Link crossing number is NP-hard

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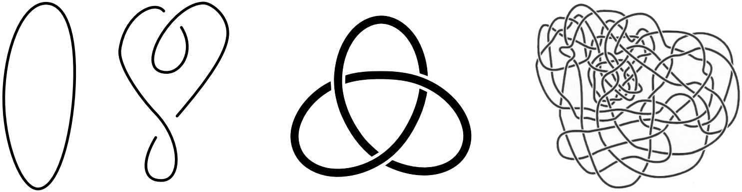
Joint work with Marcus Schaefer and Eric Sedgwick (both at DePaul University, Chicago).

Knot Theory

Knots

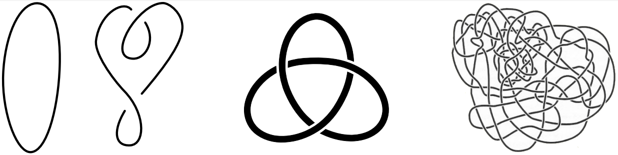
- A *knot* is a nice map $K : S^1 \to \mathbb{R}^3$.
- Two knots are *equivalent* if they are *ambient isotopic*, i.e., if there exists a continuous deformation from one into the other without crossings.
- It is not obvious that there exist non-equivalent knots.



Knot diagrams

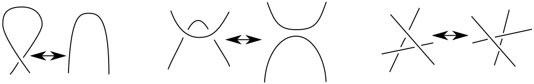
Diagrams

- A *knot diagram* is a 2D-projection of a knot where at every vertex, one indicates which strand goes above and below.
- The *crossing number* of a knot K is a minimum number of crossings over all knot diagrams for K.



Theorem (Reidemeister)

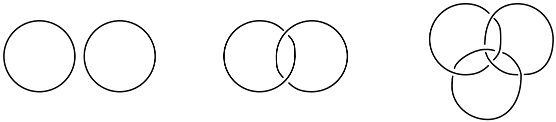
Two knot diagrams correspond to equivalent knots if and only if they can be related by a sequence of Reidemeister moves.



Links

Links

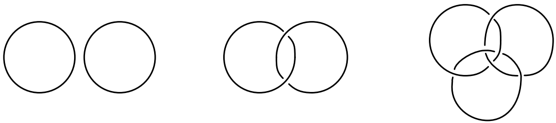
- A *link* is a disjoint union of knots.
- Two links are equivalent if they are ambient isotopic.

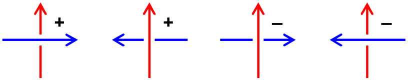


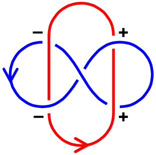
Links

Links

- A *link* is a disjoint union of knots.
- Two links are equivalent if they are ambient isotopic.
- It is easy to prove that there exists non equivalent links by using the *linking number*...
- ... but it does not work all the time







Computing the crossing number

Crossing number

Input: A knot/link diagram D.
Output: Is the crossing number of the knot/link at most k?

Theorem

The crossing number problem for links is ***NP****-hard.*

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First reaction: of course it's **NP**-hard! But:

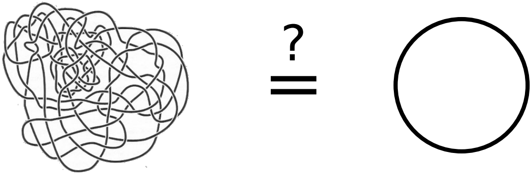
- It is open whether the crossing number of a *knot* is **NP**-hard,
- Complexity results are surprisingly hard to come by in knot theory.

Let us survey what is known.

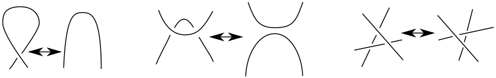
Computational problems in knot theory I

Unknot recognition

Input: A knot K represented by a diagram.
Output: Is K equivalent to the trivial knot?



- Already not obvious that this is decidable [Haken '61].
- In $\mathbf{NP} \cap co - \mathbf{NP}$ [Hass-Lagarias-Pippenger '99], [Agol'02 → Lackenby '18].
- "Easy" **NP** algorithm: guess a sequence of Reidemeister moves. $O(n^{11})$ moves are enough [Lackenby'15].



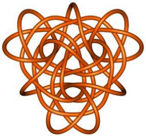
- Lower bounds: ??

