# Combinatorics and geometry of Littlewood-Richardson cones

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**Abstract**

We present several direct bijections between different combinatorial interpretations of the Littlewood-Richardson coefficients. The bijections are defined by explicit linear maps which have other applications.

## 1 Introduction

In the past decade the Littlewood-Richardson rule (LR rule) moved into a center stage in the combinatorics of Young tableaux. Much attention have received classical applications (to representation theory of the symmetric and the full linear group, to the symmetric functions, etc.) as well as more recent developments (Schubert calculus, eigenvalues of Hermitian matrices, etc.) While various combinatorial interpretations of the Littlewood-Richardson coefficients were discovered, there seems to be little understanding of how they are related to each other, and little order among them. This paper makes a new step in this direction.

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We start with three major combinatorial interpretations of the LR coefficients which we view as integer points in certain cones. We present simple linear maps between the cones which produce explicit bijections for all triples of partitions involved in the LR rule. These bijections are quite natural in this setting and in a certain sense can be shown to be unique. Below we further emphasize the importance of the linear maps.

A classical version of the LR rule, in terms of certain Young tableaux, is now well understood, and its proof has been perfected for decades. We refer to [15] for a beautifully written survey of the "classical" approach, with a historical overview and connections to the jeu-de-taquin, Schützenberger involution, etc. Unfortunately, the language of Young tableaux is often too rigid to be able to demonstrate the inherent symmetries of the LR coefficients.

A radically different combinatorial interpretation in due to Berenstein and Zelevinsky, in terms of the so called BZ triangles, which makes explicit all but one symmetry of the LR coefficients†. The authors' proof in [6] relies on a series of previous papers [10, 4, 5], a situation that is hardly satisfactory. A paper [8] establishes a technically involved bijection with the contratableaux associated with certain Yamanouchi words, which gives another combinatorial interpretation of the LR rule. This combinatorial interpretation is in fact different from the one given by LR tableaux, which makes the matter even more confusing.

In a subsequent development, Knutson and Tao introduced [14] the so called honeycombs, which are related to BZ triangles by a bijection that they sketch at the end. The paper [11] uses a related construction of "web diagrams" for a different purpose. The appendix in [14] also introduces a different language of *hives*, which proved to be more flexible to restate the Knutson-Tao proof of saturation conjecture [7].

In the appendix to [7], Fulton described in a simple language a bijection with a set of certain contratableaux, similar to that of Carré [8]. As mentioned at the end of the appendix (cf. also [9]), the latter are in a well known bijection with the classical LR tableaux. Unfortunately, this bijection is based on the Schützenberger involution, which is in fact quite involved and goes beyond the scope of this paper.

Now, let us return to the linear maps establishing the bijections. First, these maps show that the LR cones have the same combinatorial structure. Despite a visual difference between definitions of LR tableaux, hives, and BZ triangles, these combinatorial objects are essentially the same and should be treated as equivalent. In a sense, this varying nature of these combinatorial interpretations of the LR coefficients makes them "more fundamental" than others.

Let us mention here a "local" nature of the bijections we present. In general, computing the action linear maps $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ requires $O(d^2)$ arithmetic operations (multiplications and additions) to perform. In this case, however, the local nature of

†We should warn the reader that the BZ triangles presented in [19] are different, albeit strongly related.

bijections allows a $O(d)$ computation, where $d = \binom{k}{2}$, and $k$ is the number of rows in LR tableaux. This is especially striking when comparing with other Young tableau bijections, which require $O(d^{3/2})$ operations. We refer to a forthcoming paper [18] for references and details, and for a new theory explaining this phenomenon. As observed previously, the bijections in this paper combined with the symmetries of BZ triangles give nearly all the symmetries[‡] of the LR coefficients, except for one: $c^\lambda_{\mu,\nu} = c^\lambda_{\nu,\mu}$. The latter again requires $O(d^{3/2})$ operations and is in the same class as other Young tableau bijections (ibid.)

The idea of using integer points in cones is a direct descendant of the earlier papers [10, 5] and most recently has appeared in a context of integer partitions [17]. While the fact that the linear maps between cones exist at all may seem surprising, we do not claim to be the first to establish that. It is perhaps surprising that the resulting linear maps are so simple and natural in this language. We believe that this approach is perhaps more direct and fruitful when compared to other more traditional combinatorial techniques employed earlier (see above).

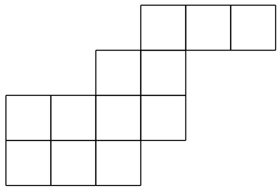
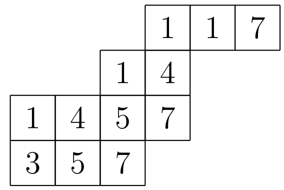
To conclude, let us describe the structure of the paper. We present in separate sections the LR tableaux, the hives of Knutson and Tao, and the BZ triangles. Along the way we establish the bijections between these combinatorial interpretations. While the linear maps which produce these bijections are easy to define, their proofs are not straightforward and are delayed until the end of the paper. We conclude with the final remarks.

