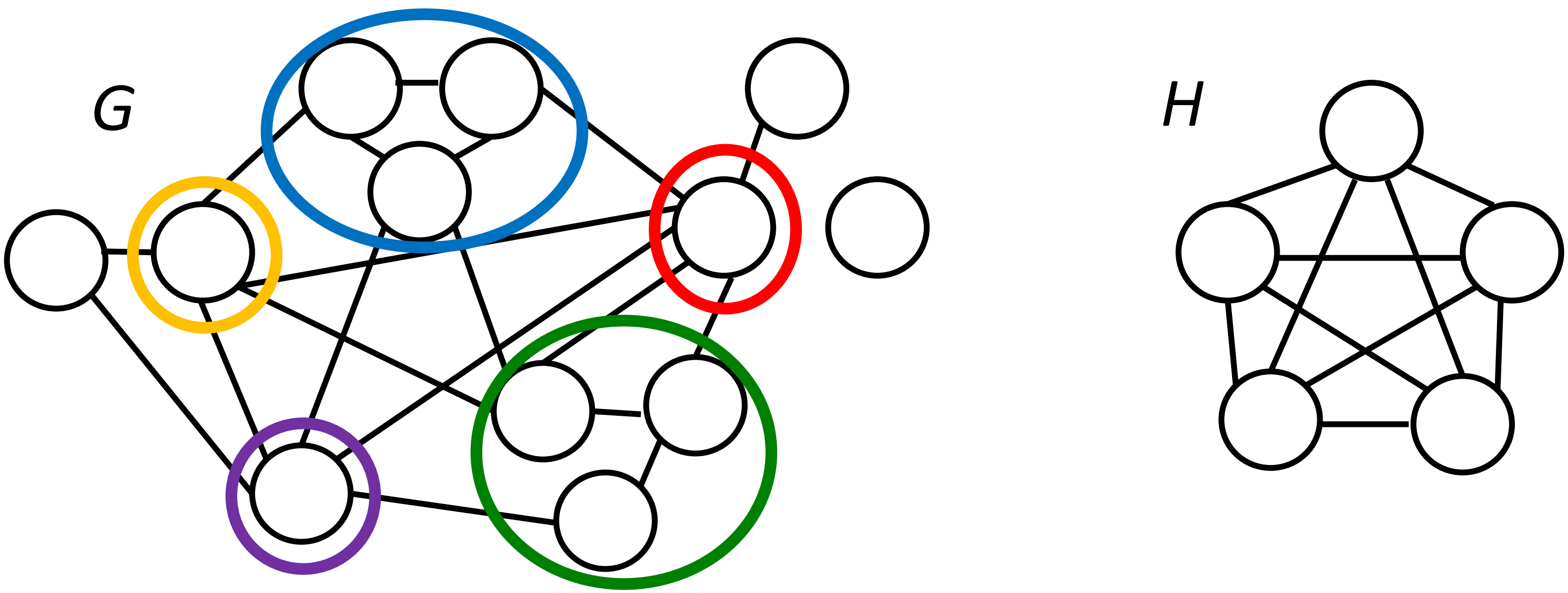


Introduction

Minor. H is a **minor** of G if there is a series of edge deletions, edge contractions and vertex deletions in G that yields H.

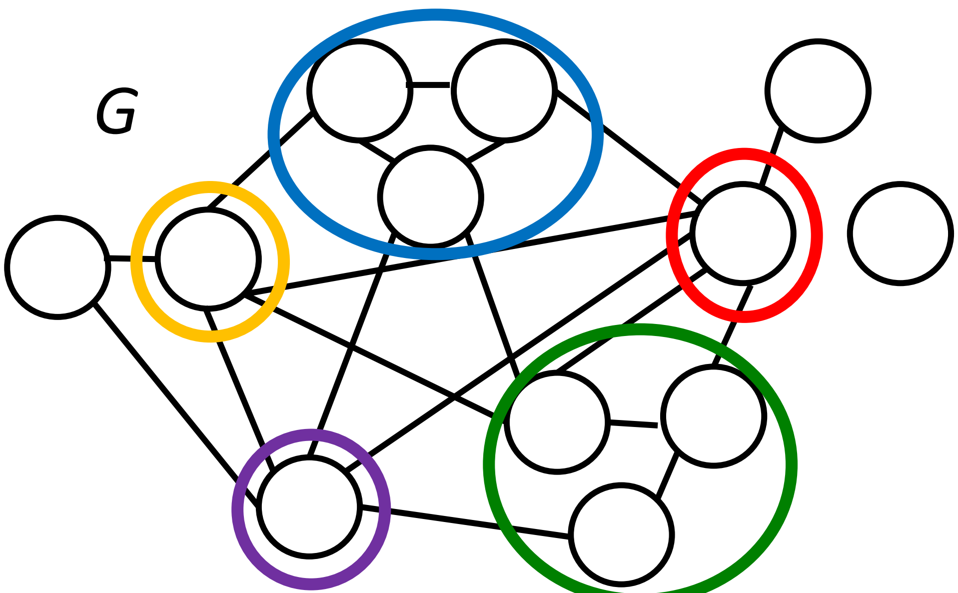
Alternatively (informal). H is a **minor** of G if we can find pairwise disjoint connected subsets of $V(G)$ to map to the vertices in H, and connect them by edges as dictated by H.



Introduction

Hadwiger number of a graph *G*. The largest h such that K_h (the clique on h vertices) is a minor of G.

G



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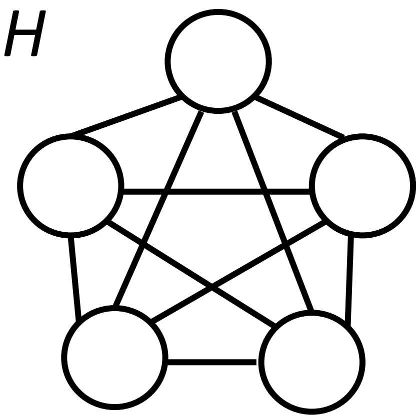

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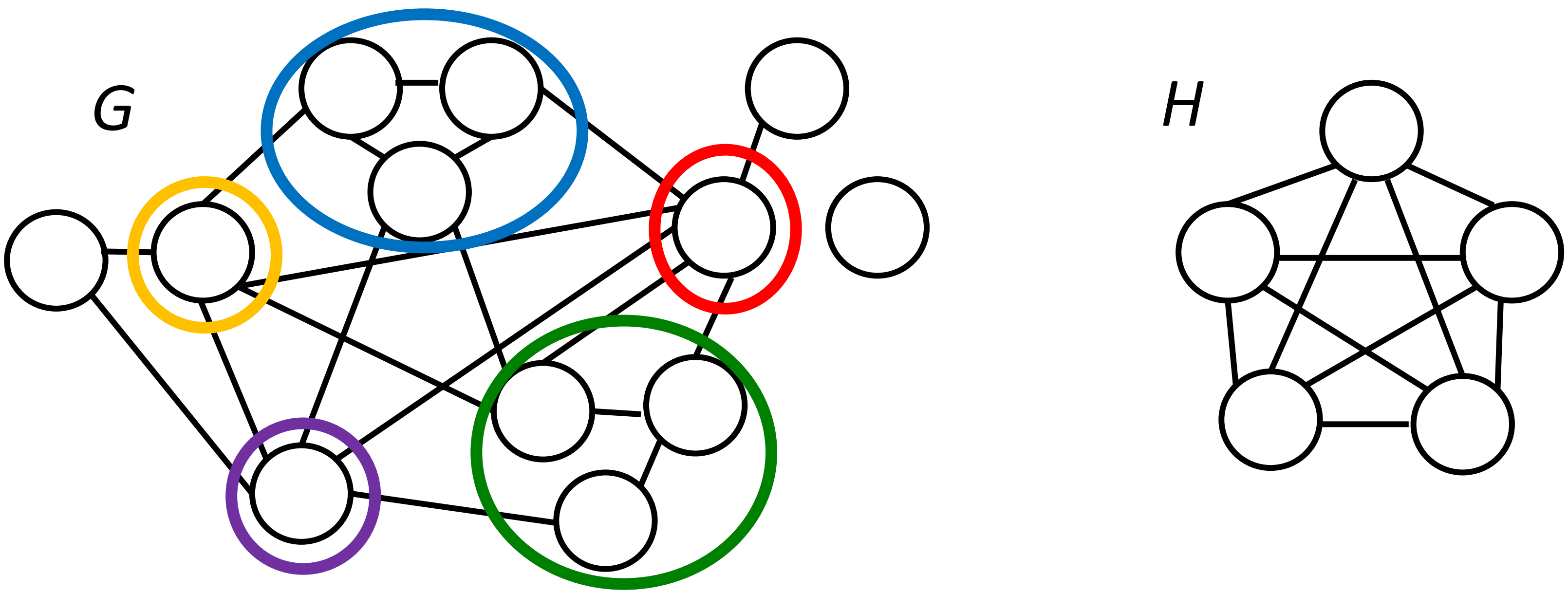
Hadwiger number of a graph *G*. The largest h such that K_h (the clique on h vertices) is a minor of G.

Can be trivially solved in time $n^{O(n)}$.

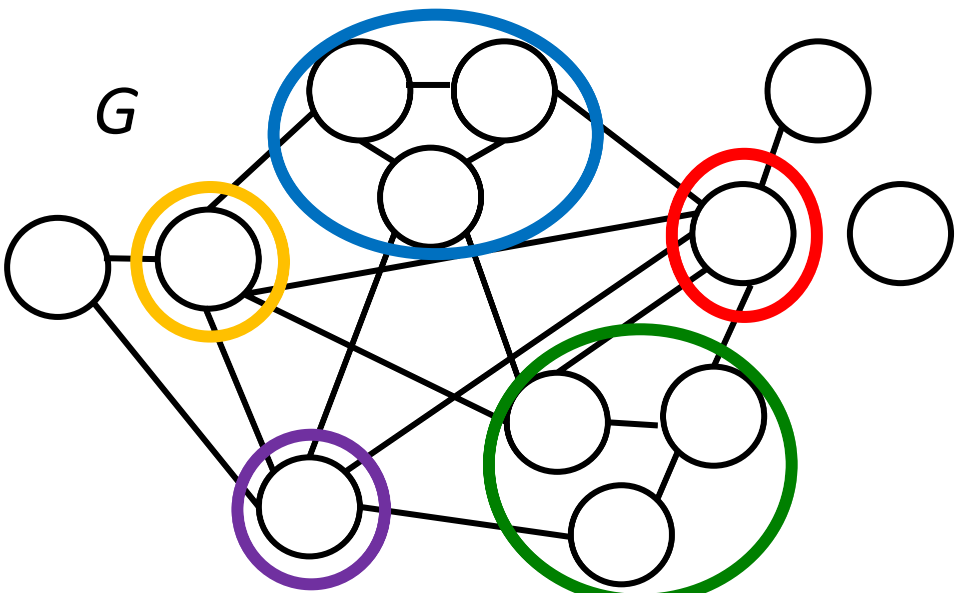


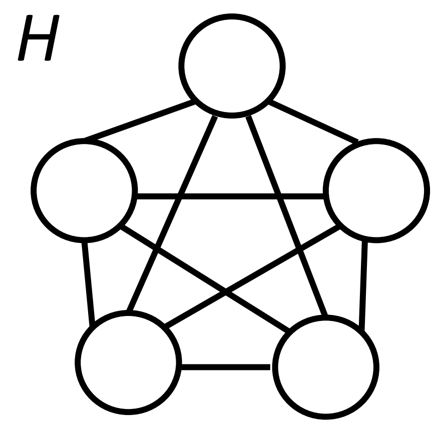
Introduction

Hadwiger number of a graph *G*. The largest h such that K_h (the clique on h vertices) is a minor of G.

Can be trivially solved in time $n^{O(n)}$.

