The Mathematics of Partisan Gerrymandering

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LA Combinatorics and Complexity Seminar

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Gerrymander detection by shape

Long history of detecting gerrymandering by shape

Boston Gazette, 1812

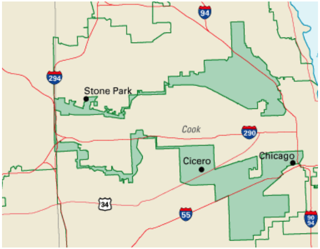
Washington Post, 2014

- "bizarre shape" quantified by "geographic compactness"
- constraints imposed by bipartisan redistricting commissions

Gerrymander detection by shape

Does shape provide enough information?

NC-12



IL-4

What about context? Does intent matter?

images from Wikipedia

Gerrymander detection by voting results

One modern approach: compare votes vs. seats

- proportionality (not a valid constitutional standard)
- efficiency gap (subject of *Gill v. Whitford*)

images from Wikipedia

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Do voting results provide enough information?

United States presidential election in Massachusetts, 2016

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Is partisan fairness at odds with geometry?

images from Wikipedia

Joint work with

Boris Alexeev
`borisalexeev.com`

Richard Kueng
Caltech

Soledad Villar
NYU

Part I

compact gerrymandering

How to quantify "bizarre shape"?

image from Wikipedia

Classical notions of geographic compactness

Isoperimetry. wasted perimeter

- perimeter
- Polsby–Popper score $\quad \propto \quad$ (area) / (perimeter)^2

Duchin, `sites.duke.edu/gerrymandering/files/2017/11/MD-duke.pdf`

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Dispersion. second moment, *sprawl*

- average distance between pairs of points
- moment of inertia

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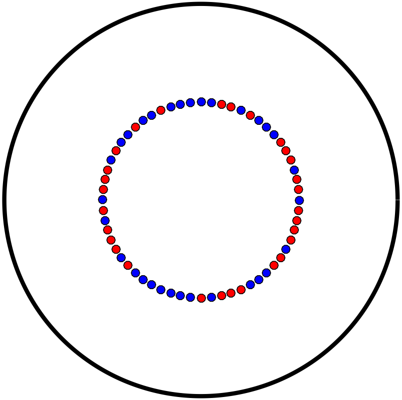
Question 1: Can we gerrymander with nice shapes?

Duchin, `sites.duke.edu/gerrymandering/files/2017/11/MD-duke.pdf`

Passing to a (cartoon) model

- state = closed disk
- $2n$ voters equispaced along concentric circle
- voter preferences = iid uniform from $\{\pm 1\}$



Passing to a (cartoon) model

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- $2n$ voters equispaced along concentric circle
- voter preferences $=$ iid uniform from $\{\pm 1\}$

Theorem

Partition a closed disk C of unit radius into two regions A, B whose closures are homeomorphic to C. Then

$$\max\left\{|\partial A|, |\partial B|\right\} \geq \pi + 2, \quad \min\left\{\frac{|A|}{|\operatorname{hull}(A)|}, \frac{|B|}{|\operatorname{hull}(B)|}\right\} \leq 1, \quad I_A + I_B \geq \frac{\pi}{2} - \frac{16}{9\pi}.$$

Equality is achieved in all three when A and B are complementary half-disks.

Proof ingredients: Jordan curve theorem, basic convexity, calculus

Alexeev, M., J. Appl. Probab., 2018

Passing to a (cartoon) model

- state = closed disk
- $2n$ voters equispaced along concentric circle
- voter preferences = iid uniform from $\{\pm 1\}$

Theorem

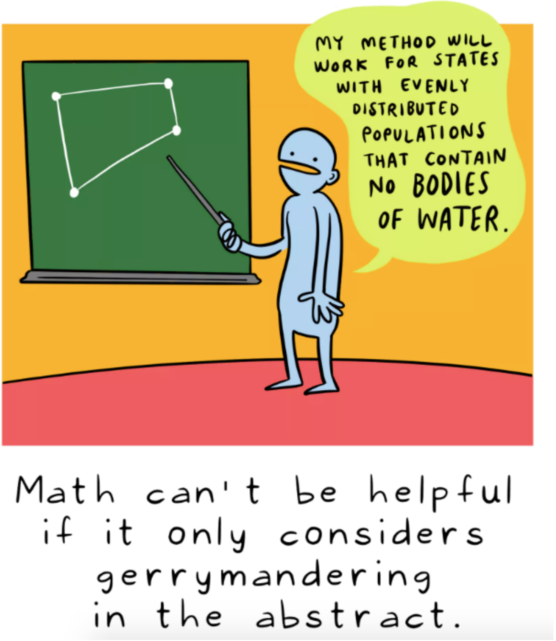
Let D_n denote the random number of majority-positive districts in an optimal partisan gerrymander by complementary half-disks. Then