## 2 Littlewood-Richardson tableaux

Let $\lambda = (\lambda_1, \dots, \lambda_k)$ be a partition of a positive integer $n$, that is, a sequence of integers whose sum is $n$ and satisfy $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0$. Its diagram is the set of pairs of positive integers $\{\, (i,j) \mid 1 \leq i \leq k,\ 1 \leq j \leq \lambda_i \,\}$, which we also denote by $\lambda$. If $\mu$ is another partition and the diagram of $\mu$ is a subset of the diagram of $\lambda$, in symbols $\mu \subseteq \lambda$, we denote by $\lambda/\mu$ the *skew diagram* consisting of the points in $\lambda$ that are not in $\mu$, and by $|\lambda/\mu|$ its cardinality. It is customary to represent diagrams pictorially as a collection of boxes [9, 16, 19]. Any filling $T$ of a skew diagram $\lambda/\mu$ with positive integers, formally a map $T : \lambda/\mu \longrightarrow \mathbb{N}$, will be called a *Young tableau* or just a *tableau* of *shape* $\lambda/\mu$. A Young tableau $T$ is called *semistandard* if its rows are weakly increasing from left to right and its columns are strictly increasing from top to bottom. The *content* of $T$ is the composition $\gamma(T) = (\gamma_1, \dots, \gamma_c)$, where $\gamma_i$ is the number of $i$'s in $T$. The *word* of $T$, denoted by $w(T)$ , is obtained from $T$ by reading its entries from right to left, in successive rows, starting with the top row and moving

[‡]There is one additional "symmetry" $c^\lambda_{\mu,\nu} = c^{\lambda'}_{\mu',\nu'}$; which doesn't seem to have a "geometric interpretation". For a combinatorial proof see [12].

down. For example, let

$D =$  and $T =$ 


then $D$ is a diagram of shape $(6, 4, 4, 3)/(3, 2)$ and $T$ is a semistandard tableaux of this shape, has content $(4, 0, 1, 2, 2, 0, 3)$ and its word is $w(T) = 711417541753$. Finally, a word $w = w_1 \cdots w_k$ in the alphabet $1, \ldots, n$ is called a *lattice permutation* if for all $1 \leq j \leq k$ and all $1 \leq i \leq n - 1$ the number of occurrences of $i$ in $w_1 \cdots w_j$ is not less than the number of occurrences of $i + 1$ in $w_1 \cdots w_j$. A semistandard tableau $T$ of skew shape is called a *Littlewood-Richardson* tableau if its word $w(T)$ is a lattice permutation. Note that the content of a Littlewood-Richardson tableau is always a partition. Given three partitions $\lambda$, $\mu$, $\nu$ such that $\mu \subseteq \lambda$ and $|\lambda| = |\mu| + |\nu|$, we denote by $c^{\lambda}_{\mu\,\nu}$ the number of Littlewood-Richardson tableaux of shape $\lambda/\mu$ and content $\nu$. We will use the following example throughout the paper. Let

$$\lambda = (23, 18, 15, 11, 8),\ \ \mu = (15, 9, 5, 2, 0) \text{ and } \nu = (16, 11, 10, 5, 2), \tag{1}$$

then the tableau in Figure 1 is an example of a Littlewood-Richardson tableau of shape $\lambda/\mu$ and content $\nu$.

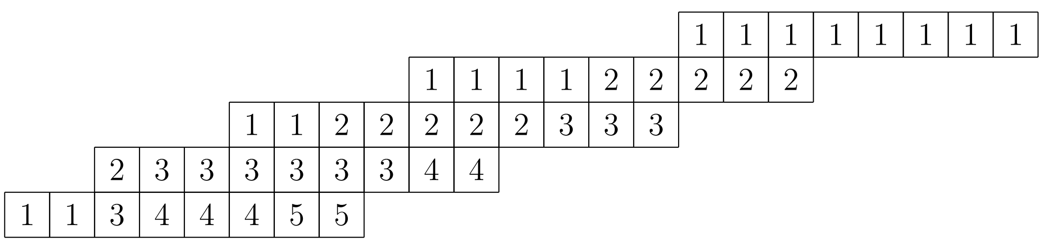



Figure 1: Littlewood-Richardson tableau

## 3 Littlewood-Richardson triangles

The *hive graph* $\Delta_k$ *of size* $k$ is a graph in the plane with $\binom{k+2}{2}$ vertices arranged in a triangular grid consisting of $k^2$ small equilateral triangles, as shown in Figure 2. Let $T_k$ denote the vector space of all labelings $A = (a_{ij})_{0 \leq i \leq j \leq k}$ of the vertices of $\Delta_k$ with real numbers such that $a_{00} = 0$. We will write such labelings as triangular arrays of real numbers in the way shown in Figure 3. The dimension of $T_k$ is clearly $\binom{k+2}{2} - 1$.

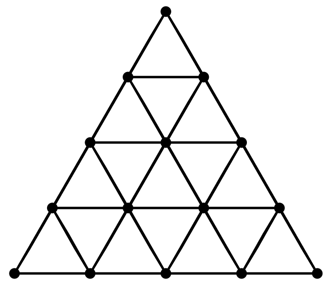


Figure 2: Hive graph $\Delta_4$.

$$
\begin{array}{ccccccc}
 & & & a_{0\,0} & & & \\
 & & a_{0\,1} & & a_{1\,1} & & \\
 & a_{0\,2} & & a_{1\,2} & & a_{2\,2} & \\
a_{0\,3} & & a_{1\,3} & & a_{2\,3} & & a_{3\,3}
\end{array}
$$

Figure 3: Triangular array of size 3.

