Removing partisan bias in redistricting: computational complexity meets the science of gerrymandering↑

Bhaskar DasGupta♥
Department of Computer Science
University of Illinois at Chicago
Chicago, IL 60607
bdasgup@uic.edu

↑ Joint result with Tanima Chatterjee, Laura Palmieri, Zainab Al-Qurashi and Anastasios Sidiropoulos

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Gerrymandering
Creation of district plans with highly asymmetric electoral outcomes to disenfranchise voters

- Long history starting from as early as 1812
  1812: shape of South Essex district (Massachusetts) resembling a salamander created to favor selected candidates

- Extensive legal history too!

  1986: US Supreme Court: gerrymandering is justiciable
  2006: US Supreme Court: some measure of partisan symmetry may be used to remedy gerrymandering
    Which measure? Court did not say. Depends case by case.
  2019: US Supreme Court: best settled at the legislative and political level (ALAS!)

- Major impediment to removing gerrymandering
  How to formulate an effective and precise measure for partisan bias that will be acceptable in courts?
Some tools politicians use for partisan gerrymandering in 2-party system

- **Packing** → concentrate voters of opposition party in a single district

- **Cracking** → spread voters of opposition party across many districts

Other methods include
- Hijacking
- Kidnapping etc.
“Efficiency Gap” measure for partisan gerrymandering

- Introduced by Stephanopoulos and McGhee in 2014 for a 2-party system (such as USA)
- Minimizes absolute difference of total “wasted votes” between the parties
- Very promising in several aspects, e.g.,
  - provides a “mathematically precise” measure of gerrymandering with desirable properties
  - was found legally convincing in a US appeals court case
    - ALAS, Supreme Court overturned the ruling in 2019
“Wasted votes” for a district

- Total votes 100 (need 51 to win)
  - Party A vote 59
  - Party B vote 41

- Wasted votes for Party A 59-51=8
- Wasted votes for Party B 41
“Efficiency gap” measure for the whole map

\[
\text{Efficiency gap} = \frac{|\text{sum of Party A wasted votes over all districts} - \text{sum of Party B wasted votes over all districts}|}{\text{Total votes over all districts}}
\]
Formalization of the efficiency gap calculation problem

Basic assumption: *only two* parties: Party A and Party B
(3rd party votes are negligible, like in USA)

Topological part of an input: a “map” $\mathcal{P}$
- partitioned into *atomic elements* or *cells*, e.g., subdivisions of counties

Two possible types of maps:

**Rectilinear polygon $\mathcal{P}$ without holes**
- $\mathcal{P}$ placed on a unit grid of size $m \times n$
- atomic elements (cells) $\Rightarrow$ unit squares of grid inside $\mathcal{P}$
- $v_{i,j}$: cell on $i^{th}$ row and $j^{th}$ column

**Arbitrary polygon $\mathcal{P}$ without holes:**
- atomic elements (cells) $\Rightarrow$ sub-polygons (without holes) inside $\mathcal{P}$
- *Alternate* way of looking: planar graph $G(\mathcal{P})$
  - nodes are cells
  - edge connects two cells if they share boundary
Formalization of the efficiency gap calculation problem

*only two* parties: Party A and Party B

- **Parameters of our gerrymandering problem**

- **Map** $\mathcal{P}$:
  - *size* $|\mathcal{P}|$: number of cells or nodes in $\mathcal{P}$

- **Cell or node** $y$ of $\mathcal{P}$:
  - $\text{PartyA}(y)$: total number of voters for Party A
  - $\text{PartyB}(y)$: total number of voters for Party B
  - $\text{Pop}(y) = \text{PartyA}(y) + \text{PartyB}(y)$: total number of voters

- **Global**:
  - $\kappa$: *required* (legally mandated) number of districts ($1 < \kappa < |\mathcal{P}|$)
    - Hard constraint: solution with different value of $\kappa$ would be *illegal*
    - precludes designing approximation algorithm in which the value of $\kappa$ changes even by just $\pm 1$
    - computational hardness for a value of $\kappa$ may *not* necessarily imply hardness for another value of $\kappa
Formalization of the efficiency gap calculation problem
only two parties: Party A and Party B

