Max-plus Polynomials and Their Roots

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Max-plus Semiring

Max-plus semiring (tropical semiring):

$$(K, \oplus, \odot),$$

where $K = \mathbb{R}$ or $K = \mathbb{Q}$ and

$$x \oplus y = \max\{x, y\},$$

$$x \odot y = x + y$$

Min-plus semiring: completely analogous

Max-plus Polynomials

Monomials:

$$M = c \odot x_1^{\odot i_1} \odot \ldots \odot x_n^{\odot i_n} = c + i_1 x_1 + \ldots + i_n x_n,$$

where $c \in \mathbb{K}$ and $i_1, \ldots, i_n \in \mathbb{Z}_+$

Notation: $\vec{x}^I = x_1^{\odot i_1} \odot \ldots \odot x_n^{\odot i_n}$

Polynomials:

$$f = \bigoplus_i M_i = \max_i M_i$$

Degree:
$\deg M = i_1 + \ldots + i_n$,
$\deg f = \max_i \deg(M_i)$

Roots

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A point $\vec{a} \in \mathbb{K}^n$ is a root of the polynomial f if the maximum $\max_i \{M_i(\vec{a})\}$ is attained on at least two different monomials M_i

A max-plus polynomial $p(\vec{x})$ is a convex piece-wise linear function

The roots of p are non-smoothness points of this function

Example 1

$$f = 1 \oplus 2 \odot x \oplus 0 \odot x^{\odot 2} = \max(1, x + 2, 2x)$$

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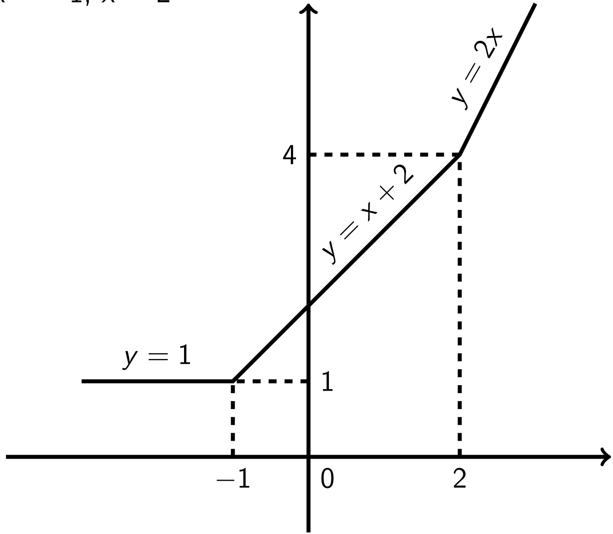

Example 2

$$f = 2 \oplus 0 \odot x \oplus 1 \odot y = \max(2, x, y+1)$$

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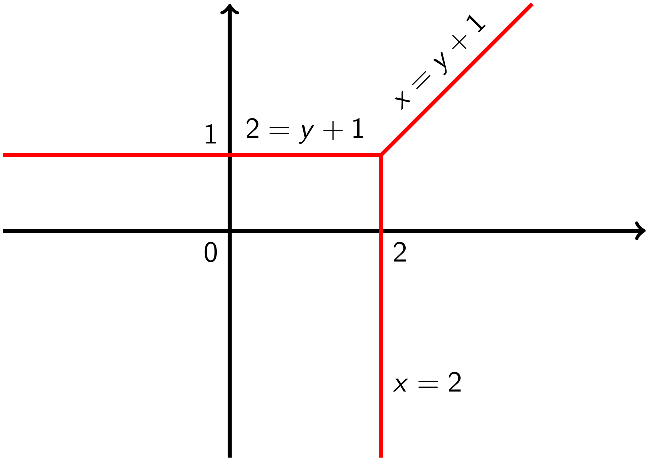

Motivation

- Mathematical physics (Cuninghame-Green, Maslov, many others, since 1970s)
- Combinatorial optimization, scheduling problems (Butkovič, many others, since 1990s)
- Algebraic geometry (Sturmfels, Mikhalkin, many others, since 1990s)
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Why useful?

