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Reduced decompositions of permutations in terms of star transpositions, generalized Catalan numbers and k-ARY trees

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Abstract

In this paper we compute the number of reduced decompositions of certain permutations $\sigma \in S_n$ as a product of transpositions $(1,2)$, $(1,3)$, …, $(1,n)$. We present several combinatorial correspondences between these decompositions and combinatorial objects such as Catalan paths and k-ary trees. © 1999 Elsevier Science B.V. All rights reserved

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1. Introduction

Let G be a finite group and B its set of generators. Call a product $g = b_1 \cdot \ldots \cdot b_l$, where each $b_i \in B$ a *decomposition* of $g \in G$ of length l. The *length* $l(g) = l_B(g)$ of an element g is the minimum length of its decomposition. We say that a decomposition of g is *reduced* if it has length $l(g)$. By $r(g) = r_B(g)$ we denote the number of reduced decompositions of $g \in G$ in term of generators in B.

Let $G = S_n$ be a symmetric group on n elements. In this paper we find the number reduced decompositions of certain permutations in terms of star transpositions. Namely, denote by $B = B_n$ the set of *star transpositions* $(1,i) \in S_n$, $2 \leqslant i \leqslant n$. It is easy to see that B generates the whole symmetric group S_n.

Denote by $L(q; p_1, p_2, \ldots,)$ the set of permutations $\sigma \in S_n$ with cycles of length q, p_1, p_2, …, $q + p_1 + p_2 + \ldots = n$ and such that the first element belongs to a cycle of length q. Observe that B_n is fixed under a permutation of the last $n-1$ elements. Therefore, both the length $l_* = l_B$ and the number of reduced decompositions

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$r_* = r_B$ are constant on the subsets $L(q; p_1, p_2, \ldots)$. We write $L(q; p^m)$ instead of $L(q; p, p, \ldots, p)$ (m times).

Theorem 1.1. *Let $n = km + 1$, $k \geqslant 2$. Then the number of reduced decompositions of a permutation $\sigma \in L(1; k^m)$, $\sigma \in S_n$, in terms of star transpositions in B_n is given by the formula*:

$$r_*(\sigma) = r_*(k, m) = \frac{k^m \cdot (km + m)!}{(km + 1)!.}.$$

For example, let $\sigma = (1)(2,3)(4,5) \cdots \in S_{2m+1}$ Then the length of σ is $l_*(\sigma) = 3 \cdot m$, the maximal length of all elements in S_{2m+1}. The Theorem claims that the number of reduced decompositions as a product of star transpositions is equal to

$$r_*(\sigma) = \frac{2^m\,(3m)!}{(2m + 1)!}.$$

In particular, when $m = 1$ we have two decompositions:

$$(2,3) = (1,2) \cdot (1,3) \cdot (1,2) = (1,3) \cdot (1,2) \cdot (1,3).$$

When $m = 2$ we have 24 reduced decompositions of (23)(45) which can be also checked directly.

In this paper we give two combinatorial proofs of Theorem 1.1. We relate reduced decompositions to generalized Dyck sequences, k-Catalan paths on a square grid (Section 3), rooted plane trees and $(k + 1)$-ary trees (Section 4). We also introduce certain bracket sequences as an intermediary (Section 2). Many of these combinatorial objects have been studied earlier, which simplifies our task.

Remark 1.2. The analogous problem has been studied for various other generating sets. See [10] for the case of adjacent transpositions and [2] for the case of all transpositions. Other generating sets include cycles of bigger length (see [9,4]).

Note that Theorem 1.1 gives the number of reduced decompositions only in a special case. Finding a general formula is an interesting open problem.

2. Reduced decompositions and bracket sequences

We think of elements of S_n as of permutations that permute elements according to their places. For example, multiplying a star transposition $(1, i)$ from the right to a permutation σ means to exchange elements $\sigma(1)$ and $\sigma(i)$. We also say that we *touch* element $\sigma(i)$ and *hit* place i. Now we can view each decomposition as a straight line algorithm which exchanges pairs of elements, one at a time.

Let us first compute the lengths of permutations.

Lemma 2.1. *Let $\sigma \in L(q; p_1, p_2, \ldots, p_m, 1^a)$, $p_1, \ldots, p_m \geqslant 2$, $q + p_1 + \ldots + p_m + a = n$. Then $l_*(\sigma) = n + m - a - 1$.*

Of course, this would immediately imply that $L(1;2^m)$ contains permutations $\sigma \in S_{2m+1}$ with maximum length (see above).