Computational problems in knot theory II

Knot equivalence

Input: Two knots K_1 and K_2 represented by diagrams.
Output: Is K_1 equivalent to K_2?

$$\stackrel{?}{=}$$



- Even harder to prove that it is decidable [Hemion '79, Matveev '07].
- Best bound on Reidemeister moves is from [Lackenby-Coward '14]:

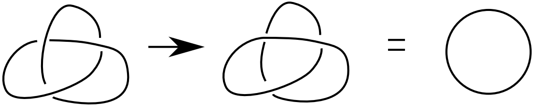
$$\left. 2^{2^{\cdot^{\cdot^{\cdot^{2^{n_1+n_2}}}}}} \right\} \text{height } c^{n_1+n_2} \text{ where } c = 10^{1000000}.$$

- [Kuperberg '19] provides an elementary algorithm, i.e., with a tower of exponentials of constant size.
- Lower bounds: ???

Computational problems in knot theory III

Unknotting number

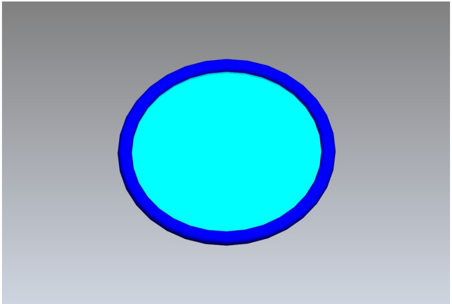
Input: A knot K represented by a diagram.
Output: Can I make k crossing changes to K to transform it into a trivial knot?

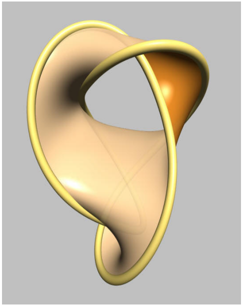


- Crossing changes are allowed in *any* diagram of K.
- Not known to be decidable. Even if k is fixed to be 1!
- No known lower bound.

Some previous hardness results

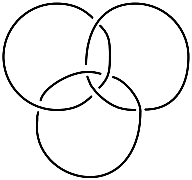
- Knot genus in a 3-manifold [Agol-Hass-Thurston '05]. Thurston norm of a link [Lackenby '17].



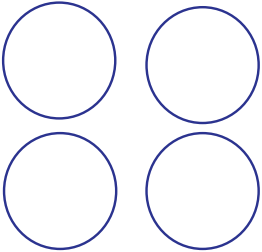


Some previous hardness results

Using Borromean rings:



- Finding a sublink [Lackenby '17], finding a trivial sublink [dM-Rieck-Sedgwick-Tancer '19].



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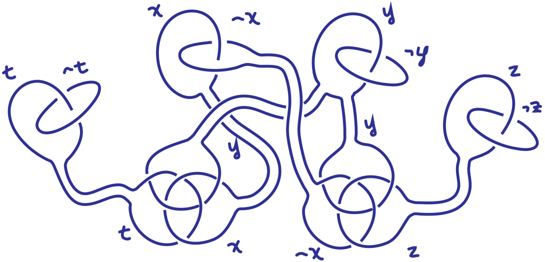

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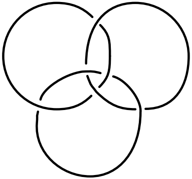
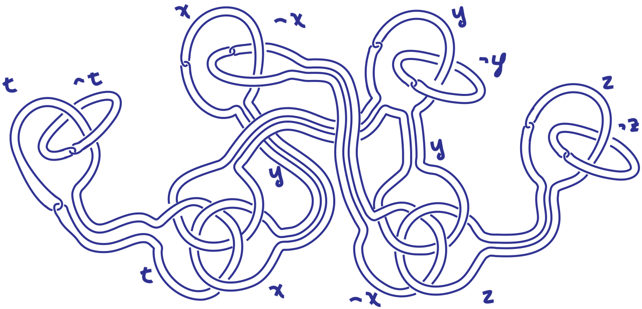


- Unlinking number [dM-Rieck-Sedgwick-Tancer-Wagner '19].

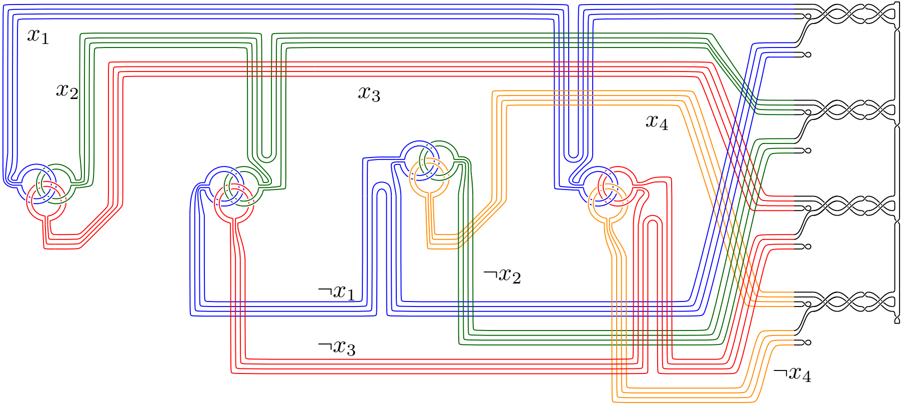


Some previous hardness results

Using Borromean rings:

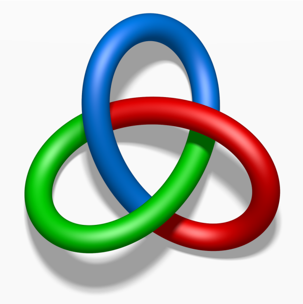
- Deciding whether a knot can be turned into a trivial diagram using at most *k* Reidemeister moves [dM-Rieck-Sedgwick-Tancer '19].

$$\Phi = (x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_4) \wedge (x_1 \vee \neg x_3 \vee \neg x_4)$$



Some previous hardness results

- Jones polynomial [Jaeger-Vertigan-Welsh '90, Kuperberg '15], coloring invariants [Kuperberg- Samperton '19].



Back to the crossing number of a link

Best known algorithm to decide whether the crossing number of a link L is at most k:

for $i = 1 \dots k$ **do**
  for All the link diagrams D with i crossings **do**
    Test whether D and L are the same link.
  end for
end for

Marc Lackenby

"[This algorithm] is obviously not very efficient but it seems unlikely that there is any quicker way of determining a link's crossing number in general."

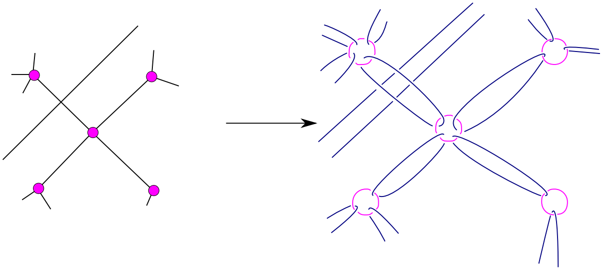
$\rightarrow$ Small progress on this by proving **NP**-hardness.

A naive reduction

Theorem (Garey-Johnson '83)

Computing the crossing number of a graph is ***NP****-hard.*

- Transforming a graph into a link.



- But when changing the cyclic ordering around the vertex, the link gets all tangled up.

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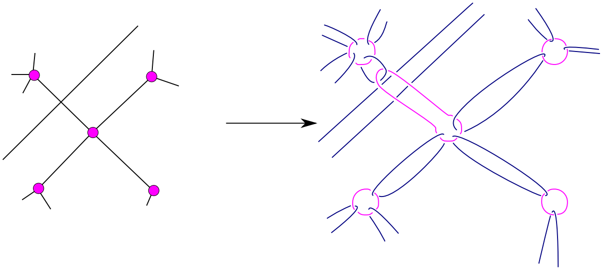


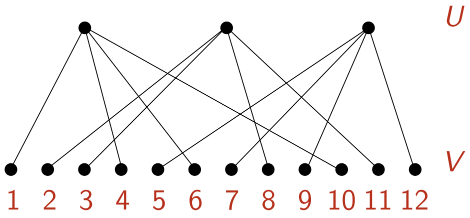
- But when changing the cyclic ordering around the vertex, the link gets all tangled up.
- We must prevent the components corresponding to vertices from stretching.

The actual reduction

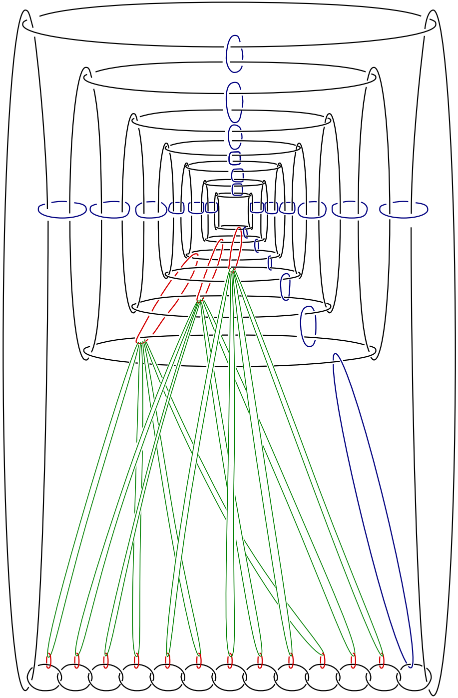
We reduce from a specific variant of the graph crossing number, where the cyclic orderings are fixed:

Theorem (Muñoz-Unger-Vrťo '02)

Determining the bipartite crossing number of a bipartite graph $G = (U \cup V, E)$ in which all vertices in U have degree 4, all vertices in V have degree 1, and the order of the V-vertices along their line is fixed, is ***NP****-complete.*



which we transform into...