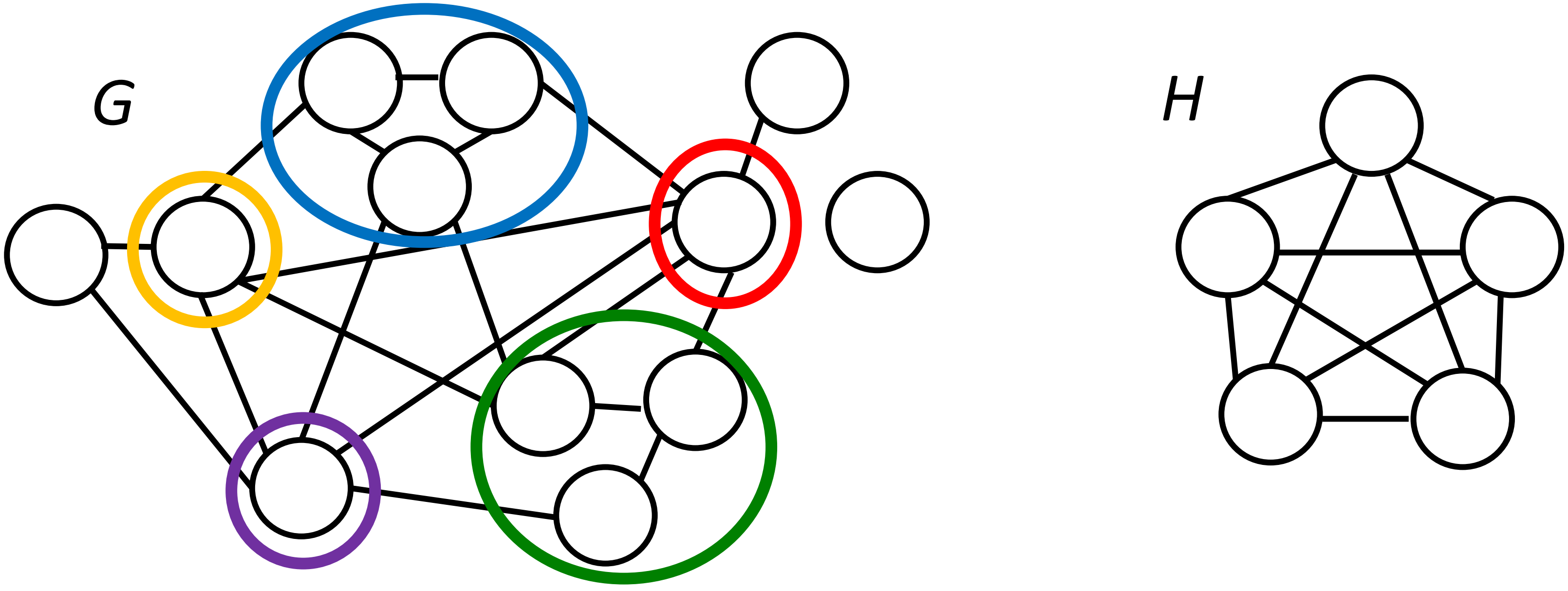
Open question. Can it be solved in time $2^{O(n)}$?
(Asked in several venues.)





Introduction

Graph Minor problem. Given two graphs *G* and *H*, is *H* a minor of *G*?

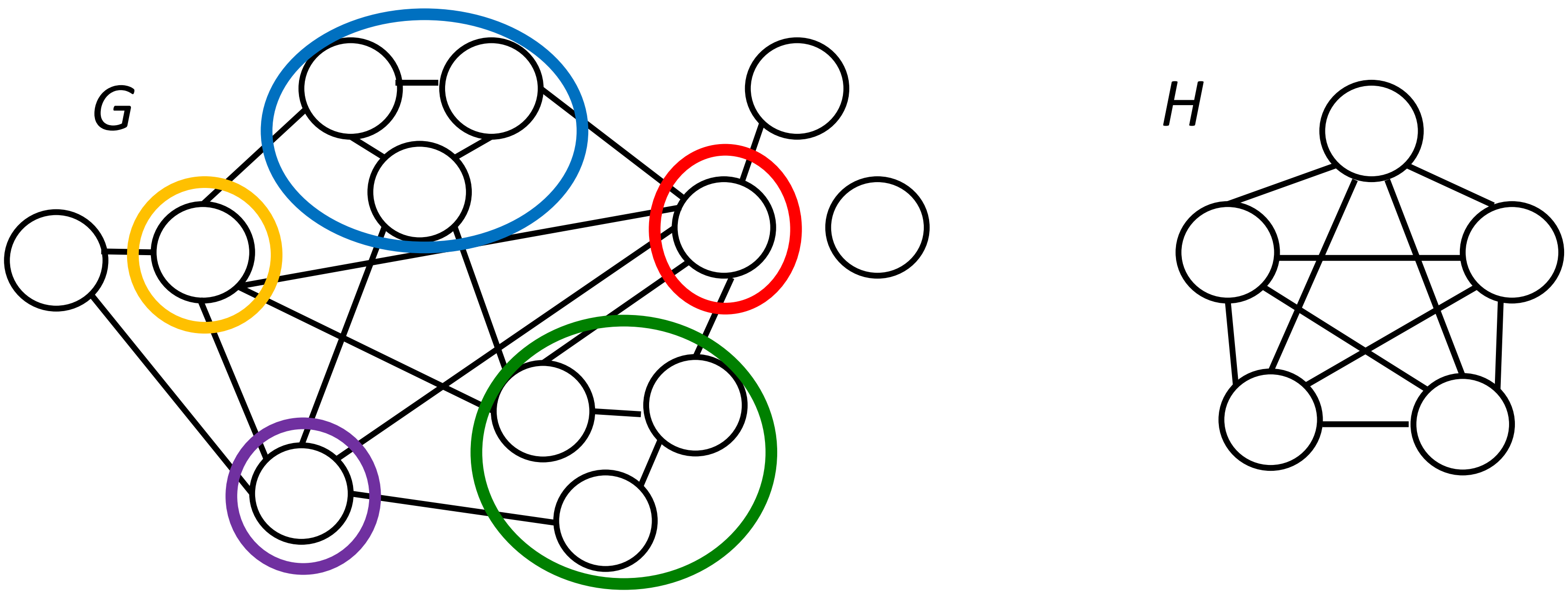


Introduction

Graph Minor problem. Given two graphs *G* and *H*, is *H* a minor of *G*?

The Graph Minor project is the inspiration behind Parameterized Complexity, and central to other research areas as well.

Survey. LSZ: *Efficient Graph Minors Theory and Parameterized Algorithms for (Planar) Disjoint Paths.* Treewidth, Kernels and Algorithms, 2020.

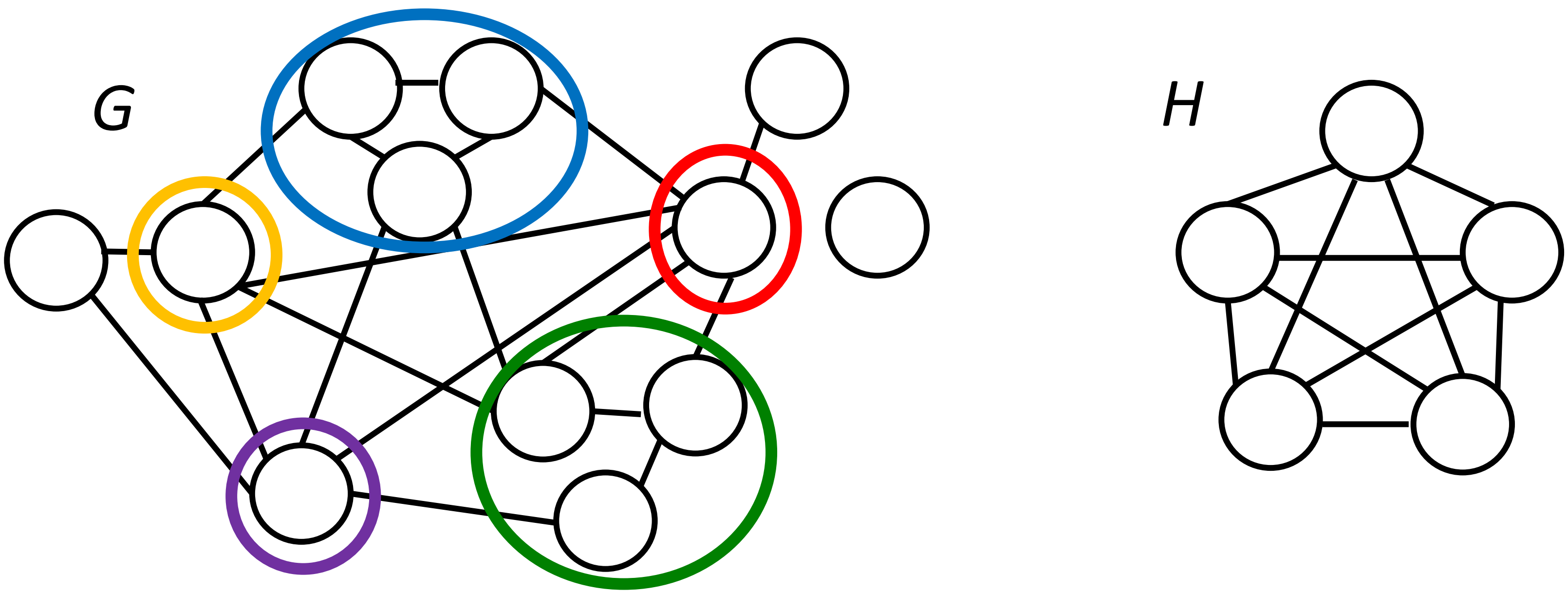


Introduction

Graph Minor problem. Given two graphs *G* and *H*, is *H* a minor of *G*?

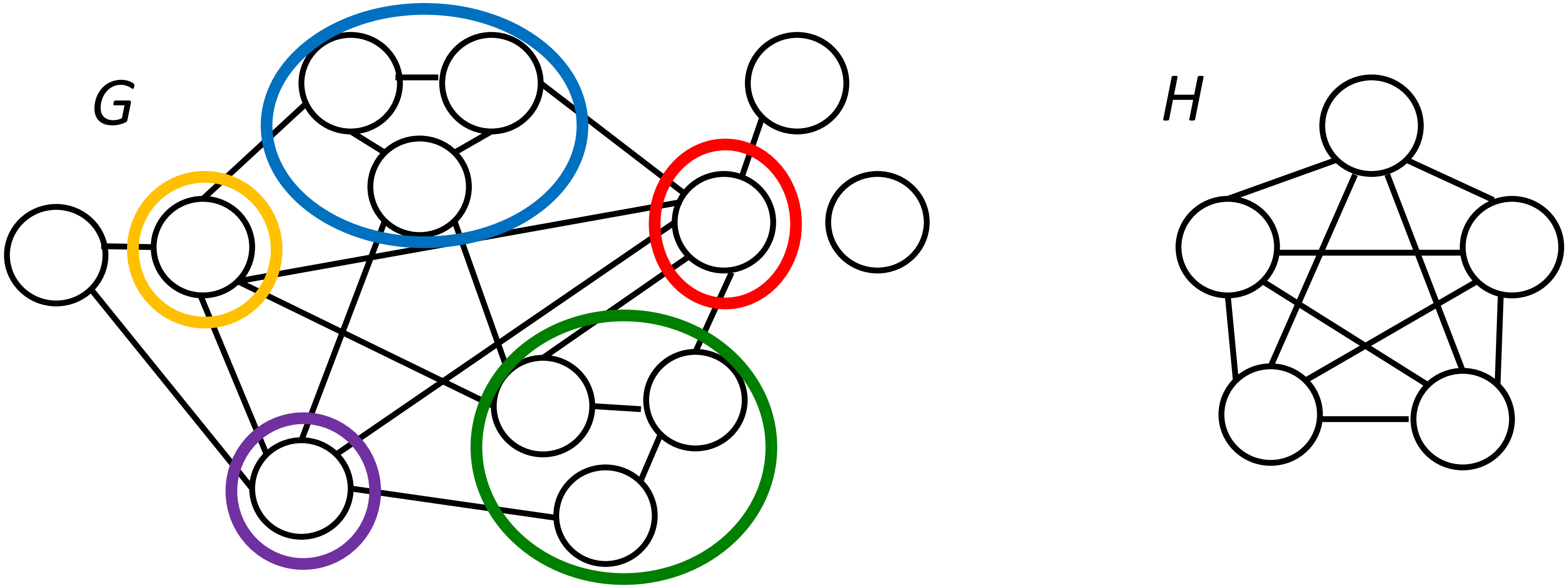
On general graphs.
- FPT, that is, solvable in time $f(h)n^{O(1)}$. [RS'95]
- Unless the ETH fails, not solvable in time $n^{o(n)}$ where n=h. [CFGKMPS'16]. Tight.

We consider the case where *H* is a clique.