Max-plus analogs of classical objects are

- complex enough to reflect properties of classical objects;
- simple enough to be computationally accessible

Origin: Algebraic Geometry

Consider the algebraic closure of the field of complex rational functions $\mathbb{C}(t)$. Its elements can be represented by Puiseux series locally at zero:

$$c_1 t^{d_1} + c_2 t^{d_2} + \dots,$$

where $d_1 < d_2 < \dots$ are rationals. The order of the series above is d_1.

Consider polynomials in $\mathbb{C}(t)[x_1, \dots, x_n]$. Then if $(a_1(t), \dots, a_n(t)) \in \mathbb{C}(t)$ is a root for some polynomial, then the sequence of orders is a root for the corresponding min-plus polynomial.

Notable Application: Counting Plain Algebraic Curves

Algebraic curves over complex numbers.

Line on the plain — can be specified by two points.

In general, fix

- the degree d,
- the number of "double points" k.

Consider degree-d complex algebraic equations $f(x, y) = 0$. If you fix certain number of points $c(d, k)$ in generic position, then there will be some certain number $m(d, k)$ of equations satisfied by them.
The problem: what is $m(d, k)$?

Example

$d = 2$, $k = 1$.

$f(x, y) = ax^2 + bxy + cy^2 + dx + ey + f$

There are essentially 5 parameters. $k = 1$ reduce the number of parameters by 1. So the curve is specified by 4 points.

Degree 2 curve with an intersection is a pair of lines. Given four points in general position there are 3 ways to draw a pair of lines through them.

So $m(2, 1) = 3$.

Algebraic Curves

The problem: what is $m(d, k)$?
The solution plan (Mikhalkin, 2003):

1. Show that $m(d, k)$ indeed does not depend on the particular choice of points.
2. Consider points of the form $(x, y) = (\phi t^{x'}, \psi t^{y'})$, where $|\phi| = |\psi| = 1$ and t is a parameter.
3. Send t to infinity and get the max-plus polynomial.
4. Solve the max-plus problem arising. This can be done combinatorially.

More Motivation

Max-plus semiring is a natural example of an algebraic structure with no subtraction

It is often important

For example:

- Matrix multiplication problem: given two $n \times n$ matrices A and B how many algebraic operations are needed do compute $A \cdot B$?

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- Max-plus case: the best known bound is $\frac{n^3}{2^{C \log^{1/2} n}}$ for some $C > 0$ (R. Williams'14)

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The philosophical goal is to understand the importance of subtraction

What is Known?

Linear polynomials:

Analogs of the rank of matricies (Sturmfels, Izhakian, Guterman and others)

Analog of matrix determinant (Sturmfels, Akian, Gaubert and others)

Analog of Gauss triangular form (Grigoriev'13)

Complexity of solvability problem: polynomially equivalent to mean payoff games (is in $\mathrm{NP} \cap \mathrm{coNP}$, not known to be in P) (Grigoriev, P.'15)

General polynomials:

Radical of the max-pus ideal studied (Shustin, Izhakian'07)

Bezout bound (Davydow, Grigoriev'17)

Analog of Hilbert's Nullstellensatz (Grigoriev, P.'18)

Complexity of solvability problem: NP-complete (Theobald'06)

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3. What is the size of the minimal set of points $R \subseteq \mathbb{K}^n$ such that any non-trivial polynomial with at most k monomials has a non-root in one of the points of R?

Combinatorial Nulstellensatz

Question 1 Given finite sets $R \subseteq \mathbb{R}^n$ and $S \subseteq \mathbb{Z}_+^n$, is there a max-plus polynomial p with $\mathrm{Supp}(p) \subseteq S$ and roots in all points of R?