We now proceed to explain how Littlewood-Richardson tableaux can be coded in a simple way as elements in $T_k$ satisfying certain inequalities. A *Littlewood-Richardson triangle of size* $k$ is an element $A = (a_{ij}) \in T_k$ that satisfies the following conditions:

(P) $a_{ij} \geq 0$, for all $1 \leq i < j \leq k$.

(CS) $\sum_{p=0}^{i-1} a_{pj} \geq \sum_{p=0}^{i} a_{p\,j+1}$, for all $1 \leq i \leq j < k$.

(LR) $\sum_{q=i}^{j} a_{iq} \geq \sum_{q=i+1}^{j+1} a_{i+1\,q}$, for all $1 \leq i \leq j < k$.

Note that the inequality

$$\sum_{p=0}^{j} a_{pj} \geq \sum_{p=0}^{j+1} a_{p\,j+1}, \text{ for } 1 \leq j < k. \tag{2}$$

follows from (CS) with $i = j$ and (LR) with $i = j$; also note that $a_{0j}$ and $a_{jj}$ could be negative. We denote by $\mathsf{LR}_k$ the cone of all Littlewood-Richardson triangles in $T_k$, and call it a *Littlewood-Richardson cone*; this has the same dimension as $T_k$. Also let $\mathsf{D}_k$ denote the set of all $k$-tuples $\lambda = (\lambda_1, \ldots, \lambda_k)$ of real numbers such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$, and $|\lambda|$ the sum of its entries, that is, $|\lambda| = \sum_{i=1}^{k} \lambda_i$. To each $A = (a_{ij}) \in \mathsf{LR}_k$ we associate the following numbers:

(B1) $\mu_j = a_{0j}$, for all $1 \leq j \leq k$.

(B2) $\lambda_j = \sum_{p=0}^{j} a_{pj}$, for all $1 \leq j \leq k$.

(B3) $\nu_i = \sum_{q=i}^{k} a_{iq}$, for all $1 \leq i \leq k$.

Then it follows from (P), (CS) and (LR) that the vectors $\lambda = (\lambda_1, \ldots, \lambda_k)$, $\mu = (\mu_1, \ldots, \mu_k)$ and $\nu = (\nu_1, \ldots, \nu_k)$ are in $\mathsf{D}_k$ and that $|\lambda| = |\mu| + |\nu|$. We call $(\lambda, \mu, \nu)$

the *type* of $A$, and denote by $\mathsf{LR}_k(\lambda,\mu,\nu)$ the set of all Littlewood-Richardson triangles of type $(\lambda,\mu,\nu)$; this is a convex polytope. For example, let $\lambda$, $\mu$, $\nu$ be as in (1), then the triangle in Figure 4 is in $\mathsf{LR}_5(\lambda,\mu,\nu)$.

$$\begin{array}{ccccccccccc} & & & & & 0 & & & & & \\ & & & & 15 & & 8 & & & & \\ & & & 9 & & 4 & & 5 & & & \\ & & 5 & & 2 & & 5 & & 3 & & \\ & 2 & & 0 & & 1 & & 6 & & 2 & \\ 0 & & 2 & & 0 & & 1 & & 3 & & 2 \end{array}$$

Figure 4: Littlewood-Richardson triangle of size 5

Let $\lambda$, $\mu$, $\nu \in \mathsf{D}_k$ be partitions, that is $\lambda$, $\mu$ and $\nu$ have non-negative integer coefficients, and suppose that $|\lambda| = |\mu| + |\nu|$. To each Littlewood-Richardson tableau $T$ of shape $\lambda/\mu$ and content $\nu$ we associate a triangular array $A_T = (a_{ij}) \in T_k$ by defining

(i) $a_{00} = 0$, $a_{0j} = \mu_j$ for $1 \leq j \leq k$, and

(ii) $a_{ij}$ equal to the number of $i$'s in row $j$ of $T$ for $1 \leq i \leq j \leq k$.

Note that the Littlewood-Richardson triangle in Figure 4 corresponds to the Littlewood-Richardson tableau in Figure 1.

**3.1. Lemma.** *Let $\lambda$, $\mu$, $\nu \in \mathsf{D}_k$ be partitions such that $|\lambda| = |\mu| + |\nu|$. Then the correspondence $T \longmapsto A_T$ is a bijection between the set of all Littlewood-Richardson tableaux of shape $\lambda/\mu$ and content $\nu$ and the set of all Littlewood-Richardson triangles of type $(\lambda,\mu,\nu)$ with integer entries. In particular $\mathsf{LR}_k(\lambda,\mu,\nu)$ has $c^{\lambda}_{\mu\,\nu}$ integer points.*

In effect, Lemma 3.1 translates combinatorics of Littlewood-Richardson tableaux into the language of integer points in polyhedra. Various other translations of this kind appear in the literature and are more or less equivalent to ours. A short "verification style" proof is given in Section 6.

## 4 Hives

The hive graph $\Delta_k$ of size $k$ is divided into $k^2$ small equilateral triangles. Each two adjacent such triangles form a rhombus with two obtuse angles and two acute angles.

There are three types of rhombi: tilted to the right, vertical and tilted to the left. They are shown in Figure 5.