Granularities of numeric parameters

- **Course granularity:**
  - \( \text{Pop}(y) \)'s are numbers of arbitrary size
  - total number of bits contributes to input size
  - data at the “county” level or “census block group” level

- **Fine granularity:**
  - \( \forall \text{ cell or node } y: \quad 0 < \text{Pop}(y) \leq c \) for some fixed constant \( c \)
  - data at the “Voting Tabulation District” (VTD) level or “census block” level

- **Ultra-fine granularity:**
  - \( \forall \text{ cell or node } y: \quad \text{Pop}(y) = c \) for some fixed constant \( c \)
  - theoretically interesting case, but practically a bit unrealistic
**Formalization of the efficiency gap calculation problem**

*only two* parties: Party A and Party B

<table>
<thead>
<tr>
<th>Notations for each $S_j$</th>
<th>Party affiliations in $S_j$</th>
<th>Population of $S_j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\kappa$ number of districts</td>
<td>$\text{PartyA}(S_j) = \sum_{y \in S_j} \text{PartyA}(y)$</td>
<td>$\text{Pop}(S_j) = \text{PartyA}(S_j) + \text{PartyB}(S_j)$</td>
</tr>
<tr>
<td>$\mathcal{S}$ set of all cells in given polygonal map $\mathcal{P}$</td>
<td>$\text{PartyB}(S_j) = \sum_{y \in S_j} \text{PartyB}(y)$</td>
<td></td>
</tr>
<tr>
<td>or, set of all nodes in given planar graph $G(\mathcal{P})$</td>
<td>$\text{Pop}(S_j)$</td>
<td></td>
</tr>
<tr>
<td>Districting scheme partition of $\mathcal{S}$ into $\kappa$ subsets $\mathcal{S}<em>1, \ldots, \mathcal{S}</em>\kappa$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Legal requirements for valid re-districting plans**

- Every $S_j$ must be a connected polygon
- Populations of different $S_j$’s must be as equal as possible
Formalization of the efficiency gap calculation problem

*only two* parties: Party A and Party B

- **Legal requirements for valid re-districting plans**
  - Every $S_j$ must be a connected polygon
  - Populations of different $S_j$’s must be as equal as possible
    - **Strict partitioning criteria**
      \[
      \{S_1, \ldots, S_\kappa\} \text{ is an exact } \kappa\text{-equipartition of } S, \text{ i.e., } \forall j : \text{ Pop}(S_j) \in \left\lfloor \frac{\text{Pop}(S)}{\kappa} \right\rfloor, \left\lceil \frac{\text{Pop}(S)}{\kappa} \right\rceil
      \]
    - **(Multiplicatively) approximate partitioning criteria**
      \[
      \{S_1, \ldots, S_\kappa\} \text{ is a } \varepsilon\text{-approximate } \kappa\text{-equipartition of } S, \text{ i.e., } \frac{\max \{\text{Pop}(S_j)\}}{\min \{\text{Pop}(S_j)\}} \leq 1 + \varepsilon
      \]
      courts may allow a maximum value of $\varepsilon$ in the range of 0.05 to 0.1
      e.g., (US Supreme Court ruling in Karcher v. Daggett, 1983)
    - **Additively approximate partitioning criteria**
      \[
      \{S_1, \ldots, S_\kappa\} \text{ is an additive } \varepsilon\text{-approximate } \kappa\text{-equipartition of } S, \text{ i.e., } \max \{\text{Pop}(S_j)\} \leq \min \{\text{Pop}(S_j)\} + \varepsilon
      \]
Formalization of the efficiency gap calculation problem

*only two* parties: Party A and Party B

```
“Wasted votes” for a district

➢ Total votes 100 (Party A needs 50 to win)        Pop(\(S_j\))
  ➢ Party A vote 59       PartyA(\(S_j\))
  ➢ Party B vote 41       PartyB(\(S_j\))

➢ Wasted votes for Party A 59 – 50 = 9       PartyA(\(S_j\)) – \(\frac{1}{2}\)Pop(\(S_j\))

➢ Wasted votes for Party B 41       PartyB(\(S_j\))

➢ Efficiency gap for \(S_j\) 9 – 41 = -32

\[\text{Effgap}(S_j) = \begin{cases} 
\left(\text{PartyA}(S_j) - \frac{1}{2}\text{Pop}(S_j)\right) - \text{PartyB}(S_j) & \text{if } \text{PartyA}(S_j) \geq \frac{1}{2}\text{Pop}(S_j) \\
2\text{PartyA}(S_j) - \frac{3}{2}\text{Pop}(S_j) & \text{otherwise}
\end{cases}\]

from the point of view of Party A (the victim party of gerrymandering)
```
Formalization of the efficiency gap calculation problem

