Oscillating Tableaux, $S_p \times S_q$-modules, and Robinson-Schensted-Knuth correspondence

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1 Introduction

In the recent time in the works of different authors [4, 5, 6, 7, 11, 14, 17] arose a new interest to the classical Robinson-Schensted-Knuth correspondence [9].

The Robinson-Schensted-Knuth correspondence (RSK) is a bijection between pairs $(P, Q)$ of semi-standard Young tableaux and matrices $M$ with nonnegative integer entries such that the column sums of $M$ give weight of $P$ and the row sums of $M$ give weight of $Q$ (see Corollary 4.5). This correspondence is important in representation theory of the general linear
group $GL(N)$ and the symmetric group $S_n$ and in the theory of symmetric functions.

We can view a pair of tableaux $(P, Q)$ of the same shape as a sequence of Young diagrams $\alpha(0) = \emptyset \subset \alpha(1) \subset \ldots \subset \alpha(p) \supset \alpha(p+1) \supset \ldots \supset \alpha(k) = \emptyset$. In general, consider a sequence of diagrams $\alpha = (\alpha(0), \alpha(1), \ldots, \alpha(k))$ such that for all $i$ either $\alpha(i) / \alpha(i+1)$ or $\alpha(i+1) / \alpha(i)$ is a horizontal stripe. Such objects generalize semi-standard Young tableaux (and pairs $(P, Q)$) and they are called oscillating tableaux.

In this paper we use the following notation: $N := \{0, 1, 2, \ldots\}$; $s(\beta) := \beta_1 + \beta_2 + \ldots + \beta_k$ for $\beta = (\beta_1, \beta_2, \ldots, \beta_k) \in \mathbb{Z}^k$.

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2 Diagrams and tableaux

Recall basic definitions from combinatorics of Young diagrams (see [10]).

A partition $\lambda$ of $n$ is a sequence of positive integers $(\lambda_1, \lambda_2, \ldots, \lambda_l)$ such that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_l > 0$ and $|\lambda| := \lambda_1 + \lambda_2 + \ldots + \lambda_l = n$. We will also write $\lambda \vdash n$. Let $\mathcal{P}$ denote the set of all partitions. By $\emptyset$ denote a unique partition of zero.

With each partitions $\lambda$ we can associate its Young diagram which is the set of pairs $(i, j) \in \mathbb{N}^2$ such that $1 \leq j \leq \lambda_i$, $i = 1, 2, \ldots, l$. Pairs $(i, j)$ are arranged on the plane $\mathbb{R}^2$ with $i$ increasing downwards and $j$ increasing from left to right. Young diagrams will be presented in the form of sets of $1 \times 1$-boxes centered at $(i, j)$. We denote partitions and the associated Young diagrams by the same letter $\lambda$.

Let “$\supset$” be the partial order on $\mathcal{P}$ by inclusion of Young diagrams, i.e., $\lambda \supset \mu$ if $\lambda_i \geq \mu_i$ for all $i$. For $\lambda \supset \mu$, skew Young diagram $\lambda/\mu$ is the set theoretical difference of the Young diagrams $\lambda$ and $\mu$. For example, if $\lambda = (6, 4, 4, 1), \mu = (4, 3, 2)$ then the skew Young diagram $\lambda/\mu$ is the shaded region in Figure 1.

A partition $\lambda' = (\lambda'_1, \ldots, \lambda'_l)$ is conjugate to a partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ if their Young diagrams are symmetric to each other with respect to the principal diagonal.

A horizontal (respectively, vertical) $m$-stripe is a skew Young diagram $\lambda/\mu$ such that every column (respectively, row) contains at most one box of $\lambda/\mu$ and $|\lambda| - |\mu| = m$.

Let $\beta = (\beta_1, \beta_2, \ldots, \beta_k) \in \mathbb{N}^k$. A Young tableau of shape $\lambda/\mu$ and weight $s(\beta)$ is a filling of $\lambda/\mu$ where $\beta_1$ is placed in $\lambda_1$, $\beta_2$ is placed in $\lambda_2$, and so on. The filling is subject to the condition that for each $i$, the numbers $1, 2, \ldots, \beta_i$ are placed in the $i$th row from left to right, and the numbers $\beta_i + 1, \beta_i + 2, \ldots, \lambda_i$ are placed in the $i$th row from right to left.
Figure 1: A skew Young diagram $\lambda/\mu$

\[\begin{array}{ccc}
 & 3 & 3 \\
1 & & \\
2 & 2 & \\
3 & & \\
\end{array}\quad \begin{array}{ccc}
 & 1 & 1 \\
 & 3 & \\
2 & 3 & \\
3 & & \\
\end{array}\]

Figure 2: A tableau $T$ and a supertableau $S$

$\beta$ is a sequence of partitions $(\alpha_{(0)} = \lambda, \alpha_{(1)}, \alpha_{(2)}, \ldots, \alpha_{(k)} = \mu)$ such that $\alpha_{(i-1)} \supset \alpha_{(i)}$ and $\alpha_{(i-1)}/\alpha_{(i)}$ is a horizontal $\beta_i$-stripe for all $i = 1, 2, \ldots, k$. Let $YT(\lambda/\mu, \beta)$ denote the set of all Young tableaux of shape $\lambda/\mu$ and weight $\beta$. Note that such tableaux are also called column-strict or semi-standard. A Young tableau is said to be standard if it has weight $\beta = (1,1,\ldots,1)$.

Let $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k) \in \{1,-1\}^k$ and $\beta^\varepsilon$ denote the sequence $b = (b_1, b_2, \ldots, b_k)$ in the alphabet $\{m, \overline{m} \mid m \in \mathbb{Z}\}$ such that $b_i = \beta_i$ (respectively $b_i = \overline{\beta_i}$) if $\varepsilon_i = 1$ (respectively $\varepsilon_i = -1$).

A supertableau (see [2]) of shape $\lambda/\mu$ and superweight $\beta^\varepsilon$ is a sequence of partitions $(\alpha_{(0)} = \lambda, \alpha_{(1)}, \alpha_{(2)}, \ldots, \alpha_{(k)} = \mu)$ such that $\alpha_{(i-1)} \supset \alpha_{(i)}$ and if $\varepsilon_i = 1$ (respectively $\varepsilon_i = -1$) then $\alpha_{(i-1)}/\alpha_{(i)}$ is a horizontal (respectively, vertical) $\beta_i$-stripe for all $i = 1, 2, \ldots, k$. Let $ST(\lambda/\mu, b)$ denote the set of all supertableaux of shape $\lambda/\mu$ and superweight $b = \beta^\varepsilon$. It is clear that $ST(\lambda/\mu, \beta^{(1,1,\ldots,1)}) = YT(\lambda/\mu, \beta)$.

When we present tableaux and supertableaux, we insert the integers $k - i + 1$ into the boxes of $\alpha_{(i-1)}/\alpha_{(i)}$ for $i = 1, 2, \ldots, k$. Figure 2 shows examples of a tableau $T \in YT(\lambda/\mu, (1,2,3))$ and a supertableau $S \in ST(\lambda/\mu, (2,1,\overline{3}))$. 
3 Oscillating tableaux

We can view tableaux as paths in certain graph $\mathcal{Y}$. The vertices of $\mathcal{Y}$ are Young diagrams and diagrams $\lambda$ and $\mu$ are connected by an edge in $\mathcal{Y}$ if $\lambda/\mu$ (or $\mu/\lambda$) is a horizontal stripe. Let $\mathcal{Y}_n$ denote the $n$th level of $\mathcal{Y}$, i.e., $\mathcal{Y}_n$ is the set of all diagrams $\lambda$ with $|\lambda| = n$. We call $\mathcal{Y}$ the extended Young graph because it is obtained from the Young graph by adding some edges connecting non-adjacent levels.

