SFU

On Percolation and NP hardness

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joint work with Huck Bennett and Daniel Reichman

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Coloring Random Subgraphs of a Fixed Graph

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Random graphs

- $G(n, p)$ is the standard model of random graphs:
  - start with the complete graph K_n
  - keep each edge with probability p.
- We understand well the standard graph related quantities.
- For example, for $G(n, 0.5)$:
  - maximum clique = $\Theta(\log(n))$
  - maximum independent set = $\Theta(\log(n))$
  - Chromatic number = $\Omega\left(\frac{n}{\log(n)}\right)$

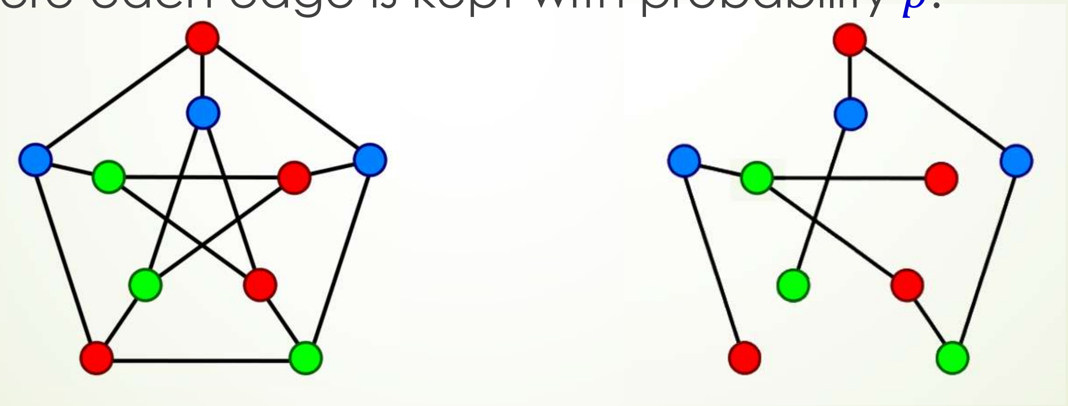
Random graphs

- $G(n,p)$ is the standard model of random graphs:
  - start with the complete graph K_n
  - keep each edge with probability p.
- What about the algorithmic problems on $G(n,0.5)$?
  - Find maximum clique in $G(n,0.5)$
  - Find the optimal coloring of $G(n,0.5)$
- We have efficient *approximation* algorithms, but we don't know optimal algorithms.

Random subgraphs of a fixed graph

For the rest of the talk p =0.5.

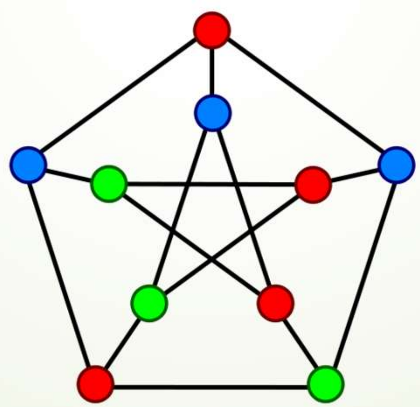
- Fix a graph $G = (V, E)$ and a parameter $p \in (0,1)$.
- Denote by $G_p = (V, E')$ a random subgraph of G, where each edge is kept with probability p.



- What about the algorithmic problems on G_p?

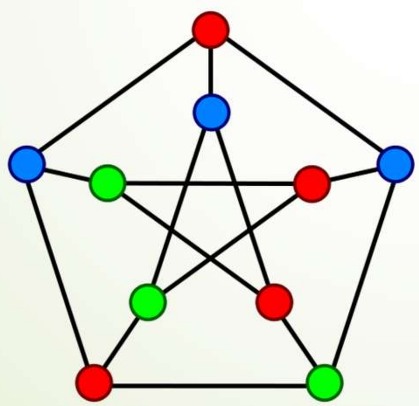
Graph coloring

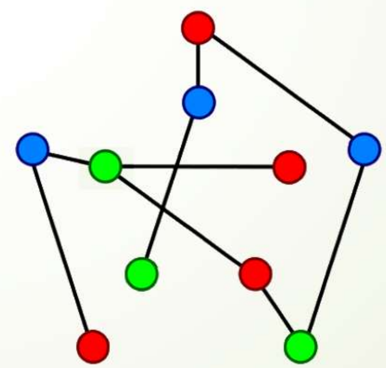
- Given a graph G we want to assign colors to the vertices so that no two adjacent vertices share the same color.
- The smallest number of colors needed to color a graph G is called its **chromatic number**, $\chi(G)$.