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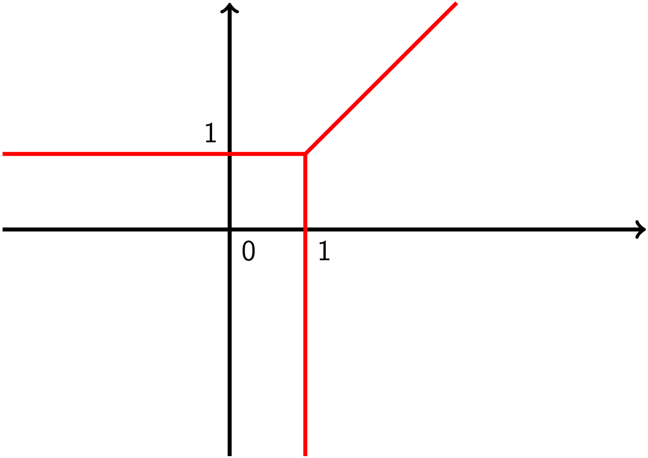
Can be extended to any $R = S = \mathrm{Supp}(p)$. Open in the classical setting!

Example, $n = 2, d = 1$

$f = 1 \oplus 0 \odot x \oplus 0 \odot y = \max(1, x, y)$.

Roots:



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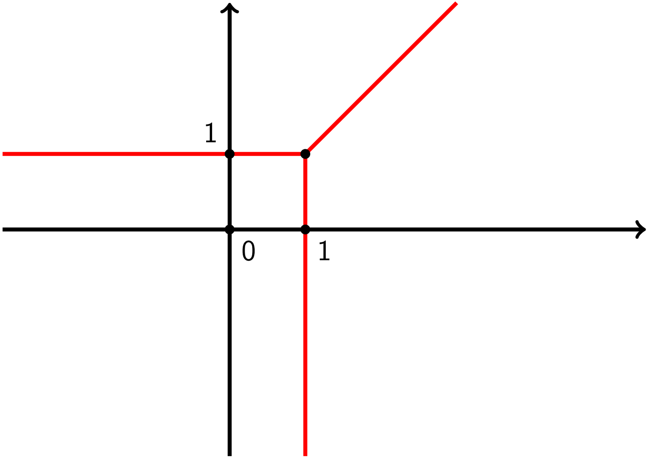

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Theorem (Classical Combinatorial Nullstellensatz)

If p is of total degree at most nd and a monomial $x_1^d x_2^d \dots x_n^d$ is in p, then p has a non-root in $[d]^n$

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Not the case for max-plus polynomials!

Theorem (Max-plus)

If $|S| > |R|$, then there is a max-plus polynomial p with $\operatorname{Supp}(p) = S$ and roots in all points of R

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Proof strategy: Look at the polynomial with varying coefficients, analyze as a max-plus linear system, use known results for max-plus linear systems

Schwartz-Zippel Lemma

Question 2 Given finite sets $R \subseteq \mathbb{R}^n$ and $S \subseteq \mathbb{Z}_+^n$, how many roots can a max-plus polynomial p with $\mathrm{Supp}(p) \subseteq S$ have in the set R?

Classical case:

Theorem (Classical Schwartz-Zippel Lemma)

Let $R \subseteq \mathbb{R}$ be of size k and p be a non-zero polynomial of degree d. Then p has roots in at most dk^{n-1} points in R^n.

Max-plus Schwartz-Zippel Lemma

Theorem

- *Let $R \subseteq \mathbb{R}$ be of size k and p be a non-zero max-plus polynomial of degree d. Then p has roots in at most*

$$k^n - (k-d)^n \approx ndk^{n-1}$$

points in R^n

- *Exactly the same statement is true for the polynomials with individual degree of each variable at most d*
- *The bound is optimal*