Proof. Indeed, break σ into a product of cycles. Each cycle of length p not containing the first element 1, can be decomposed into a product of $p+1$ star transpositions. If a cycle of length q contains 1, $q-1$ star transpositions suffice. Therefore, $l_* \leqslant (q-1) + (p_1+1) + \cdots + (p_m+1) = n+m-a-1$. The opposite inequality follows from the following observation. We need to touch $n-a-1$ elements that are not fixed points or 1. At any time, we say that an element is *untouched* if it is not 1 and we have not touched it before. Since each time we transpose an element at the first place with some other element, at a time we cannot touch more than one untouched element. In addition to that, the first time we touch an element in a cycle some already touched element which is not in that cycle, gets inside that cycle. At one point this element must get back to the first place and when this happens no new elements are touched. Therefore, we need to use at least $(n+m-a-1)$ transposition which proves the claim. □

Denote by $R_*(\sigma)$ the set of star decompositions of $\sigma \in S_n$. Let $\sigma \in L(1;k^m)$ be a permutation with m cycles of length k and a fixed point 1. Fix any ordering of cycles in σ. By a *symbol* with index i, $1 \leqslant i \leqslant m$, we mean either a left bracket $[_i$, or a right bracket $]_i$, or a vertical line $|_i$. Define a map κ which maps reduced decompositions of σ into a sequence of symbols by the following rule:

- Each star transposition which hits the ith cycle corresponds to a *symbol* with index i. The transposition that hits the ith cycle for the first time corresponds to the left bracket $[_i$, for the last time, to the right bracket $]_i$, and to the vertical line $|_i$ in between.

  For example, κ maps the reduced decomposition

$$(1,2)(1,7)(1,6)(1,7)(1,3)(1,2)(1,5)(1,4)(1,5)$$

of an element $\sigma = (2,3)(4,5)(6,7) \in L(1;2^3)$ to

$$[_1\ [_3\ |_3\ ]_3\ |_1\ ]_1\ [_2\ |_2\ ]_2.$$

We call such sequences *bracket sequences*.

Lemma 2.2. *Let ξ be a bracket sequence obtained as an image of κ. Then ξ satisfies the following conditions*:

(1) *ξ has $(k+1)m$ symbols, $k+1$ times of each of the indices $1,\ldots,m$. Among symbols with the same index i, the left bracket $[_i$ is to the left of $k-1$ vertical lines $|_i$ which are to the left of the right bracket $]_i$.*

(2) *If a symbol with index i is in between two symbols with index j, so are all symbols with index i, $1 \leqslant i,j \leqslant m$.*

Proof. The first condition follows immediately from the proof of Lemma 2.1. To prove the second condition, observe that the first time we hit a cycle, we get an element of that cycle at the first place. Every next time we hit that cycle we must have an element of that cycle in the first place or otherwise we would need more than $(k+1)m$ transpositions. Only after we hit the cycle for the last, $(k+1)$st time, we get an element on the first place that was there before the first hit. Now, find the first symbol that lies between two symbols with the same index j, different from the index i of the former symbol. From (1) it must be left bracket $[_i$. Therefore, in the corresponding decomposition the jth cycle is now in the ith cycle and the only way we can hit the jth cycle again is by having it back. But before that, we must make the last hit of the jth cycle. In terms of symbols, it means that we must also have $]_i$ before the next symbol with index j. This proves the second condition. □

Lemma 2.3. *Let $\Xi(k,m)$ be the set of bracket sequences described by conditions* (1), (2) *in Lemma 2.2. Then the map $\kappa : R_*(\sigma) \to \Xi(k,m)$, $\sigma \in L(1;k^m)$ is surjective. Moreover the preimage of each sequence $\xi \in \Xi(k,m)$ contains exactly k^m reduced decompositions*:

$$|\kappa^{-1}(\xi)| = k^m.$$

For example, $\Xi(2,2)$ *contains six sequences*: $[_1\ |_1\ ]_1\ [_2\ |_2\ ]_2$, $[_1\ |_1\ [_2\ |_2\ ]_2\ ]_1$, $[_1\ [_2\ |_2\ ]_2\ |_1\ ]_1$, $[_2\ |_2\ ]_2\ [_1\ |_1\ ]_1$, $[_2\ |_2\ [_1\ |_1\ ]_1\ ]_2$ *and* $[_2\ [_1\ |_1\ ]_1\ |_2\ ]_2$. *Each of them is an image of* 4 *reduced decompositions.*

Proof. In order to find an element of the preimage $\kappa^{-1}(\xi)$ we need to assign to each symbol in ξ with index i a transposition which hits the ith cycle. There are k ways to do that since every such a transposition is determined by the transposition assigned to the left bracket $[_i$. Recall that σ has m cycles, which implies that $|\kappa^{-1}(\xi)| \leqslant k^m$.