Graph coloring

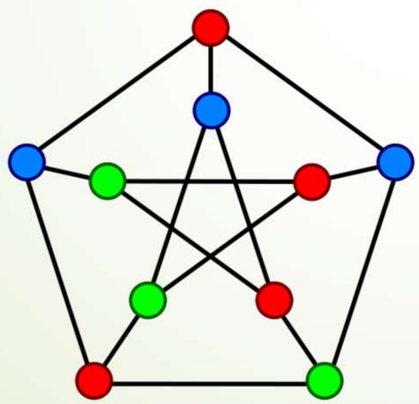
- Theorem: Computing $\chi(G)$ is NP-hard.
- Theorem: It is NP-hard to tell if $\chi(G) = 3$ or $\chi(G) \geq 4$.
- Question: What about computing $\chi(G_p)$? Does this problem become easier than the standard worst-case problem?

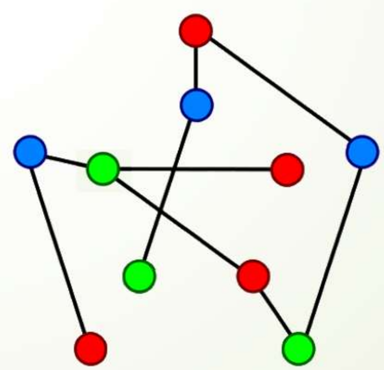




Graph coloring

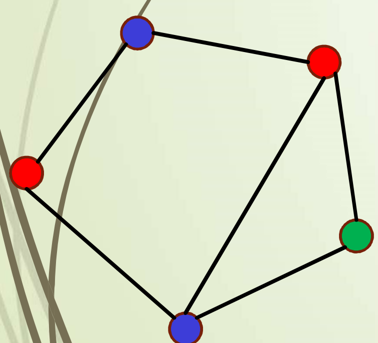
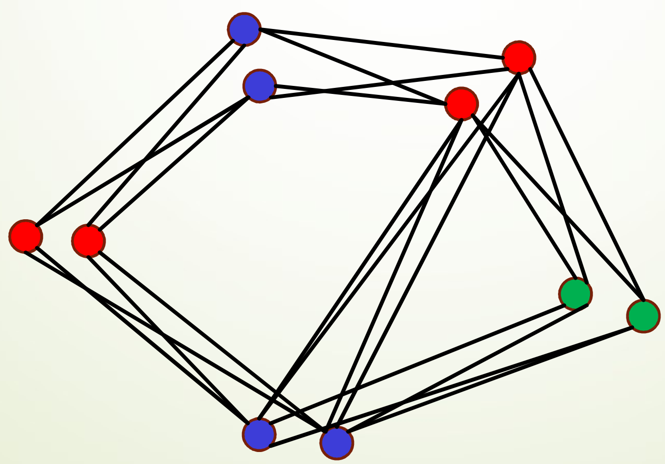
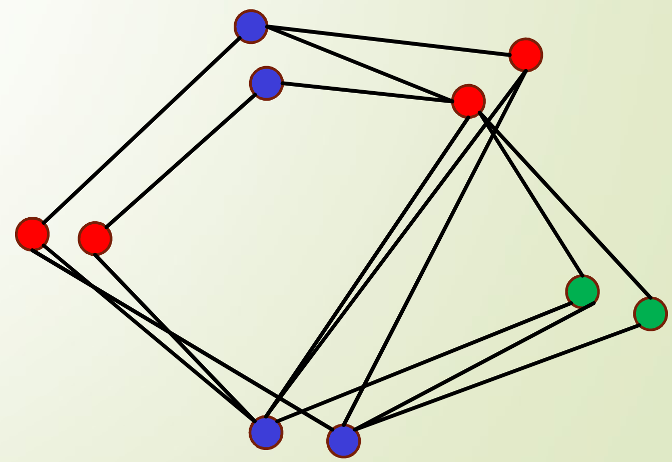
- <u>Theorem</u>: Given an input graph $G = (V, E)$ it is NP-hard to compute $\chi(G_{0.5})$ with high probability.
- That is, if we can compute $\chi(G_{0.5})$ with high probability, then we can compute $\chi(G)$ with high probability.





