On the Odd Area of Planar Sets

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Abstract

The main result in the paper is a construction of a simple (in fact, just a union of two squares) set T in the plane with the following property. For every $\epsilon > 0$ there is a family $\mathcal{F}$ of an odd number of translates of T such that the area of those points in the plane that belong to an odd number of sets in $\mathcal{F}$ is smaller than ϵ.

1 Introduction

The following puzzle appeared in the fall's contest of 'the tournament of towns' for the year 2009 [4]:

On an infinite chessboard are placed 2009 $n \times n$ *cardboard pieces such that each of them covers exactly* n^2 *cells of the chessboard. Prove that the number of cells of the chessboard which are covered by odd numbers of cardboard pieces is at least* n^2.

As for the history of the problem, this puzzle was originated by Uri Rabinovich and communicated to Igor Pak who communicated it to Arseniy V. Akopyan, and then it found its way to the organizers of the Tournament of Towns in Moscow.

Perhaps the shortest solution to this puzzle is an elegant use of the coloring technique: Color the grid square (a, b) by the color $((a \mod n), (b \mod n))$. We thus have n^2 different colors and the crucial observation is that each big square of size $n \times n$ contains precisely one grid square of each color. It follows that there must be at least one grid square of each color class that is covered an odd number of times (because the total number of grid squares from each color is 2009 with multiple counting).

It is an immediate consequence of this puzzles that also the continuous version of this problem is true. That is, given an odd number of axis-parallel unit squares in the plane the total area of all points covered by an odd number of squares is at least 1. Equality is possible of course, for example if all squares coincide.

For two sets X and Y we denote by $X \oplus Y$ the set of points covered an odd number of times by X and Y. In other words, this is just a different notation for the symmetric difference of X

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and Y. However, we extend this notation for more than two sets. Therefore, for sets $X_1, \ldots, X_t$ we will denote by $X_1 \oplus \ldots \oplus X_t$ (or sometimes by $\bigoplus_{i=1}^{t} X_i$) the set of all points that belong to an odd number of the sets $X_1, \ldots, X_t$. Notice that we have for any three sets X, Y, Z: $(X \oplus Y) \oplus Z = X \oplus (Y \oplus Z)$. For a set X and a vector v we denote by $X + v$ the Minkowsky sum of X and v, that is, $X + v = \{x + v \mid x \in X\}$.

Definition 1. For a measurable set S we denote by $A(S)$ the measure of S. By a *shape* we refer to any measurable compact set T in the plane (or, depending on the context, in $\mathbb{R}^d$) with $A(T) > 0$. For a shape T we define $OA(T)$, that we call the *odd area* of T, as follows:

$$OA(T) = \inf\{A(\bigoplus_{i=1}^{2k-1}(T + v_i)) \mid k \in \mathbb{N}, \quad v_1, \ldots, v_{2k-1} \in \mathbb{R}^2\}.$$

In other words, the odd area of T is the maximum number $OA(T)$ such that for any collection $\mathcal{F}$ of a finite and odd number of translates of T, the set of all points in the plane that belong to an odd number of members from $\mathcal{F}$ has area of at least $OA(T)$.

Clearly, $OA(T) \leq A(T)$ for any shape T, as we can place all the translated copies of T one on the top of the other, or even just consider a single copy of T. Consequently, $OA(T)/A(T) \in [0, 1]$ for any T. Notice that the definition of the odd area is valid in any dimension, although in this paper we will focus more on the plane $\mathbb{R}^2$ and on the line $\mathbb{R}$.

In [2], the exact value of $OA(T)$ is determined when T is a triangle or a trapezoid in the plane. It is shown there that $OA(T) = \frac{1}{2}A(T)$ when T is a triangle and $OA(T) = \frac{1}{4}(b - a)h$ when T is a trapezoid with parallel sides of lengths a and b (and it is important that $a \neq b$, as for $a = b$ we have $OA(T) = A(T)$) whose height is equal to h. Notice in each of these cases $OA(T)$ is strictly positive. In this paper we consider related questions and develop further techniques for handling those problems. Not less importantly, we raise some new and intriguing problems waiting to be solved.