Introduction

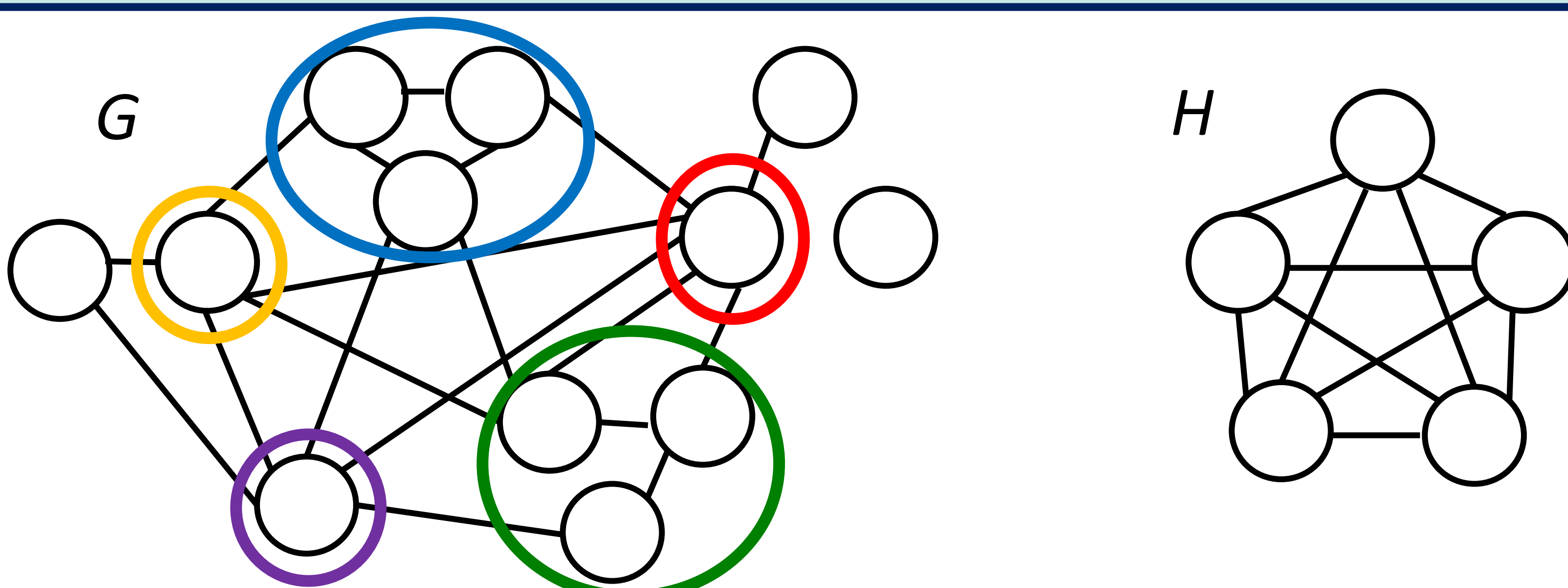
I. Subgraph Isomorphism problem. Given two graphs *G* and *H*, does *G* contain a subgraph isomorphic to *H*?



Introduction

I. Subgraph Isomorphism problem. Given two graphs G and H, does G contain a subgraph isomorphic to H?

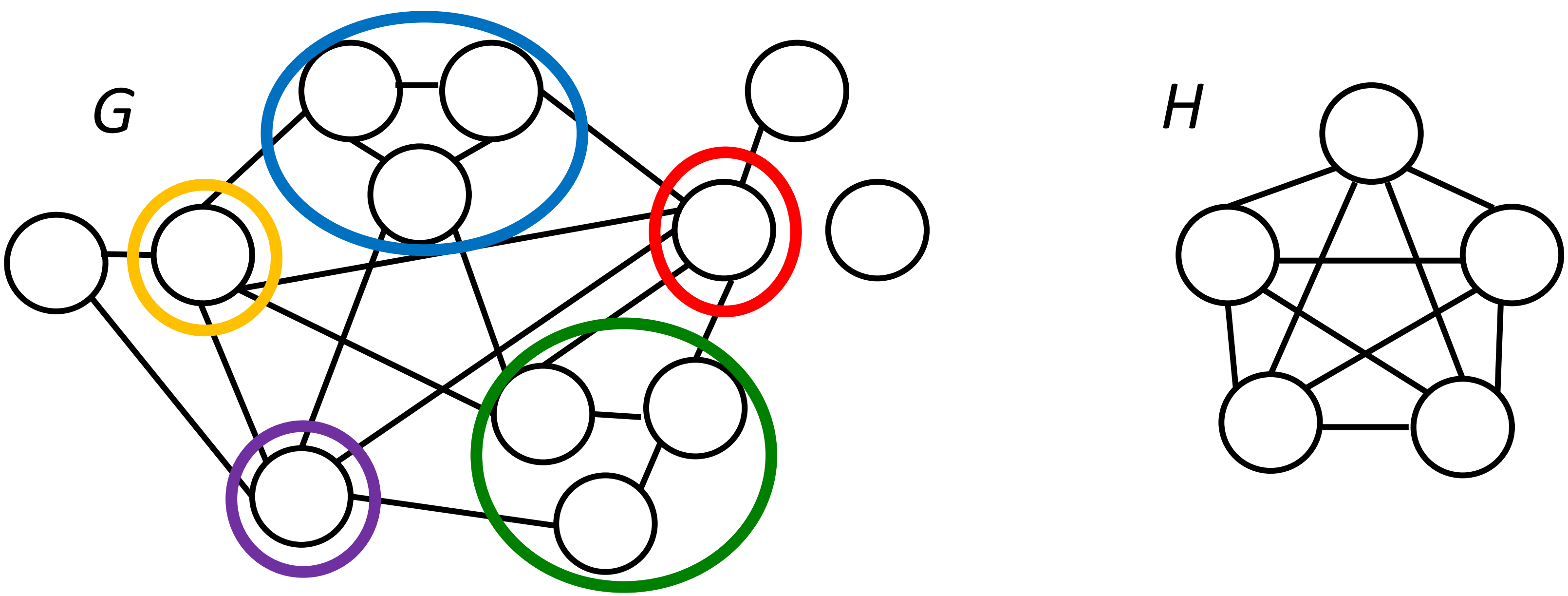
On general graphs.
- W[1]-hard (unlikely to be solved in $f(h)n^{O(1)}$).
- Can be solved in time $n^{O(h)}$.
- Unless the ETH fails, cannot be solved in time $n^{o(n)}$ where n=h. [CFGKMPS'16]



Introduction

I. Subgraph Isomorphism problem. Given two graphs *G* and *H*, does *G* contain a subgraph isomorphic to *H*?

When *H* is a clique.
- W[1]-hard.
- Can be solved in time $2^{O(n)}$.



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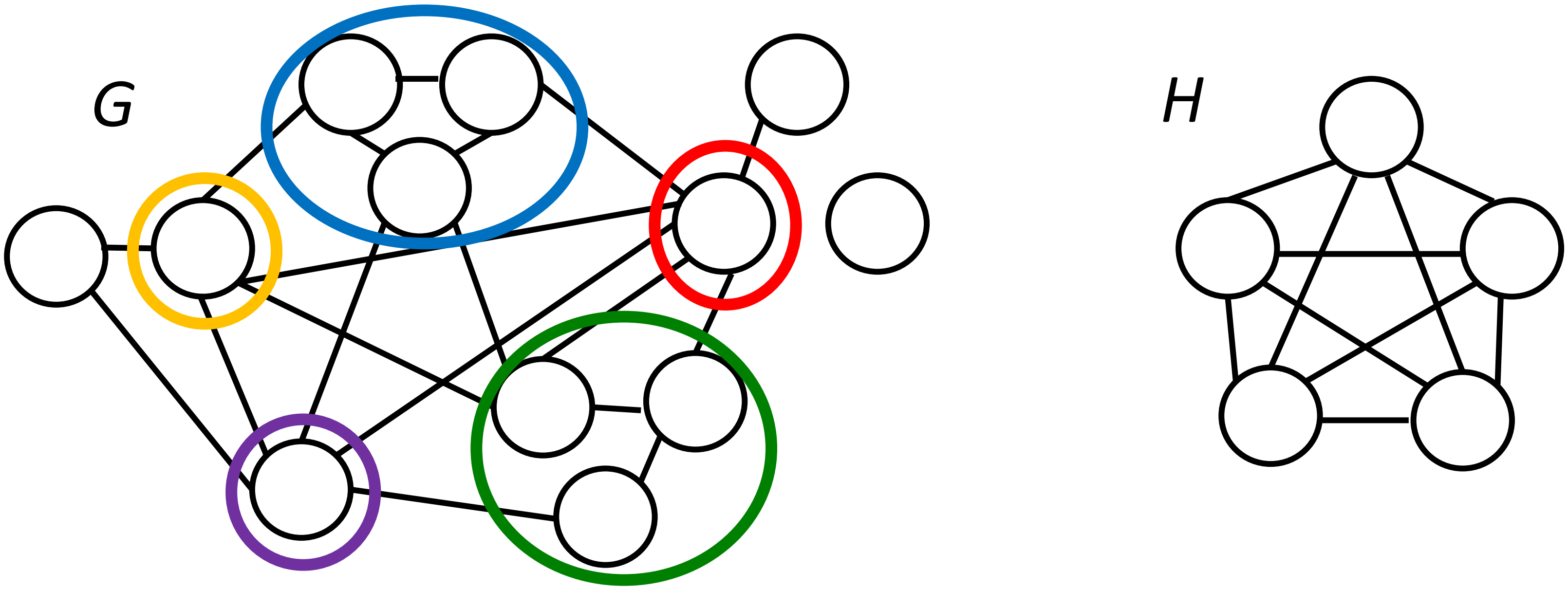
II. Graph Homomorphism problem. Given two graphs *G* and *H*, does there exist a homomorphism from *G* to *H*?

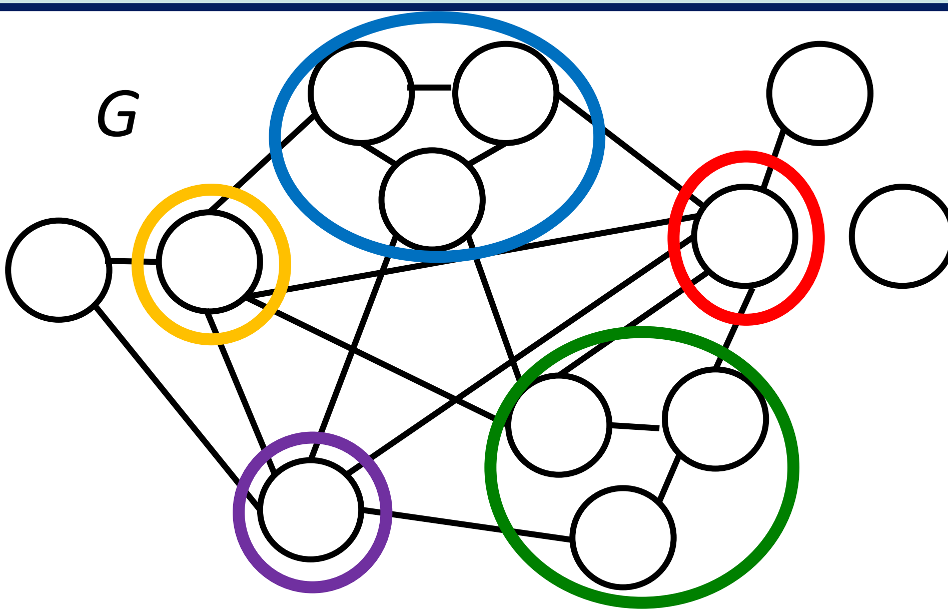


Introduction

II. Graph Homomorphism problem. Given two graphs G and H, does there exist a homomorphism from G to H?

On general graphs.
- para-NP-hard (NP-hard even when h=3).
- Can be solved in time $h^{O(n)}$.
- Unless the ETH fails, cannot be solved in time $h^{o(n)}$ where n=h. [CFGKMPS'16]