It is clear that Young tableaux correspond to decreasing paths in the graph $\mathcal{Y}$. An oscillating tableau is an arbitrary path in $\mathcal{Y}$.

**Definition 3.1** Let $\lambda, \mu$ be partitions and $\beta = (\beta_1, \beta_2, \ldots, \beta_k) \in \mathbb{Z}^k$. An oscillating tableau of shape $(\lambda, \mu)$ and weight $\beta$ is a sequence of partitions $\alpha = (\alpha_0 = \lambda, \alpha_1, \alpha_2, \ldots, \alpha_k = \mu)$ such that for all $i = 1, 2, \ldots, k$ the following conditions hold:

1. If $\beta_i \geq 0$ then $\alpha_{i-1} \supset \alpha_i$ and $\alpha_{i-1}/\alpha_i$ is a horizontal $\beta_i$-stripe;
2. If $\beta_i < 0$ then $\alpha_i \supset \alpha_{i-1}$ and $\alpha_i/\alpha_{i-1}$ is a horizontal $(-\beta_i)$-stripe.

By $OT(\lambda, \mu, \beta)$ denote the set of all oscillating tableaux of shape $(\lambda, \mu)$ and weight $\beta$.

It is clear that $OT(\lambda, \mu, \beta)$ is nonempty only when $|\lambda| - s(\beta) = |\mu|$. If all $\beta_i$ are nonnegative then $OT(\lambda, \mu, \beta) = YT(\lambda/\mu, \beta)$.

4 Intransitive graphs

**Definition 4.1** Let $\delta = (\delta_1, \delta_2, \ldots, \delta_k) \in \mathbb{Z}^k$ be a sequence such that $s(\delta) = 0$. An intransitive graph of type $\delta$ is an oriented graph $\gamma$ on the vertices $\{1, 2, \ldots, k\}$ (multiple edges allowed) such that:

1. If $(i, j)$ is an edge of $\gamma$ then $i < j$.
2. If $\delta_i \geq 0$ then indegree of $i$ is $\delta_i$ and outdegree of $i$ is 0.
3. If $\delta_i \leq 0$ then outdegree of $i$ is $-\delta_i$ and indegree of $i$ is 0.

Denote by $G(\delta)$ the set of all intransitive graphs of type $\delta$. 
Figure 3: An intransitive graph $\gamma \in G(-2,1,-2,0,-2,2,3)$

Note that $G(\delta)$ is nonempty if and only if $\sum_{j=1}^l \delta_i \leq 0$ for $l = 1, 2, \ldots, k$. Figure 3 shows an example of an intransitive graph.

Remark 4.2 Let $x_1, x_2, \ldots, x_k$ be variables. Consider the following q-analogue of Kostant’s partition function

$$P_q = \prod_{i>j} (1 - qe^{x_i-x_j})^{-1} = \sum_{\delta: s(\delta) = 0} P_q(\delta) e^{\delta_1 x_1 + \ldots + x_k \delta_k}.$$ 

Then the number $G(\delta)$ of intransitive graphs of type $\delta$ is equal to the coefficient of the least power of $q$ in $P_q(\delta)$. So we can view the number $G(\delta)$ as an analogue of $P_q(\delta)$ as $q \to 0$, i.e., “christal analogue of $P_q(\delta)$”.

Intransitive graphs are closely related to oscillating tableaux. In Sections 5 and 7 we present several theorem illustrating this connection. Here we formulate a special case which is especially clear.

Theorem 4.3 Let $\beta \in \mathbb{Z}^k$ be such that $s(\beta) = 0$. Then the number of oscillating tableaux of shape $(\hat{0},\hat{0})$ and weight $\beta$ is equal to the number of intransitive graphs of type $\beta$

$$|OT(\hat{0},\hat{0},\beta)| = |G(\beta)|.$$ 

In Section ?? we construct a bijection $\Phi_{\lambda\mu\beta}$ which in the case $\lambda = \mu = \hat{0}$ is is a bijection between $OT(\hat{0},\hat{0},\beta)$ and $G(\beta)$.

We call an oscillating tableau of weight $\beta = (\beta_1,\ldots,\beta_k)$ standard if $\beta_i = \pm 1$ for all $i$. Clearly, standard oscillating tableux correspond to paths in the Young graph.

Corollary 4.4 The number of paths in the Young graph from $\hat{0}$ to $\hat{0}$ of length $2k$ is equal to $(2k-1)!! = (2k-1)(2k-3)\ldots 1$.

Proof — If $\beta_i = \pm 1$ for all $i$ then an intransitive graph of type $\beta$ is a perfect matching. Therefore, by Theorem 4.3 the number of standard tableux of shape $(\hat{0},\hat{0})$ with weight of length $2k$ is equal to the number perfect matchings on the set of vertices $\{1, 2, \ldots, 2k\}$ which is equal to $(2k-1)!!$. □
In the end of this section we show how oscillation tableaux and intransitive graphs are connected with classical Robinson-Shensted-Knuth correspondence [9].

Let \( \beta' = (\beta'_1, \beta'_2, \ldots, \beta'_s) \in \mathbb{N}^s \), \( \beta'' = (\beta''_1, \beta''_2, \ldots, \beta''_t) \in \mathbb{N}^t \), and \( \beta \) be the sequence \((-\beta'_s, -\beta'_{s-1}, \ldots, -\beta'_1, \beta''_1, \beta''_2, \ldots, \beta''_t) \in \mathbb{Z}^{s+t} \). It is clear that every oscillating tableau \( \alpha \in OT(\hat{0}, \hat{0}, \beta) \) can be presented by a pair \((P, Q)\) of Young tableaux of the same shape and with weights \(\beta'\) and \(\beta''\) respectively. We can associate with an intransitive graph \( \gamma \in G(\beta) \) the \( s \times t \)-matrix \( A = (a_{ij}) \) such that \( a_{ij} \) is equal to the multiplicity of the edge \( (s+1-i, s+j) \) in \( \gamma \). We get the following corollary of Theorem 4.3.

**Corollary 4.5** Let \( \beta' \in \mathbb{N}^s \) and \( \beta'' \in \mathbb{N}^t \). Then the number of pairs \((P, Q)\) of Young tableaux of the same shape and with weights \(\beta'\) and \(\beta''\) respectively is equal to the number of \( s \times t \)-matrices \( A = (a_{ij}) \) such that

1. \( a_{ij} \in \mathbb{N} \) for \( i = 1, 2, \ldots, s, \ j = 1, 2, \ldots, t \),
2. \( \sum_j a_{ij} = \beta'_i \) for \( i = 1, 2, \ldots, s \),
3. \( \sum_i a_{ij} = \beta''_j \) for \( j = 1, 2, \ldots, t \).