*only two* parties: Party A and Party B

“Wasted votes” for a district

- Total votes **100** (Party A needs 50 to win)
  - Party A vote **41**
  - Party B vote **59**

- Wasted votes for Party A **41**

- Wasted votes for Party B **59 - 50 = 9**

- Efficiency gap for **$S_j$** **41 - 9 = -32**

\[
\text{Effgap}(S_j) = \begin{cases} 
\text{PartyA}(S_j) - \left( \text{PartyB}(S_j) - \frac{1}{2} \text{Pop}(S_j) \right) & \text{if PartyA}(S_j) < \frac{1}{2} \text{Pop}(S_j) \\
2\text{PartyA}(S_j) - \frac{1}{2} \text{Pop}(S_j) & \text{otherwise}
\end{cases}
\]

from the point of view of Party A (the victim party of gerrymandering)
Formalization of the efficiency gap calculation problem

*only two* parties: Party A and Party B

\[ \kappa \quad \text{number of districts} \]
\[ S \quad \text{set of all cells in given polygonal map } \mathcal{P} \]
\[ \text{or, set of all nodes in given planar graph } G(\mathcal{P}) \]
\[ \text{districting scheme} \quad \text{partition of } S \text{ into } \kappa \text{ subsets } S_1, \ldots, S_\kappa \]

\[ \text{Effgap}_\kappa(\mathcal{P}, S_1, \ldots, S_\kappa) = \left| \sum_{j=1}^{\kappa} \text{Effgap}(S_j) \right| \]

(to be minimized)

from the point of view of Party A (the victim party of gerrymandering)
Formalization of the efficiency gap calculation problem

*only two* parties: Party A and Party B

### $\kappa$-district Minimum Wasted Vote Problem ($\text{MIN-WVP}_\kappa$)

#### Input
- map $\mathcal{P}$ with $\text{Pop}(y)$, $\text{PartyA}(y)$, $\text{PartyB}(y)$ for every cell $y \in \mathcal{P}$
- integer $1 < \kappa \leq |\mathcal{P}|$

#### Assumption
$\mathcal{P}$ has at least one $\kappa$-equipartition

#### Valid solution
Any $\kappa$-equipartition $S_1, \ldots, S_\kappa$ of $\mathcal{P}$

#### Objective
Minimize $\text{Effgap}_\kappa(\mathcal{P}, S_1, \ldots, S_\kappa) = |\sum_{j=1}^{\kappa} \text{Effgap}(S_j)|$

#### Notation
$\text{OPT}_{\kappa}(\mathcal{P}) \overset{\text{def}}{=} \min \{ \text{Effgap}_\kappa(\mathcal{P}, S_1, \ldots, S_\kappa) \mid S_1, \ldots, S_\kappa \text{ is a } \kappa\text{-equipartition of } \mathcal{P} \}$

‡ in exact or approximate sense
A numerical example to illustrate efficiency gap calculation problem

Two possible district maps

<table>
<thead>
<tr>
<th>PartyA(Q)</th>
<th>PartyB(Q)</th>
<th>Effgap(Q)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q₁</td>
<td>208</td>
<td>192</td>
</tr>
<tr>
<td>Q₂</td>
<td>170</td>
<td>230</td>
</tr>
<tr>
<td>Q₃</td>
<td>88</td>
<td>312</td>
</tr>
</tbody>
</table>

\[
\text{Effgap}(Q, Q_{1}, Q_{2}, Q_{3}) = | −184 + 140 \) − 24| = 68
\]

<table>
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<tr>
<th>PartyA(Q)</th>
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<tr>
<td>Q₁</td>
<td>208</td>
<td>192</td>
</tr>
<tr>
<td>Q₂</td>
<td>134</td>
<td>266</td>
</tr>
<tr>
<td>Q₃</td>
<td>124</td>
<td>276</td>
</tr>
</tbody>
</table>

\[
\text{Effgap}(Q, Q_{1}, Q_{2}, Q_{3}) = | −184 + 58 + 48| = 78
\]
Mathematical properties of $\text{Effgap}_\kappa(\mathcal{P}, S_1, \ldots, S_\kappa)$: set of attainable values

assume strict $\kappa$-equipartition, i.e., $\text{Pop}(S_1) = \cdots = \text{Pop}(S_\kappa)$