Graph coloring

- Theorem: Given an input graph $G = (V, E)$ it is NP-hard to compute $\chi(G_{0.5})$ with high probability.
- Proof: We show a polytime reduction that gets G and outputs G' s.t.
  - $\chi(G') = \chi(G)$, and with high probability $\chi(G_{0.5}) = \chi(G)$.
  - How? Graph blow-up: “make the edges thicker"

What about other NP-hard problems on G_p?

Some problems become easier

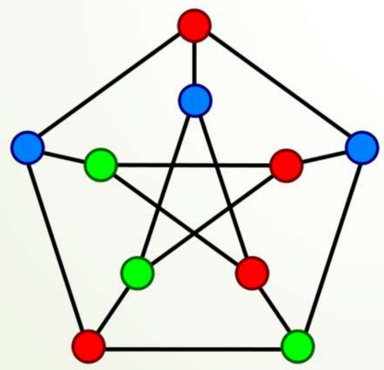
- What about other NP-hard problems?
- Are they all NP-hard for G_p
- Theorem: Given an input graph $G = (V, E)$ we can find the maximum clique in $G_{0.5}$ with high probability in time $n^{O(\log(n))}$.
- Proof: Max clique in $G_{0.5}$ has size at most $O(\log(n))$.

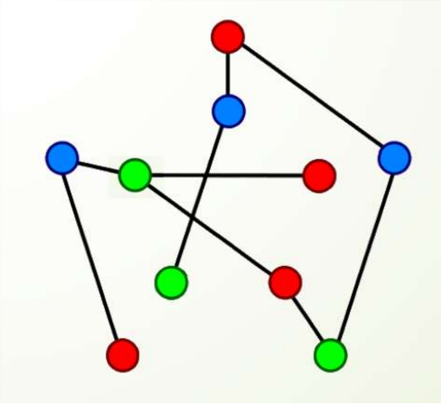
Max-Clique is NP-hard, but
Max-Clique on G_p is much easier

Chromatic number of G_p
The question of B. Bukh

Random subgraphs of a fixed graph

- Fix a graph $G = (V, E)$ and consider $G_{0.5} = (V, E')$.
- Q: What can we say about $\chi(G_p)$ as a function of $\chi(G)$?
- Q: What can we say about $E[\chi(G_p)]$ as a function of $\chi(G)$?





Some trivial facts

Clearly $Pr[\chi(G_p) \leq \chi(G)] = 1$.
If G is a union of many k-cliques, then $Pr[\chi(G_p) = \chi(G)] \approx 1$.

If $G = K_n$ then $G_p = G(n,p)$.
We know $\mathrm{E}[\chi(G(n,p))] = \Theta_p(n/\log(n))$.

The question [Bukh]

- **Question**: Let G be a graph with $\chi(\mathrm{G}) = k$.
  Prove that $\mathrm{E}[\chi(G_{0.5})] \geq \Omega(k/\log(k))$.

Some facts

- **Theorem 1**[*Easy*]:
  Let G be a graph with $\chi(G) = k$. Then $\mathrm{E}[\chi(G_{0.5})] \geq \Omega(k/\log(n))$.

Answers the question for $k > n^{0.1}$.

But what if $k \ll n^{o(1)}$?

- **Theorem 2**[illegible]:
  Let G be a graph with $\chi(G) = k$ and $\alpha(G) \leq O(n/k)$.
  Then $\mathrm{E}[\chi(G_{0.5})] \geq \Omega(k/\log(k))$.

Holds for most graphs,
e.g., $G(n,p)$ $G(n,d)$

- **Theorem 3**[*Mohar, Wu 18*]:
  Let G be a graph with $\chi_f(G) = k$. Then $\chi_f(G_{0.5}) \geq \Omega(k/\log(k))$.

$$\frac{\chi(G)}{\log(n)} \leq \chi_f(G) \leq \chi(G)$$

More facts

- **<u>Theorem</u>** <u>*[Easy]*</u>:
  Let G be a graph with $\chi(G) = k$. Then $E[\chi(G_{0.5})] \geq \sqrt{k}$.
- In fact, $Pr\left[\chi(G_{0.5}) \geq \sqrt{k}\right] \geq 0.5$.

