

ON THE COMPLEXITY OF COMPUTING KRONECKER COEFFICIENTS

IGOR PAK* AND GRETA PANOVA*

ABSTRACT. We study the complexity of computing *Kronecker coefficients* $g(\lambda, \mu, \nu)$. We give explicit bounds in terms of the number of parts ℓ in the partitions, their largest part size N and the smallest second part M of the three partitions. When $M = O(1)$, i.e. one of the partitions is *hook-like*, the bounds are linear in $\log N$, but depend exponentially on ℓ . Moreover, similar bounds hold even when $M = e^{O(\ell)}$. By a separate argument, we show that the positivity of Kronecker coefficients can be decided in $O(\log N)$ time for a bounded number ℓ of parts and without restriction on M . Related problems of computing Kronecker coefficients when one partition is a hook, and computing characters of S_n are also considered.

1. INTRODUCTION AND MAIN RESULTS

The study of *Kronecker coefficients* $g(\lambda, \mu, \nu)$ of the symmetric group S_n has rare qualities of being both classical, highly technical, and largely mysterious. The area was initiated by Murnaghan 75 years ago [Mu1], and continued to be active for decades, with scores of interesting connections to other areas. Despite a large body of work on the Kronecker coefficients, both classical and very recent, it is universally agreed that “frustratingly little is known about them” [Bür]. The problem of finding a combinatorial interpretation for $g(\lambda, \mu, \nu)$, can be restated as whether computing the coefficients is in $\#P$. It remains a major open problem, one of the oldest unsolved problems in Algebraic Combinatorics.

More recently, the interest in computing Kronecker coefficients has intensified in connection with *Geometric Complexity Theory*, pioneered recently as an approach to the P vs. NP problem (see [M2, MS, R2]). With Valiant’s theory of determinant computations as its starting point, their approach relies, among other things, on the (conjectural) ability to decide in polynomial time the positivity of Kronecker coefficients and their plethystic generalizations. Envisioned as a far reaching mathematical program requiring over 100 years to complete [F2], this approach led to a flurry of activity in an attempt to understand and establish some critical combinatorial and computational properties of Kronecker coefficients (see [BOR1, BI1, CDW, Ike, M1]). This paper is a new advance in this direction.

While we present several algorithmic and complexity results, they are centered around a single unifying problem. We are trying to understand what exactly makes the Kronecker coefficients hard to compute. Since the problem is $\#P$ -hard in general (see [BI1]), a polynomial time algorithm for deciding positivity is unlikely to exist. On the other hand, the problem can be viewed as a generalization of LR coefficients $c_{\beta\gamma}^\alpha$, another $\#P$ -complete problem. In view of a nice geometric interpretation of the latter, it can be computed using *Barvinok’s algorithm* in polynomial time for any fixed ℓ (see §8.4). We show that a similar analysis applies to Kronecker coefficients. In other words, it is not the large part sizes that make an obstacle, but the number ℓ of parts in the partitions.

Date: April 13, 2014.

*Department of Mathematics, UCLA, Los Angeles, CA 90095, {pak,panova}@math.ucla.edu.

Our main result is the following theorem, which introduces a new parameter M , the smallest number among the second parts of the three partitions. Our complexity bound is general, but is especially sharp for triples of partitions when one is *hook-like*. We state the theorem here in a somewhat abbreviated form, as we postpone the definitions and details.

Main Theorem 3.5. *Let $\lambda, \mu, \nu \vdash n$ be partitions with lengths $\ell(\lambda), \ell(\mu), \ell(\nu) \leq \ell$, the largest parts $\lambda_1, \mu_1, \nu_1 \leq N$, and $\nu_2 \leq M$. Then the Kronecker coefficients $g(\lambda, \mu, \nu)$ can be computed in time*

$$O(\ell \log N) + (\ell \log M)^{O(\ell^3 \log \ell)}.$$

To illustrate the result, consider several special cases of the theorem. First, when ℓ is fixed, the recent breakthrough in [CDW] gives a polynomial time algorithm for computing Kronecker coefficients. The main theorem in this case is Main Lemma 5.4 which gives explicit bounds on the dependence of ℓ . Curiously, even for a simpler problem of deciding *positivity* $g(\lambda, \mu, \nu) > 0$ this gives the best known general bound. Our Theorem 6.1 uses the *semigroup property* of Kronecker coefficients to give a surprisingly powerful *linear* bound, but without giving explicitly the dependence on ℓ .