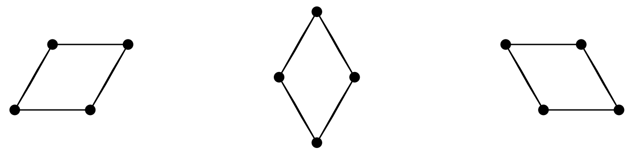


Figure 5: Types of rhombi in a hive graph.

A *hive of size k* is a labeling $H = (h_{ij})_{0\le i\le j\le k}$ of the vertices of the hive graph $\Delta_k$ with real numbers such that for each rhombus the sum of the labels at obtuse vertices is bigger than or equal to the sum of the labels at acute vertices; equivalently, $H = (h_{ij})$ satisfies the following inequalities:

(R) $h_{i\,j} - h_{i\,j-1} \ge h_{i-1\,j} - h_{i-1\,j-1}$, for $1 \le i < j \le k$.
(V) $h_{i-1\,j} - h_{i-1\,j-1} \ge h_{i\,j+1} - h_{i\,j}$, for $1 \le i \le j < k$.
(L) $h_{i\,j} - h_{i-1\,j} \ge h_{i+1\,j+1} - h_{i\,j+1}$, for $1 \le i \le j < k$.

We denote by $\mathsf{H}_k$ the cone of all hives of size $k$ that satisfy the extra condition $h_{00} = 0$, and call it a *hive cone.* As we did for Littlewood-Richardson triangles, we associate to each hive $H = (h_{ij}) \in \mathsf{H}_k$ numbers:

(B1′) $\mu_j = h_{0\,j} - h_{0\,j-1}$, for $1 \le j \le k$.
(B2′) $\lambda_j = h_{j\,j} - h_{j-1\,j-1}$, for $1 \le j \le k$.
(B3′) $\nu_i = h_{i\,k} - h_{i-1\,k}$, for $1 \le i \le k$.

Then it follows from (R), (V) and (L) that the vectors $\lambda = (\lambda_1, \dots, \lambda_k)$, $\mu = (\mu_1, \dots, \mu_k)$ and $\nu = (\nu_1, \dots, \nu_k)$ are in $\mathsf{D}_k$ and that $|\lambda| = |\mu| + |\nu|$. For example,

$$\mu_j = h_{0\,j} - h_{0\,j-1} \ge h_{1\,j+1} - h_{1\,j} \ge h_{0\,j+1} - h_{0\,j} = \mu_{j+1}.$$

We call $(\lambda, \mu, \nu)$ the *type* of $A$, and denote by $\mathsf{H}_k(\lambda, \mu, \nu)$ the set of all hives of type $(\lambda, \mu, \nu)$; this is a convex polytope. For example, let $\lambda$, $\mu$ and $\nu$ be as in (1), then the triangle in Figure 6 is in $\mathsf{H}_5(\lambda, \mu, \nu)$.

For any positive integer $k$, we define a linear map $\Phi_k \colon T_k \longrightarrow T_k$ by

$$\Phi_k(a_{i\,j}) = (h_{i\,j}), \quad \text{where} \quad h_{i\,j} = \sum_{p=0}^{i} \sum_{q=p}^{j} a_{p\,q}\,.$$

Note that the hive in Figure 6 is the image under $\Phi_5$ of the Littlewood-Richardson triangle in Figure 4. We have the following theorem.

**4.1. Theorem.** *The map $\Phi_k$ defined above is a volume preserving linear operator which maps $\mathsf{LR}_k$ bijectively onto $\mathsf{H}_k$, and $\mathsf{LR}_k(\lambda, \mu, \nu)$ onto $\mathsf{H}_k(\lambda, \mu, \nu)$, for all $\lambda$, $\mu$, $\nu \in \mathsf{D}_k$.*

0

15 23

24 36 41

29 43 53 56

31 45 56 65 67

31 47 58 68 73 75

Figure 6: Hive of size 5

As mentioned in the introduction, the proof can be found in section 6. Let us mention here two important corollaries. For any polytope $P$ let $e(P)$ denote the number of integer points in $P$.

**4.2. Corollary.** $e(\mathsf{H}_k(\lambda,\mu,\nu)) = c^{\lambda}_{\mu\,\nu}$, *for all* $\lambda$, $\mu$, $\nu \in \mathsf{D}_k$ *with non-negative integer coefficients.*

**4.3. Corollary.** $\mathrm{Vol}(\mathsf{H}_k(\lambda,\mu,\nu)) = \mathrm{Vol}(\mathsf{LR}_k(\lambda,\mu,\nu))$, *for all* $\lambda$, $\mu$, $\nu \in \mathsf{D}_k$.

