# LARGE SIMPLE CYCLES IN SYMPLICIAL COMPLEXES

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Based on a joint work with

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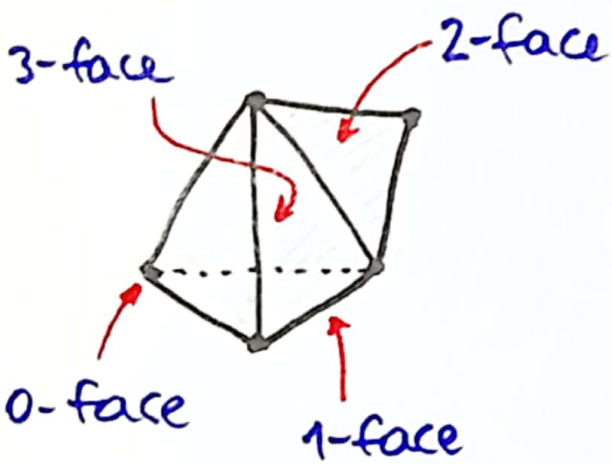
# Symplicial Complexes




K

d-cycle: all (d-1)-faces have even degree.

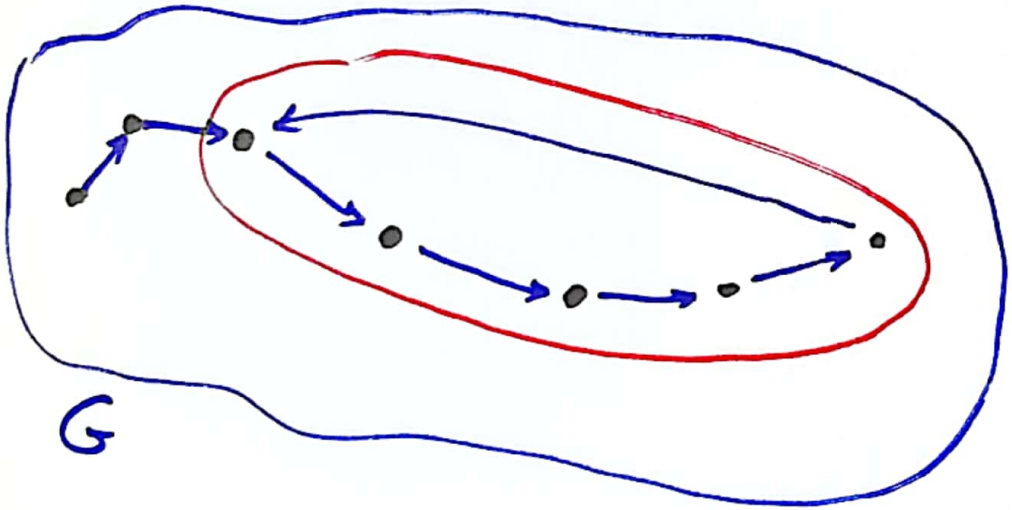
($\mathbb{F}_2$)

$K^{(i)}$: The i-faces of K.

# Erdős - Gallai Th. in graphs

$|E| > 2k(|V|-1) \implies$ G contains a cycle of size $\geq k$.

Proof: (assuming degs $> k$)




Perform a walk on G without ever returning to k previously visited vertices.

# Our Goal

E-G says $c(G) \gtrsim |E|/|V|$. the density of $G$.

size of max. cycle

$K$: symplicial complex. Show:

$$\boxed{c_d(K) \overset{?}{\gtrsim} \text{some function of } |K^{(d)}|/|K^{(d-1)}|.}$$

size of max. simple $d$-cycle

density

# Other ideas for a proof?

* $\{G \mid c(G) \le k\}$ is closed under minors.

  $\Rightarrow c(G) \ge \text{density}(G)/\log(\text{density}(G))$.

  What are the minors of simplicial complexes?

* Instead of $c(G) \ge \frac{1}{2}\frac{|E|}{|V|-1}$, write $c(G) \ge \frac{1}{2}\frac{|E|}{\text{rank}(G)}$,

  and further $c(G) \ge \frac{1}{2}\max_{G' \subseteq G}\frac{|E(G')|}{\text{rank}(G')}$

  $\gamma(G)$

  size of max. acyclic subset of edges.

Appears in Nash-Williams Th.

$\boxed{\text{Matroids?}}$

# Matroids

$M = (E, \mathcal{I})$ — $E$: elements; $\mathcal{I}$: independent sets

$\mathcal{I}$ closed under containment; has the exchange property.

$G$ graph $\longrightarrow$ $M(G)$: $E$: $E[G]$ $\mathcal{I}$: forests

$\mathbb{F}_2^n \supseteq S$ $\longrightarrow$ Linear Matroid: $E$: $S$ $\mathcal{I}$: linearly ind. subsets

$K$ simp. complex $\longrightarrow$ $M(K)$ Simplicial Matroids $E$: $K^{(d)}$ $\mathcal{I}$: acyclic subsets of $K^{(d)}$

$\cup\!\!\mid$

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$c(M) =$ max. size circuit in $M$

$\gamma(M) = \max_{E' \subseteq E} |E'| / \mathrm{rank}(E')$ — size of max. acyclic set in $E'$.

**Goal:** Lower-bound $c(M)$ in terms of $\gamma(M)$

# Matroidal Tools

[Gillen] Linear ($\mathbb{F}_2$) matroid lacks a size $k$ (graphic) minor $\Rightarrow$

$$\gamma(M) \leq 2^{2^{3k}}.$$

$\Rightarrow$ $c(M) \geq \frac{1}{3}\log_2\log_2 \gamma(M)$.

too weak, too heavy...