G



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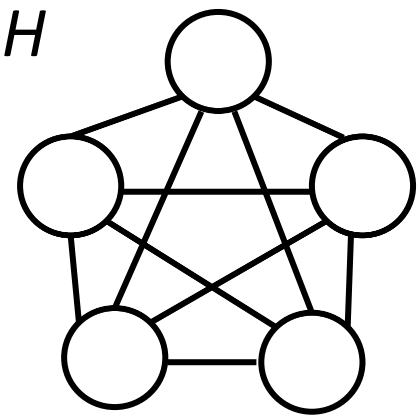
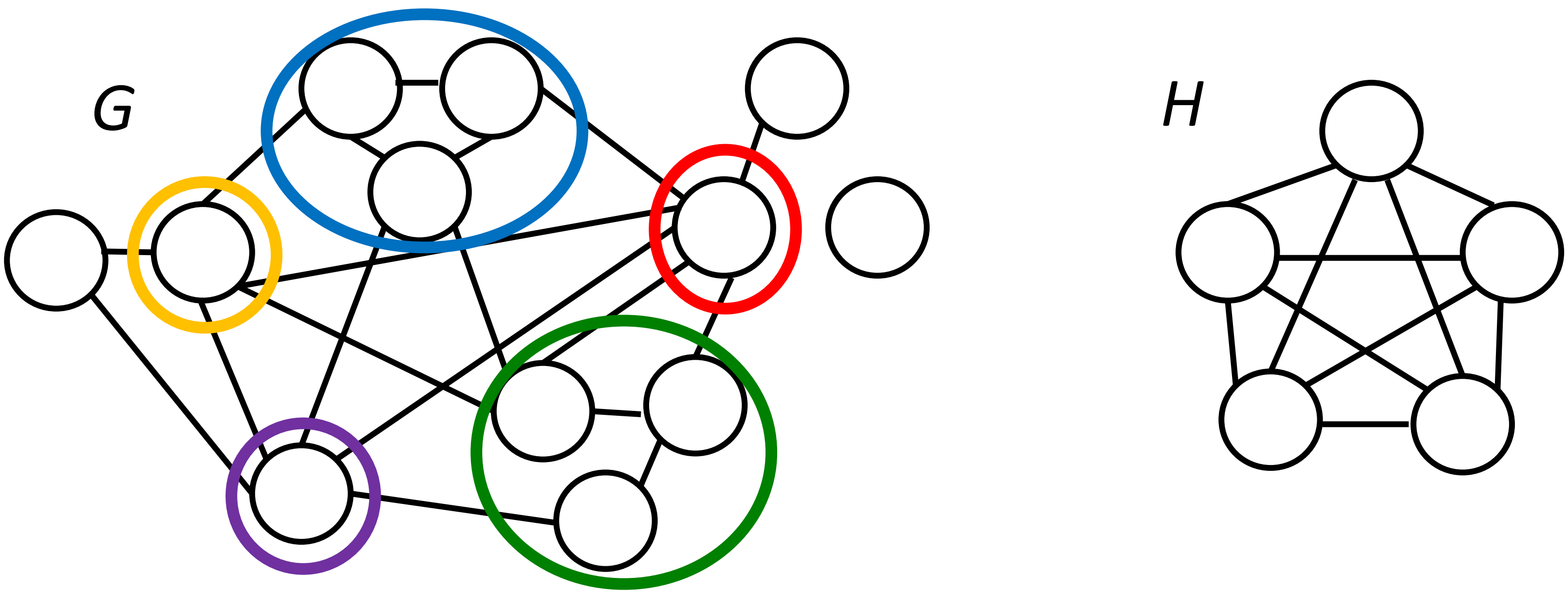


Introduction

II. Graph Homomorphism problem. Given two graphs G and H, does there exist a homomorphism from G to H?

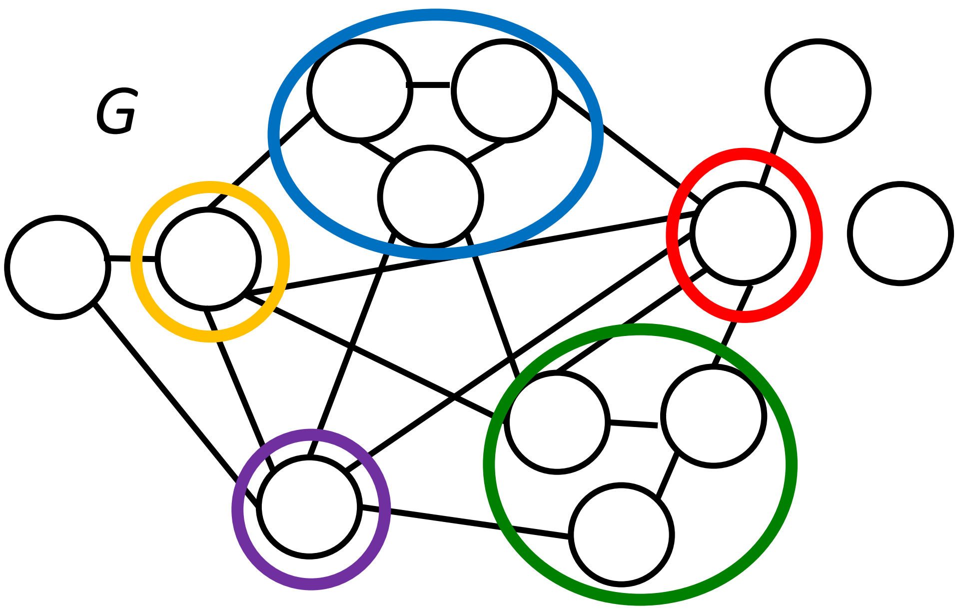
When H is a clique.
- Equivalent to h-Coloring.
- para-NP-hard (NP-hard even when h=3).
- Can be solved in time $2^{O(n)}$. [BHK'09, L'76]

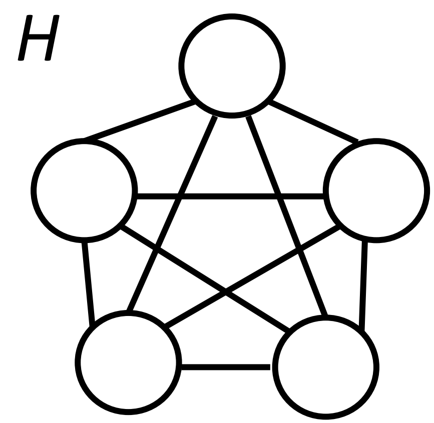


Introduction

II. Graph Homomorphism problem. Given two graphs *G* and *H*, does there exist a homomorphism from *G* to *H*?

When *G* is a clique.
- Equivalent to Subgraph Isomorphism where we swap the roles of *G* and *H*.
- Can be solved in time $2^{O(h)}$.





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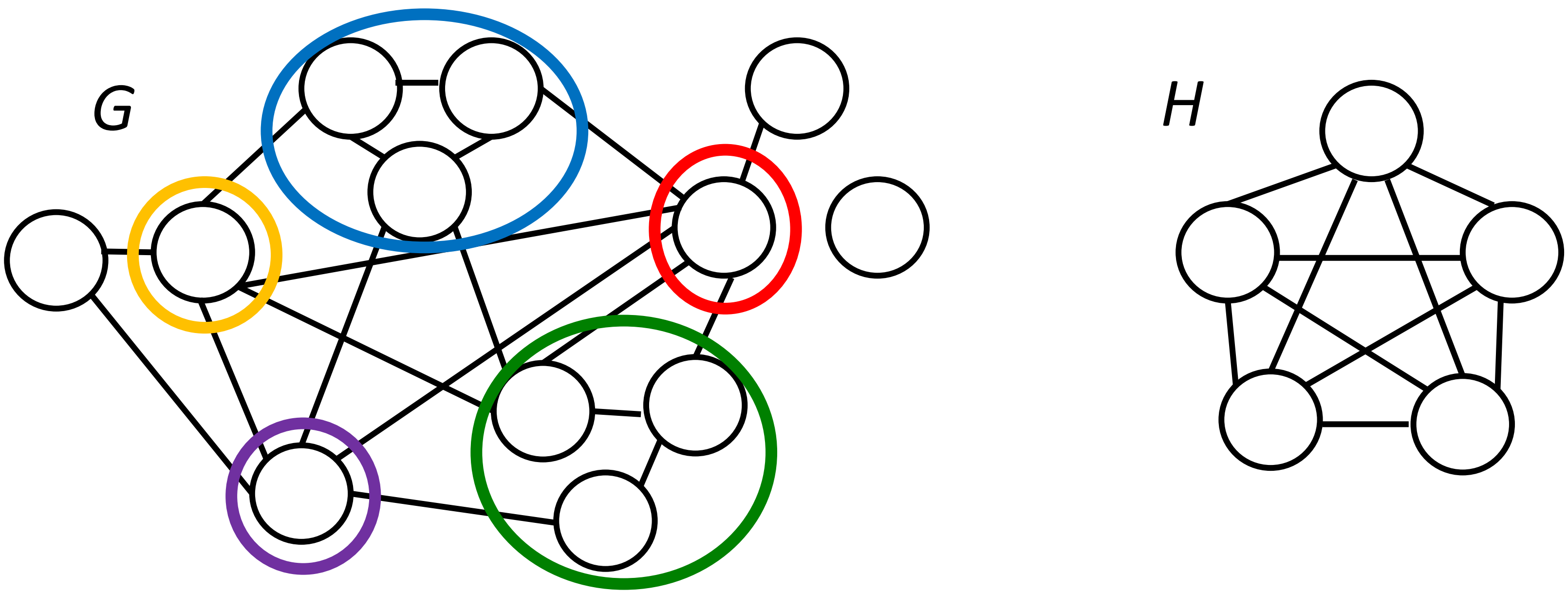
III. Topological Graph Minor problem. Given two graphs *G* and *H,* is *H* a **topological minor** of *G*?

- **Only contract edges incident to at least one degree-2 vertex.** Alternatively, the connected subsets are singletons, and pairwise vertex disjoint paths map to edges.
- Perhaps the closest relative to the Graph Minor problem.



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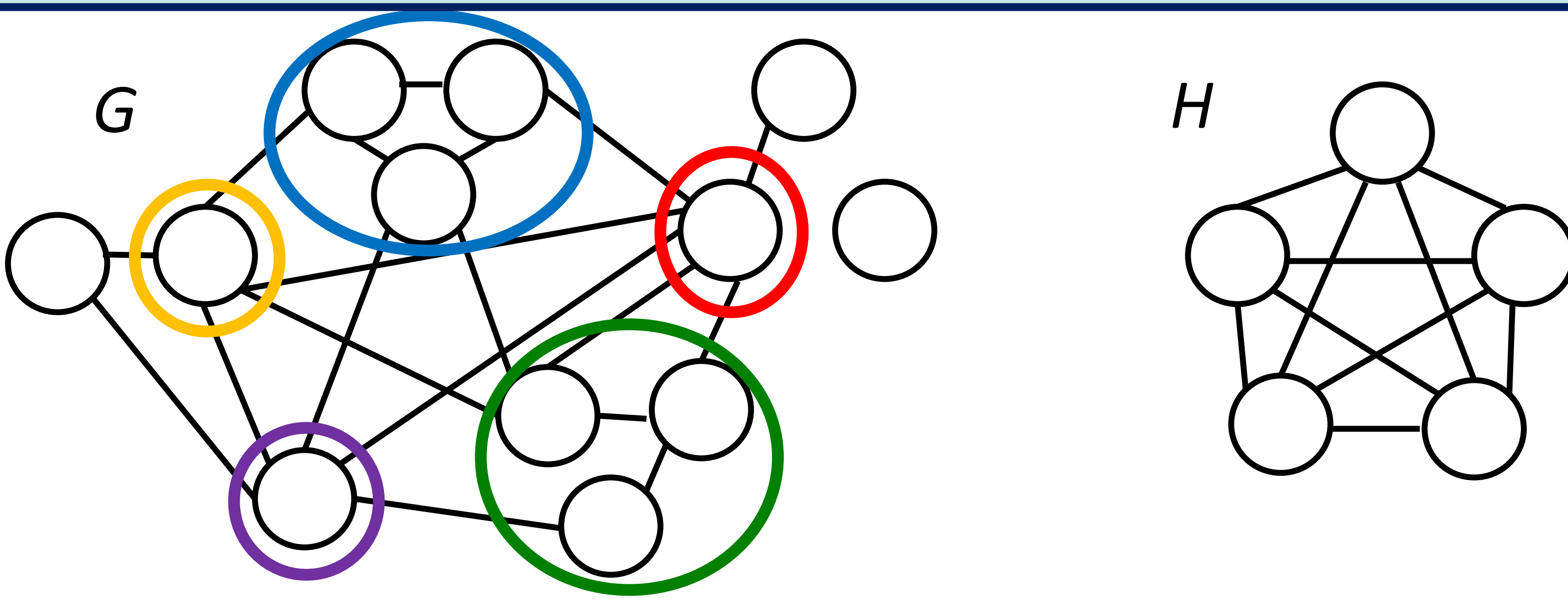
III. Topological Graph Minor problem. Given two graphs *G* and *H*, is *H* a topological minor of *G*?