## 5 Berenstein-Zelevinsky triangles

For any integer $k \geq 1$ we construct a graph $\Gamma_k$ from the hive graph $\Delta_{k+1}$ in the following way: Its vertices are the middle points of the edges of the hive graph that do not lie on the boundary, and their edges are those joining pairs of middle points on edges lying on small triangles of $\Delta_{k+1}$, see Figure 7. We call $\Gamma_k$ the Berenstein-Zelevinsky graph of size $k$. The vertices of the Berenstein-Zelevinsky graph are partitioned into disjoint blocks of cardinality three, each block corresponding to a small equilateral triangle; these triangles are distributed in the graph: one on the first (top) level, two on the second level, three on the third level, and so on. Let $V_k$ denote the vector space of all labelings $X = (x_{i\,j}, y_{i\,j}, z_{i\,j})_{1\leq i\leq j\leq k}$ of $\Gamma_k$ with real numbers. The labelings are carried out in such a way that the vertices of the $i$-th triangle on the $j$-th level are labeled with $x_{i\,j}$, $y_{i\,j}$, $z_{i\,j}$ as indicated in Figure 8. The dimension of $V_k$ is $3\binom{k+1}{2}$. Note that the labels $y_{i\,j}$, $z_{i\,j}$, $x_{i+1\,j+1}$, $y_{i+1\,j+1}$, $z_{i\,j+1}$, $x_{i\,j+1}$ form an hexagon for each $1 \leq i \leq j < k$ and hence there are $\binom{k}{2}$ hexagons in $\Gamma_k$. We will be interested in the subspace $W_k$ of $V_k$ consisting of all labelings such that for each hexagon in $\Gamma_k$ the sum of the labels in each edge equals the sum of the labels of the diametrically opposite edge, that is

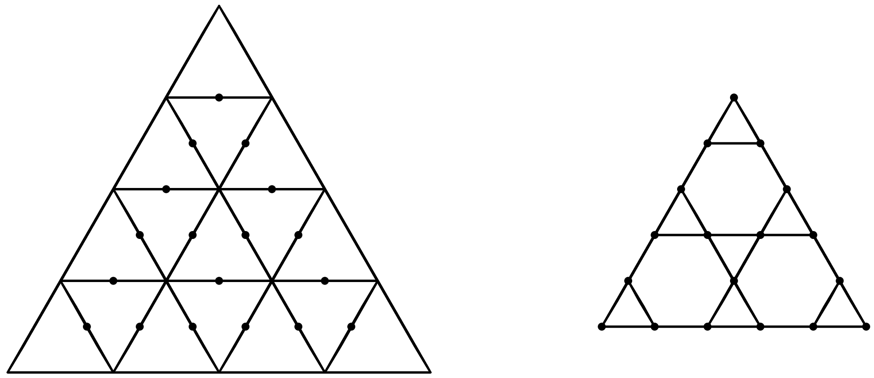


Figure 7: Hive graph $\Delta_4$ and the corresponding Berenstein-Zelevinsky graph $\Gamma_3$.

$$x_{1\,1}$$
$$y_{1\,1} \quad z_{1\,1}$$
$$x_{1\,2} \qquad\qquad x_{2\,2}$$
$$y_{1\,2} \quad z_{1\,2} \quad y_{2\,2} \quad z_{2\,2}$$
$$x_{1\,3} \qquad\qquad x_{2\,3} \qquad\qquad x_{3\,3}$$
$$y_{1\,3} \quad z_{1\,3} \quad y_{2\,3} \quad z_{2\,3} \quad y_{3\,3} \quad z_{3\,3}$$

Figure 8: Labeling of $\Gamma_3$

(BZ1) $y_{i\,j} + z_{i\,j} = y_{i+1\,j+1} + z_{i\,j+1}$, for all $1 \leq i \leq j < k$.
(BZ2) $x_{i\,j+1} + y_{i\,j} = x_{i+1\,j+1} + y_{i+1\,j+1}$, for all $1 \leq i \leq j < k$.
(BZ3) $x_{i\,j+1} + z_{i\,j+1} = x_{i+1\,j+1} + z_{i\,j}$, for all $1 \leq i \leq j < k$.

Observe that any of these three equalities follows from the other two.

**5.1. Lemma.** *The vector space $W_k$ has dimension $\frac{1}{2}k(k+5) = \dim T_{k+1} - 2$.*

A *Berenstein-Zelevinsky triangle of size $k$* is any labeling of $\Gamma_k$ in $W_k$ with non-negative entries. Let $\mathsf{BZ}_k$ denote the cone of all Berenstein-Zelevinsky triangles of size $k$. Let $\lambda$, $\mu$, $\nu \in \mathsf{D}_{k+1}$, then we say that a Berenstein-Zelevinsky triangle is of type $(\lambda, \mu, \nu)$ is it satisfies the following conditions:
(B1$''$) $x_{1\,j} + y_{1\,j} = \mu_j - \mu_{j+1}$, for $1 \leq j \leq k$.
(B2$''$) $x_{j\,j} + z_{j\,j} = \lambda_j - \lambda_{j+1}$, for $1 \leq j \leq k$.
(B3$''$) $y_{i\,k} + z_{i\,k} = \nu_i - \nu_{i+1}$, for $1 \leq i \leq k$.

Note that, in contrast to Littlewood-Richardson triangles and hives, a Berenstein-Zelevinsky triangle has many different types. Let $\mathsf{BZ}_k(\lambda, \mu, \nu)$ denote the set of all

Berenstein-Zelevinsky triangles of type $(\lambda, \mu, \nu)$; this is a convex polytope. For example, let $\lambda$, $\mu$ and $\nu$ be as in (1), then the triangle in Figure 9 is in $\mathsf{BZ}_4(\lambda, \mu, \nu)$. Here the $x_{i\,j}$'s are written with roman numerals, the $y_{i\,j}$'s by **boldface** numerals and the $z_{i\,j}$'s by *italic* numerals.