The main result in the current paper is a construction of a (simple) shape T with $OA(T) = 0$. To this end we will develop new tools for studying the function $OA(\cdot)$.

Definition 2. Let T be a shape. We define $m_T(n)$ to be the minimum cardinality of a finite family $\mathcal{F}$ of translated copies of T such that $\bigoplus_{F \in \mathcal{F}} F$ contains almost (that is, up to a set of measure zero) every point in a given $n \times n$ square. If such a finite family $\mathcal{F}$ does not exist we set $m_T(n) = \infty$. We define $\beta(T) = \limsup \frac{n^2}{m_T(n)}$.

Observe that if one can tile the plane with a shape T, then $m_T(n) = \frac{n^2}{A(T)} + o(n^2)$ and consequently $\beta(T) = A(T)$.

The following lemma will be useful in bounding $OA(T)$ from below.

Lemma 1. *For every shape T we have $OA(T) \geq \beta(T)$.*

Proof. If $\beta(T) = 0$, there is nothing to prove. Therefore, we can assume $\beta(T) > 0$ and $m_n(T) < \infty$ for every n. Let d denote the diameter of T, that is the distance between the two furthest points in T. Let $\epsilon > 0$ be given. Find odd number k and k vectors $v_1, \ldots, v_k \in \mathbb{R}^2$ such that the area of $D = \bigoplus_{i=1}^{k}(T + v_i)$ is at most $OA(T) + \epsilon$. Find n large enough such that $kA(T)d/n < \epsilon$ and such that $\frac{n^2}{m_T(n)} > \beta - \epsilon$. Set $m = m_T(n)$. Let $u_1, \ldots, u_m \in \mathbb{R}^2$ be vectors such that $C = \bigoplus_{i=1}^{m}(T + u_i)$

contains almost every point of an $n \times n$ square B. Notice that $C \setminus B$ is a set of area at most $4d(n+d)$, as each of $T+u_1, \ldots, T+u_m$ has a nonempty intersection with B.

Consider now the set $E = \bigoplus_{i=1}^{m}(D+u_i)$. Clearly, the set E has area at most $m(OA(T)+\epsilon)$. We claim that E is exactly the set $\bigoplus_{i=1}^{k}(C+v_i)$. Indeed,

$$\begin{aligned} E &= \bigoplus_{i=1}^{m}(D+u_i) = \bigoplus_{i=1}^{m}((\bigoplus_{j=1}^{k} T+v_j)+u_i) = \bigoplus_{i=1}^{m}\bigoplus_{j=1}^{k}(T+v_j+u_i) = \bigoplus_{j=1}^{k}\bigoplus_{i=1}^{m}(T+v_j+u_i) \\ &= \bigoplus_{j=1}^{k}((\bigoplus_{i=1}^{m}(T+u_i))+v_j) = \bigoplus_{j=1}^{k}(C+v_j). \end{aligned} \tag{1}$$

We notice that

$$E = \bigoplus_{j=1}^{k}(C+v_j) = \bigoplus_{j=1}^{k}((B \oplus (C \setminus B))+v_j) = \bigoplus_{j=1}^{k}(B+v_j) \oplus \bigoplus_{j=1}^{k}((C \setminus B))+v_j).$$

The area of $\bigoplus_{j=1}^{k}(B+v_j)$ is at least the area of B (i.e., n^2) because B is a box and we have $OA(B) = A(B)$. The area of $\bigoplus_{j=1}^{k}((C \setminus B))+v_j)$ is bounded from above by the sum of the areas of $(C \setminus B)+v_j$, that is, by $k4d(n+d)$. Hence the area of E is at least $n^2 - 4kd(n+d)$.