Second, when we have $\lambda_2, \mu_2, \nu_2 \leq M$ and N is large compared to $(M\ell)$, the Kronecker coefficients stabilize to *reduced Kronecker coefficients*, which generalize LR coefficients and are believed to be easier to compute (see § 8.10). Our Main Theorem lends further support to this conjecture.

Finally, when $M = O(1)$ the theorem gives a new type complexity bound of computing $g(\lambda, \mu, \nu) > 0$ when ν is *hook-like*. This may seem surprising, as already the case of *hooks* (i.e. when $M = 1$), received considerable attention in the literature (see e.g. [Las, Rem, Ros]). There, even in the simplest cases, the resulting formulas for Kronecker coefficients seem rather difficult, and the recent combinatorial interpretation by Blasiak unsuitable for efficient computation [Bla]. Curiously, we use Blasiak's combinatorial interpretation of Kronecker coefficients to show that computing $g(\lambda, \mu, \nu)$ is in $\#\text{P}$ when ν is a hook (Theorem 4.3).

Corollary 1.1 *In the notation of the Main Theorem, suppose*

$$\log M, \ell = O\left(\frac{(\log \log N)^{1/3}}{(\log \log \log N)^{2/3}}\right).$$

Then there is a polynomial time algorithm to compute $g(\lambda, \mu, \nu)$.

The proofs are based on two main tools. The first is the *Reduction Lemma* (Lemma 5.1), which implies that when ν_2 is small, we either immediately have $g(\lambda, \mu, \nu) = 0$, or else there are partitions $\varphi(\lambda), \varphi(\mu), \varphi(\nu)$ of size $O(\ell^3 \nu_2)$, such that $g(\lambda, \mu, \nu) = g(\varphi(\lambda), \varphi(\mu), \varphi(\nu))$. In other words, we reduce the problem from binary input to unary input, and apply Lemma 5.4 to the latter case.

The second tool is the *Main Lemma* 5.4 mentioned above, which gives an effective bound on the complexity of computing Kronecker coefficients without any restrictions on M . It coincides with the Main Theorem 3.5 when $M = N$, and states that the Kronecker coefficients can be computed in time $\text{Poly}((\ell \log N)^{\ell^3 \log \ell})$. This is achieved by separating the algebraic and complexity parts; the latter is reduced to counting integer points in certain 3-way statistical tables via Barvinok's algorithm (see §2.5 and §8.4).

The rest of this paper is structured as follows. We begin with basic definitions in Section 2 and proceed to state our new complexity results in Section 3. In Section 4, we discuss Blasiak's combinatorial interpretation and its implications. This section is largely separate from the rest of the paper and uses some background in Algebraic Combinatorics.

The main results of this paper, notably the Main Lemma and the Reduction Lemma are proved in Section 5. We follow with two short sections 6 and 7 discussing the case of bounded ℓ and the complexity of computing the characters of S_n , respectively. Namely, we prove that the problem of deciding whether $\chi^\lambda[\nu] = 0$ is NP-hard, extending earlier easy results by Hepler [Hep] (see §8.11 for the connection with Kronecker coefficients). We conclude with final remarks and open problems in Section 8.

2. DEFINITIONS AND BACKGROUND

We briefly remind the reader of basic definitions, standard notations and several claims which will be used throughout the paper. For more background on the representation theory of the symmetric group and related combinatorics, see e.g. [Mac, Sag, Sta].

2.1. Partitions and characters. Let $\lambda = (\lambda_1, \lambda_2, \dots) \vdash n$ be a partition of n , and let P_n denote the set of partitions of n . Denote by λ' the conjugate partition of λ . Denote by $\ell(\lambda) = \lambda'_1$ the number of parts in λ . We use Young diagram $[\lambda]$ to represent a partition λ . Partitions $(n - k, 1^k)$ are called *hooks*; partitions $(n - k, k)$ with two parts will also play a major role. We also define the union and intersection of two partitions as the union or intersection of their Young diagrams. In other words, $\pi = \lambda \cup \mu$ mean that $\pi_i = \max(\lambda_i, \mu_i)$, and $\rho = \lambda \cap \mu$ means that $\rho_i = \min(\lambda_i, \mu_i)$. Denote by $\lambda + \mu$ the partition $(\lambda_1 + \mu_1, \lambda_2 + \mu_2, \dots)$.