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## Seymour Lemma:

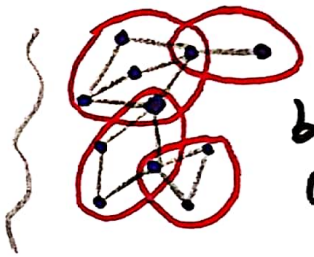
Let $M$ be loopless and connected, $C$ max. circuit in $M$.

Then,

$$c(M/C) < c(M) = C.$$

# Matroids: Connectedness and Minors

* $e \sim f$ in $M$ if both lie in some circuit.
  $\sim$ is an equivalence relation;
  equiv. classes: components.



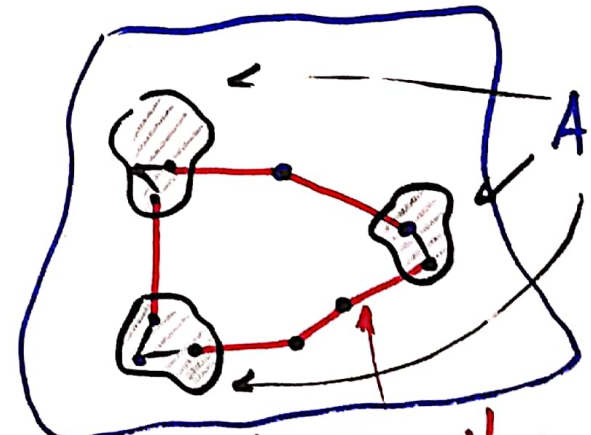
in graphs: biconnected components.

* $A \subseteq E(M)$

deletion: $M \setminus A$ $\quad E: E \setminus A$, induced matroid

contraction: $M/A$ $\quad E: E \setminus A$

ind. sets of $M/A$
extend to ind. sets of $M$

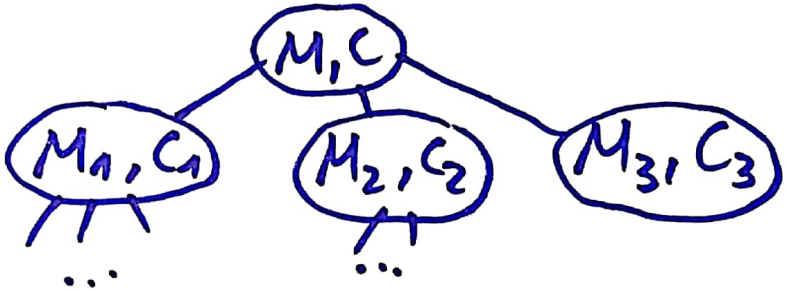



$M$ $\quad$ a circuit in $M/A$

# Decomposition Tree of M

M: connected, loopless

$T_M$:




* Every vertex $x \in T_M$ has associated $M_x, C_x$
  $M_x$: a minor of $M$;
  $C_x$: max. circuit in $M_x$
* The root has $M, C$.
* The children of $x$ are the components of $M_x / C_x$ after the removal of the loops.

# The Matroidal Theorem

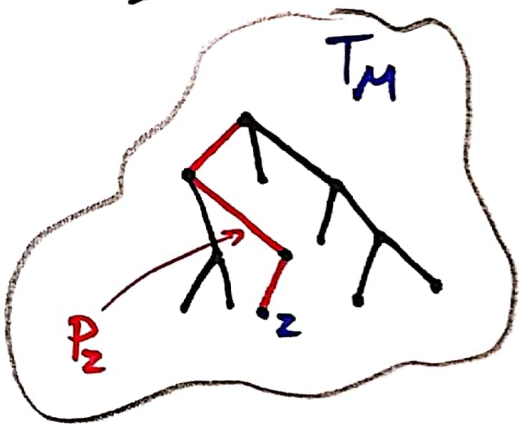
Let $S_M(t) = \max\{|E'|,\ E' \subseteq E,\ \mathrm{rank}(E') \leq t\}$.

**Theorem:**

$$\gamma(M) \leq S_M\left(c(M)\cdot(c(M)-1)/2\right).$$

**Proof ideas:**

- $T_M$ has at most $\mathrm{rank}(M)$ leaves;
- $\bigcup_{z:\text{leaves of } T_M} \mathrm{span}\left(\bigcup_{x\in P_z} C_x\right) = E(M)$
- $\mathrm{rank}\left(\bigcup_{x\in P_z} C_x\right) = \sum_{x\in P_z} \overset{\text{rank}}{(C_x)} \leq c_M + (c_M-1) + \dots$




$\Downarrow$

$$|E(M)| \leq \mathrm{rank}(M)\cdot S_M\left(\qquad\qquad\right).$$

# Back to Simplicial Complexes

Let $A \subseteq K^{(d)}$ be a set of $d$-simplices on $[n]$.

**Theorem:**
The extremal $A$ is shifted, and $\mathrm{rank}_d(A) \gtrsim |A|^{1-\frac{1}{d+1}}$.

$\Rightarrow \quad s_k(t) \lesssim t^{1+1/d}$

$\Rightarrow \quad \boxed{c_d(K) \gtrsim \gamma_d(K)^{d/2d+2}}$

With further effort, and more simplicial complexes,

$$\boxed{c_d(K) \geq \sqrt{\frac{2}{d+1}\,\gamma_d(K)} - 1}$$

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