Combining the lower and upper bound on the area of E we get

$$(OA(T)+\epsilon)m \geq A(E) \geq n^2 - 4kd(n+d).$$

Recall that $n^2/m > \beta(T) - \epsilon$ and observe that $m = m(n) \geq \frac{n^2}{A(T)}$. Hence,

$$OA(T)+\epsilon \geq n^2/m - 4kd(n+d)/m \geq \beta(T)-\epsilon-4kd(n+d)A(T)/n^2 \geq \beta(T)-\epsilon-8kdA(T)/n \geq \beta(T)-9\epsilon.$$

Since this is true for every $\epsilon > 0$ we deduce that $OA(T) \geq \beta(T)$. ∎

As an immediate consequence we get the following theorem:

Theorem 1. *Suppose $\mathbb{R}^2$ can be tiled with translates of a shape T, then $OA(T) = A(T)$.*

Indeed, if we can tile the plane with translated copies of T, then $m_T(n) \leq (n^2+4nd)/A(T)$, where d is the diameter of T. It follows that $\beta(T) = \limsup n^2/m_T(n) \geq A(T)$ (the last inequality is in fact an equality) and consequently, by Lemma 1, $OA(T) = A(T)$.

We will now bring another application of Lemma 1. This application is a stand-alone result that will not be used later in this paper.

Definition 3. A *grid animal* is a shape T that is the union of $B+v_1, \ldots B+v_t$, where B is the unit square $[0,1] \times [0,1]$ and $v_1, \ldots, v_t$ are vectors with integers coordinates.

Theorem 2. *Let T be any grid animal. Then $OA(T) \geq 1$.*

Remark: If T is a grid animal and $T_1, \ldots, T_k$ are translates of T that sit on the integer grid squares (that is, if each T_i is a translate of T by an integer vector), then it is immediate (regardless of whether k is odd or not) that the area of $\bigoplus_{i=1}^k T_i$ is at least 1. This is true because we can consider the 1×1 square with integer coordinates (a, b) that is smallest with respect to the lexicographic order and belongs to some T_j. Then (assuming no two of $T_1, \ldots, T_k$ are equal, or else we may delete them in pairs) it is easy to see that it belongs to precisely one of the T_j's and in particular the area of $\bigoplus_{i=1}^k T_i$ is at least 1. Therefore, it is important to notice that in Theorem 2 we are allowed to consider translates of T by vectors with non-integer coordinates.

Proof of Theorem 2. This is in fact a corollary of Lemma 1. To see this we need the following simple lemma:

Lemma 2. *Let T be a grid animal, then $m_T(n) \leq n^2$.*

Proof. We provide a simple algorithm for finding a family $\mathcal{F}$ of at most n^2 translates of T that together cover almost every point of an $n \times n$ square an odd number of times.

We start with $\mathcal{F} = \emptyset$. Let A be the leftmost grid square composing the shape T in the bottom most row where there is at least one square in T (in other words, A is the grid square composing T that is smallest in the appropriate lexicographic order). Take a translated copy of T so that A coincides with the square $B = [0,1] \times [0,1]$ and add this translate of T to $\mathcal{F}$. Look on the grid square $B + (1,0)$ if it is already covered an odd number of times by the members in $\mathcal{F}$ continue. Otherwise add the copy $T + (1,0)$ to $\mathcal{F}$. Move on to consider the square $B + (2,0)$ and decide whether to include $T + (2,0)$ in $\mathcal{F}$. Then continue to the square $B + (3,0)$ and so on until $B + (n-1, 0)$. Then scan the second row of 1×1 squares: $B + (0,1), \ldots, B + (n-1, 1)$ and continue like this until the whole $[0,n] \times [0,n]$ square is covered an odd number of times. ∎

It follows from Lemma 2 that $\beta(T) \geq 1$. Lemma 1 implies now that $OA(T) \geq \beta(T) \geq 1$. ∎

Remark: The case where T equals a unit square shows that Theorem 2 is (at least in one case) best possible.