In [9] D. E. Knuth generalized the constructions of G. de B. Robinson [12] and C. Schensted [13] and obtained a one-to-one correspondence between such pairs \((P, Q)\) and matrices \( A \). In this special case the bijection \( \Phi_{\lambda, \mu, \beta} \) (see Section ??) coincides with Robinson-Schensted-Knuth correspondence.

### 5 \( S_p \times S_q \)-module \( M(p, \beta, q) \)

In this section we consider a permutational representation of \( S_p \times S_q \) in the linear space generated by intransitive graphs. Multiplicities of irreducible components in this representation are given by the numbers of oscillating tableaux.

Let \( p, q \in \mathbb{N}, \ \beta = (\beta_1, \ldots, \beta_k) \in \mathbb{Z}^k \) such that \( p - s(\beta) = q, r = p + k, \) and \( n = p + k + q \). Let \( G(p, \beta, q) \) be the set of intransitive graphs of type \( \delta = (\delta_1, \delta_2, \ldots, \delta_n) \), where

\[
\delta_i = \begin{cases} 
-1 & \text{for } i = 1, \ldots, p, \\
\beta_{i-p} & \text{for } i = p + 1, \ldots, r, \\
1 & \text{for } i = r + 1, \ldots, n.
\end{cases}
\]
The direct product of two symmetric groups $S_p \times S_q$ acts on the graphs $\gamma \in G(p, \beta, q)$ as follows: the group $S_p$ permutes the first $p$ vertices in $\gamma$ and the group $S_q$ permutes the last $q$ vertices in $\gamma$. More precisely, if $g = (\sigma, \rho) \in S_p \times S_q$, $\gamma \in G(p, \beta, q)$ then $(i, j)$ is an edge of graph $g \cdot \gamma$ if and only if $(g^{-1}(i), g^{-1}(j))$ is an edge of $\gamma$, where

$$g(s) = \begin{cases} 
\sigma(s) & s = 1, \ldots, p, \\
\rho(s-r) & s = p+1, \ldots, r, \\
\rho(s-r)+r & s = r+1, \ldots, n.
\end{cases}$$

Let $M(p, \beta, q)$ be the linear space over $\mathbb{C}$ with basis $\{v_\gamma\}$, $\gamma \in G(p, \beta, q)$. The action of the group $S_p \times S_q$ on $G(p, \beta, q)$ gives a linear representation $M(p, \beta, q)$ of $S_p \times S_q$.

**Example 5.1** Let $p = q$ and $\beta = \emptyset$ be the empty sequence. Then graphs from $G(p, \emptyset, p)$ can be identified with permutations in $S_p$. In this case $M(p, \emptyset, p)$ is the regular representation $\text{Reg}(S_p)$ of $S_p \times S_p$. That is $M(p, \emptyset, p)$ is isomorphic to the group algebra $\mathbb{C}[S_p]$ on which one copy of $S_p$ acts by left multiplications and the other copy of $S_p$ acts by right multiplications.

**Example 5.2** Let $q = 0$ and $\beta_i \geq 0$ for all $i = 1, 2, \ldots, k$. Then a graph $\gamma \in G(p, \beta, 0)$ can be identified with the word $w = w_1 w_2 \ldots w_p$ in the alphabet $\{1, 2, \ldots, k\}$ where $w_i = j$ if $(i, p+j)$ is an edge of $\gamma$. Clearly, the word $w$ contains $\beta_1$ 1’s, $\beta_2$ 2’s, etc. The symmetric group $S_p$ acts on such words $w$ by permutation of letters $w_i$. The representation $M_\beta = M(p, \beta, 0)$ is the well-known monomial representation, see [8],

$$M_\beta = \text{Ind}_{S_\beta_1 \times \ldots \times S_\beta_k}^{S_p} \text{Id},$$

where $\text{Id}$ is the identity representation of $S_\beta_1 \times \ldots \times S_\beta_k$.

Now we can give a combinatorial interpretation of multiplicities of irreducible components in $M(p, \beta, q)$ in terms of oscillating tableaux.

Let $\pi_\lambda$ be the irreducible $S_n$-module associated with a partition $\lambda \vdash n$ (see [8, 10]). Every irreducible representation of the group $S_p \times S_q$ is of the form $\pi_\lambda \otimes \pi_\mu$, where $|\lambda| = p$ and $|\mu| = q$.

**Theorem 5.3**

$$M(p, \beta, q) \cong \sum |\text{OT}(\lambda, \mu, \beta)| \cdot \pi_\lambda \otimes \pi_\mu,$$

where the sum is over all partitions $\lambda \vdash p$ and $\mu \vdash q$. 

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The following two Corollaries present classical identities.

For \( p, q, \beta \) such as in Example 5.1 Theorem 5.3 gives

**Corollary 5.4**

\[
\text{Reg}(S_p) = \sum_{\lambda \vdash p} \pi_\lambda \otimes \pi_\lambda.
\]

This is a standard fact from representation theory of finite groups.

For \( p, q, \beta \) such as in Example 5.2 Theorem 5.3 gives

**Corollary 5.5**

\[
M_\beta = M(p, \beta, 0) = \sum_{\lambda \vdash p} |YT(\lambda, \beta)| \cdot \pi_\lambda
\]

This is the classical Young rule for decomposition of monomial representations \( M_\beta \) of symmetric groups (see [18, 8, 10]).

Clearly, Theorem 4.3 is a special case of Theorem 5.3 for \( p = q = 0 \).

### 6 Proof of Theorem 5.3

Let \( \mathcal{M} \) be the category whose objects \( \text{Ob}_\mathcal{M} \) are finite groups and morphisms \( \text{Mor}_\mathcal{M}(G, H) \) (or simply \( \text{Mor}(G, H) \)) from a group \( G \) to a group \( H \) are equivalence classes of complex finite dimensional \( G \times H \)-modules. Let \( V \in \text{Mor}(G, H) \) and \( W \in \text{Mor}(H, K) \). \( G, H, K \in \text{Ob}_\mathcal{M} \), then composition \( V \circ W \) of morphisms \( V \) and \( W \) is the following \( G \times K \)-module

\[
V \circ W = V \otimes_{\mathbb{C}[H]} W
\]

(the tensor product over the group algebra \( \mathbb{C}[H] \)). In other words, the tensor product \( V \otimes_{\mathbb{C}} W \) is a \( G \times H \times H \times K \)-module. Then \( V \circ W \) is the \( G \times K \)-module of \( H \)-invariants in \( V \otimes_{\mathbb{C}} W \) (with the diagonal action of \( H \) on \( V \otimes_{\mathbb{C}} W \)). The composition is a bilinear operation with respect to the direct sum of modules.

Let \( \hat{G} \) denote the set of equivalence classes of irreducible representations of \( G \). Then any irreducible \( G \times H \)-module is of the form \( \alpha \otimes \beta^* \), where \( \alpha \in \hat{G} \), \( \beta \in \hat{H} \) and \( \beta^* \) denotes the conjugate to \( \beta \) (which is also irreducible). It is clear that these irreducible modules form a \( \mathbb{N} \)-basis of \( \text{Mor}(G, H) \).