On general graphs. • FPT. [GKMW’11]

- Can be solved in time $n^{O(n)}$.
- Unless the ETH fails, cannot be solved in time $n^{o(n)}$ where n=h. [CFGKMPS’16]

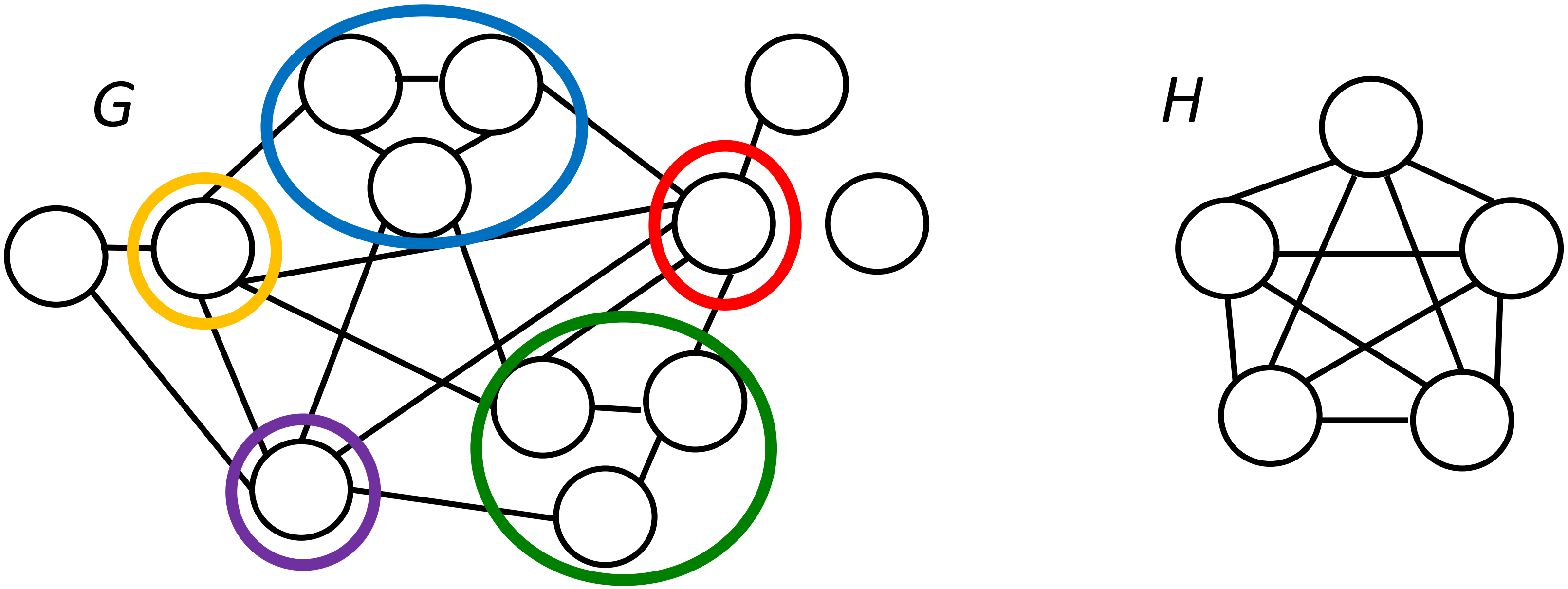


Introduction

III. Topological Graph Minor problem. Given two graphs G and H, is H a topological minor of G?

When H is a clique. • FPT. [GKMW'11]

• Can be solved in time $2^{O(n)}$. [LW'09]



Introduction

H is a clique	Subgraph Isomorphism	Graph Homomorphism	Topological Graph Minor	Graph Minor (Hadwiger Number)
FPT?	No	No	Yes	Yes
$2^{O(n)}$?	Yes	Yes	Yes	???

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Overview of Our Contribution

H is a clique	Subgraph Isomorphism	Graph Homomorphism	Topological Graph Minor	Graph Minor (Hadwiger Number)
FPT?	No	No	Yes	Yes
$2^{O(n)}$?	Yes	Yes	Yes	No

Theorem. Unless the ETH fails, the Hadwiger Number problem cannot be solved in time $n^{o(n)}$.

Solves the open question in the beginning of this presentation.

Overview of Our Contribution

H is a clique	Subgraph Isomorphism	Graph Homomorphism	Topological Graph Minor	Graph Minor (Hadwiger Number)
FPT?	No	No	Yes	Yes
$2^{O(n)}$?	Yes	Yes	Yes	No

Corollary. Unless the ETH fails, the Clique Contraction problem (can we contract at most k edges in a given graph G to obtain a clique) cannot be solved in time $n^{o(n)}$.

Solves an open question by [CFGKMPS'16].

Overview of Our Contribution

***F*-Contraction.** Given a graph G and non-negative integer t, can we contract at most t edges in G to obtain a graph in F?

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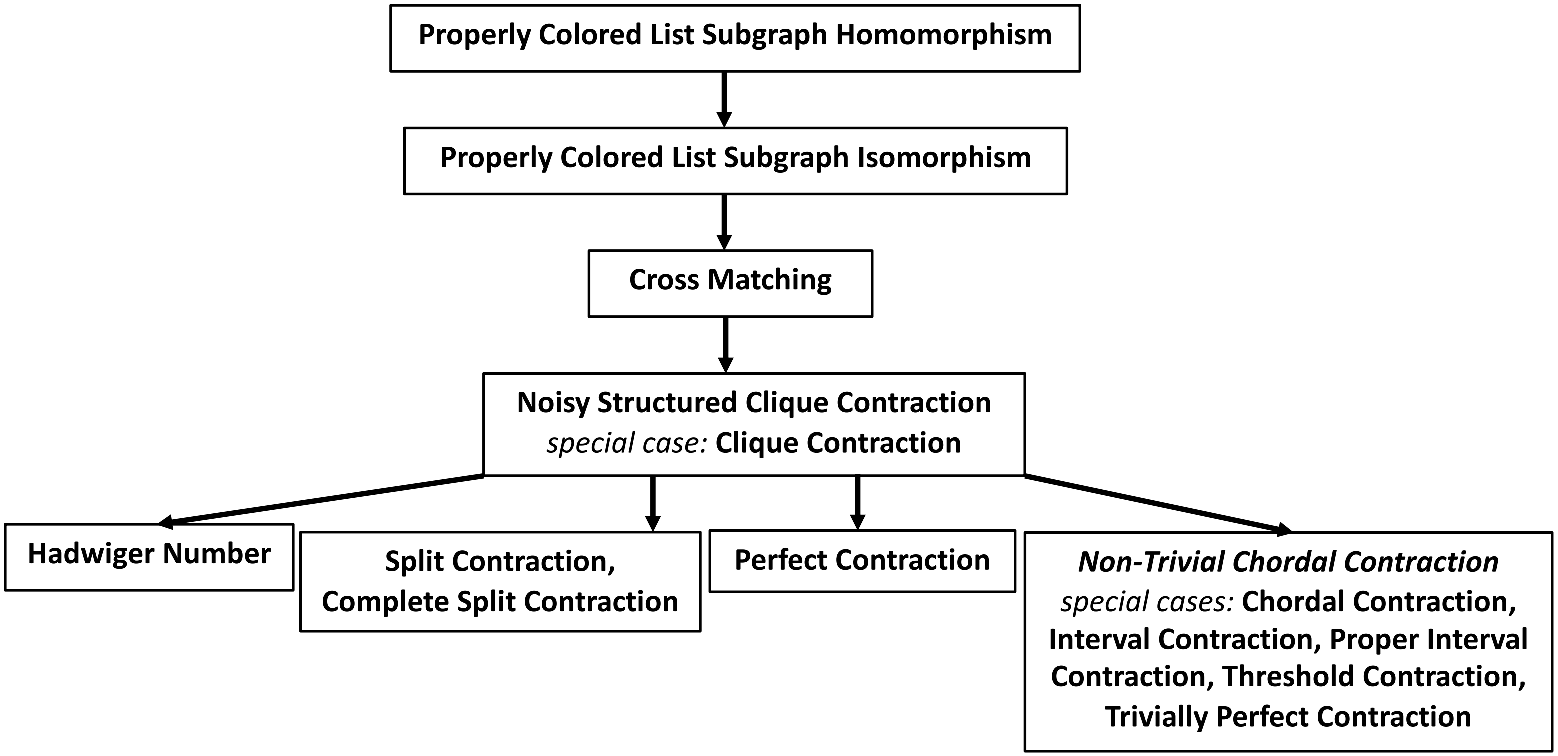
***F*-Contraction.** Given a graph G and non-negative integer t, can we contract at most t edges in G to obtain a graph in F?

Consequences of our approach. Unless the ETH fails, none of the following problems can be solved in time $n^{o(n)}$:

- Clique Contraction.
- Chordal Graph Contraction.
- Interval Graph Contraction.
- Proper Interval Graph Contraction.
- Threshold Graph Contraction.
- Perfect Graph Contraction.
- Trivially Perfect Graph Contraction.
- Split Graph Contraction.
- Perfect Split Graph Contraction.

Overview of Our Contribution



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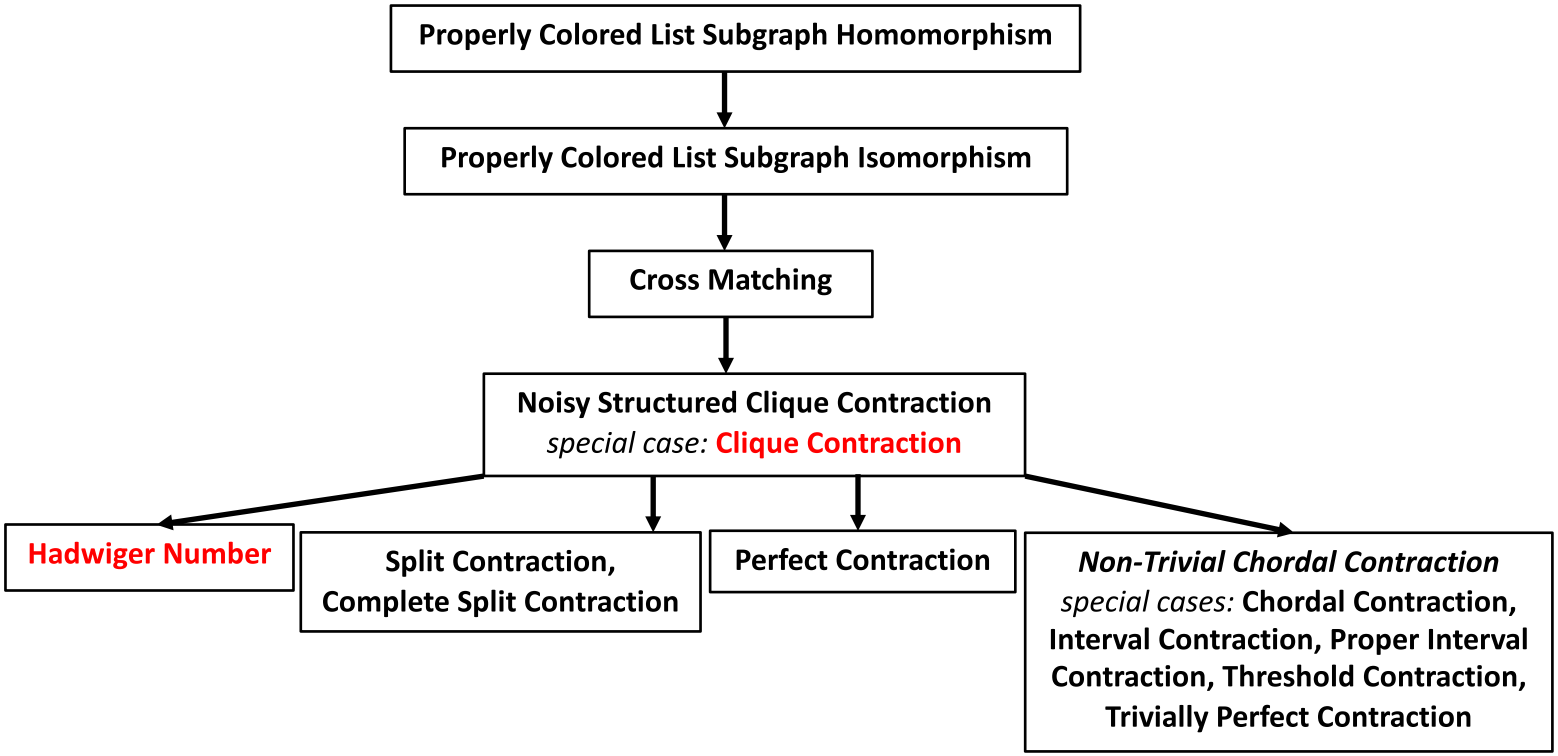

If G is connected, then we can contract at most k edges to obtain a clique if and only the Hadwiger number of G is n-k.