2 Construction of a shape T with $OA(T) = 0$

In this section we will construct a shape T such that $OA(T) = 0$. Our construction will in fact be a one dimensional construction that will yield a construction in any dimension. Indeed, if T is a shape in $\mathbb{R}$ such that $OA(T) = 0$, then $T \times [0,1]^{d-1}$ is a shape in $\mathbb{R}^d$ satisfying $OA(T \times [0,1]^{d-1}) = 0$. Therefore, for the rest of this section all of our shapes will lie in $\mathbb{R}$ and the notion of length will replace that of an area.

Definition 4. Let a, b, and c be positive real numbers. We denote by $[a, b, c]$ the shape that is the disjoint union of the two segments $[0, a]$ and $[a + b, a + b + c]$. In other words it is a segment of length a and then a gap of length b and then another segment of length c.

The next lemma is general and applies to shapes in any dimension.

Lemma 3. *Let T be a shape and $v_1, \ldots, v_t$ vectors. Let $D = \bigoplus_{i=1}^t (T + v_i)$. Then $OA(T) \geq \frac{1}{t} OA(D)$.*

Proof. Let $\epsilon > 0$ be given. There exist an odd number k and vectors $u_1, \ldots, u_k$ such that the area of $\bigoplus_{i=1}^{k}(T + u_i)$ is smaller than $OA(T) + \epsilon$.

Notice that

$$\bigoplus_{i=1}^{k}(D + u_i) = \bigoplus_{i=1}^{k}((\bigoplus_{j=1}^{t}(T + v_j)) + u_i) = \bigoplus_{j=1}^{t}((\bigoplus_{i=1}^{k}(T + u_i)) + v_j).$$

Therefore, the area of $\bigoplus_{i=1}^{k}(D+u_i)$ is at most $t(OA(T)+\epsilon)$. On the other hand this area must be at least $OA(D)$. Therefore, $t(OA(T) + \epsilon) \geq OA(D)$ yielding $OA(T) \geq \frac{1}{t}OA(D) - \epsilon$. As this is true for every $\epsilon > 0$, we conclude that $OA(T) \geq \frac{1}{t}OA(D)$. ∎

The next lemma is quite easy and intuitive. We leave the proof to the reader.

Lemma 4. *Let T be a shape in $\mathbb{R}$ and let $c > 0$ be any real number. We denote by cT the set $cT = \{cx \mid x \in T\}$. Then $OA(cT) = c \cdot OA(T)$.*

Remark. Lemma 4 is stated for shapes in $\mathbb{R}$ but of course a similar lemma is valid in any dimension d, where the statement in the lemma should be modified to $OA(cT) = c^d \cdot OA(T)$.

Lemma 5. *Let $T = [1, b, 1]$ for some positive integer b. Then $OA(T) = 2$.*

Proof. To see that $OA(T) \leq 2$ notice that $OA(T) \leq A(T) = 2$. To see that $OA(T) \geq 2$, consider the following $b + 1$ translates of T: $T, T + 1, T + 2 \ldots, T + b$. The disjoint union of all these copies is the interval $I = [0, 2b + 2]$ of length $2b + 2$. As $OA(I) = 2b + 2$, we conclude from Lemma 3 that $OA(T) \geq \frac{1}{b+1}OA(I) = 2$. ∎

Notice the following simple corollary of Lemma 5 and Lemma 4:

Corollary 1. *Suppose $T = [q, bq, q]$ for positive integers b and q, then $OA(T) = 2q$.*

The next theorem will be our main tool in proving Theorem 4.