We denote by χ^λ , $\lambda \vdash n$, the irreducible character of the symmetric group S_n , and $\chi^\lambda[\mu]$ be its value $\chi^\lambda(u)$ on any permutation u of cycle type μ . For a general character η , the multiplicity of χ^λ in η is given by the scalar product:

$$c(\chi^\lambda, \eta) = \frac{1}{n!} \sum_{u \in S_n} \chi^\lambda(u) \eta(u).$$

We use “sign” to denote character corresponding to partition (1^n) .

The characters can be computed by the *Murnaghan–Nakayama rule* (see e.g. [Sag, Sta]). Briefly, it says that

$$\chi^\lambda[\mu] = \sum_{B \in \text{MN}_\mu^\lambda} (-1)^{ht(B) - \ell(\mu)},$$

where MN_μ^λ is the set of all border-strip tableaux of shape λ and type μ and $ht(B)$ is the sum of the number of rows (height) in each border-strip of B .

2.2. Kronecker coefficients. We use $\chi \otimes \eta$ to denote the tensor product of characters. The *Kronecker coefficients* $g(\lambda, \mu, \nu)$, where $\lambda, \mu, \nu \vdash n$ are given by

$$\chi^\lambda \otimes \chi^\mu = \sum_{\nu \vdash n} g(\lambda, \mu, \nu) \chi^\nu.$$

It is well known that

$$g(\lambda, \mu, \nu) = \frac{1}{n!} \sum_{\omega \in S_n} \chi^\lambda(\omega) \chi^\mu(\omega) \chi^\nu(\omega).$$

This implies that Kronecker coefficients have full S_3 group of symmetry:

$$g(\lambda, \mu, \nu) = g(\mu, \lambda, \nu) = g(\lambda, \nu, \mu) = \dots$$

In addition, recall that $\chi^\lambda \otimes \text{sign} = \chi^{\lambda'}$. This implies

$$g(\lambda, \mu, \nu) = g(\lambda', \mu', \nu) = g(\lambda', \mu, \nu') = g(\lambda, \mu', \nu').$$

2.3. Symmetric functions. We denote by h_n and $h_\lambda = h_{\lambda_1} h_{\lambda_2} \dots$ the *homogeneous symmetric functions*, and by s_λ the *Schur functions* (see e.g. [Mac, Sta]). The *Littlewood–Richardson (LR) coefficients* are denoted by $LR(\lambda, \mu, \nu) = c_{\mu, \nu}^\lambda$, where $|\lambda| = |\mu| + |\nu| = n$. They are given by

$$s_\mu \cdot s_\nu = \sum_{\lambda \vdash n} c_{\mu, \nu}^\lambda s_\lambda.$$

The integers $c_{\mu, \nu}^\lambda$ have a combinatorial interpretation in term of certain semistandard Young tableaux (see [Sag, Sta]) and *BZ triangles* (see e.g. [PV1]).

Define the *Kronecker product* of symmetric functions as follows:

$$s_\mu * s_\nu = \sum_{\lambda \vdash n} g(\lambda, \mu, \nu) s_\lambda.$$

The following *Littlewood’s identity* (see [L1]) is crucial for our study:

$$(1) \quad s_\lambda * (s_\tau s_\theta) = \sum_{\alpha \vdash |\tau|, \beta \vdash |\theta|} c_{\alpha\beta}^\lambda (s_\alpha * s_\tau) (s_\beta * s_\theta).$$

We also need the *generalized Cauchy’s identity* (see [Mac, Ex I.7.10] and [Sta, Ex 7.78])

$$(2) \quad \sum_{\lambda, \mu, \nu} g(\lambda, \mu, \nu) s_\lambda(x) s_\mu(y) s_\nu(z) = \prod_{i, j, \ell} \frac{1}{1 - x_i y_j z_\ell}.$$

Given a power series $F = a_0 + a_1 t + a_2 t^2 + \dots$, denote by $[t^i]F$ the coefficient a_i . Similarly, when F is a symmetric function and A is a Schur function, denote by $[A]F$ the coefficient of A in the expansion of F in the linear basis of Schur functions. By a slight abuse of notation, we use $[A]F$ for other bases of symmetric functions as well.