The opposite inequality is proved by the following argument. Consider the right-most left bracket in ξ. By condition (2), this and the following k symbols must be associated with the last cycle of σ to be hit. These symbols can be replaced by the corresponding k star transpositions in exactly k ways, since once the first transposition is chosen the positions of the others are fixed. Now remove this cycle and use induction. □

3. k-Catalan paths and Dyck sequences

Fix $m \geqslant 1$. Define the set of *k-Catalan paths* $C(k,m)$ to be the set of paths on a square grid from $(0,0)$ to $(m \cdot k, m)$ that stay weakly below the line $y = x/k$. For example, there are 12 elements in $C(2,3)$.

The k-Catalan paths are well studied. It is known that

$$|C(k,m)| = \frac{1}{km+1}\binom{(k+1)m}{m}$$

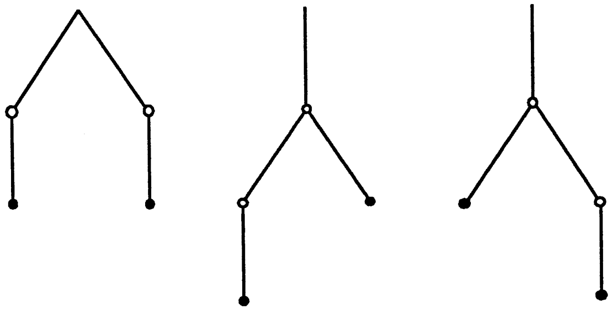

Fig. 1. Set of plane trees $P(2,2)$.

(see e.g. [3]). There is a short combinatorial proof which uses the Cycle Lemma (see [12]). When $k = 1$ we get the ordinary Catalan paths and Catalan numbers (see e.g. [3,11]).

There is an easy way to code k-Catalan paths in terms of *Dyck sequences*. Put 0 when the path goes right and 1 when the path goes up. Formally, define the set of Dyck sequences $D(k,m)$ to be the set of sequences $(a_1,\dots,a_{(k+1)m})$ of 0 and 1 with km zeroes and m ones and such that for every $1 \leqslant i \leqslant (k+1)m$ we have $a_1 + \cdots + a_i \leqslant i/(k+1)$. Then $|D(k,m)| = |C(k,m)|$. For example the three Dyck sequences in $D(2,2)$ are $(0,0,1,0,0,1)$, $(0,0,0,1,0,1)$ and $(0,0,0,0,1,1)$.

Define a map $\varphi : \Xi(k,m) \to D(k,m)$ as follows. Take a bracket sequence and put 0 instead of each left bracket or a vertical line, put 1 instead of each right bracket.

Lemma 3.1. *A map φ defined above is surjective. Moreover, the preimage of each Dyck sequence in $D(k,m)$ contains exactly $m!$ bracket sequences.*

Note that this immediately implies Theorem 1.1. Indeed, together with Lemma 2.3 it shows that the surjection $\varphi \circ \kappa$ maps reduced decompositions onto Dyck sequences such that a preimage of each Dyck sequence in $D(k,m)$ contains exactly $k^m m!$ reduced decompositions. Together with the formula for $|D(k,m)| = |C(k,m)|$ this proves the result.

Proof of Lemma 3.1. First we need to prove that φ is well defined, i.e. $\varphi(\xi) \in D(k,m)$ for every $\xi \in \Xi(k,m)$. Indeed, before each right bracket in a sequence there must be a left bracket and $k-1$ vertical lines. Therefore, in a $0-1$ sequence $\varphi(\xi)$ among the first i elements there are at least k times as many zeroes as ones. This proves the claim.