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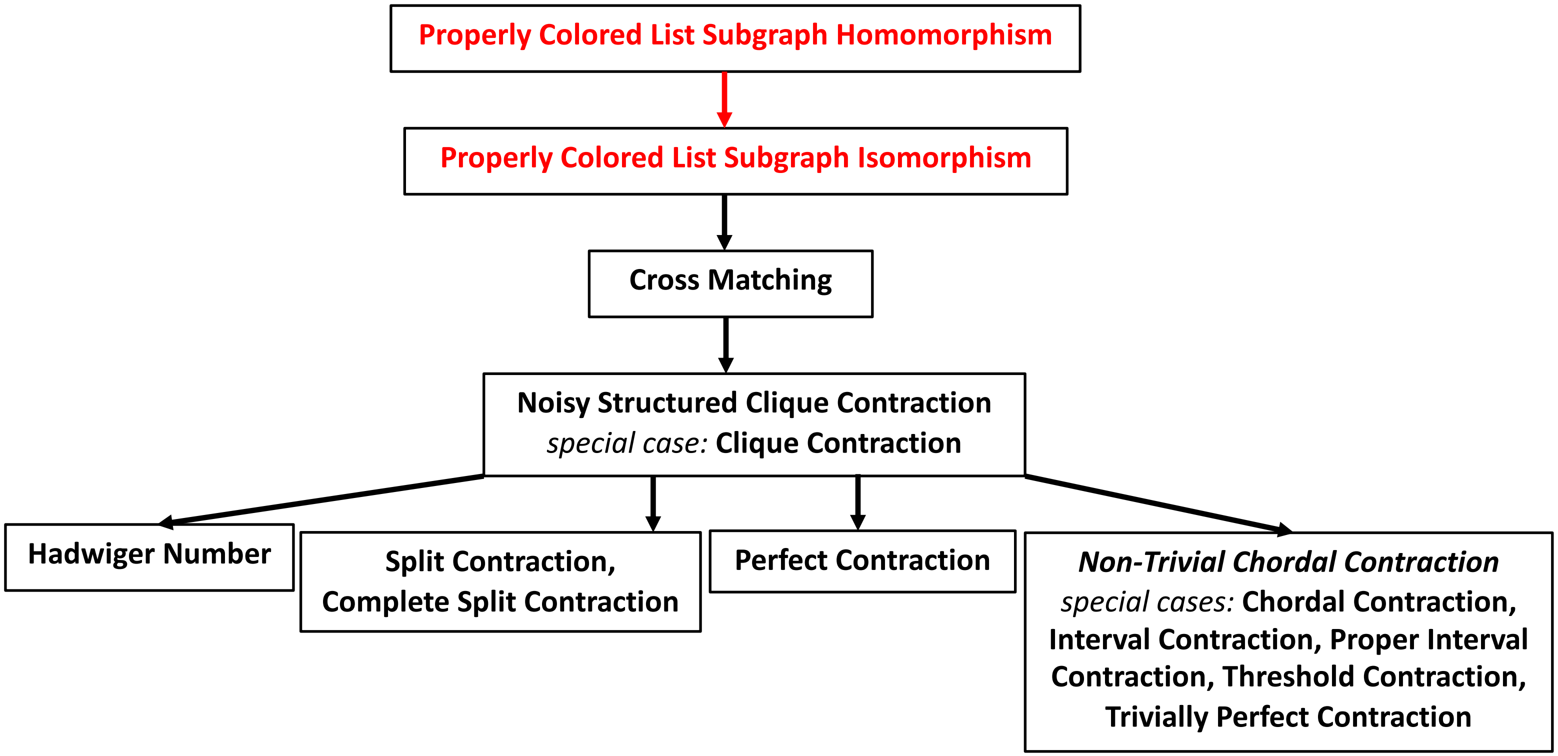

Some Technical Details

List Subgraph Isomorphism problem. Given two graphs G and H where $n=h$, as well as a list of vertices in H for each vertex in G, does G contain a subgraph isomorphic to H where each vertex in G is mapped to a vertex in its list?

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[CFGKMPS'16]. Unless the ETH fails, List Subgraph Isomorphism cannot be solved in time $n^{o(n)}$.

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[CFGKMPS'16]. Unless the ETH fails, List Subgraph Isomorphism cannot be solved in time $n^{o(n)}$.

Properly Colored List Subgraph Isomorphism problem. G and H are properly colored, and for every vertex v in G, all vertices (in H) in the list of v have the same color as v.

Some Technical Details

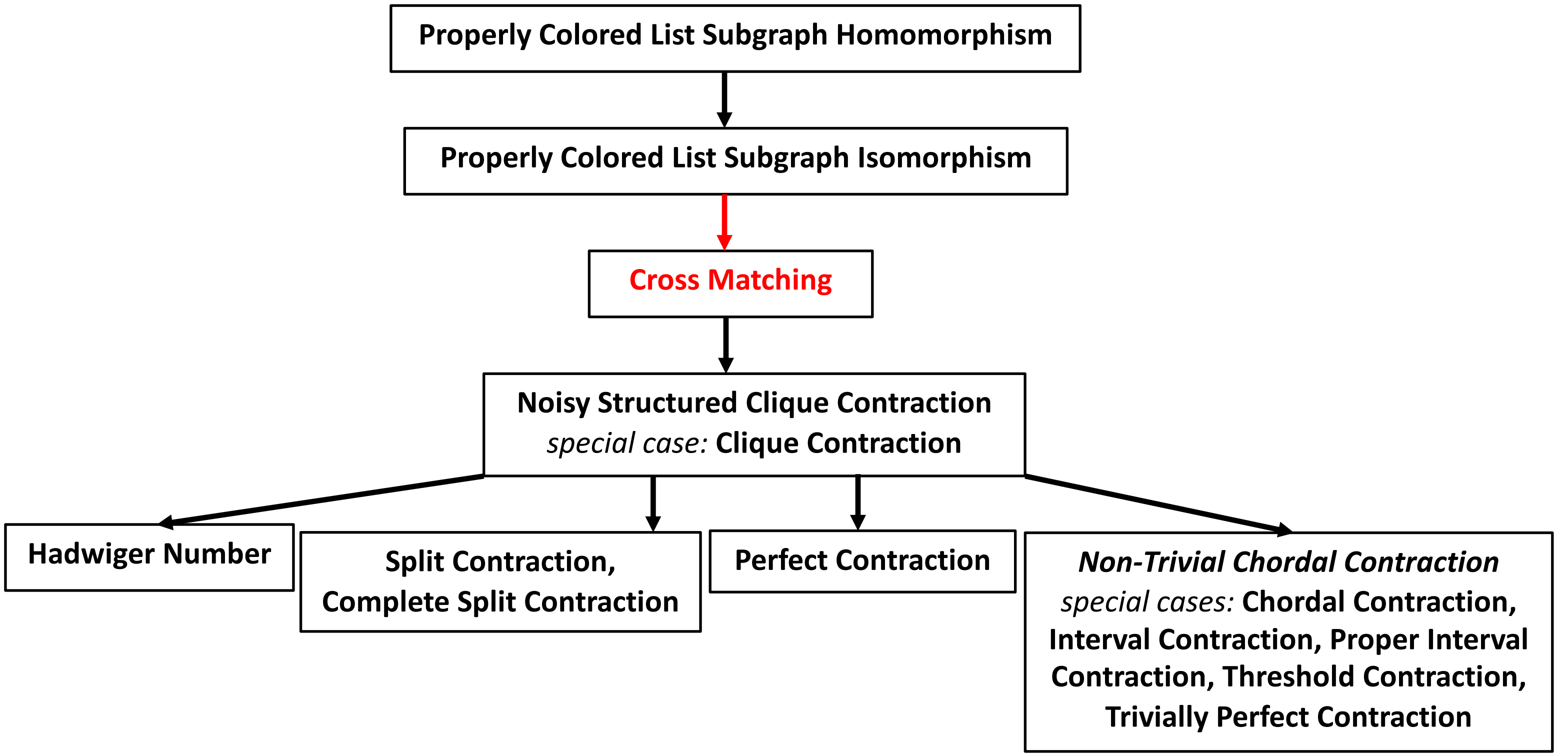
List Subgraph Isomorphism problem. Given two graphs G and H where $n=h$, as well as a list of vertices in H for each vertex in G, does G contain a subgraph isomorphic to H where each vertex in G is mapped to a vertex in its list?

[CFGKMPS'16]. Unless the ETH fails, List Subgraph Isomorphism cannot be solved in time $n^{o(n)}$.

Properly Colored List Subgraph Isomorphism problem. G and H are properly colored, and for every vertex v in G, all vertices (in H) in the list of v have the same color as v.

Our refinement. Unless the ETH fails, Properly Colored List Subgraph Isomorphism cannot be solved in time $n^{o(n)}$.

Some Technical Details



Some Technical Details

Cross Matching problem. Given a graph L with a partition (A,B) of its vertex set, does there exist a perfect matching between A and B whose contraction in G yields a clique?

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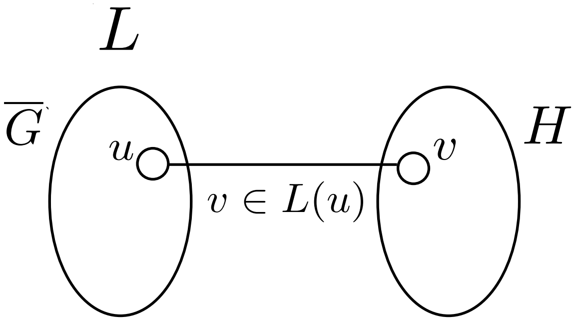
Cross Matching problem. Given a graph *L* with a partition (*A*,*B*) of its vertex set, does there exist a perfect matching between *A* and *B* whose contraction in *G* yields a clique?

Relation to List Subgraph Isomorphism?

Think of the following construction of *L*:



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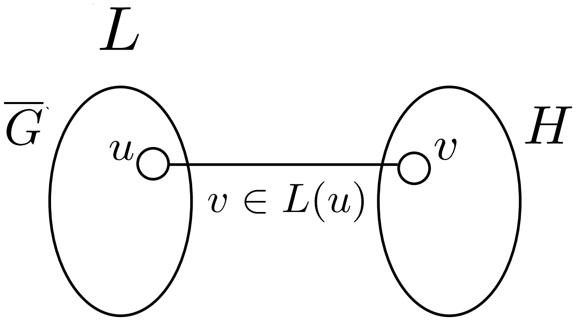

- A perfect matching between *A* and *B* can be thought of as a mapping of *G* to *H*.

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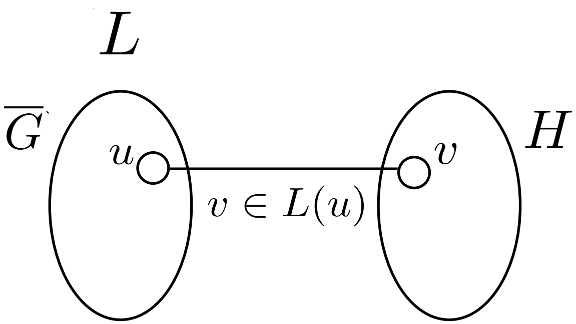

- A perfect matching between *A* and *B* can be thought of as a mapping of *G* to *H*.
- To obtain a clique, we would like to map non-edges in the complement of *G*, being edges in *G*, to edges in *H*.

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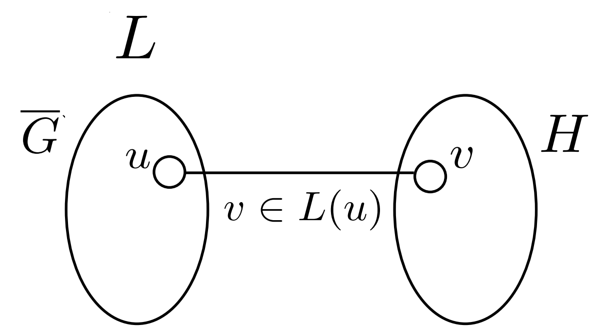
Relation to List Subgraph Isomorphism?



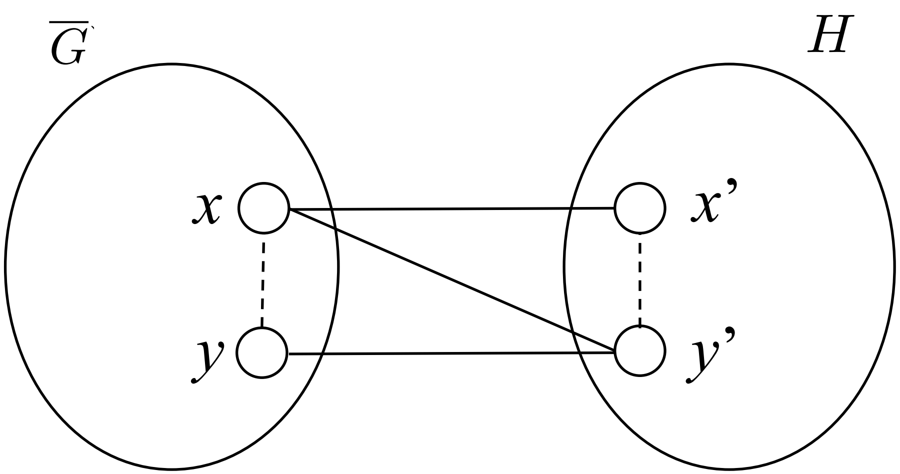
Think of the following construction of *L*:

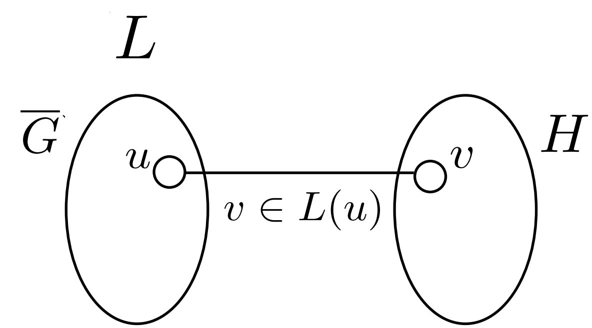
- A perfect matching between *A* and *B* can be thought of as a mapping of *G* to *H*.
- To obtain a clique, we would like to map non-edges in the complement of *G*, being edges in *G*, to edges in *H*.
- The difficulty is that non-edges in the complement of *G* can be ``filled'' by crossing edges. To argue that this cannot happen, we critically use the proper colorings of *G* and *H*.