$$\Pr(D_n = 2) = \frac{1}{2} - \Theta\left(\frac{1}{\sqrt{n}}\right), \qquad \lim_{n\to\infty} \Pr(D_n = 0) = \frac{1}{1+e^{\pi}}.$$

Proof ingredients: IVT, Donsker's invariance principle, Brownian bridges

Upshot: 73% of seats (on average) from 50% of votes (on average)

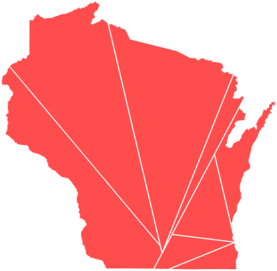
Alexeev, M., J. Appl. Probab., 2018

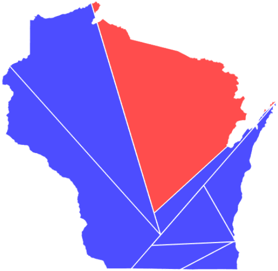


Walch, `thenib.com/changing-the-math-on-gerrymandering`

Beyond the model

Apply gerrymandered version of Smith's split-line algorithm:





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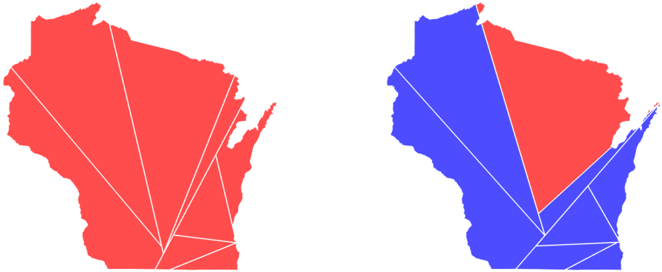

In general, vote majority $\mapsto$ seat unanimity (ham sandwich theorem)

Why not impose partisan fairness?

Smith, `rangevoting.org/SplitLR.html`

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Part II

an impossibility theorem

How to measure partisan fairness?

Proportionality. prop votes $\approx$ prop seats

Stephanopoulos, McGhee, Univ. Chic. Law Rev., 2015

How to measure partisan fairness?

Proportionality. prop votes $\approx$ prop seats

Efficiency gap. $\#\Big($ red wasted votes $\Big) \approx \#\Big($ blue wasted votes $\Big)$

- voter locations $= A, B \subseteq [0,1]^2$
- districts $= D_1 \sqcup \cdots \sqcup D_k = [0,1]^2$
- wasted votes by A in $i \in [k]$:

$$w_{A,i} := \begin{cases} |A \cap D_i| - \lceil \frac{1}{2} |(A \cup B) \cap D_i| \rceil & \text{if } |A \cap D_i| > |B \cap D_i| \\ |A \cap D_i| & \text{else} \end{cases}$$

- efficiency gap:

$$\mathrm{EG}(D_1, \dots, D_k; A, B) := \frac{1}{|A \cup B|} \sum_{i=1}^{k} \left(w_{A,i} - w_{B,i} \right)$$

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Technical definitions

Districting system. $\textsc{dist}\colon (A, B, k) \mapsto (D_1, \ldots, D_k) =: D$

Alexeev, M., Amer. Math. Mo., 2018

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One person, one vote. $\exists \delta \in [0, 1)$, $\forall (A, B, k)$, $D = \textsc{dist}(A, B, k)$ satisfies

$$(1-\delta)\left\lfloor \frac{|A \cup B|}{k} \right\rfloor \leq \left|(A \cup B) \cap D_i\right| \leq (1+\delta)\left\lceil \frac{|A \cup B|}{k} \right\rceil \qquad \forall i \in [k]$$