Lemma 1:

$\triangleright$ $\text{Effgap}_\kappa(\mathcal{P}, S_1, \ldots, S_\kappa)$ assumes one of the $\kappa + 1$ values:

$$\left| 2 \times \text{PartyA}(\mathcal{P}) - \left( z + \frac{\kappa}{2} \right) \frac{\text{Pop}(\mathcal{P})}{\kappa} \right| \quad \text{for } z = 0, 1, \ldots, \kappa$$

$\triangleright$ If $\text{Effgap}_\kappa(\mathcal{P}, S_1, \ldots, S_\kappa) = \left| 2 \text{PartyA}(\mathcal{P}) - \left( z + \frac{\kappa}{2} \right) \frac{\text{Pop}(\mathcal{P})}{\kappa} \right|$ then

$$\frac{\text{Pop}(\mathcal{P})}{2\kappa} z \leq \text{PartyA}(\mathcal{P}) \leq \frac{\text{Pop}(\mathcal{P})}{2\kappa} z + \frac{1}{2} \text{Pop}(\mathcal{P})$$

Illustrative example: $\kappa = 2$

only 3 possible values of $\text{Effgap}_2(\mathcal{P}, S_1, S_2)$

$$\left| 2 \times \text{PartyA}(\mathcal{P}) - \frac{1}{2} \text{Pop}(\mathcal{P}) \right| \quad \text{or} \quad \left| 2 \times \text{PartyA}(\mathcal{P}) - \text{Pop}(\mathcal{P}) \right| \quad \text{or} \quad \left| 2 \times \text{PartyA}(\mathcal{P}) - \frac{3}{2} \text{Pop}(\mathcal{P}) \right|$$
First a “somewhat” bad news
(worst-case computational complexity meets gerrymandering)

Theorem (informal description)

Not only calculation of efficiency gap is NP-complete, but

assuming $P \neq NP$, no non-trivial approximation is possible in polynomial time

But, have no fear!
We have only shown hardness in theoretical worst-case
Worst-case computational complexity meets gerrymandering

Assumptions

- Map $\mathcal{P}$: rectilinear polygon without holes
- Strict partitioning criteria: $\{S_1, \ldots, S_\kappa\}$ is exact $\kappa$-equipartition of $S$
- Course granularity: $\text{Pop}(y)$’s are numbers of arbitrary size
- $P \neq NP$

Theorem 1

For any rational constant $\varepsilon \in (0, 1)$, for any $\rho$ and all $2 \leq \kappa \leq \varepsilon|\mathcal{P}|$, the $\text{MIN-WVP}_\kappa$ problem for rectilinear polygon $\mathcal{P}$ does not admit a $\rho$-approximation algorithm.

Reduction: from PARTITION problem
Worst-case computational complexity meets gerrymandering

Assumptions

- Map $\mathcal{P}$: planar graph $G = (V, E)$
- (Multiplicatively) approximate partitioning criteria:
  \[
  \{S_1, \ldots, S_\kappa\} \text{ is a } \varepsilon\text{-approximate } \kappa\text{-equipartition of } S, \text{ i.e., } \frac{\max\{\text{Pop}(S_j)\}}{\min\{\text{Pop}(S_j)\}} \leq 1 + \varepsilon
  \]
- Fine granularity: $\forall$ node $y$: $0 < \text{Pop}(y) \leq c$ for some fixed constant $c$

**Theorem 2**

For any constant $0 < \varepsilon < 1/2$,

computing an exact solution of the $\text{MIN-WVP}_\kappa$ problem

is NP-complete

Proof does *not* provide any non-trivial inapproximability ratio

Reduction: from maximum independent set for planar cubic graphs
However, even in theory, we can efficiently compute efficiency gap under “reasonable” assumptions

*e.g.*, with these assumptions:

- Input map: a rectilinear polygon $\mathcal{P}$ (without holes)
- Every district must have a “nice” shape (*y-convex* shape)
- $\kappa$ (number of districts) is *constant*
- Total population $\text{Pop}(\mathcal{P})$ is *polynomial* in number of cells $|\mathcal{P}|$
We developed and implemented a simple heuristic algorithm based on “local search” method

- Start with some existing or random valid solution
- Search for nearby valid solutions by randomly “swapping” local regions among various districts
  - Pitfall: can get stuck with far-away local optima but, does not seem to often occur for real maps

➢ Next few slides: results for real maps
Local search algorithm

START

i-th iteration i<100?