Some Technical Details

Cross Matching problem. Given a graph *L* with a partition (*A*,*B*) of its vertex set, does there exist a perfect matching between *A* and *B* whose contraction in *G* yields a clique?

Relation to List Subgraph Isomorphism?





x and y non-adjacent in $\overline{G}$

color(x') = color(x)

color(y') = color(y) = color(x)

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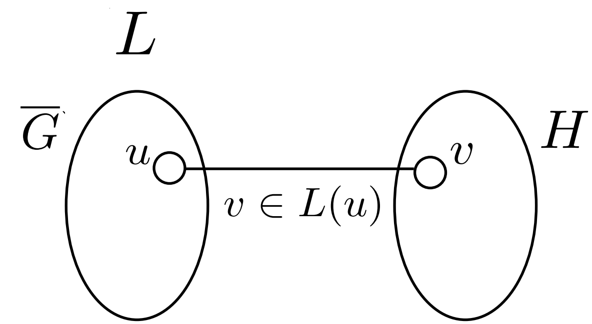
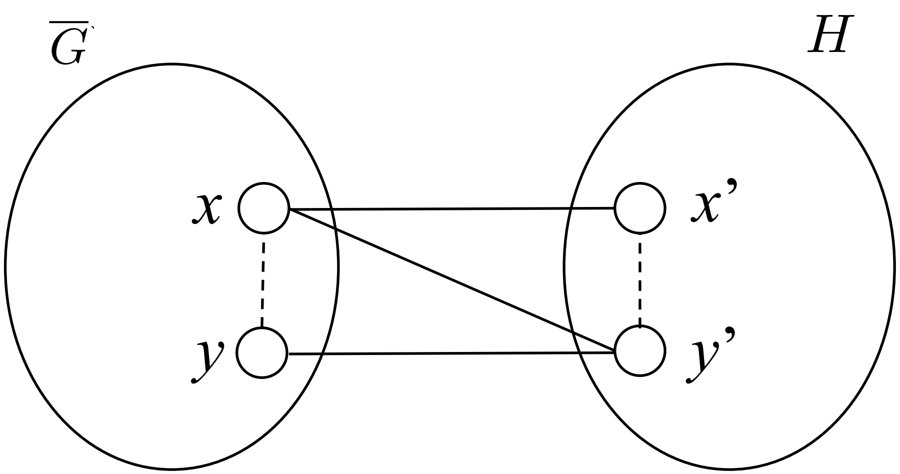
Relation to List Subgraph Isomorphism?





x and y non-adjacent in $\overline{G}$

$\Rightarrow$ x and y adjacent in G

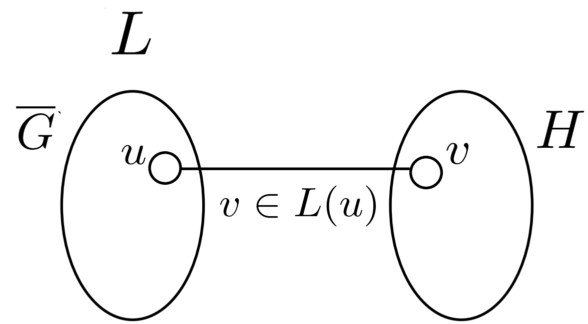
$\text{color}(x') = \text{color}(x)$

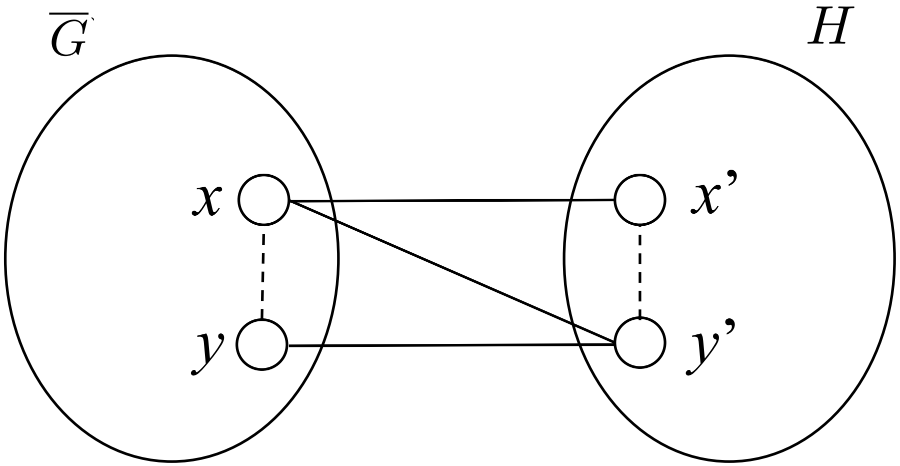
$\text{color}(y') = \text{color}(y) = \text{color}(x)$

Some Technical Details

Cross Matching problem. Given a graph *L* with a partition (*A*,*B*) of its vertex set, does there exist a perfect matching between *A* and *B* whose contraction in *G* yields a clique?

Relation to List Subgraph Isomorphism?





x and y non-adjacent in $\overline{G}$

⇒ x and y adjacent in G

⇒ x and y have different colors

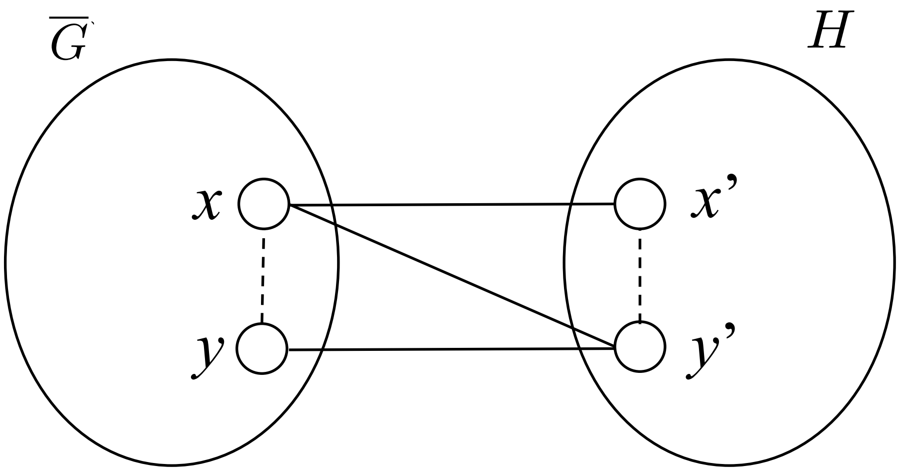
color(x') = color(x)

color(y') = color(y) = color(x)

Some Technical Details

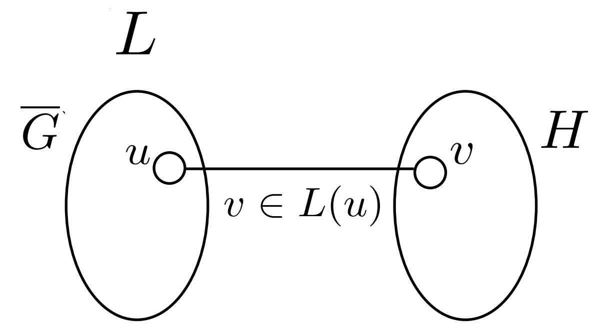
Cross Matching problem. Given a graph *L* with a partition (*A*,*B*) of its vertex set, does there exist a perfect matching between *A* and *B* whose contraction in *G* yields a clique?

Relation to List Subgraph Isomorphism?



color(x') = color(x)

color(y') = color(y) = color(x)



x and y non-adjacent in $\overline{G}$

⇒ x and y adjacent in G

⇒ x and y have different colors

⇒ contradiction

Some Technical Details

Properly Colored List Subgraph Homomorphism

Properly Colored List Subgraph Isomorphism

Cross Matching

Noisy Structured Clique Contraction
special case: **Clique Contraction**

Hadwiger Number

Split Contraction, Complete Split Contraction

Perfect Contraction

Non-Trivial Chordal Contraction
special cases: **Chordal Contraction, Interval Contraction, Proper Interval Contraction, Threshold Contraction, Trivially Perfect Contraction**

Some Technical Details

Cross Matching problem. Given a graph L with a partition (A,B) of its vertex set, does there exist a perfect matching between A and B whose contraction in G yields a clique?

Relation to Clique Contraction?

In one direction, a solution to Cross Matching is clearly a solution to Clique Contraction.

Some Technical Details

In the other direction, a solution to Clique Contraction might **not** be a solution to Cross Matching. We add vertices and edges to the graph in an instance of Cross Matching to enforce all solutions to be perfect matchings between *A* and *B*.
Further, we show that the addition of “noise” (extra vertices and edges incident to them) to the core graph has limited “effect”: It is not helpful to contract them in order to “fill” non-edges in the core.

Some Technical Details

Properly Colored List Subgraph Homomorphism

↓

Properly Colored List Subgraph Isomorphism

↓

Cross Matching

↓

Noisy Structured Clique Contraction
special case: **Clique Contraction**

↓

- **Hadwiger Number**
- **Split Contraction, Complete Split Contraction**
- **Perfect Contraction**
- ***Non-Trivial Chordal Contraction***
  special cases: **Chordal Contraction, Interval Contraction, Proper Interval Contraction, Threshold Contraction, Trivially Perfect Contraction**

Some Technical Details

In the other direction, a solution to Clique Contraction might **not** be a solution to Cross Matching. We add vertices and edges to the graph in an instance of Cross Matching to enforce all solutions to be perfect matchings between *A* and *B*.
Further, we show that the addition of “noise” (extra vertices and edges incident to them) to the core graph has limited “effect”: It is not helpful to contract them in order to “fill” non-edges in the core.

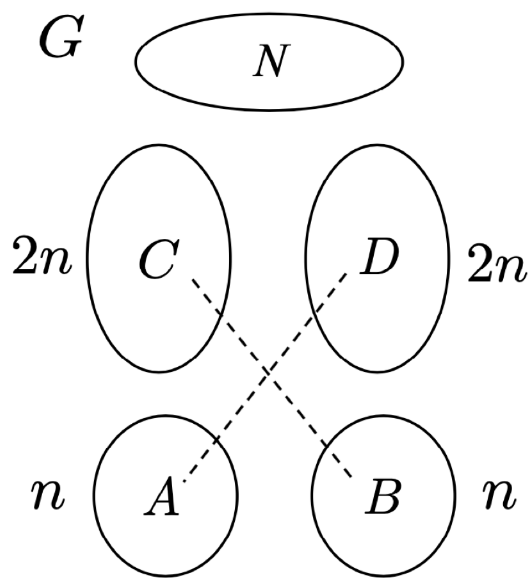
Depending on the contraction problem at hand, the noise is slightly different, but the proof technique is essentially the same: First show that the core must yield a clique (e.g., to obtain a chordal graph), and then show that the noise is “irrelevant”.

Some Technical Details

Noisy Structured Clique Contraction

Input: A graph G on at least $6n$ vertices for some $n \in \mathbb{N}$, and a partition (A, B, C, D, N) of $V(G)$ such that $|A| = |B| = n$, $|C| = |D| = 2n$, no vertex in A is adjacent to any vertex in D, and no vertex in B is adjacent to any vertex in C.

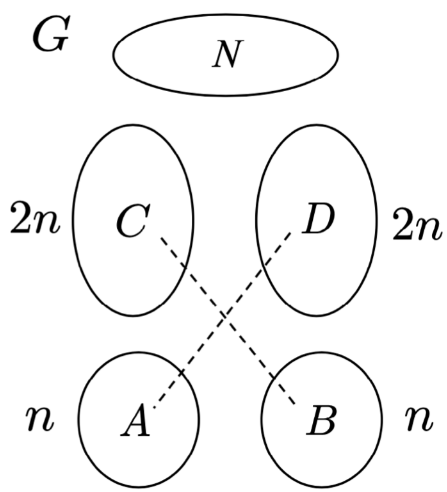
Question: Does there exist a subset $F \subseteq E(G)$ of size at most n such that $G[A \cup B \cup C \cup D \cup X]/F$ is a clique,[a] where $X = \{u \in N : \text{there exists a vertex } v \in A \cup B \cup C \cup D$ such that u and v belong to the same connected component of $G[F]\}$?