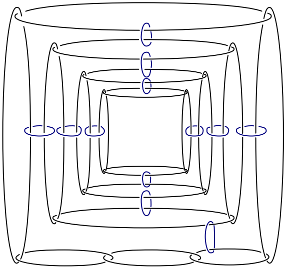


Why does this work?

One direction is immediate: from a graph drawing with low crossing number we get a link diagram with low crossing number.

For the other direction, we want to prove that in any diagram of low crossing number, things are as we would expect:

- the frame is rigid and

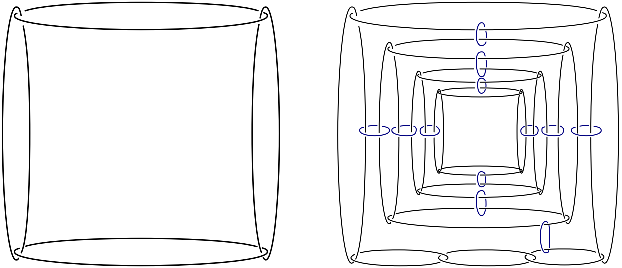


- the only things moving are the red curves.

Using linking numbers

Main tool: Linking numbers.

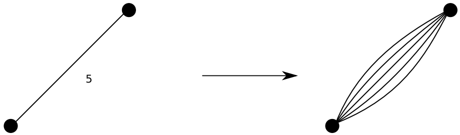
- With linking numbers, we can prove that this diagram of the frame is the unique one with a minimal number of crossings.



- Then the hope is that the placement of the frame forces other crossings (even those not forced by linking numbers).
- But adding the other gadgets may break the rigidity of the frame.

Weighted crossings

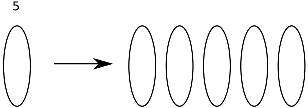
- This is a common issue in reductions involving crossing numbers.
- We can gain rigidity by putting big *weights*: each edges has a weight w_e, and the weighted crossing number of e and f crossing is $w_e w_f$.
- This can be easily simulated by using multiple edges.



- In the setting of graphs, it is immediate that all the multiple edges will be drawn the same way in some crossing-minimal drawing.
- Big weights can enforce rigidity.

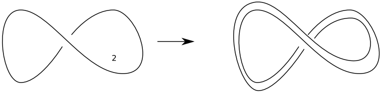
Weighted knots

Likewise, we can use multiple copies of knot to represent weights:

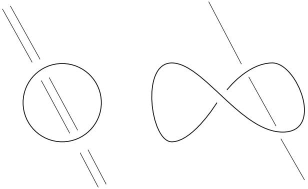


However:

- Self-crossings throw off the accounting, thus we only use unknots.



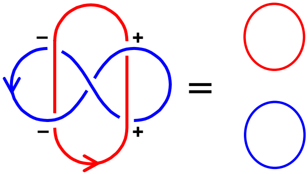
- We can not argue that in a crossing-minimal drawing, all the copies of a knot will be drawn the same way.



Our solution

- We do use weighted knots, and choose weights wisely.
- When arguing that things look like we want them to look, we use a relaxed notion of equivalence.

Two links are *parity-link equivalent* if the parity of the linking number between pairs of components is the same in both crossings.

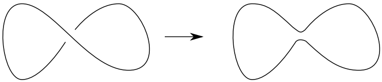


Parity-link equivalence is simpler to handle

Lemma

For any link L, let D' be a diagram with a minimum number of crossings of a link L' which is parity-link equivalent to L. Then no link component in D' has self-crossings.

Proof:



Working from a different link.

- The argument showing that the frame is rigid is only based on linking numbers!

So, if L has a drawing with a low crossing number:

- We look at the crossing-minimal drawing D of a link L' that is parity-link equivalent to our link L. It also has a low number of crossings.
- L' might be different from L, but it does not matter:
- There, the frame is rigid.
- Likewise, the only non-rigid pieces are the moving red curves.
- We can find a drawing of our original bipartite graph from D with few crossings.

Some perspectives

- We also get **NP**-hardness for the minimal crossing number under other notions of equivalence: parity-link equivalence, linking-number equivalence, link-homotopy and link concordance.
- How to adapt this to knots? Or links with a bounded number of components? *Alternating knots* might help but the weighting issue is problematic.
- Is it still hard for a fixed value of the crossing number?
- Hardness of the main knot theory problems?
- What about the bridge number?

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Thank you! Questions?