Let \( \text{Reg}(G) \) be the regular representation of \( G \times G \), i.e. \( \text{Reg}(G) \) is the group algebra \( \mathbb{C}[G] \) on which one copy of \( G \) acts by left multiplications and the other copy of \( G \) acts by right multiplications.
The following proposition presents two simple facts from representation theory of finite groups:

**Proposition 6.1**  
1. Let $\alpha \in G$, $\beta \in H$, $\gamma \in H$, $\delta \in K$. Then

$$(\alpha \otimes \beta^*) \circ (\gamma \otimes \delta^*) = \begin{cases} 
\alpha \otimes \delta^* & \text{if } \beta = \gamma, \\
0 & \text{if } \beta \neq \gamma.
\end{cases}$$

2. The regular representation $\text{Reg}(G) = \sum_{\alpha \in G} \alpha \otimes \alpha^*$ is the identity morphism in the category $\mathcal{M}$ from $G$ to $G$.

Now construct a category $\mathcal{T}$. The objects of $\mathcal{T}$ are nonnegative integers $\text{Ob}_\mathcal{T} = \mathbb{N}$ and for $p, q \in \text{Ob}_\mathcal{T}$, morphisms $\text{Mor}_\mathcal{T}(p, q)$ from $p$ to $q$ are sequences $\beta = (\beta_1, \ldots, \beta_k)$ of integers such that $p - s(\beta) = q$ and $p - \sum_{i=1}^k \beta_i \geq 0$ for $j = 1, 2, \ldots, k$. The composition of morphisms $\beta' = (\beta'_1, \ldots, \beta'_l)$ and $\beta'' = (\beta''_1, \ldots, \beta''_m)$ is the sequence $\beta' \circ \beta'' = (\beta'_1, \ldots, \beta'_l, \beta''_1, \ldots, \beta''_m)$.

Consider the following maps from $\text{Ob}_\mathcal{T}$ to $\text{Ob}_{\mathcal{M}}$ and from $\text{Mor}_\mathcal{T}$ to $\text{Mor}_{\mathcal{M}}$

$$M_{\text{ob}} : \ p \in \text{Ob}_\mathcal{T} \rightarrow S_p \in \text{Ob}_{\mathcal{M}},$$

$$M_{\text{mor}} : \ \beta \in \text{Mor}_\mathcal{T}(p, q) \rightarrow M(p, \beta, q) \in \text{Mor}_{\mathcal{M}}(S_p, S_q).$$

**Theorem 6.2** These maps give a functor $\mathcal{M}$ from category $\mathcal{T}$ to category $\mathcal{M}$. In other words, if $\beta' \in \text{Mor}_\mathcal{T}(p, q)$ and $\beta'' \in \text{Mor}_\mathcal{T}(q, r)$ then $M(p, \beta', q) \circ M(q, \beta'', r) = M(p, \beta' \circ \beta'', r)$.

**Proof** — Define an operation of “composition” for intransitive graphs. Let $\gamma' \in G(p, \beta', q)$, $\gamma'' \in G(q, \beta'', r)$, the sequence $\beta'$ has $k$ elements, and $\beta''$ has $l$ elements. Join the vertex $p+k+i$ of the graph $\gamma'$ with the vertex $i$ of graph $\gamma''$ for $i = 1, 2, \ldots, q$. Delete all these vertices and renumber the remaining vertices by the numbers $1$ through $p+k+l+r$ (all vertices of $\gamma'$ are less than the vertices of $\gamma''$). As a result we get the graph $\gamma' \circ \gamma'' \in G(p, \beta' \circ \beta'', r)$. See an example on Figure 4.

Let $\{v_{\gamma}\}$, $\gamma' \in G(p, \beta', q)$ be the basis of $M(p, \beta', q)$ and $\{v_{\gamma''}\}$, $\gamma'' \in G(q, \beta'', r)$ be the basis of $M(q, \beta'', r)$. Then vectors $v_{\gamma'} \otimes v_{\gamma''}$ form a basis of $M(p, \beta', q) \otimes_{\mathbb{C}} M(q, \beta'', r)$. We must select $S_q$-invariants in this space. To do this we should symmetrize the space $M(p, \beta', q) \otimes_{\mathbb{C}} M(q, \beta'', r)$ by diagonal
Let Sym denote this symmetrization. Then we can identify 
Sym($v_{\gamma'} \otimes v_{\gamma''}$) with $v_{\gamma' \circ \gamma''}$. Hence vectors of the type $v_{\gamma' \circ \gamma''}$ generate the representation $M(p, \beta', q) \circ M(q, \beta'', r)$. On the other hand, it is clear that every element of $G(p, \beta' \circ \beta'', r)$ is of the form $\gamma' \circ \gamma''$ and vice versa.

Therefore, $M(p, \beta', q) \circ M(q, \beta'', r) \simeq M(p, \beta' \circ \beta'', r)$. \hfill \Box

Now we are able to prove Theorem 5.3. We will do it in two steps. First, we prove it in the case when the sequence $\beta$ consists of one number $\beta = (b)$. Then we prove it for arbitrary $\beta$.

1. Let $\beta = (-b)$ and $b \geq 0$ (the case when $b \leq 0$ is dual). Then $q = p + b$ and

\[
M(p, (-b), q) = \text{Ind}_{S_p \times S_{p+b}}^{S_{p+b}} \text{Reg}(S_p) \otimes \text{Id}_b,
\]

where Id$_b$ is the identity representation of $S_b$. Now

\[
M(p, (-b), q) = \text{Ind}_{S_p \times S_{p+b}}^{S_{p+b}} \sum_{\lambda \vdash p} \pi_{\lambda} \otimes \pi_{\lambda} \otimes \text{Id}_b
\]

\[= \sum_{\lambda \vdash p} \pi_{\lambda} \otimes \text{Ind}_{S_p \times S_{p+b}}^{S_{p+b}} \pi_{\lambda} =^* \sum_{\lambda \vdash p, \mu \vdash q} |OT(\lambda, (-b), \mu)| \cdot \pi_{\lambda} \otimes \pi_{\mu}.
\]

The first equality is true by Proposition 6.1(2) and the fact that for the symmetric group we have $\pi_{\lambda}^* = \pi_{\lambda}$. The equality $(\ast)$ uses the Pieri rule:

\[
\text{Ind}_{S_p \times S_{p+b}}^{S_{p+b}} \pi_{\lambda} = \sum_{\mu} \pi_{\mu},
\]

where the sum is over all $\mu$ such that $\mu/\lambda$ is a horizontal $b$-stripe, see [8].

2. Let $\beta = (\beta_1, \ldots, \beta_k)$ be a sequence of integers and $p_i = p - \sum_{j=1}^{i-1} \beta_j$, $q = p_k$. Then

\[
M(p, \beta, q) =^{(1)} M(p_0, (\beta_1), p_1) \circ \ldots \circ M(p_{k-1}, (\beta_k), p_k)
\]

\[=^{(2)} \left( \sum_{\lambda \vdash p, \mu \vdash q} \pi_{\lambda(1)} \otimes \pi_{\mu(1)} \right) \circ \ldots \circ \left( \sum_{\lambda \vdash p, \mu \vdash q} \pi_{\lambda(k)} \otimes \pi_{\mu(k)} \right)
\]

\[=^{(3)} \sum_{\lambda \vdash p, \mu \vdash q} |OT(\lambda, \mu, \beta)| \cdot \pi_{\lambda} \otimes \pi_{\mu},
\]

where in the second line the direct sums are over $\lambda_{(i)} \vdash p_{i-1}$ and $\mu_{(i)} \vdash p_i$ such that $\lambda_{(i)}/\mu_{(i)}$ is a horizontal $\beta_i$-stripe (if $\beta_i \geq 0$) or $\mu_{(i)}/\lambda_{(i)}$ is a horizontal $(-\beta_i)$-stripe (if $\beta_i \leq 0$) for all $i = 1, 2, \ldots, k$.