T

Select n random counties from the original dataset. n \in (1,K) with K< tot_counties

F

Return solution with min EG. If EG>7%, re-execute the algorithm

STOP

Next input: new EG and new map

Go to (B)

F: Go to (B)

Calculate new districts and compute new EG

Pop dev < 10 % AND new EG < old EG?

F

Calculate new districts and compute new EG

T

i-th iteration i<100?

Select n random counties from the original dataset. n \in (1,K) with K< tot_counties

T

Return solution with min EG. If EG>7%, re-execute the algorithm

F

Go to (B)

Shift jth county to kth district. Disconnected map?

T

Shift jth county to kth district. Disconnected map?

F

k-th neighb k<tot_n?

T

k-th neighb k<tot_n?

F

Go to (A)

County already analyzed?

T

County already analyzed?

F

j-th county on district boundary?

T

j-th county on district boundary?

F

j-th county j<n?

T

j-th county j<n?

F

Pop dev < 10 % AND new EG < old EG?

F

Pop dev < 10 % AND new EG < old EG?

T

Next input: new EG and new map

Go to (B)

F: Go to (B)
Wisconsin

Total votes: 2,841,407
Dem votes: 1,441,804 ~ 51%
Rep votes: 1,399,603 ~ 49%

Current EG: 14.8%
Dem #seats: 3
Rep #seats: 5

New EG: 3.8%
Dem #seats: 3
Rep #seats: 5
Virginia

Total votes: 3,569,498
Dem votes: 1,736,164 ~ 49%
Rep votes: 1,833,334 ~ 51%

Current EG: 22%
Dem #seats: 3
Rep #seats: 8

New EG: 3.6%
Dem #seats: 5
Rep #seats: 6
Texas

Total votes: 7,379,170
Dem votes: 2,949,900 ~ 40%
Rep votes: 4,429,270 ~ 60%

Current EG: 4.01%
Dem #seats: 12
Rep #seats: 24

New EG: 3.3%
Dem #seats: 12
Rep #seats: 24
Pennsylvania

2012 House Elections

Total votes: 5,374,461
Dem votes: 2,722,560 ~ 51%
Rep votes: 2,651,901 ~ 49%

Current EG: 23.8%
Dem #seats: 5
Rep #seats: 13

New EG: 8.64%
Dem #seats: 6
Rep #seats: 12
Pennsylvania

2016 Presidential Elections

Total votes: 5,896,628
Dem votes: 2,925,776 ~ 50%
Rep votes: 2,970,852 ~ 50%

New EG: 8.05%
Dem #seats: 7
Rep #seats: 11

Current EG: 14.34%
Dem #seats: 6
Rep #seats: 12

New EG: 3%
Dem #seats: 8
Rep #seats: 10

Created on Feb. 2018 (by a local court in PA) and based on symmetry between seat share and vote share.
Some Interesting Insights based on simulation results

- **Seat gain vs. efficiency gap**
  - lower efficiency gap does not necessarily lead to seat gains for the loosing party

- **Compactness vs. efficiency gap**
  - Our new district map have fewer districts that are oddly shaped compared to the current gerrymandered maps

- **How natural are current gerrymandered districts?**
  - It seems that original gerrymandered districts are far from being a product of arbitrarily random decisions
Science of gerrymandering is a huge garden with so many unknown fruits for hungry theoretical computer scientists!

So many questions, so few answers

- Define and analyze new quantitative measures of gerrymandering
  - What about 3 or more party systems?
- Analyze computational complexities of existing measures of gerrymandering
- Join court cases as an expert witness and convince judges that computational complexity matters

Data files: https://www.cs.uic.edu/~dasgupta/gerrymander/

Thank you for your attention!

Questions?