2.4. Semigroup property. The triples (λ, μ, ν) for which $g(\lambda, \mu, \nu) > 0$ form a semigroup in the following sense.

Theorem 2.1 (Semigroup property). *Suppose $\lambda, \mu, \nu, \alpha, \beta, \gamma$ are partitions of n , such that $g(\lambda, \mu, \nu) > 0$ and $g(\alpha, \beta, \gamma) > 0$. Then $g(\lambda + \alpha, \mu + \beta, \nu + \gamma) > 0$.*

This leads to the following definition of the *Kronecker semigroup*. Let \mathcal{K}_ℓ denote the set of triples of partitions (λ, μ, ν) written as vectors

$$(\lambda_1, \dots, \lambda_\ell, \mu_1, \dots, \mu_\ell, \nu_1, \dots, \nu_\ell),$$

such that $g(\lambda, \mu, \nu) > 0$. The theorem implies that \mathcal{K}_ℓ is a semigroup under addition.

Corollary 2.2. *The Kronecker semigroup \mathcal{K}_ℓ is finitely generated.*

Both results are proved in [CHM] (see also §8.8 and [Ike, §4.4]). We use them crucially in Section 6.

2.5. Barvinok’s algorithm. Let $P \subset \mathbb{R}^d$ be a convex polytope given by a system of linear equations and inequalities over integers. Denote by L the size of the input.

Theorem 2.3 (Barvinok). *For every fixed d , there is a polynomial algorithm computing the number of integer points in P . Furthermore, for general d , the algorithm works in $L^{O(d \log d)}$ time.*

The original algorithm by Barvinok required $L^{O(d^2)}$ time, which was subsequently reduced to that in the theorem. We refer to [Bar, BP] for the proof, detailed surveys and further references (see also [DHTY, DK]).¹

3. COMPLEXITY PROBLEMS

3.1. Decision problems. We are interested in deciding whether Kronecker coefficients $g(\lambda, \mu, \nu)$ are strictly positive.

POSITIVITY OF KRONECKER COEFFICIENTS (KP):

Input: Integers N, ℓ , partitions $\lambda = (\lambda_1, \dots, \lambda_\ell)$, $\mu = (\mu_1, \dots, \mu_\ell)$, $\nu = (\nu_1, \dots, \nu_\ell)$, where $0 \leq \lambda_i, \mu_i, \nu_i \leq N$, and $|\lambda| = |\mu| = |\nu|$.

Decide: whether $g(\lambda, \mu, \nu) > 0$.

Recall that the two ways to present the input: in *binary* and in *unary*. The difference is in the input size, denoted $\text{size}(\lambda, \mu, \nu)$: in the binary case we have $\text{size}(\lambda, \mu, \nu) = \Theta(\ell \log N)$, and in the unary case $\text{size}(\lambda, \mu, \nu) = \Theta(\ell N)$. Throughout the paper we assume the input is in binary, unless specified otherwise. The problem then becomes a well known conjecture:

Conjecture 3.1 (Mulmuley). $\text{KP} \in \text{P}$.

Note that it is not even known whether $\text{KP} \in \text{NP}$, except in a few special cases, such as when one of the input partitions is a hook. This case is considered in Section 4.

3.2. Counting problems. There are analogous problems about computing the exact values of the coefficients mentioned above.

KRONECKER COEFFICIENTS (KRON):

Input: Integers N, ℓ , partitions $\lambda = (\lambda_1, \dots, \lambda_\ell)$, $\mu = (\mu_1, \dots, \mu_\ell)$, $\nu = (\nu_1, \dots, \nu_\ell)$, where $0 \leq \lambda_i, \mu_i, \nu_i \leq N$, and $|\lambda| = |\mu| = |\nu|$.

Compute: the Kronecker coefficient $g(\lambda, \mu, \nu)$.

The main complexity result in the area is the following recent theorem.