Equality (1) follows from Theorem 6.2; (2) follows from p. 1; (3) follows from Proposition 6.1(1) and definition of oscillating tableaux. \hfill \Box
7 Combinatorial theorem

In this section we give a combinatorial analogue of Theorem 5.3.

A sequence $\tau = (\tau_1, \tau_2, \ldots, \tau_k) \in \mathbb{Z}^k$ is called normal if there exist $0 \leq r \leq l \leq k$ such that $\tau_1, \tau_2, \ldots, \tau_r > 0$; $\tau_{r+1} = \cdots = \tau_l = 0$; $\tau_{l+1}, \ldots, \tau_k < 0$.

For a sequence $\beta \in \mathbb{Z}^k$, let $\text{nor}(\beta)$ denote the normal sequence obtained from $\beta$ by shuffling all positive entries of $\beta$ into the beginning and all negative entries into the end. For example, $\text{nor}(0, -3, 1, -1, 0, -2, 0, 1, 3) = (1, 1, 3, 0, 0, 0, -3, -1, -2)$.

For $\beta, \delta \in \mathbb{Z}^k$ the expression $\delta \prec \beta$ means that for all $i = 1, 2, \ldots, k$ either $0 \leq \delta_i \leq \beta_i$ or $0 \geq \delta_i \geq \beta_i$.

Now we can state the combinatorial theorem.

**Theorem 7.1** Let $\lambda, \mu \in \mathcal{P}$ be some partitions, $\beta \in \mathbb{Z}^k$. Then

$$|OT(\lambda, \mu, \beta)| = \sum |G(\delta)| \cdot |OT(\lambda, \mu, \text{nor}(\beta - \delta))|,$$

where the sum is over all $\delta \in \mathbb{Z}^k$ such that $s(\delta) = 0$ and $\delta \prec \beta$.

In order to deduce Theorem 7.1 from Theorem 5.3 we need one simple lemma.

**Lemma 7.2** Let $p, q \in \mathbb{N}$, $\beta \in \mathbb{Z}^k$ be such that $p - s(\beta) = q$. Then

$$M(p, \beta, q) = \sum |G(\delta)| \cdot M(p, \text{nor}(\beta - \delta), q),$$

where the direct sum is over all $\delta \in \mathbb{Z}^k$ such that $s(\delta) = 0$ and $\delta \prec \beta$.

**Proof** — Let $\xi \in G(\delta)$, where $\delta = (\delta_1, \ldots, \delta_k) \in \mathbb{Z}^k$, $s(\delta) = 0$. Let $G(p, \delta, q)_\xi$ be the set of graphs from $G(p, \delta, q)$ whose restriction on the vertices $p+1, p+2, \ldots, p+k$ is the graph $\xi$. If $G(p, \beta, q)_\xi$ is nonempty then $\delta \prec \beta$.

It is clear that when $\delta \prec \beta$ and $\xi \in G(\delta)$ the submodule in $M(p, \beta, q)$ generated by $\{v, \gamma \in G(p, \beta, q)_\xi\}$ is equivalent to $M(p, \text{nor}(\beta - \delta), q)$. \qed

Now Theorem 7.1 immediately follows from Theorem 5.3 and Lemma 7.2.

We will give a combinatorial proof of Theorem 7.1. In Section ?? we will construct a bijection $\Phi_{\lambda, \mu, \beta}$ which establishes a one-to-one correspondence between the following two sets.

$$\Phi_{\lambda, \mu, \beta} : OT(\lambda, \mu, \beta) \rightarrow \prod G(\delta) \times OT(\lambda, \mu, \text{nor}(\beta - \delta)).$$
Let $\lambda = \mu = \hat{0}$. Then there is a unique oscillating tableau of shape $(\hat{0}, \hat{0})$ of normal weight. Namely, $(\hat{0}, \hat{0}, \ldots, \hat{0}) \in OT(\hat{0}, \hat{0}, (0, 0, \ldots, 0))$. We have $\delta = \beta$ in Theorem 7.1. Hence Theorem 4.3 is a special case of Theorem 7.1.

8 Superanalogue

In this section we give superanalogues of definitions and theorems from Sections 4-7.

Definition 8.1 Let $\lambda, \mu$ be partitions, $\beta \in \mathbb{Z}^k$, $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k) \in \{1, -1\}^k$. An oscillating supertableau of shape $(\lambda, \mu)$ and superweight $b = \beta^\varepsilon$ (see Section 2) is a sequence of partitions $(\alpha(0) = \lambda, \alpha(1), \alpha(2), \ldots, \alpha(k) = \mu)$ such that for all $i = 1, 2, \ldots, k$ the following conditions hold.

1. If $\varepsilon_i = 1$ and $\beta_i \geq 0$ then $\alpha(i-1) \supseteq \alpha(i)$ and $\alpha(i)/\alpha(i)$ is a horizontal $\beta_i$-stripe;

2. If $\varepsilon_i = 1$ and $\beta_i < 0$ then $\alpha(i) \supseteq \alpha(i-1)$ and $\alpha(i)/\alpha(i-1)$ is a horizontal $(-\beta_i)$-stripe;

3. If $\varepsilon_i = -1$ and $\beta_i \geq 0$ then $\alpha(i-1) \supseteq \alpha(i)$ and $\alpha(i-1)/\alpha(i)$ is a vertical $\beta_i$-stripe;

4. If $\varepsilon_i = -1$ and $\beta_i < 0$ then $\alpha(i) \supseteq \alpha(i-1)$ and $\alpha(i)/\alpha(i-1)$ is a vertical $(-\beta_i)$-stripe.

The set of all oscillating supertableaux of shape $(\lambda, \mu)$ and superweight $b = \beta^\varepsilon$ is denoted by $OST(\lambda, \mu, b)$.

It is clear that $OST(\lambda, \mu, b)$ is nonempty only when $|\lambda| - s(\beta) = |\mu|$. If all $\beta_i \geq 0$ then $OST(\lambda, \mu, \beta^\varepsilon) = ST(\lambda/\mu, \beta^\varepsilon)$. And $OST(\lambda, \mu, \beta^{(1, 1, \ldots, 1)}) = OT(\lambda, \mu, \beta)$.

Definition 8.2 Let $\delta \in \mathbb{Z}^k$ be such that $s(\delta) = 0$ and $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k) \in \{1, -1\}^k$. An intransitive graph of supertype $d = \delta^\varepsilon$ is an oriented graph $\gamma$ on the set of vertices $\{1, 2, \ldots, k\}$ satisfying the conditions 1–3 of Definition 4.1 and also the condition:

4. If $\varepsilon_i \neq \varepsilon_j$ then $\gamma$ contains at most one edge $(i, j)$.

Let $SG(\delta^\varepsilon)$ be the set of all such graphs.
The following algebra $\mathcal{A}(\epsilon)$ is closely related to Definition 8.2.

**Definition 8.3** Let $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_k) \in \{1, -1\}^k$. The algebra $\mathcal{A}(\epsilon)$ generated by variables $x_{ij}, 1 \leq i < j \leq k$ with the following relations.