Theorem 3. *Let k and z be relatively prime positive integers and let $T = [k, z, k]$. Then $OA(T) = 2$. Moreover, one can find a collection of an odd number of translates of T of no more than $2^{(k+z)!}(k+z)$ translates such that the length of the set of points that belong to an odd number of these translates is equal to 2.*

Proof. It is not hard to see that $OA(T) \geq 2$. Define

$$C = (T + 0) \oplus (T + (k + z)) \oplus (T + 2(k + z)) \oplus \ldots \oplus (T + (k - 1)(k + z)).$$

Notice (after a moment of inspection) that $C = [k, kz+(k-1)k, k]$. By Corollary 1, $OA(C) = 2k$. On the other hand C is the set of points covered an odd number of times by some k translates of T. Therefore, by Lemma 3, $OA(T) \geq \frac{1}{k}OA(C) = 2$.

It is left to show that $OA(T) \leq 2$ (this is the important and more difficult part of the theorem). To this end we will show the existence of an odd number s of real numbers $a_1, \ldots, a_s$ such that the length of $\bigoplus_{i=1}^{s}(T + a_i)$ is equal to 2. In fact, each of the numbers a_i will be an integer.

Our main tool is the following algebraic lemma:

Lemma 6. *Let k and z be relatively prime positive integers and consider the polynomial*

$$\begin{aligned} p(x) &= (1 + x + x^2 + \ldots + x^{k-1}) + (x^{k+z} + \ldots + x^{2k+z-1}) \\ &= (1 + x + x^2 + \ldots + x^{k-1})(1 + x^{k+z}) = \frac{(1+x^k)(1+x^{k+z})}{1+x} \end{aligned}$$

over $\mathbb{F}_2$. There exists a polynomial $q(x)$ of degree at most $2^{(k+z)!}(k+z)$ with $q(1) = 1$ such that $p(x)q(x) = 1 + x^m$ for some integer m.

Proof. We recall the following fact about polynomials over $\mathbb{F}_2$: For every natural number n, the polynomial $1 + x^{2^n - 1}$ is the product of all irreducible polynomials $f(x)$, different from $f(x) = x$, whose degree divides n.

We will need the following simple observations:

Claim 1. *The polynomials $1 + x^k$ and $1 + x^{k+z}$ do not have a common root other than $x = 1$, over any extension field of $\mathbb{F}_2$.*

Proof. Assume that a is a common root of both polynomials, then $a^k = a^{k+z} = 1$. As a must be different from 0, we conclude that $a^z = 1$. Since $a^k = a^z = 1$ and k and z are relatively prime, we have $1 = sz + tk$ for some integers s and t. It follows now that $a = a^1 = a^{sz+tk} = 1$. ∎

Claim 2. *If t is odd, then the polynomial $1 + x^t$ does not have a multiple root over any extension field of $\mathbb{F}_2$ (and is therefore product of distinct irreducibles).*

Proof. Easy to see by considering the derivative tx^{t-1} whose only root is $x = 0$. ∎

Claim 3. *For every nonnegative integers t and ℓ, $1 + x^{t2^\ell} = (1 + x^t)^{2^\ell}$, over any extension field of $\mathbb{F}_2$.*

Proof. Easy to see by considering the binomial coefficients $\binom{2^\ell}{j}$ and observing they are all even, unless $j = 0$ or $j = 2^\ell$. ∎

Going back to the proof of Lemma 6, we split into two cases.

Case 1. k is odd. In this case let us write $k + z = 2^\ell t$ where $\ell \geq 0$ is a nonnegative integer and t is odd.