Use induction to show that preimage of each Dyck sequence $(a_1,\dots,a_{(k+1)m}) \in D(k,m)$ contains exactly $m!$ bracket sequences. The claim is trivial when $m = 1$. For a general m, take the first 1 in a sequence. Suppose it is at the jth place. It corresponds to the first right bracket in a bracket sequence with some index i. But that means that the preceding k zeroes must correspond to the left bracket and vertical lines with the same index i. Since i could be any index, $1 \leqslant i \leqslant m$, we can just delete these $k+1$ consecutive elements and get a Dyck sequence with $m-1$ ones. This completes the step of induction and proves the lemma. □

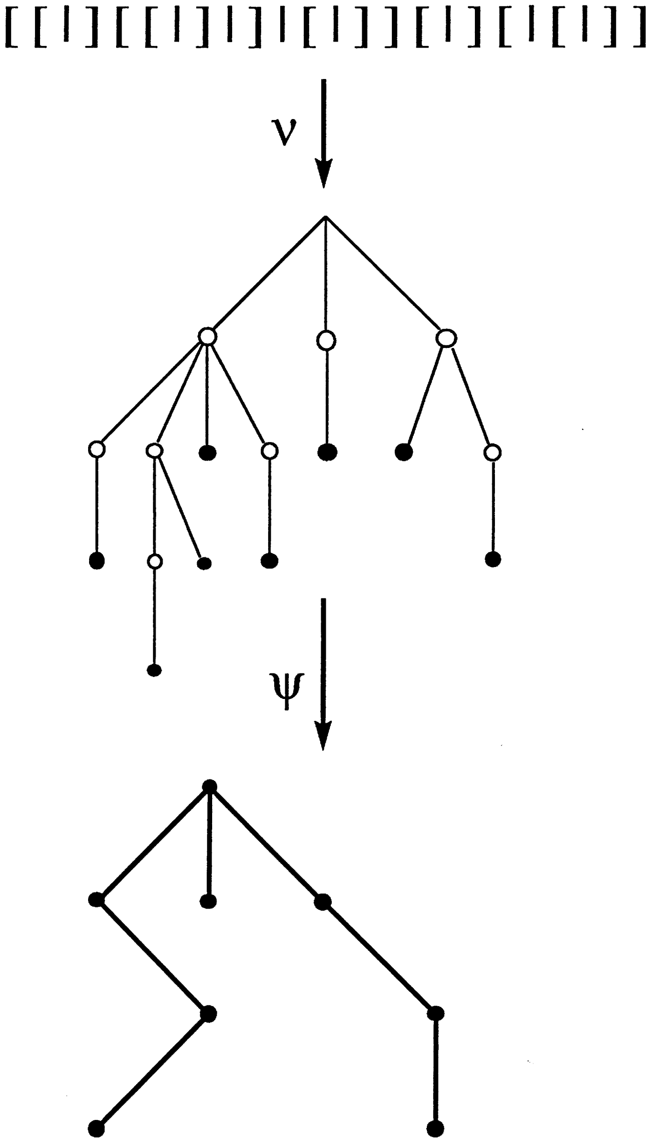

Fig. 2. Map $\nu : \Xi(2,8) \to P(2,8)$ and bijection $\psi : P(2,8) \to T(3,8)$.

4. Plane and *k*-ary trees

Define $P(k,m)$ to be a set of plane rooted trees with m white nodes, $(k-1)m$ black nodes and such that

- Every black node is a leaf. Every white node has $k-1$ black sons.

For example, there are three different plane trees in $P(2,2)$ (see Fig. 1). When $k=1$ we get ordinary plane trees.

It is easy to find a surjective map $\nu : \Xi(k,m) \to P(k,m)$. Indeed, let left brackets correspond to white nodes and vertical lines to black nodes. Whenever we have a left bracket $[_i$ between two symbols with index j, place a node corresponding to $[_i$ to be a son of a node corresponding to $[_j$ in a location which respects the left to right ordering. An example is shown in Fig. 2. We omit the details.

Note that ν disregards indices of the symbols. Thus the preimage of each plane tree in $P(k,m)$ contains exactly $m!$ bracket sequences.

Denote by $T(k,m)$ the set of k-ary trees with m vertices (see [3,11]). There is a known bijection $\psi : P(k,m) \to T(k+1,m)$ (see [5]) which generalizes the famous bijection between plane and binary trees (see [1,8,12]). An example is shown in Fig. 2.

Now, it is known that $|T(k,m)| = [1/(km-m+1)]\binom{km}{m}$ (see e.g. [7,11,3]). By the results above, the surjection $\psi \circ \nu \circ \kappa : R_*(\sigma) \to T(k+1,m)$, $\sigma \in L(1;k^m)$ contains $k^m\, m!$ reduced decompositions in each preimage. This gives another proof of Theorem 1.1.

We would like to remark that the direct bijection $\rho : D(k,m) \to P(k,n)$ is also known (see [6]).

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