1. $x_{ij} x_{jr} = 0$ for any $1 \leq i < j < r \leq k$,
2. $x_{ij} x_{lm} = (-1)^{\sigma_{ij} \sigma_{im}} x_{lm} x_{ij}$, where

   $$\sigma_{ij} = \begin{cases} 0 & \epsilon_i = \epsilon_j, \\ 1 & \epsilon_i \neq \epsilon_j. \end{cases}$$

Relation 2 implies that $x_{ij}$ with $\sigma_{ij} = 0$ are commutative variables and $x_{lm}$ with $\sigma_{lm} = 1$ are anticommutative variables.

For any oriented graph $\gamma$ on the set of vertices $\{1, 2, \ldots, k\}$ we can construct (up to a sign) a monomial $m_\gamma$ in the algebra $\mathcal{A}(\epsilon)$:

$$m_\gamma = \pm \prod x_{ij},$$

where the product is over all edges $(i, j)$ of graph $\gamma$.

Nonzero monomials in $\mathcal{A}(\epsilon)$ correspond to intransitive graphs of type $\beta^\epsilon$ with fixed $\epsilon$ and arbitrary $\beta$. Indeed, condition 4.1(2) corresponds to condition 8.3(1) and 8.2(4) corresponds to the fact that $x_{lm}^2 = 0$ for an anticommutative variable $x_{lm}$ with $\sigma_{lm} = 1$.

Let $\mathcal{A}_\delta(\epsilon)$ denote the subspace of $\mathcal{A}(\epsilon)$ which is generated (as a linear space) by monomials $m_\gamma$ for $\gamma \in \Cal{SG}(\delta^\epsilon)$. It is clear that $\mathcal{A}(\epsilon) = \bigoplus_{\delta} \mathcal{A}_\delta(\epsilon)$.

Let $p, q \in \mathbb{N}$, $\beta = (\beta_1, \ldots, \beta_k)$, $\epsilon = (\epsilon_1, \ldots, \epsilon_k) \in \{1, -1\}^k$, and $\psi, \omega \in \{1, -1\}$. Suppose that

$$\delta = (\underbrace{-1, -1, \ldots, -1}_{p \text{ times}}, \beta_1, \beta_2, \ldots, \beta_k, \underbrace{1, 1, \ldots, 1}_{q \text{ times}});$$

$$\epsilon = (\underbrace{\psi, \psi, \ldots, \psi}_{p \text{ times}}, \underbrace{\epsilon_1, \epsilon_2, \ldots, \epsilon_k, \omega, \omega, \ldots, \omega}_{q \text{ times}}).$$

Let $\Cal{SG}(p, \beta^\epsilon, q)$ be the set of intransitive graphs of supertype $\delta^\epsilon$. Denote by $M(p, \beta^\epsilon, q)$ the subspace $\mathcal{A}_\delta(\epsilon)$, where $p = p^\psi$ and $q = q^\omega$. Then $\{m_\gamma : \gamma \in \Cal{SG}(p, \beta^\epsilon, q)\}$ is a basis of the space $M(p, \beta^\epsilon, q)$.

The group $S_p \times S_q$ acts on this space, cf. Section 5. The symmetric group $S_p$ permutes the first index of variables $x_{ij}$ with $i = 1, 2, \ldots, p$ and $S_q$ permutes the second index of variables $x_{ij}$ with $j = p+k+1, \ldots, p+k+q$. 


The following example gives an odd analogue of the regular representation of $S_p$ (see Example 5.1).

**Example 8.4** Let $\beta^e = \emptyset$ be the empty sequence, $p = p$ and and $q = \overline{p}$, $p \in \mathbb{N}$. Then $M(p, \emptyset, \overline{p})$ is the representation of $S_p \times S_p$ on the group algebra $\mathbb{C}[S_p]$ given by the formula

$$(\sigma, \pi) \cdot f = \text{sgn}(\sigma^{-1}) \sigma f \pi^{-1},$$

where $(\sigma, \pi) \in S_p \times S_p$, $f \in \mathbb{C}[S_p]$ and $\text{sgn}$ denotes the sign of permutation. Denote this representation by $\text{Alt}_p$.

We use the following notation. For a partition $\lambda \in \mathcal{P}$ and $\psi \in \{1, -1\}$, $\lambda^\psi = \lambda$ if $\psi = 1$ and $\lambda^\psi = \lambda'$ (the conjugate partition) if $\psi = -1$.

Now we can present a superanalogue of Theorem 5.3.

**Theorem 8.5**

$$M(p^\psi, \beta^e, q^\omega) \simeq \sum |\text{OST}(\lambda^\psi, \mu^\omega, \beta^e)| \cdot \pi_{\lambda} \otimes \pi_{\mu},$$

where the sum is over all partitions $\lambda \vdash p$ and $\mu \vdash q$.

For $p, q, \beta^e$ such as in Example 8.4 we have by Theorem 8.5

**Corollary 8.6**

$$\text{Alt}_p = \sum_{\lambda \vdash p} \pi_{\lambda} \otimes \pi_{\lambda'}.$$

This is an odd analogue of Corollary 5.4. Of course this formula easily follows from definition of $\text{Alt}_p$.

**Sketch of proof of Theorem 8.5** — The proof is analogous to the proof of Theorem 5.3. The only difference is the definition of “composition” for intransitive graphs. If we define the composition as in Section 6 then it may happen that the composition of two graphs $\gamma' \in SG(p, b', q)$ and $\gamma'' \in SG(q, b'', r)$ is not a graph from $SG(p, b' \circ b'', r)$. We define “supercomposition” $\gamma' \circ^8 \gamma''$ of graphs $\gamma'$ and $\gamma''$ by

$$\gamma' \circ^8 \gamma'' = \begin{cases} \gamma' \circ \gamma'' & \text{if } \gamma' \circ \gamma'' \in SG(p, b' \circ b'', r), \\ 0 & \text{otherwise}. \end{cases}$$
This convention is consistent with interpretation of composition in terms of symmetrization. Indeed, if $\gamma' \circ \gamma''$ is not in $SG(p, b \circ b', r)$ then $\text{Sym}(m(\gamma') \otimes m(\gamma'')) = 0$. 

Now we give a superanalogue of Theorem 7.1. Let $b = (b_1, b_2, \ldots, b_k) = \beta^z$ (see Section 2). Let nor$(b)$ denote the word obtained from the word $b = (b_1, b_2, \ldots, b_k)$ by shuffling all negative entries into the beginning and all positive entries into the end. For example, nor$(0, 3, -1, 1, 0, -1, -3) = (-1, -1, 3, 0, 0, 0, 3, 1, 2)$.

**Theorem 8.7** Let $\lambda, \mu \in \mathcal{P}$ be some partitions, $\beta \in \mathbb{Z}^k$, $\varepsilon \in \{1, -1\}^k$. Then

$$|\text{OST}(\lambda, \mu, \beta^z)| = \sum |SG(\delta^z)| \cdot |\text{OST}(\lambda, \mu, \text{nor}((\beta - \delta)^z))|,$$

where the summation is over all $\delta \in \mathbb{Z}^k$ such that $s(\delta) = 0$ and $\delta \prec \beta$.

This theorem can be deduced from Theorem 8.5 in the same way as Theorem 7.1 from Theorem 5.3.