For $d = 1$ this is known in computational complexity as Isolation Lemma

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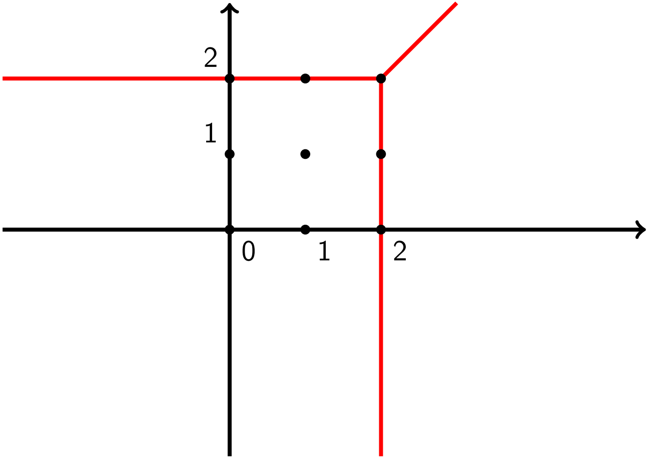
Proof Idea.

Use Max-plus Combinatorial Nullstellensatz □

Example, $n = 2, k = 3, d = 1$

Optimal polynomial:



The number of roots is $k^n - (k-d)^n = 3^2 - (3-1)^2 = 5$

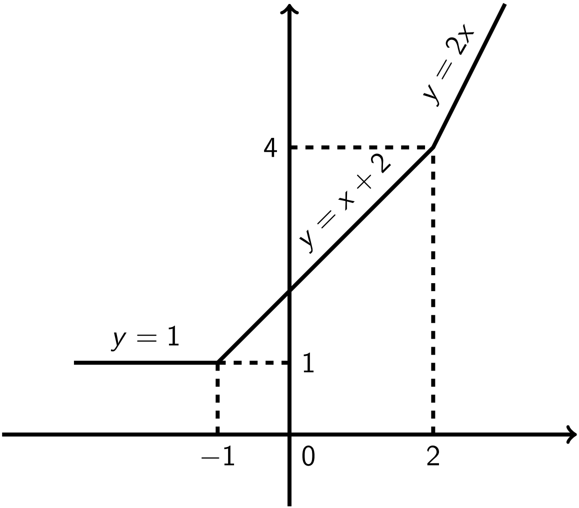
Universal Testing Set

Question 3 What is the size r of the minimal set of points $R \subseteq \mathbb{K}^n$ such that any non-trivial polynomial with at most k monomials has a non-root in one of the points of R?

Classical case: $r = k$ (Grigoriev, Karpinski, Singer, Ben-Or, Tiwari, Kaltofen, Yagati)

Example, $n = 1$

$f = 1 \oplus 1 \odot x \oplus 0 \odot x^{\odot 2} = \max(1, x + 1, 2x)$



$k + 1$ monomials are needed for k roots, so $r = k$

Max-plus Universal Testing Set, $\mathbb{K} = \mathbb{R}$

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For polynomials over $\mathbb{R}$ the minimal size r of the universal testing set for max-plus polynomials with at most k monomials is equal to k

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Proof Idea.

Universal set: pick a set R of points whose coordinates are linearly independent over $\mathbb{Q}$

Let p vanish on R. Consider a graph: vertices are monomials, edges connect monomials that both have maximums on one of the roots in R

Show that the graph can have no cycles

$\square$

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Theorem

For the size of the minimal universal testing set over $\mathbb{Q}$ the following inequalities hold:

$$\frac{(k-1)(n+1)}{2} + 1 \leq r \leq k(n+1) + 1.$$

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Proof Idea.