Theorem 3.2 (Bürgisser–Ikenmeyer). $\text{KRON} \in \text{GapP}$.

Here GapP is a class of functions obtained as a difference of two functions in $\#\text{P}$. We give a different proof of the theorem in Section 5 (cf. [CDW]). We conclude with two more conjectures by Mulmuley [M1].

Conjecture 3.3 (Mulmuley). $\text{KRON} \in \#\text{P}$.

Conjecture 3.4 (Mulmuley). *The Kronecker coefficient $g(\lambda, \mu, \nu)$ is equal to the number of integer points in convex polytope $P(\lambda, \mu, \nu)$ with a polynomial description.*

¹Occasionally, this complexity is reported as $L^{O(d)}$, but it seems a more careful accounting gives the bound as in the theorem (A. Barvinok, personal communication).

This conjecture is the counting version of Conjecture 3.1. It extends the classical result by Gelfand and Zelevinsky, expressing LR coefficients as the number of integer points in convex polytopes (see e.g. [Zel])

The main result of the paper is the following theorem (the proof is in Section 5).

Theorem 3.5 (Main Theorem). *Consider the problem KRON, where the input is integers N, ℓ and partitions $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell), \mu = (\mu_1, \mu_2, \dots, \mu_\ell), \nu = (\nu_1, \dots, \nu_\ell)$, such that $\lambda_1, \mu_1, \nu_1 \leq N$ and $|\lambda| = |\mu| = |\nu|$. Suppose that $\nu_2 \leq M$. Then $g(\lambda, \mu, \nu)$ can be computed in time*

$$O(\ell \log N) + (\ell \log M)^{O(\ell^3 \log \ell)}.$$

Corollary 3.6 ([CDW]). *When ℓ is fixed, we have $\text{KP} \in \text{P}$ and $\text{KRON} \in \text{FP}$.*

Here FP is a class of functions computable in polynomial time, a counterpart of P for decision problems. Note that in Section 6, we use the semigroup property to prove the first part of the corollary.

4. THE CASE OF A HOOK

Here we consider separately the complexity of KP and KRON when one of the partitions involved is a hook.

Let $\nu = (n-t, 1^t)$ be a hook shape, and $\lambda, \mu \vdash n$, such that $\ell(\lambda), \ell(\mu) \leq k$ and $\lambda_1, \mu_1 \leq N$. Theorem 3.5 implies the following result in this case.

Corollary 4.1. *Let $\nu = (n-t, 1^t)$ be a hook, $\lambda_1, \mu_1 \leq N$ and $\ell(\lambda), \ell(\mu) \leq k$. Then $g(\lambda, \mu, \nu)$ can be computed in time*

$$O(k^2 \log N) + k^{O(k^6 \log k)}.$$

Proof. Recall that when $\ell(\lambda)\ell(\mu) < \ell(\nu)$ we have $g(\lambda, \mu, \nu) = 0$, see [Dvir]. Thus, when $t > k^2$ this implies $g(\lambda, \mu, \nu) = 0$. On the other hand, when the height of the hook $t \leq k^2$, apply Theorem 3.5 with $\ell = k^2$ and $M = 1$, to obtain the result. \square

Remark 4.2. Note that Lemma 5.3 and subsequently Lemma 5.4 below can be easily modified for partitions of different lengths. This more careful analysis in the proofs of lemmas can reduce the exponent in Corollary 4.1 from k^6 to k^4 . We omit this improvement for the sake of clarity.

We use the recent combinatorial interpretation by Blasiak in [Bla] as outlined below, to prove the following result.

Theorem 4.3. *When ν is a hook, we have $\text{KP} \in \text{NP}$ and $\text{KRON} \in \#\text{P}$.*

We consider barred and unbarred entries $\bar{1}, \bar{2}, \dots, 1, 2, \dots$, which we use to fill a Young tableau. Such tableau is called *semi-standard* if the entries are weakly increasing in both rows and columns, with no two equal barred numbers in the same row, and no two equal unbarred numbers in the same column. The *content* of a tableau A is a sequence (m_1, m_2, \dots) , where m_r is the total number of r and \bar{r} entries in A . Two orders are considered:

$$\text{natural order : } \quad \bar{1} < 1 < \bar{2} < 2 < \dots \quad \text{and}$$

$$\text{small barred order : } \quad \bar{1} \prec \bar{2} \prec \dots \prec 1 \prec 2 \prec \dots$$