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Polsby–Popper compactness. $\exists \gamma > 0$, $\forall (A, B, k)$, $D = \textsc{dist}(A, B, k)$ satisfies

$$4\pi \cdot \frac{|D_i|}{|\partial D_i|^2} \ge \gamma \qquad \forall i \in [k]$$

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Bounded efficiency gap. $\exists \alpha, \beta > 0$, $\forall (A, B, k)$, $D = \textsc{dist}(A, B, k)$ satisfies

$$\left|\mathrm{EG}(D_1, \ldots, D_k; A, B)\right| < \frac{1}{2} - \alpha \quad \text{whenever} \quad \big||A| - |B|\big| < \beta |A \cup B|$$

Alexeev, M., Amer. Math. Mo., 2018

Impossibility

Theorem

There is no districting system that simultaneously satisfies

- one person, one vote
- Polsby–Popper compactness
- bounded efficiency gap

Alexeev, M., Amer. Math. Mo., 2018

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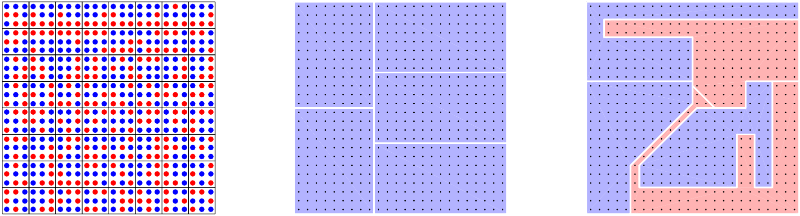
Proof idea: homogeneous mixture of voters



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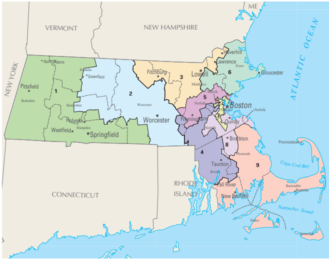
$$\text{EG} \approx 0.3 \gg 0.08$$

$$\text{MA} \notin \{\text{MD}, \text{NC}, \text{WI}, \ldots\}$$

Political geography explanation?



images from Wikipedia

Part III

fair redistricting is hard

Background

Sometimes, the set of compliant maps is **complicated**

Constraints for Wisconsin State Assembly:

- all districts have equal population
- at most 58 counties can be split in different districts
- at most 62 municipalities can be split
- the average Reock score is at least 0.39
- the average Polsby–Popper score is at least 0.28
- at least 6 districts are at least 40% black (among citizens of voting age)
- districts 8 and 9 do not change (previously ordered by a federal court)

How hard is it to find a fair map among compliant maps?

Computational complexity

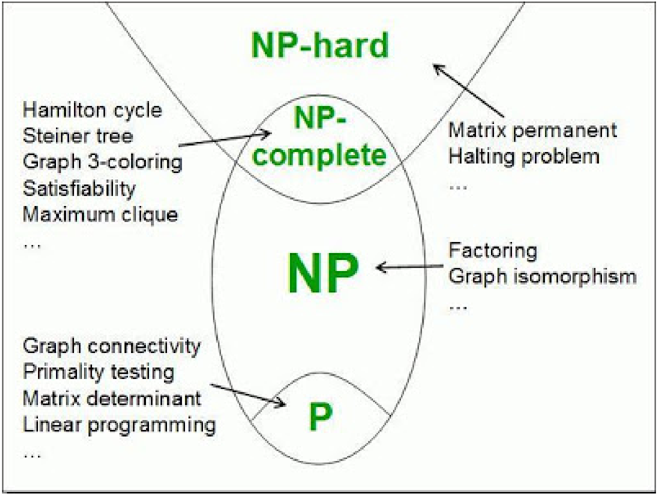


image from quantumgazette.blogspot.com

Fair maps among compliant maps

Compliant maps

- all districts have approximately the same population
- mild notion of geographic compactness

Fair maps

- both parties receive at least some level of representation

Kueng, M., Villar, Theor. Comput. Sci., 2019

Fair maps among compliant maps

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Theorem

Deciding whether there exists a fair redistricting among compliant maps is NP-hard.