Let $m = (2^{(k+z)!} - 1)2^\ell$. We claim that $1 + x^m$ is divisible by $p(x)$. To prove this it is enough to show that $1 + x^m$ is divisible by $1 + x^k$ and by $1 + x^{k+z}$. This is true because by Claim 1 the greatest common divisor of the latter two polynomials is $1 + x$ and $p(x) = \frac{(1+x^k)(1+x^{k+z})}{1+x}$. The polynomial $1 + x^m$ is divisible by $1 + x^k$ because $1 + x^k$ is a product of distinct irreducibles (by Claim 2 and the assumption that k is odd) and each such irreducible has degree that divides $(k+z)!$. Hence, $1 + x^k$ divides $1 + x^{2^{(k+z)!}-1}$ and consequently $1 + x^k$ divides also the polynomial $1 + x^m = (1 + x^{2^{(k+z)!}-1})^{2^\ell}$. To see that $1 + x^m$ is divisible by $1 + x^{k+z}$ notice that, by Claim 3, $1 + x^{k+z} = (1 + x^t)^{2^\ell}$. Now, $1 + x^t$ is a product of distinct irreducibles each of which has degree that divides $(k+z)!$. Therefore, $1 + x^t$ divides $1 + x^{2^{(k+z)!}-1}$. It follows that the polynomial $1 + x^{k+z} = (1 + x^t)^{2^\ell}$ divides $(1 + x^{2^{(k+z)!}-1})^{2^\ell} = 1 + x^m$.

Having shown that $p(x)$ divides $1 + x^m$, there exists a polynomial $q(x)$ such that $p(x)q(x) = 1 + x^m$ and the degree of q is at most $2^{(k+z)!}2^\ell \leq 2^{(k+z)!}(k+z)$, as desired. It remains to show that $q(1) = 1$. To this end notice that $x = 1$ is a root of multiplicity 2^ℓ of $p(x)$. Indeed, $x = 1$ is a root of multiplicity 2^ℓ of the polynomial $1 + x^{k+z} = 1 + x^{2^\ell t} = (1 + x^t)^{2^\ell}$ and $x = 1$ is not a root of $(1 + x^k)/(1 + x)$, as we assume k is odd. On the other hand $x = 1$ is also a root of multiplicity 2^ℓ of $(1 + x^{2^{(k+z)!}-1})^{2^\ell} = 1 + x^{(2^{(k+z)!}-1)(2^\ell)} = 1 + x^m$. Therefore, because $p(x)q(x) = 1 + x^m$, it must be that $x = 1$ is not a root of $q(x)$ and consequently $q(1) = 1$.

Case 2. k is even. Write $k = t2^\ell$ where t is odd and $\ell > 0$ is a positive integer. In this case both z and $k + z$ are odd, because k and z are relatively prime.

Let $m = (2^{(k+z)!} - 1)2^\ell$. Similar to Case 1, we will show that $p(x)$ divides $1 + x^m$. Again it is enough to show that each of $1 + x^k$ and $1 + x^{k+z}$ divides $1 + x^m$.

Because $k + z$ is odd, $1 + x^{k+z}$ is a product of distinct irreducibles (see Claim 2) and each of these irreducibles has degree that divides $(k + z)!$. Therefore, $1 + x^{k+z}$ divides $1 + x^{2^{(k+z)!}-1}$ which in turn divides $1 + x^m$. To see that $1 + x^k$ divides $1 + x^m$ notice that $1 + x^t$ divides $1 + x^{2^{(k+z)!}-1}$ and hence $1 + x^k = (1 + x^t)^{2^\ell}$ divides $(1 + x^{2^{(k+z)!}-1})^{2^\ell} = 1 + x^{(2^{(k+z)!}-1)2^\ell} = 1 + x^m$.