2

**4** *3*

2 1

**2** *2* **5** *2*

3 4 3

**0** *3* **1** *1* **6** *1*

0 0 0 3

**2** *3* **0** *1* **1** *4* **3** *0*

Figure 9: Berenstein-Zelevinsky triangle of size 4.

For any integer $k \geq 2$, we define a linear map $\Psi_k\colon T_k \longrightarrow W_{k-1}$ by setting $\Psi_k(h_{i\,j}) = (x_{i\,j}, y_{i\,j}, z_{i\,j})$ where

$$\begin{aligned} x_{i\,j} &= h_{i\,j} + h_{i-1\,j} - h_{i-1\,j-1} - h_{i\,j+1}, \\ y_{i\,j} &= h_{i-1\,j} + h_{i\,j+1} - h_{i\,j} - h_{i-1\,j+1}, \\ z_{i\,j} &= h_{i\,j} + h_{i\,j+1} - h_{i-1\,j} - h_{i+1\,j+1}, \end{aligned}$$

for all $1 \leq i \leq j < k$. Note that the values of the $x_{i\,j}$'s, $y_{i\,j}$'s and $z_{i\,j}$'s are obtained by taking, respectively, the differences of the inequalities (V), (R) and (L) used to define hives. It should be remarked that the $y_{i\,j}$'s are obtained from (R) by adding one to $j$. It is straightforward to check that the image of $\Phi_k$ is contained in $W_{k-1}$. The composition $\Psi_k \circ \Phi_k\colon T_k \longrightarrow W_{k-1}$ has also a nice description: $\Psi_k \circ \Phi_k(a_{i\,j}) = (x_{i\,j}, y_{i\,j}, z_{i\,j})$ with

$$\begin{aligned} x_{i\,j} &= \textstyle\sum_{p=0}^{i-1} a_{p\,j} - \sum_{p=0}^{i} a_{p\,j+1}, \\ y_{i\,j} &= a_{i\,j+1}, \\ z_{i\,j} &= \textstyle\sum_{q=i}^{j} a_{i\,q} - \sum_{q=i+1}^{j+1} a_{i+1\,q}, \end{aligned} \tag{3}$$

for all $1 \leq i \leq j < k$. Again, the values of the $x_{i\,j}$'s, $y_{i\,j}$'s and $z_{i\,j}$'s are obtained by taking, respectively, the differences of the left and right hand sides in the inequalities (CS), (P) and (LR) used to define Littlewood-Richardson triangles. For example, the

Berenstein-Zelevinsky triangle in Figure 9 is the image under $\Psi_5$ of the hive in Figure 6 and the image under $\Psi_5 \circ \Phi_5$ of the Littlewood-Richardson triangle in Figure 4. Note that the boldface numerals in Figure 9 are contained in the Littlewood-Richardson triangle from Figure 4.

**5.2. Theorem.** *The linear operator $\Psi_k \circ \Phi_k$ maps $\mathsf{LR}_k$ surjectively onto $\mathsf{BZ}_{k-1}$, and $\mathsf{LR}_k(\lambda, \mu, \nu)$ bijectively onto $\mathsf{BZ}_{k-1}(\lambda, \mu, \nu)$, for any $\lambda$, $\mu$, $\nu \in \mathsf{D}_k$.*

**5.3. Corollary.** *The linear operator $\Psi_k$ maps $\mathsf{H}_k$ surjectively onto $\mathsf{BZ}_{k-1}$, and $\mathsf{H}_k(\lambda, \mu, \nu)$ bijectively onto $\mathsf{BZ}_{k-1}(\lambda, \mu, \nu)$, for any $\lambda$, $\mu$, $\nu \in \mathsf{D}_k$.*

**5.4. Corollary.** $e(\mathsf{BZ}_{k-1}(\lambda, \mu, \nu)) = c^{\lambda}_{\mu\,\nu}$, *for any $\lambda$, $\mu$, $\nu \in \mathsf{D}_k$ with non-negative integer coefficients.*

It will follow from Lemma 6.1 and the proof of Theorem 5.2 that the cones $\mathsf{LR}_k$ and $\mathsf{H}_k$ are isomorphic to $\mathsf{BZ}_{k-1} \times \mathbb{R}^2$. One can embed the cone $\mathsf{BZ}_{k-1}$ into $\mathsf{LR}_k$ in the following way: For any $k \geq 2$, let $\Omega_k \colon W_{k-1} \longrightarrow T_k$ be the linear operator defined by $\Omega_k(x_{i\,j}, y_{i\,j}, z_{i\,j}) = (a_{i\,j})$ where

$$\begin{array}{rcl} a_{0\,j} & = & \sum_{l=j}^{k-1} x_{1\,l} + y_{1\,l} \quad \text{for } 1 \leq j < k, \quad \text{and} \quad a_{0\,k} = 0, \\ a_{i\,j} & = & y_{i\,j-1}, \quad \text{for } 1 \leq i < j \leq k, \\ a_{j\,j} & = & \sum_{l=j}^{k-1} z_{l\,l} \quad \text{for } 1 \leq j < k, \quad \text{and} \quad a_{k\,k} = 0. \end{array} \tag{4}$$

Then we have:

**5.5. Theorem.** *The linear operator $\Omega_k$ defined above maps $\mathsf{BZ}_{k-1}$ injectively into $\mathsf{LR}_k$, and $\mathsf{BZ}_{k-1}(\lambda, \mu, \nu)$ bijectively onto $\mathsf{LR}_k(\lambda, \mu, \nu)$ for any $\lambda$, $\mu$, $\nu \in \mathsf{D}_k$ such that $\mu_k = 0$ and $\nu_k = 0$.*

## 6 Proof of results

**Proof of Lemma 3.1.** Let $T$ be a Littlewood-Richardson tableau of shape $\lambda/\mu$ and content $\nu$, then $A_T$ satisfies (P) by definition. Since $T$ has strictly increasing columns (CS) follows, and since $w(T)$ is a lattice permutation, $A_T$ satisfies (LR). It is also clear that $A_T$ is of type $(\lambda, \mu, \nu)$. Conversely, for any Littlewood-Richardson triangle $A = (a_{ij})$ in $\mathsf{LR}_k(\lambda, \mu, \nu)$ with integer entries, we define a tableau $T_A$ of shape $\lambda/\mu$ by placing in row $j$, in weakly increasing order, $a_{ij}$ $i$'s for each $i$ and $j$. It is routine to check that $T$ is a Littlewood-Richardson tableau of shape $\lambda/\mu$ and content $\nu$, and that both constructions are inverses of each other. Here we use that (2) follows from (CS) and (LR). □

**Proof of Theorem 4.1.** Let $\{E_{ij}\}$ be the canonical basis of $T_k$, that is $E_{ij} = \left(e^{ij}_{pq}\right)$, where

$$e^{ij}_{pq} = \begin{cases} 1, & \text{if } p = i \text{ and } q = j; \\ 0, & \text{otherwise.} \end{cases}$$

We order it according to the lexicographic order of the subindices, that is,

$$\mathcal{B} = \{E_{01}, E_{02}, \ldots, E_{0k}, E_{11}, \ldots, E_{1k}, \ldots, E_{kk}\}.$$

The matrix of $\Phi_k$ with respect to $\mathcal{B}$ is lower triangular with ones on the main diagonal, therefore it has determinant one, is volume preserving, and maps $\mathbb{Z}^{\binom{k+2}{2}-1}$ bijectively onto $\mathbb{Z}^{\binom{k+2}{2}-1}$. The inverse of $\Phi_k$ is given by $\Phi_k^{-1}(h_{i\,j}) = (a_{i\,j})$ where

$$a_{i\,j} = \begin{cases} h_{0\,j} - h_{0\,j-1}, & \text{if } i = 0 \text{ and } 1 \leq j \leq k. \\ h_{j\,j} - h_{j-1\,j}, & \text{if } 1 \leq i = j \leq k. \\ h_{i\,j} - h_{i\,j-1} - h_{i-1\,j} + h_{i-1\,j-1}, & \text{if } 1 \leq i < j \leq k. \end{cases}$$

Let $(a_{i\,j}) \in \mathsf{LR}_k$ and $(h_{i\,j}) = \Phi_k(a_{i\,j})$, then we have

$$h_{s\,t} - h_{s\,t-1} = \sum_{p=0}^{s} a_{p\,t} \quad \text{and} \quad h_{s+1\,t} - h_{s\,t} = \sum_{q=s+1}^{t} a_{s+1\,q},$$

for $0 \leq s < t \leq k$. It is straightforward, using these two identities, to check that $(a_{i\,j})$ satisfies (P), (CS) or (LR), respectively, if and only if $(h_{i\,j})$ satisfies (R), (V) or (L), respectively; therefore $\Phi_k(\mathsf{LR}_k) = \mathsf{H}_k$. Also, it is straightforward to check that $(a_{i\,j})$ and $(h_{i\,j})$ have the same type; therefore $\Phi_k(\mathsf{LR}_k(\lambda,\mu,\nu)) = \mathsf{H}_k(\lambda,\mu,\nu)$, for all $\lambda$, $\mu$ and $\nu \in \mathsf{D}_k$. □

**Proof of Lemma 5.1.** We form a system of linear equations by taking, for each $1 \leq i \leq j < k$, that is, for each hexagon in $\Gamma_k$, equations (BZ2) and (BZ3). Then, after arranging the variables in the order $x_{1\,1}$, $y_{1\,1}$, $z_{1\,1}$, $x_{1\,2}$, $y_{1\,2}$, $z_{1\,2}$, $x_{2\,2}, \ldots, z_{k\,k}$, we easily check that the matrix of coefficients of the system is in echelon form and has rank $2\binom{k}{2}$. Thus $\dim W_k = 3\binom{k+1}{2} - 2\binom{k}{2} = \frac{1}{2}k(k+5)$. □

Before we prove Theorem 5.2, let us prove the following lemma.