Kueng, M., Villar, Theor. Comput. Sci., 2019

Proof idea: Reduction from planar 3-SAT

3-SAT: Decide whether there exists a boolean assignment satisfying a formula of the form

$$(\neg x_1 \vee x_2 \vee \neg x_4) \wedge (\neg x_2 \vee \neg x_4 \vee \neg x_3) \wedge \dots$$

Mahajan, Nimbhorkar, Varadarajan, Theor. Comput. Sci., 2012

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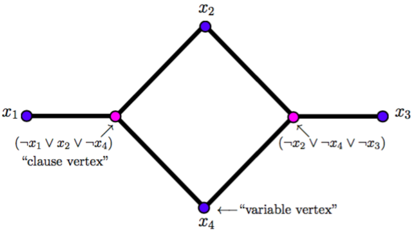
Planar 3-SAT: Bipartite graph with $V = \{$ variables, clauses $\}$



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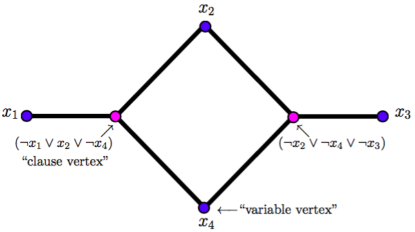
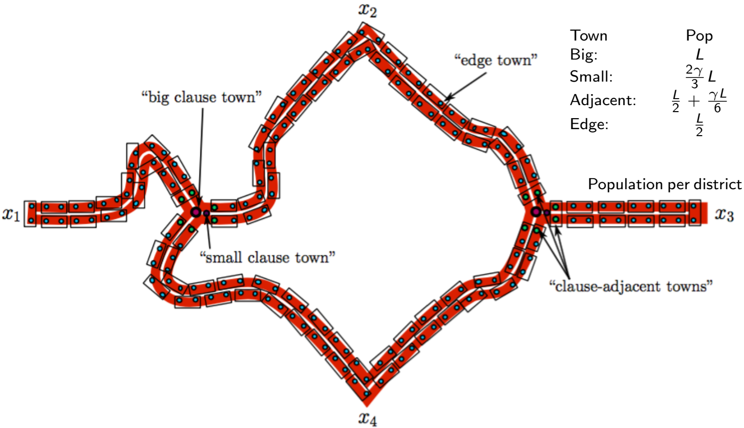


Planar 3-SAT is NP-complete

Mahajan, Nimbhorkar, Varadarajan, Theor. Comput. Sci., 2012

Proof idea: Reduction from planar 3-SAT

instance of planar 3-SAT $\quad \mapsto \quad$ instance of fair redistricting



Town	Pop	D	R
Big:	L	L	0
Small:	$\frac{2\gamma}{3}L$	$\frac{2\gamma}{3}L$	0
Adjacent:	$\frac{L}{2}+\frac{\gamma L}{6}$	$\frac{L}{4}$	$\frac{L}{4}+\frac{\gamma L}{6}$
Edge:	$\frac{L}{2}$	$\frac{L}{2}-\frac{\gamma L}{4}$	$\frac{L}{2}+\frac{\gamma L}{4}$

Population per district $\in [L, L+\gamma]$.

D wins at most $2k$ districts
even with almost half of the vote and Total pop $\gg 2k$
Formula is satisfiable iff D wins $2k$ districts.

Kueng, M., Villar, Theor. Comput. Sci., 2019

Important considerations

Worst-case complexity

- says very little about real-world maps
- identifies limitations of general-purpose redistricting protocols

Review

Political geography can bring tension between shape and fairness

- Sometimes, you can gerrymander with nice shapes
- Sometimes, you need strange shapes to obtain fairness
- Sometimes, a fair map exists, but it's hard to find

Questions?

Partisan gerrymandering with geographically compact districts
B. Alexeev, D. G. Mixon
J. Appl. Probab. 55 (2018) 1046–1059

An impossibility theorem for gerrymandering
B. Alexeev, D. G. Mixon
Amer. Math. Mo. 125 (2018) 878–884.

Fair redistricting is hard
R. Kueng, D. G. Mixon, S. Villar
Theor. Comput. Sci. 791 (2019) 28–35.

Utility Ghost: Gamified redistricting with partisan symmetry
D. G. Mixon, S. Villar
arXiv:1812.07377

Also, Google **short fat matrices** for my research blog.