Having shown that $p(x)$ divides $1+x^m$, we conclude that there exists $q(x)$ such that $p(x)q(x) = 1 + x^m = (1 + x^{2^{(k+z)!}-1})^{2^\ell}$. It remains to show that $q(1) = 1$. We observe that $x = 1$ is a root of multiplicity 2^ℓ of $p(x)$. Indeed, $p(x) = (1 + x^k)(1 + x^{k+z})/(1 + x) = (1 + x^t)^{2^\ell}(1 + x^{k+z})/(1 + x)$. By Claim 3 (and the fact that t is odd), $x = 1$ is a root of multiplicity 2^ℓ of $(1 + x^t)^{2^\ell}$, while it is not a root of $(1 + x^{k+z})/(1 + x)$ because $k + z$ is odd. On the other hand $x = 1$ is also a root of multiplicity 2^ℓ of the polynomial $1 + x^m = (1 + x^{2^{(k+z)!}-1})^{2^\ell}$. Because $p(x)q(x) = 1 + x^m$, it follows that $x = 1$ is not a root of $q(x)$ and consequently $q(1) = 1$. ∎

Let us now go back to the proof of Theorem 3. We need to show that if $T = [k, z, k]$, then $OA(T) \leq 2$. To see this we just need to interpret Lemma 6 geometrically. Let $p(x) = (1 + x + x^2 + \ldots + x^{k-1}) + (x^{k+z} + \ldots + x^{k+z-1})$. By Lemma 6, there exists $q(x)$, a polynomial over $\mathbb{F}_2$ of degree at most $2^{(k+z)!}(k + z)$, such that $p(x)q(x) = 1 + x^m$ for some integer $m > 1$ and moreover $q(1) = 1$. Write $q(x) = x^{i_1} + x^{i_2} + \ldots + x^{i_s}$, where $0 \leq i_1 < i_2 < \ldots < i_s$ and $s \leq 2^{(k+z)!}(k + z)$. Then it is not hard to see that the set $(T + i_1) \oplus (T + i_2) \oplus \ldots \oplus (T + i_s)$ has length precisely 2 and in fact this set consists of a union of precisely two unit segments namely $[0, 1]$ and $[m - 1, m]$. $q(1) = 1$ implies that s is odd and this shows that $OA(T) \leq 2$. ∎

Corollary 2. *Let k and z be relatively prime positive integers and let $T = [1, \frac{z}{k}, 1]$. Then $OA(T) = \frac{2}{k}$.*

Proof. Observe that $k \cdot T = [k, z, k]$ and therefore, by Theorem 3, $OA(k \cdot T) = 2$. By Lemma 4, we have: $OA(T) = \frac{1}{k} OA(k \cdot T) = \frac{2}{k}$. ∎

It follows from Theorem 3 that in Corollary 2 one can construct a family $\mathcal{F}$ of an odd number s of translates of T, where $s \leq 2^{(k+z)!}(k + z)$, such that $A(\bigoplus_{F \in \mathcal{F}} F) = \frac{2}{k}$.

Before turning to the proof of Theorem 4 we need one more easy observation (this observation is true in any dimension).

Claim 4. *Let T and T' be two shapes in $\mathbb{R}$. Assume that $A(T \oplus T') < \epsilon$ where $\epsilon > 0$ is given. Let $v_1, \ldots, v_s \in \mathbb{R}$, where s is odd. Then $OA(T) \leq A(\bigoplus_{i=1}^{s}(T' + v_i)) + s\epsilon$.*

Proof. We have

$$\begin{aligned} OA(T) &\leq A(\bigoplus_{i=1}^{s}(T+v_i)) = A(\bigoplus_{i=1}^{s}((T'\oplus T\oplus T')+v_i)) \\ &= A((\bigoplus_{i=1}^{s}(T'+v_i))\oplus(\bigoplus_{i=1}^{s}((T\oplus T')+v_i))) \leq A((\bigoplus_{i=1}^{s}(T'+v_i))+A(\bigoplus_{i=1}^{s}((T\oplus T')+v_i))) \\ &\leq A(\bigoplus_{i=1}^{s}(T'+v_i))+s\cdot A(T\oplus T') = A(\bigoplus_{i=1}^{s}(T'+v_i))+s\epsilon. \end{aligned}$$

∎

We are now ready to prove our main result.