In Section 4 we will construct a bijection

$$\Phi_{\lambda, \beta}^{\text{super}} : \text{OST}(\lambda, \mu, \beta^z) \rightarrow \prod_{\delta \prec \beta} SG(\delta^z) \times \text{OST}(\lambda, \mu, \text{nor}((\beta - \delta)^z)).$$

This will give a combinatorial proof of Theorem 8.5.

If $\lambda = \mu = \hat{0}$ then Theorem 8.7 implies the following

**Corollary 8.8** Let $\beta \in \mathbb{Z}^k$ be such that $s(\beta) = 0$, $\varepsilon \in \{1, -1\}^k$. Then the number of oscillating tableaux of shape $(\hat{0}, 0)$ and superweight $b = \beta^z$ is equal to the number of intransitive graphs of supertype $b$

$$|\text{OST}(\hat{0}, \hat{0}, b)| = |G(b)|.$$
Corollary 8.9 Let $\beta' \in \mathbb{N}^s$ and $\beta'' \in \mathbb{N}^t$. Then the number of pairs of tableaux $(P, Q)$ with conjugated shapes and with weights $\beta'$ and $\beta''$ respectively is equal to the number of $s \times t$-matrices satisfying the conditions 1–3 of Corollary 4.5 with all entries equal to 0 or 1.

Knuth in [9] construct also an odd analogue of RSK-correspondence which is a bijection between the set of such $s \times t$-matrices and the set of such pairs of tableaux $(P, Q)$. In this special case the bijection $\Phi^{\text{super}}_{\mathfrak{N}_{\mu \nu}}$ coincides with Knuth’s correspondence.

9 Increasing and decreasing operators

First we give another description of the category $\mathcal{M}$ from Section 6.

Let $G$ be a finite group. By $\text{Rep}(G)$ denote the set of equivalence classes of complex finite dimensional representations of $G$. It is clear that $\text{Rep}(G) = \text{Mor}_\mathcal{M}(\{\text{id}\}, G)$ (see Section 6), where $\{\text{id}\}$ denote the group with one element $\text{id}$.

Let $W \in \text{Mor}_\mathcal{M}(G, H)$. Consider the $\mathbb{N}$-linear map $\langle W \rangle$ from $\text{Rep}(G)$ to $\text{Rep}(H)$ which is defined by $\langle W \rangle V = V \circ W$, where $V \in \text{Rep}(G) = \text{Mor}_\mathcal{M}(\{\text{id}\}, G)$. On the other hand, if we know a map $\langle W \rangle$ then we can reconstruct the morphism $W$ in $\mathcal{M}$.

By $R$ denote the direct sum $R = \text{Rep}(S_0) \oplus \text{Rep}(S_1) \oplus \text{Rep}(S_2) \oplus \ldots$

Let $\langle M(p, b, q) \rangle$ be the operator from $\text{Rep}(S_p)$ to $\text{Rep}(S_q)$ which corresponds to $S_p \times S_q$-module $M(p, b, q)$. Recall that $b = \beta^*$ is a sequence in the alphabet $\{m, \overline{m} \mid m \in \mathbb{Z}\}$. Let $\langle b \rangle$ be the endomorphism of $R$ such that $\langle b \rangle = \sum \langle M(p, b, q) \rangle$, where the sum is over $p - s(\beta) = q$. In the case when the sequence $b$ has only one element $m$ or $\overline{m}$, $m \in \mathbb{Z}$, we denote these operators by $\langle m \rangle$ or $\langle \overline{m} \rangle$. It is clear from Section 8 that $\langle (b_1, b_2, \ldots, b_k) \rangle = \langle b_1 \rangle \cdot \langle b_2 \rangle \cdot \ldots \cdot \langle b_k \rangle$.

If $n \in \mathbb{N}$ then we call operators $\langle n \rangle$ and $\langle \overline{n} \rangle$ increasing and denote them by $I(n)$ or $I(\overline{n})$. If $-n \in \mathbb{N}$ then we call operators $\langle n \rangle$ and $\langle \overline{n} \rangle$ decreasing and denote them $D(n)$ or $D(\overline{n})$. The following description of operators $I(n)$, $I(\overline{n})$, $D(n)$, and $D(\overline{n})$ follows from Sections 6 and 8.

Let $V \in \text{Rep}(S_p)$. Then

$$I(n) \cdot V = \text{Ind}_{S_p}^{S_{p+n}} V;$$

$$I(\overline{n}) \cdot V = \text{Ind}_{S_p \times S_n}^{S_{p+n}} (V \otimes \text{sgn}_n),$$

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where \( \text{sgn}_n \) is the sign representation of \( S_n \).

Let \( V \in \text{Rep}(S_{p+n}) \). Then

\[
D(n) \cdot V = \text{Inv}_n(\text{Res}_{S_p \times S_n} V);
\]

\[
D(\pi) \cdot V = \text{Skew}_n(\text{Res}_{S_p \times S_n} V),
\]

where \( \text{Inv}_n \) is the space of \( S_n \)-invariants and \( \text{Skew}_n \) is the space of skew invariants of \( S_n \).

The space \( R \) has the basis \( \{ \pi_\lambda \mid \lambda \in \mathcal{P} \} \) consisting of all irreducible representations of all symmetric groups. Therefore a linear operator on the space \( R \) can be represented as an infinite matrix indexed by partitions.

All increasing and decreasing operators in coordinates are given below.

\[
I(n)_{\lambda\mu} = \begin{cases} 1 & \text{if } \lambda \supset \mu \text{ and } \lambda/\mu \text{ is a horizontal } n\text{-stripe}, \\ 0 & \text{otherwise}, \end{cases}
\]

\[
D(n)_{\lambda\mu} = \begin{cases} 1 & \text{if } \mu \supset \lambda \text{ and } \mu/\lambda \text{ is a horizontal } n\text{-stripe}, \\ 0 & \text{otherwise}, \end{cases}
\]

\[
I(\pi)_{\lambda\mu} = \begin{cases} 1 & \text{if } \lambda \supset \mu \text{ and } \lambda/\mu \text{ is a vertical } n\text{-stripe}, \\ 0 & \text{otherwise}, \end{cases}
\]

\[
D(\pi)_{\lambda\mu} = \begin{cases} 1 & \text{if } \mu \supset \lambda \text{ and } \mu/\lambda \text{ is a vertical } n\text{-stripe}, \\ 0 & \text{otherwise}. \end{cases}
\]

It is clear that \( \langle b \rangle_{\lambda\mu} = (\langle b_1 \rangle \cdot \langle b_2 \rangle \cdot \ldots \cdot \langle b_k \rangle)_{\lambda\mu} = |OST(\lambda, b, \mu)|. \)

All increasing operators commute and all decreasing operators commute. But increasing and decreasing operators do not commute with each other. The following theorem gives the relations between these operators. Here \( [a, b] = ab - ba \) denotes the commutator of operators.