Upper bound: Count the dimension of semialgebraic set of sets of roots of max-plus polynomials

Lower bound: Given set of points R construct polynomial with roots in all points of R inductively $\square$

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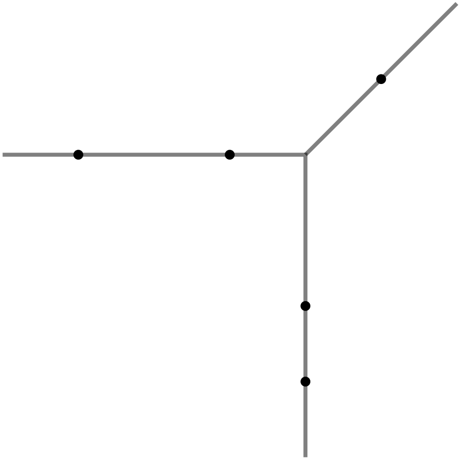
Connection to covering points by polytopes $\square$

Connection to Coverings

Suppose some points are roots for some polynomial



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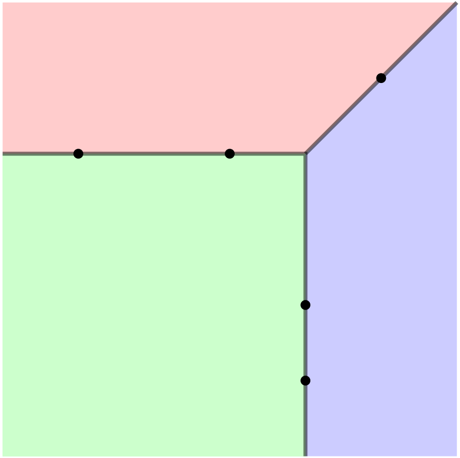

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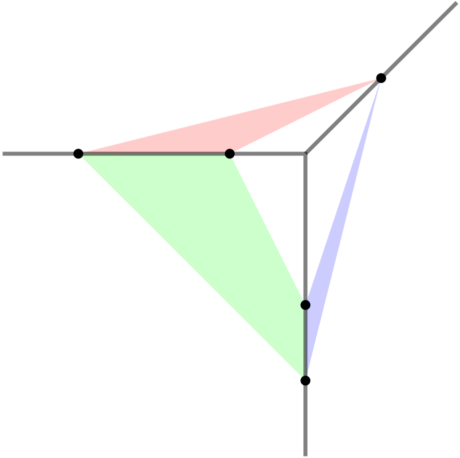

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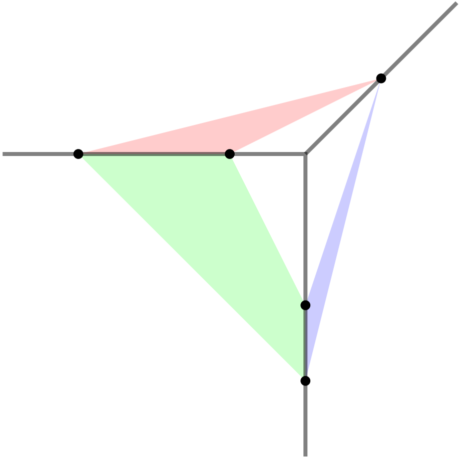

We get a double covering of points by polytopes

Connection to Coverings

- $k(s,n)$ minimal number k such that any s points in $\mathbb{Q}^n$ are roots of some polynomial with k monomials
- $k_2(s,n)$ minimal number k such that any s points in $\mathbb{Q}^n$ can be doubly covered by k polytopes
- $k_1(s,n)$ minimal number k such that any s points in $\mathbb{Q}^n$ can be covered by k polytopes

Connection to Coverings

- Observation: $k(s,n) \geq k_2(s,n) \geq k_1(s,n) \geq k_2(\frac{s}{n+2}, n)$
- It is known that $k_1(s,n) \leq \frac{2s}{2n+3}$ (Urabe '99)
- It is conjectured that $k_1(s,n) = \lceil \frac{s}{2n} \rceil$ (Urabe '99)
- We show $k_2(s,2) \geq \lceil \frac{s}{2} \rceil + 1$
- This gives an upper bound on r for $n = 2$

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- Max-plus Universal Testing Set
  Completely different for $\mathbb{R}$ and $\mathbb{Q}$
  Gap between lower and upper bound for $\mathbb{Q}$
  Connection to coverings