Flip Distances between Graph Orientations

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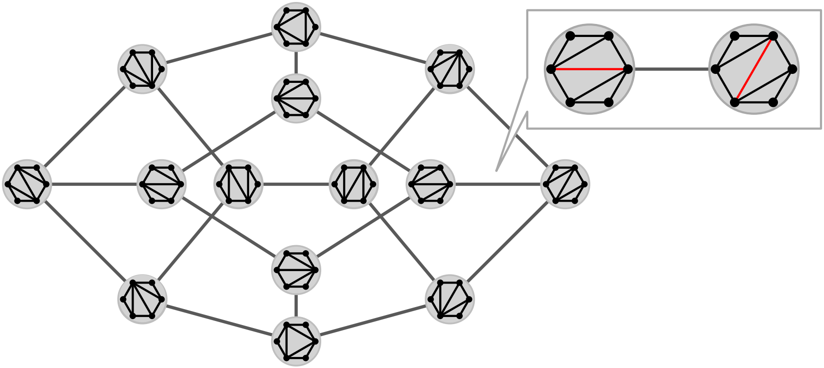
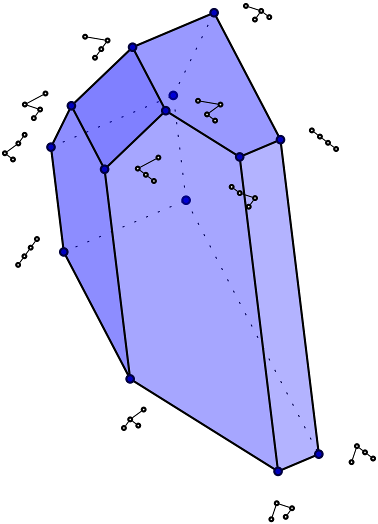
Flip Graphs



Polytope



Flip Distances

- Diameter of associahedra

  Sleator, Tarjan, Thurston 1988, Pournin 2014

- Computational question: given two objects, what is their flip distance?
- Flip distance between triangulations of point sets is NP-hard.

  Lubiw and Pathak 2015

- Flip distance between triangulations of simple polygons is NP-hard.

  Aichholzer, Mulzer, Pilz 2015

- Flip distance between triangulations of a convex polygon: Major open question.

Flip Distances

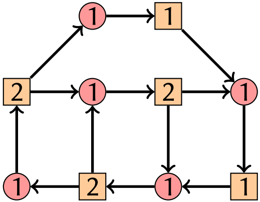
Complexity of flip distances on “nice” combinatorial polytopes?

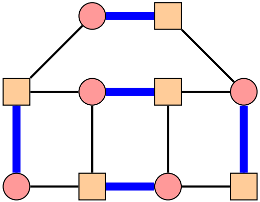
- Matroid polytopes: easy
- Associahedra and polymatroids: open
- Intersection of two matroids?

α-orientations

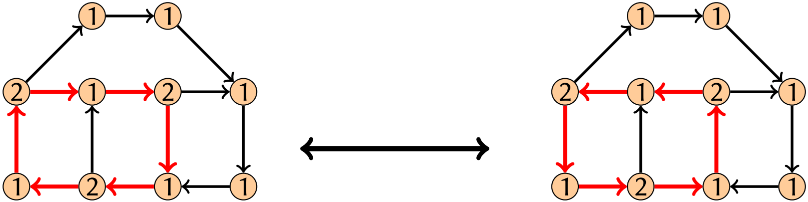
An α-orientation of G is an orientation of the edges of G in which every vertex v has outdegree $\alpha(v)$.

Perfect Matchings in Bipartite Graphs

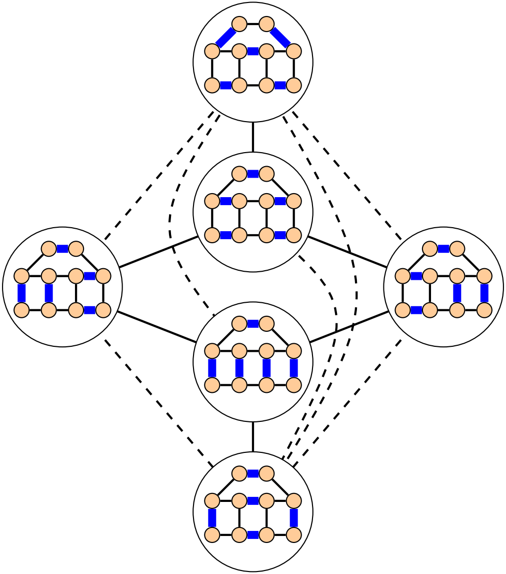




Flips in α-orientations



Flip Graph on Perfect Matchings



Adjacency

- The common base polytope of two matroids is the intersection of the two matroid polytopes.
- α-orientations are intersections of two partition matroids.
- Adjacency is characterized by cycle exchanges.

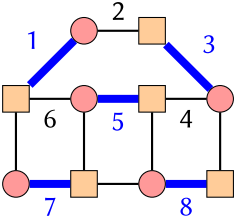
  Frank and Tardos 1988, Iwata 2002

- Adjacency on perfect matching polytope: symmetric difference induces a single cycle.