**Theorem 9.1** Let \( m, n \in \mathbb{N} \). The following relations hold.

1. \( [I(m), I(n)] = [I(\pi), I(\pi)] = [D(m), D(n)] = [D(\pi), D(\pi)] = 0. \)
2. \( [I(m), I(\pi)] = [D(m), D(\pi)] = 0. \)
3. \( [I(m+1), D(n+1)] = I(m)D(n), \ [I(m+1), D(n+1)] = I(\pi)D(\pi). \)
4. \( [I(m+1), D(n+1)] = D(\pi)I(m), \ [I(m+1), D(n+1)] = D(n)I(\pi). \)

In the following section we give a combinatorial proof of Theorem 9.1.
10 Local bijections

Let \( m, n \in \mathbb{N} \). In this section we construct the following four bijections:

1. \( \psi_1 : YT(\lambda/\nu, (m, n)) \to YT(\lambda/\nu, (n, m)) \),
2. \( \psi_2 : ST(\lambda/\mu, (m, \bar{n})) \to ST(\lambda/\nu, (\bar{n}, m)) \),
3. \( \psi_3 : OT(\lambda, \nu, (-m, n)) \to \prod_{0 \leq k \leq \min(m, n)} OT(\lambda, \nu, (n-k, -m+k)) \),
4. \( \psi_4 : OST(\lambda, \nu, (-m, \bar{n})) \to \prod_{0 \leq k \leq \min[1, m, n]} OST(\lambda, \nu, (\bar{n}-k, -m+k)) \).

It is clear that these bijections are sufficient to prove Theorem 9.1. Later we will use bijections \( \psi_3 \) and \( \psi_4 \) in combinatorial proofs of Theorems 7.1 and 8.7.

In all examples, when displaying an (oscillating) (super)tableau \( \sigma = (\lambda, \mu, \nu) \), we insert 2's into the boxes of the skew diagram \( \lambda/\mu \) (or \( \mu/\lambda \)) and 1's into the boxes of \( \mu/\nu \) (or \( \nu/\mu \)). The symbol \( 1/2 \) in a box means that we insert simultaneously integers 1 and 2 into this box.

We say that a skew diagram \( \lambda/\mu \) falls into a disjoint union of skew diagrams \( \tau_1, \tau_2, \ldots, \tau_l \) if \( \lambda/\mu = \bigcup_i \tau_i \) and for all \( 1 \leq i < j \leq l \) any box of \( \tau_j \) is below and to the left of any box of \( \tau_i \). For example, the skew diagram on Figure 1 falls into a disjoint union of three diagrams. We also say that a (super)tableau of shape \( \lambda/\mu \) falls into a disjoint union of so does the shape \( \lambda/\mu \).

Constructions:

1. Let \( \alpha = (\lambda, \mu, \nu) \in YT(\lambda/\mu, (m, n)), \lambda = (\lambda_1, \lambda_2, \ldots), \mu = (\mu_1, \mu_2, \ldots), \) and \( \nu = (\nu_1, \nu_2, \ldots). \) Then we have \( \lambda_i \geq \mu_i \geq \lambda_{i+1}, i = 1, 2 \ldots \); and \( \mu_i \geq \nu_i \geq \mu_{i+1}, i = 1, 2 \ldots. \) Set by convention \( \nu_0 = \infty. \) On the following diagram arrow \( x \to y \) denotes the inequality \( x \geq y. \)

\[
\begin{array}{ccccccc}
\lambda_1 & \lambda_2 & \lambda_3 & \ldots & \lambda_i & \lambda_{i+1} & \ldots \\
\downarrow & \downarrow & \downarrow & \ldots & \downarrow & \uparrow \\
\mu_1 & \mu_2 & \ldots & \mu_i & \ldots \\
\uparrow & \downarrow & \ldots & \uparrow & \downarrow \\
\infty & \nu_1 & \nu_2 & \ldots & \nu_{i-1} & \nu_i & \ldots
\end{array}
\]

Let \( a_i = \min(\lambda_i, \nu_{i-1}) \) and \( b_i = \max(\lambda_{i+1}, \nu_i), i = 1, 2 \ldots. \) Then \( a_i \geq \mu_i \geq b_i. \) Set \( \tilde{\mu}_i = a_i + b_i - \mu_i, i = 1, 2 \ldots, \) i.e., \( \tilde{\mu}_i \) is symmetric to \( \mu_i \) in the interval \( (b_i, a_i). \)
Let $\psi_1 : \alpha \mapsto \widetilde{\alpha}$. It is easy to see that $\psi_1$ is a bijection between the sets $YT(\lambda/\mu, (m,n))$ and $YT(\lambda/\mu, (n,m))$. Figure 5 shows an example of the bijection $\psi_1$.

2. Let $\alpha = (\lambda, \mu, \nu) \in ST(\lambda/\mu, (m, \overline{n}))$ ...

3. Let $\alpha = (\lambda, \mu, \nu) \in OT(\lambda, \mu, (-m, n)$, $\lambda = (\lambda_1, \lambda_2, \ldots)$, $\mu = (\mu_1, \mu_2, \ldots)$, and $\nu = (\nu_1, \nu_2, \ldots)$. Then we have $\mu_i \geq \lambda_i \geq \mu_{i+1}$, $\mu_i \geq \nu_i \geq \mu_{i+1}$, $i = 1, 2, \ldots$; $|\mu| - |\lambda| = m$, and $|\mu| - |\nu| = n$.

4. Let $\alpha = (\lambda, \mu, \nu) \in OST(\lambda, \nu, (-m, \overline{n}))$ ...

11 Generalized Gelfand-Tsetlin patterns

Let $\alpha = (\alpha_{(0)}, \alpha_{(1)}, \ldots, \alpha_{(k)}) \in OT(\lambda, \mu, \beta)$ be an oscillating tableau of weight $\beta = (\beta_1, \beta_2, \ldots, \beta_k)$. Let $w = w_1 w_2 \ldots w_k$ be a word in the alphabet $\{+, -\}$
such that if $\beta_i$ is positive (negative) then $w_i = + (w_i = -)$, $i = 1, 2, \ldots, k$. Let $\rho(i)$ be the number of $+$’s in the word $w_1w_2\ldots w_i$, $i = 1, 2, \ldots, k$.

The *generalized Gelfand-Tsetlin pattern* $P$ of type $w$ corresponding to the oscillating tableau $\alpha$ is the two-dimensional array $P = \{p_{ij}\}$, where $i = 1, 2, \ldots, k$, $j \geq \rho(i)$, and $p_{ij} = \alpha_{(i)j-\rho(i)}$. For example, a generalized Gelfand-Tsetlin pattern of type $w = ++-\ldots$ is an array of the following form (as above $x \to y$ means $x \geq y$).

$$
\begin{array}{cccccc}
\alpha_{(0)1} & \alpha_{(0)2} & \alpha_{(0)3} & \alpha_{(0)4} & \cdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \cdots \\
\alpha_{(1)1} & \alpha_{(1)2} & \alpha_{(1)3} & \cdots \\
\downarrow & \downarrow & \downarrow & \cdots \\
\alpha_{(2)1} & \alpha_{(2)2} & \alpha_{(2)3} & \cdots \\
\downarrow & \downarrow & \downarrow & \cdots \\
\alpha_{(3)1} & \alpha_{(3)2} & \alpha_{(3)3} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{array}
$$

Note that standard Gelfand-Tsetlin patterns have type $w = +++++\ldots$ in our terminology.

We can present a generalized Gelfand-Tsetlin pattern $P$ (and the corresponding oscillating tableau) in more convinient form as a plane partition with cutted off corners. For example, Figure 7 presents the oscillating tableau

$$
((211), (3211), (221), (211), (421), (321)).
$$
References