Theorem 4. *There exist shapes T with $OA(T) = 0$.*

Proof. We will show that $OA(T) = 0$ for the shape $T = [1, \gamma, 1]$ where γ is a real number defined as follows: Let $\{a_n\}_{n=1}^{\infty}$ be the sequence defined by $a_1 = 2$ and for $n > 1$: $a_n = 2^{2(2a_{n-1})!}$. We set $\gamma = \sum_{n=1}^{\infty} \frac{1}{a_n}$.

Let $\epsilon > 0$ be given. Let N be such that $\frac{1}{a_N} < \epsilon/4$ (N clearly exists as $\{a_n\}$ goes, very fast, to infinity). Let z and k be relatively prime positive integers such that $\sum_{i=1}^{N} \frac{1}{a_i} = \frac{z}{k}$. Observe that we must have $k = a_N$ while z is odd and satisfies $z < k$ (the latter is because $\gamma < 1$).

Consider the shape T' defined as $T' = [1, \frac{z}{k}, 1]$. By Corollary 2, $OA(T') = \frac{2}{a_N} < \epsilon/2$. Moreover, there are $v_1, \ldots, v_s \in \mathbb{R}$, where s is an odd number smaller than $2^{(2a_N)!}(2a_N)$, such that $A(\bigoplus_{i=1}^{s}(T'+v_i)) \leq \epsilon/2$. Observe that $T \oplus T'$, that is the symmetric difference of T and T', has area equal to $2\sum_{i=N+1}^{\infty} \frac{1}{a_i}$ and this is strictly smaller than $\frac{4}{a_{N+1}}$ (because of the fast rate at which $\{a_n\}$ goes to infinity). It follows, by Claim 4, that

$$OA(T) \leq A(\bigoplus_{i=1}^{s}(T'+v_i)) + s\frac{4}{a_{N+1}} \leq \epsilon/2 + 2^{(2a_N)!}(2a_N)\frac{4}{a_{N+1}}.$$

However, the latter summand $2^{(2a_N)!}(2a_N)\frac{4}{a_{N+1}}$ in the right hand side of the inequality satisfies

$$2^{(2a_N)!}(2a_N)\frac{4}{a_{N+1}} = 8a_N 2^{(2a_N)!-2(2a_N)!} = \frac{8a_N}{2^{(2a_N)!}} \leq \frac{1}{a_N} \leq \epsilon/4.$$

We conclude that $OA(T) \leq \epsilon/2 + \epsilon/4 < \epsilon$. ∎

Remark. It is not hard to see that the number γ defined in the proof of Theorem 4 is irrational. This is because of the fast rate at which $\{a_n\}$ goes to infinity and the fact that each a_n is a power of 2 (we leave it to the reader as an amusing exercise). As a consequence, the shape T constructed in Theorem 4, with the property that $OA(T) = 0$ yields in two (and similarly in all) dimensions a shape, namely $P = T \times [0, 1] \subset \mathbb{R}^2$ with $OA(P) = 0$. Notice that P is just a union of two $1 times 1$ squares with a gap of length (width) γ between them. Because γ is irrational, P cannot have rational vertices under any affine transformation in the plane. In an on going work [3] we show that this is not a coincidence. It turns out that if P is a polygon (or union of polygons) with rational vertices, then necessarily $OA(P) > 0$.

3 Open problems

There are many challenging problems related to the notion of odd area that we cannot solve. We bring here some of which that are most related to Theorem 4.

- Perhaps one of the most interesting problem related to odd area is to determine the odd area of a circular disc. In particular it is not even known whether the odd area of a disc is at all positive.
- For a positive odd integer n define $OA_n(T)$ to be the minimum value of $\bigoplus_{i=1}^{n}(T+v_i)$, where each v_i is any vector in $\mathbb{R}^2$. Suppose that $OA(T)=0$. How fast does $OA_n(T)$ goes to 0 as n goes to infinity?
- Is there a *convex* set T such that $OA(T)=0$? Is there such a *convex polygon* T?

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