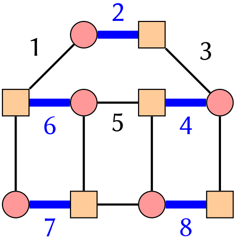
  Balinski and Russakoff 1974

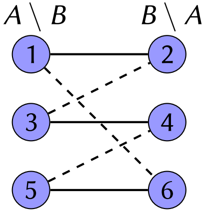
Adjacency

$A = \{1, 3, 5, 7, 8\}$

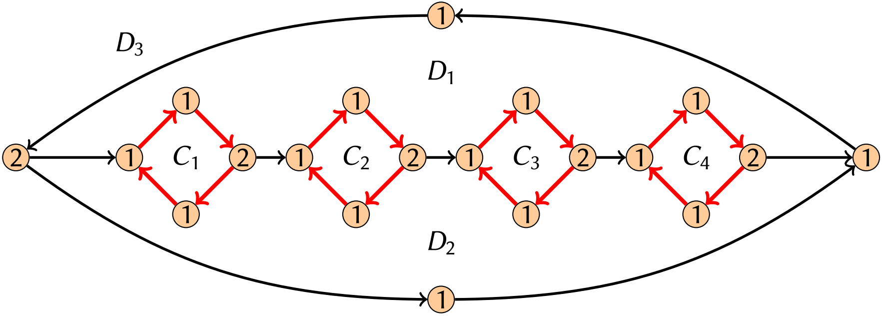


$B = \{2, 4, 6, 7, 8\}$





Flip Distance



- Symmetric difference composed of four cycles: C_1, C_2, C_3, C_4
- Flip distance three: D_1, D_2, D_3

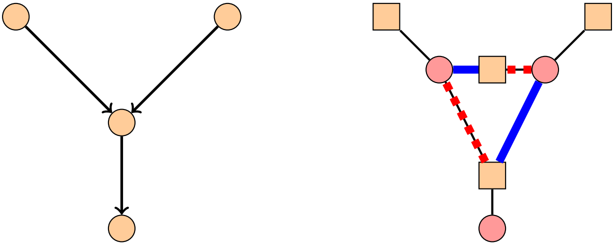
NP-completeness

Theorem

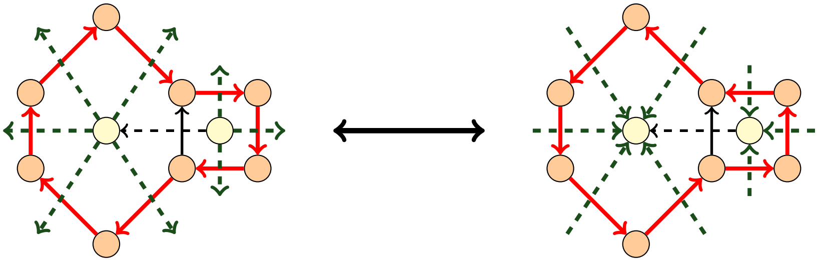
Given a two-connected bipartite subcubic planar graph G and a pair X, Y of perfect matchings in G, deciding whether the flip distance between X and Y is at most two is NP*-complete.*

NP-hardness

Reduction from Hamiltonian cycle in planar, degree 1/2-2/1 digraphs.



Planar Duality



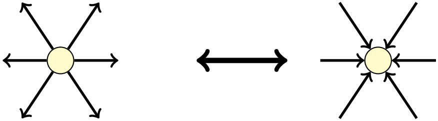
Dual Flips

A *c-orientation* is an orientation in which the number of forward edges in any cycle C is equal to $c(C)$.

Propp 2002, Knauer 2008

primal	dual
α-orientation	c-orientation
directed cycle flip	directed cut flip
facial cycle flip	source/sink flip

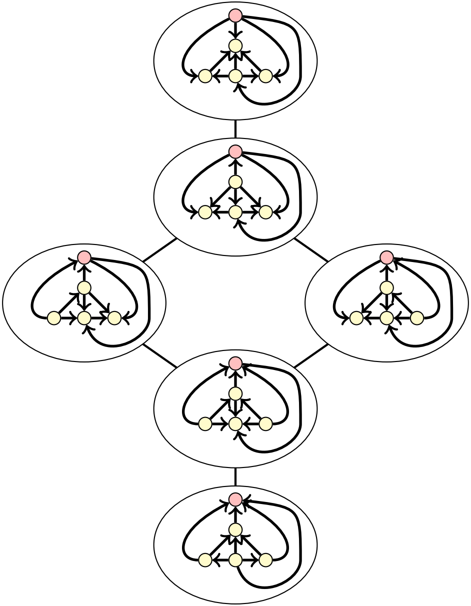
Source/sink flips



- What is the complexity of computing the source/sink flip distance?
- Here the flip graphs is known to have a nice structure.

Propp 2002, Felsner and Knauer 2009,2011

A Distributive Lattice



Source/sink flip distance

Theorem

There is an algorithm that, given a graph G with a fixed vertex $\top$ and a pair X, Y of c-orientations of G, outputs a shortest source/sink flip sequence between X and Y, and runs in time $O(m^3)$ where m is the number of edges of G.

Flip distance with Larger Cut Sets

Theorem

Let X, Y be c-orientations of a connected graph G with fixed vertex $\top$. It is NP*-hard to determine the length of a shortest cut flip sequence transforming X into Y, which consists only of minimal directed cuts with interiors of order at most two.*

Reduction from the problem of computing the jump number of a poset.

Thank you!

Algorithmica
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