**6.1. Lemma.** *The linear operators $\Psi_k$ and $\Psi_k \circ \Phi_k$ are surjective. Moreover, equations (5) give a full description of $(\Psi_k \circ \Phi_k)^{-1}(X)$ for any $X \in W_{k-1}$.*

**Proof.** It is enough to show that $\Psi_k \circ \Phi_k$ is surjective. Let $X = (x_{i\,j}, y_{i\,j}, z_{i\,j}) \in W_{k-1}$. For each $s$, $t \in \mathbb{R}$ we define an element $A_{s\,t} = (a_{i\,j}) \in T_k$ by

$$\begin{array}{rcl} a_{0\,j} & = & s + \sum_{l=j}^{k-1} x_{1\,l} + y_{1\,l} \quad \text{for } 1 \leq j < k, \quad \text{and} \quad a_{0\,k} = s, \\ a_{i\,j} & = & y_{i\,j-1}, \quad \text{for } 1 \leq i < j \leq k, \\ a_{j\,j} & = & t + \sum_{l=j}^{k-1} z_{l\,l} \quad \text{for } 1 \leq j < k, \quad \text{and} \quad a_{k\,k} = t. \end{array} \tag{5}$$

Let $X' = (x'_{i\,j}, y'_{i\,j}, z'_{i\,j}) = \Psi_k \circ \Phi_k(A_{s\,t})$. We claim that $X' = X$. By definition, $x'_{i\,j}$, $y'_{i\,j}$ and $z'_{i\,j}$ satisfy equations (3). Combining (3) and (5) we get that $y'_{i\,j} = a_{i\,j+1} = y_{i\,j}$ for all $1 \leq i \leq j < k$. Again, combining (3) and (5), we obtain

$$\begin{array}{rcl} x'_{i\,j} & = & (x_{1\,j} + y_{1\,j}) + \displaystyle\sum_{p=1}^{i-1} y_{p\,j-1} - \sum_{p=1}^{i} y_{p\,j} \\ & = & (x_{1\,j} + y_{1\,j-1} - y_{2\,j}) + \displaystyle\sum_{p=2}^{i-1} y_{p\,j-1} - \sum_{p=3}^{i} y_{p\,j}. \end{array}$$

Condition (BZ2) implies that $x_{1\,j} + y_{1\,j-1} - y_{2\,j} = x_{2\,j}$; and repeated application of (BZ2) yields $x'_{i\,j} = x_{i\,j}$ for all $1 \leq i \leq j < k$. Finally, the equality $z'_{i\,j} = z_{i\,j}$, is obtained in a similar way from (BZ1). Thus $\Psi_k \circ \Phi_k$ is surjective. The last statement follows from the identity $\dim T_k = \dim W_{k-1} + 2$. □

**Proof of Theorem 5.2.** It follows from (3) that $\Psi_k \circ \Phi_k(\mathsf{LR}_k) = \mathsf{BZ}_{k-1}$; and it follows from (3) and (B1)-(B3) that $A$ and $\Psi_k \circ \Phi_k(A)$ have the same type, for any $A \in \mathsf{LR}_k$, thus $\Psi_k \circ \Phi_k(\mathsf{LR}_k(\lambda, \mu, \nu)) = \mathsf{BZ}_k(\lambda, \mu, \nu)$. The last claim follows from the remark that different elements in the preimage of an $X \in \mathsf{BZ}_k(\lambda, \mu, \nu)$ have different types. □

**Proof of Theorem 5.5.** It follows from (3), (4) and the proof of Lemma 6.1 that $\Psi_k \circ \Phi_k \circ \Omega_k$ is the identity map on $W_{k-1}$, and that $\Omega(\mathsf{BZ}_{k-1}) \subseteq \mathsf{LR}_k$. The last statement follows from Theorem 5.2. □

## 7 Final remarks

Let us start with the complexity issues. Recall that the LR triangles, hives, and BZ triangles, all of size $k$, are given by $\theta(k^2)$ entries. As defined, maps $\Phi^{-1}$ and $\Psi$ require only a constant number of arithmetic operations per entry, and thus have $O(k^2)$ complexity. It is an easy exercise in dynamic programming to show that $\Phi$ and $\Psi^{-1}$ have the same complexity, linear in the input.

The complexity $O(k^2)$ is in stark contrast with the $O(k^3)$ complexity required by the jeu-de-taquin and Schützenberger involution (cf. [9, 19, 18]). This explains why Fulton's map in [7] has the same complexity. In fact, Fulton reworks the bijection of Carré [8] which establishes a combinatorial map $\Upsilon : e(\mathsf{LR}_k(\lambda, \mu, \nu)) \to e(\mathsf{H}_k(\lambda, \nu, \mu))$. As we mentioned in the introduction and will reiterate below, there is no linear map establishing the symmetry $\mathsf{H}_k(\lambda, \nu, \mu) \to \mathsf{H}_k(\lambda, \mu, \nu)$. One can use a more complicated map called tableaux switching to demonstrate this symmetry [3] (see also [15, 18]).

Now, the symmetries of the LR coefficients are quite intriguing in a sense that most of them can be established by simple means. If one operates with LR tableaux, one simply has to map them into BZ triangles (which takes $O(k^2)$ steps), perform the symmetry, and return back to LR tableaux (which takes $O(k^2)$ steps again). For the remaining $\mu \leftrightarrow \nu$ symmetry several authors found an explicit map (in different languages) [1, 2, 15, 20] but all of them use $\theta(k^3)$ steps (see [18] for the theory and the explanation). It would be interesting to prove the lower bound $\Omega(k^3)$ but we are doubtful such result is feasible at the moment. What one can show, however, is that this 'last' symmetry cannot be performed by a linear map already for $k = 4$. We leave this statement as an interesting exercise to the reader, and refer to a sequel paper [18] for further results in this direction.

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