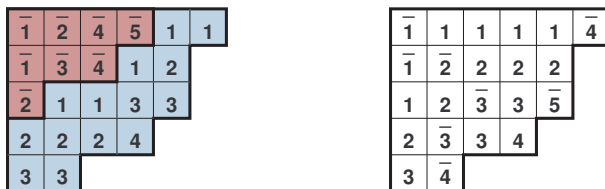


FIGURE 1. Semi-standard Young tableaux of shape $(6, 5^2, 4, 2)$ with small barred and natural order, of the same content $(7, 6, 5, 3, 2)$.

An example of two tableaux with different orders but the same shape and content is given in Figure 1.

There is a natural *tableau switching* bijection $A \leftrightarrow \varkappa(A)$ between semi-standard tableaux with natural and small bar ordering, preserving the shape and content. The idea is to make a number of jeu-de-taquin slides exchanging barred and unbarred entries. We skip the rather technical definition.

Note that one can view a small bar order as a pair of two semi-standard Young tableaux A_1 and A_2 , one of shape λ/ν and one of shape ν' , respectively. Here A_1 is the subtableau consisting of the unbarred entries of A , and A_2 is the conjugate of the subtableau of barred entries of A . Denote by $w(A)$ the word obtained by reading A_2 right to left, top row to bottom row, concatenated with the word obtained by reading A_1 left to right, bottom row to top row. For example, for A as in the figure, we have

$$w(A) = 11.21.3311.4222.33..2.134.1245.$$

By $m_i(w)$ denote the number of i -s in w . Word w is called a *ballot sequence* if in its every prefix w' we have $m_1(w') \geq m_2(w') \geq \dots$. For example, $w(A)$ as above is not a ballot sequence as can be seen for the prefix $w' = 112133$, with $m_2(w') = 1 < m_3(w') = 2$.

Theorem 4.4 (Blasiak). *Let $\lambda, \mu \vdash n$ and $\nu = (n - k, 1^k)$. Then $g(\lambda, \mu, \nu)$ is equal to the number of small bar semi-standard tableaux A of shape λ , content μ , with k barred entries, such that $w(A)$ is a ballot sequence, and such that the lower left corner of $\varkappa(A)$ is unbarred.*

Although we never defined tableaux switching, as originally defined in [BSS] only gives a pseudo-polynomial time algorithm. Below we show that one can speed up the tableaux switching to be able to check the validity of Blasiak's tableaux in polynomial time (cf. [PV2]).

Proof of Theorem 4.3. First, we check whether $\ell(\nu) = k + 1$ is greater than $\ell(\lambda)\ell(\mu)$. It follows from [Dvir] that in the case we have $g(\lambda, \mu, \nu) = 0$.

Now, if this is not the case, we must have $k \leq \ell(\lambda)\ell(\mu)$, so k is polynomial in the size of the input. Proceed as follows.

The polynomial witness for KP is a small-bar semi-standard tableaux A of shape λ , content μ and k barred entries. We encode the tableaux as a pair of one $\ell(\mu) \times \ell(\lambda)$ matrix and one $2 \times k$ matrix. In the first matrix, the entry at position (i, j) records the number of i 's in row j . In the second matrix, each column corresponds to a single barred letter and contains the coordinates of its position in the tableaux. This encoding requires $\ell(\mu)\ell(\lambda) \log N + 2k \log N$ bits. As noted above, this is polynomial in the input size.

We show that tableau-switching can be done in polynomial time. Indeed, switching a single \bar{i} and all unbarred letters j can be done in one step, as follows. If in the process, \bar{i} becomes adjacent to a letter j for the first time then the letter \bar{i} should move either to the

row below or to the end of the horizontal strip of letters j . Since there are $k = \ell(\nu) - 1$ barred letters, the tableaux-switching is done in polynomially many operations.