- **<u>Proof</u>** <u>*[Easy]*</u>: Let $H \sim G_{0.5}$. Note that $G \backslash H$ is distributed like $G_{0.5}$.
- Then $\chi(H) \cdot \chi(G \backslash H) \geq \chi(G) = k$.
- *Hence* $E[\chi(G_{0.5})] = E\left[\frac{\chi(H)+\chi(G\backslash H)}{2}\right] \geq E\left[\sqrt{\chi(H) \cdot \chi(G\backslash H)}\right] \geq \sqrt{k}$.

A more refined question

- Let G be a graph with $\chi(\mathrm{G}) = k$. And let $d < k$.
- **Question**: What is the probability that $\chi(G) < d$?

- Is it true that $Pr[\chi(G_{0.5}) \leq d] \leq Pr[\chi(G(k, 0.5)) \leq d]$?
- Is it true that $Pr[\chi(G_{0.5}) \leq d] \leq poly(Pr[\chi(G(k, 0.5)) \leq d])$?

- **Special case**: What is the probability that $\chi(\mathrm{G}_{0.5}) = 2$?
- **Fact**: For $G = K_k$ we have $Pr[\chi(G_{0.5}) = 2] = exp(-k^2)$.

Probability that $G_{0.5}$ is bipartite

- **Theorem**: Let G be a graph with $\chi(G) = k$ for $k > 8$.
Then $Pr[\chi(G_{0.5}) = 2] \leq \exp(-ck^2)$.

Lemma: Suppose that every 8-coloring of G leaves $\geq t$ monochromatic edges.
Then $Pr[\chi(G_{0.5}) = 2] \leq \exp(-ct)$.

- I
- As a hint we use the following lemma.
- **Proof of Theorem**: *If $\chi(G) = k$, then in any 8-coloring of G* one of the colors induces a subgraph G' with $\chi(G') \geq \lceil k/8 \rceil$.
- *Fact: G' contains $\geq \binom{\chi(G')}{2} = \Omega(k^2)$ edges.*
- *Using the lemma we get that $Pr[\chi(G_{0.5}) = 2] \leq \exp(-ck^2)$.* ■

Probability that $G_{0.5}$ is bipartite

- **Theorem**: Let G be a graph with $\chi(G) = k$ for $k > 8$.
  Then $Pr[\chi(G_{0.5}) = 2] \leq \exp(-ck^2)$.

  This should be compared to $Pr[\chi(G(k, 0.5)) = 2] \leq \exp(-c'k^2)$.

- Therefore, for all G with $\chi(G) = k$ we have
  - $Pr[\chi(G_{0.5}) \leq 2] \leq poly(Pr[\chi(G(k, 0.5)) \leq 2])$

Probability that $\chi(G_{0.5})$ is small

- **Theorem**: Let G be a graph with $\chi(G) = k$ and let $d < k^{1/3}$.
  Then $Pr[\chi(G_{0.5}) \leq d] \leq \exp(-\frac{k(k-d^3)}{d^3})$.

For $G(n,p)$ *we have* $Pr[\chi(G(n,0.5)) < d] = \exp(-\frac{k(k-d\log(d))}{d})$.

- In particular, $Pr\left[\chi(G_{0.5}) < 0.9k^{1/3}\right] \leq \exp(-k)$.

Open problems:

1. Prove/disprove: For all G with $\chi(G) = k$
   $$E[\chi(G_p)] \geq \Omega_{\mathrm{p}}(k/\log(k)).$$

2. Prove/disprove: For all G with $\chi(G) = k$ and $d \leq k$
   $$Pr[\chi(G_p) \leq d] \leq poly(Pr[\chi(G(k,p)) \leq d]).$$

3. Prove/disprove: For all G with $\chi(G) = k$ and $p \in (0,1)$
   $$E[\chi(G_{p/2})] \geq c_p E[\chi(G_p)].$$

Open problems:

1. Find maximal $p_3 \in (0,1)$ such that for all planar G
   $E[\chi(G_p)] \leq 3$ for all $p < p_3$.

   Fact: $p_3 \leq n^{-1/6}$. Is this tight?

2. Find maximal $p_2 \in (0,1)$ such that for all planar G
   w.h.p. G_{p_2} is bipartite.

3. Find maximal $p_{tree} \in (0,1)$ such that for all planar G
   w.h.p. $G_{p_{tree}}$ has no cycles.

   Fact: $p_{tree} \geq n^{-1/2}$. Is this tight?

Thank you