Some Technical Details

Theorem. Unless the ETH is false, Structured Clique Contraction cannot be solved in $n^{o(n)}$ time.

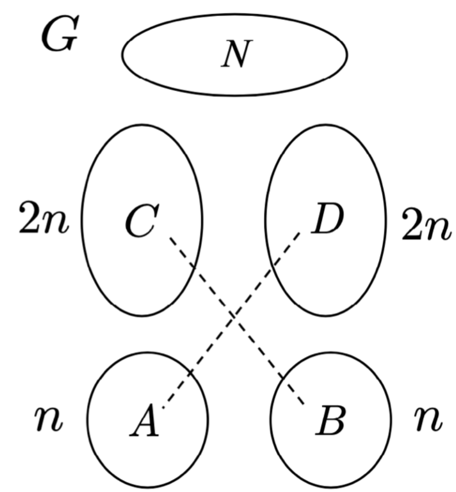
Corollary. Unless the ETH is false, Clique Contraction cannot be solved in $n^{o(n)}$ time.



Some Technical Details

Lemma. Let *F* be a solution to an instance (*G*,*A*,*B*,*C*,*D*,*N*,*n*) of Structured Clique Contraction. Then, *F* is a matching of size *n*, where each edge has one endpoint in *A* and the other in *B*.

In proofs. Let Π be a contraction problem. Given an instance *I*=(*G*,*A*,*B*,*C*,*D*,{},*n*) of Structured Clique Contraction, construct an instance *I'* of Π (that can be viewed as an instance (*G*,*A*,*B*,*C*,*D*,*N*,*n*) of Structured Clique Contraction).



Some Technical Details

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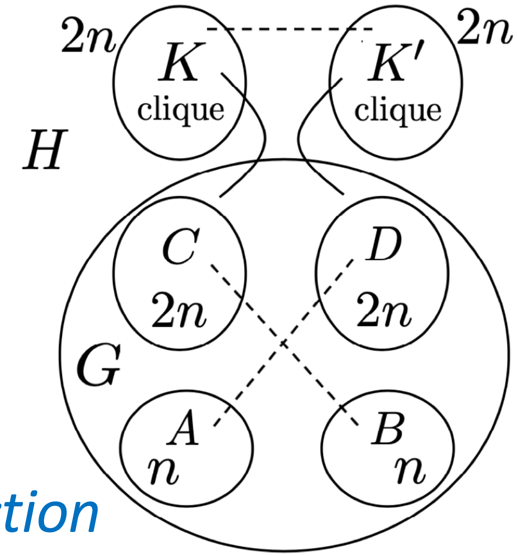


Chordal Contraction

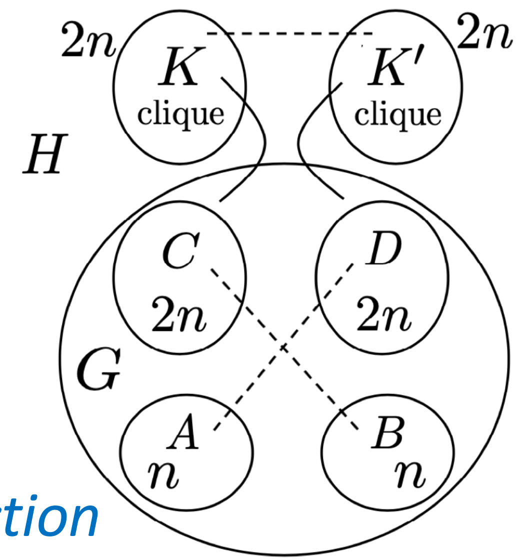
Some Technical Details

Lemma. Let F be a solution to an instance (G,A,B,C,D,N,n) of Structured Clique Contraction. Then, F is a matching of size n, where each edge has one endpoint in A and the other in B.

In proofs. Let Π be a contraction problem. Given an instance I=($G,A,B,C,D,\{\},n$) of Structured Clique Contraction, construct an instance I' of Π (that can be viewed as an instance (G,A,B,C,D,N,n) of Structured Clique Contraction).

I. I has a solution.

→ I' has the same solution.



Chordal Contraction

Some Technical Details

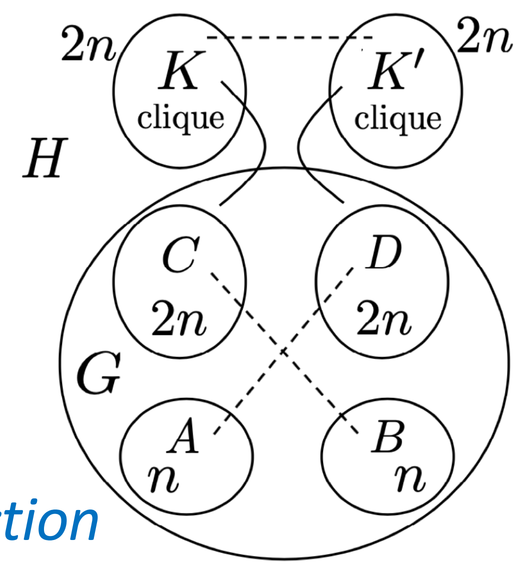
Lemma. Let *F* be a solution to an instance (*G,A,B,C,D,N,n*) of Structured Clique Contraction. Then, *F* is a matching of size *n*, where each edge has one endpoint in *A* and the other in *B*.

In proofs. Let Π be a contraction problem. Given an instance *I*=(*G,A,B,C,D,*{}*,n*) of Structured Clique Contraction, construct an instance *I'* of Π (that can be viewed as an instance (*G,A,B,C,D,N,n*) of Structured Clique Contraction).

II. *I'* has a solution.

→ …

→ *I'* as an instance of SCC has a solution.



Chordal Contraction

Some Technical Details

Lemma. Let *F* be a solution to an instance (*G*,*A*,*B*,*C*,*D*,*N*,*n*) of Structured Clique Contraction. Then, *F* is a matching of size *n*, where each edge has one endpoint in *A* and the other in *B*.

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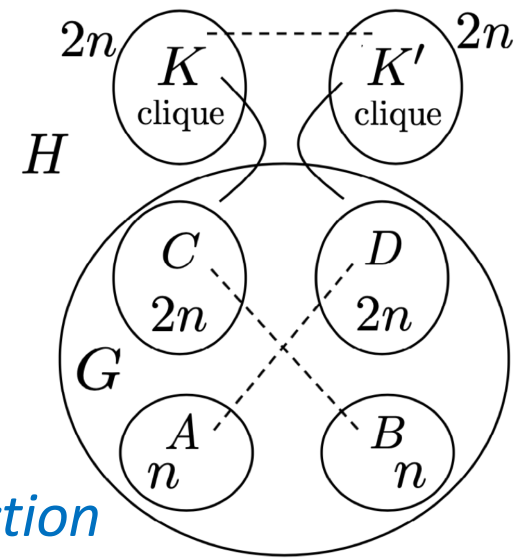
II. *I'* has a solution.

→ …

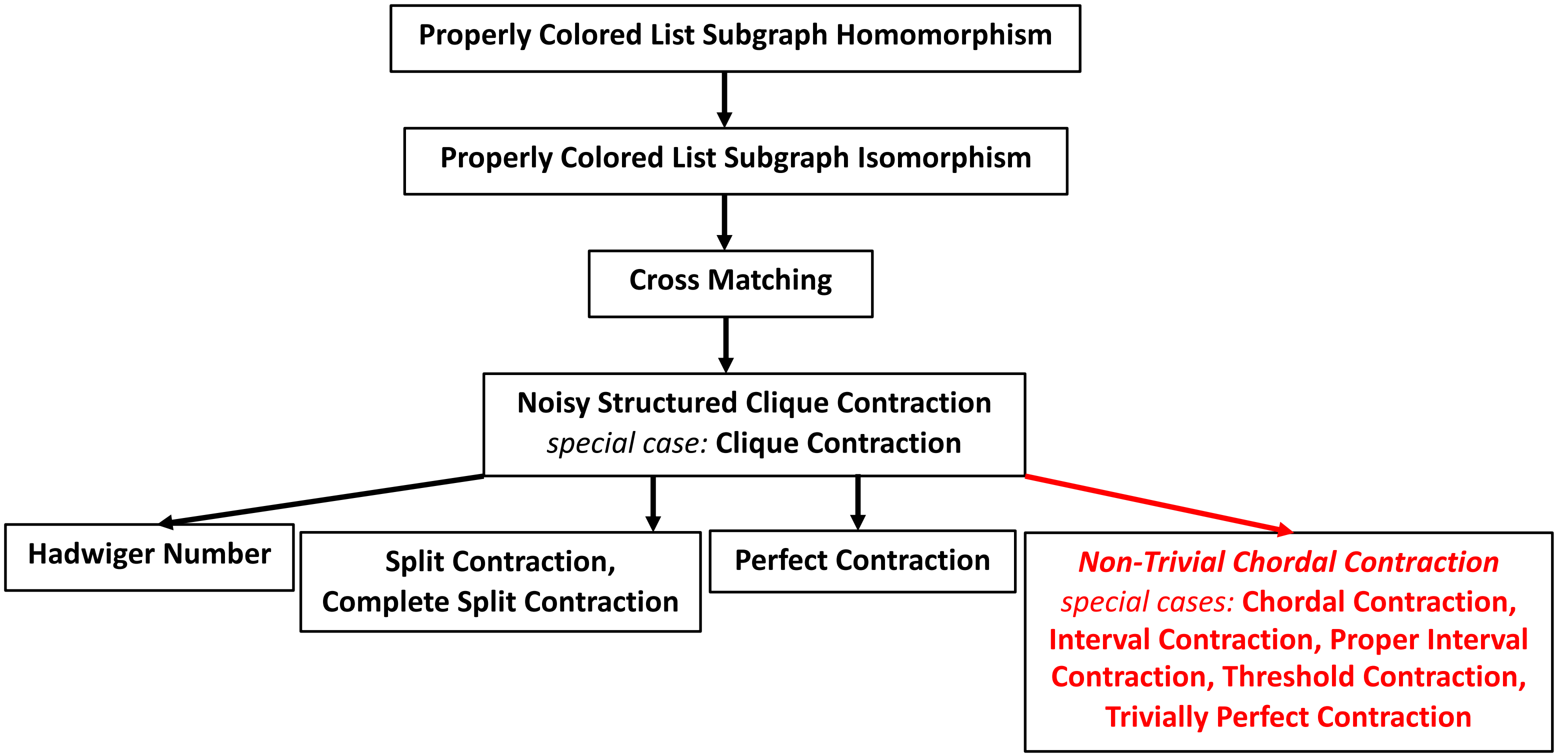
→ *I'* as an instance of SCC has a solution.

→ By the lemma, it is also a solution to *I*.



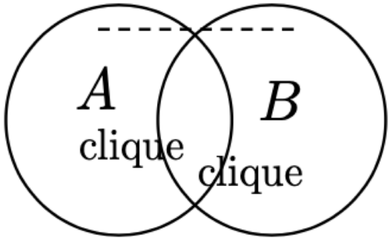
Chordal Contraction

Some Technical Details



Some Technical Details

Non-trivial Chordal Graph Class. A graph class is a **non-trivial chordal class** if it is a subclass of the class of chordal graphs, and a superclass of the class of graphs that are the union of two cliques.



Some Technical Details

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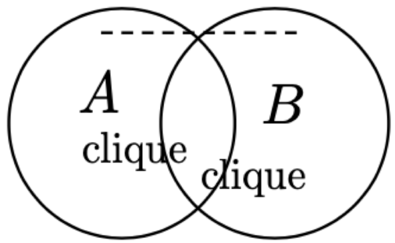
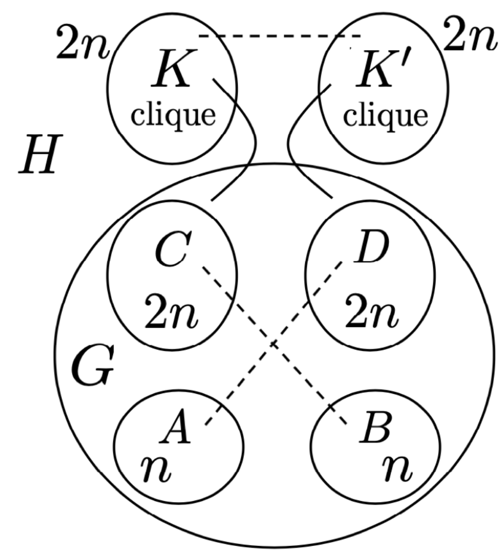


Theorem. Let C be a non-trivial graph class. Unless the ETH is false, C-Contraction cannot be solved in $n^{o(n)}$ time.



Outline

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Overview of Our Contribution

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