By Theorem 4.4, KRON counts the number of tableaux as in the theorem. From the above argument, we can verify that they satisfy the condition in the Theorem in polynomial time. Moreover, the number of such tableaux is at most exponential in the input size. Since $\#\mathbf{P}$ is closed under exponential summation [F1], this implies the result. \square

5. COMPLEXITY OF KRON AND THE PROOFS

We turn towards the computational complexity of KRON as defined in Section 3. Here we prove Theorem 3.5 and related results. We first establish our main tool for this, the Reduction Lemma.

5.1. The Reduction Lemma. In order to prove these statements we will need the following Lemma. Informally, it states that under the conditions of Theorem 3.5 we can either conclude that $g(\lambda, \mu, \nu) = 0$ or reduce the computation of $g(\lambda, \mu, \nu)$ to the computation of a Kronecker coefficient for much smaller partitions.

Lemma 5.1 (Reduction Lemma). *Given integers n, ℓ , let $\lambda, \mu, \nu \vdash n$ and $\ell(\lambda), \ell(\mu), \ell(\nu) \leq \ell$. We have the following two cases:*

- (i) *If $|\lambda_i - \mu_i| > n - \nu_1$ for some i , then $g(\lambda, \mu, \nu) = 0$,*
- (ii) *If $|\lambda_i - \mu_i| \leq n - \nu_1$ for all i then $g(\lambda, \mu, \nu) = g(\varphi(\lambda), \varphi(\mu), \varphi(\nu))$, for certain partitions $\varphi(\lambda), \varphi(\mu), \varphi(\nu)$ (defined in the proof), for which $|\varphi(\lambda)| = |\varphi(\mu)| = |\varphi(\nu)| \leq 2(n - \nu_1)\ell^2$.*

The rest of this subsection is the proof of this lemma.

Preliminaries. We start with a few observations and a setup for the proof.

Let $t = n - \nu_1$. Clearly, by the definition of $*$, we have $s_\alpha * s_{n-r} = s_\alpha$ for all $\alpha \vdash n - r$. Then Littlewood's identity (1), gives

$$[s_\tau](s_\sigma * (s_{n-r} s_\zeta)) = \sum_{\alpha \vdash n-r, \beta \vdash r} c_{\alpha\beta}^\sigma [s_\tau] s_\alpha (s_\beta * s_\zeta)$$

for any three partitions $\tau, \sigma, (n - r, \zeta) \vdash n$. Expressing the remaining Kronecker products as Schur functions via

$$s_\beta * s_\zeta = \sum_{\gamma \vdash r} g(\beta, \zeta, \gamma) s_\gamma,$$

we conclude that the coefficient of s_τ can be obtained as

$$(3) \quad [s_\tau](s_\sigma * (s_{n-r} s_\zeta)) = \sum_{\alpha \vdash n-r, \beta \vdash r, \gamma \vdash r} c_{\alpha\beta}^\sigma c_{\alpha\gamma}^\tau g(\beta, \zeta, \gamma)$$

Let also $s = n - \nu_1$ and let $\pi = (\nu_2, \dots, \nu_\ell)$, so that $\nu = (n - s, \pi)$.

Proof of part (i). The Pieri rule gives

$$s_{n-s} s_\pi = s_{(n-s, \pi)} + \sum_{\eta \in P_\nu} s_\eta,$$

where P_ν is the set of partitions η obtained by adding a horizontal strip of length $n - s$ to π and such that $\eta \neq \nu$. Observe that in this case the last condition is equivalent to $\eta_1 > n - s$. Taking a Kronecker product with s_λ gives

$$s_\lambda * (s_{n-s} s_\pi) = s_\lambda * s_\nu + \sum_{\eta \in P_\nu} s_\lambda * s_\eta.$$

Extracting the coefficient at s_μ in the Schur function expansion, we get

$$(4) \quad [s_\mu](s_\lambda * (s_{n-s} s_\pi)) = g(\lambda, \mu, \nu) + \sum_{\eta \in P_\nu} g(\lambda, \mu, \eta).$$

Suppose that $|\lambda_i - \mu_i| > t \geq |\pi| = s$ and $g(\lambda, \mu, \nu) > 0$. Substituting $\tau \leftarrow \mu$, $\sigma \leftarrow \lambda$, $r \leftarrow s$ and $\zeta \leftarrow \pi$ in equation (3), and comparing the two equations we get

$$0 < g(\lambda, \mu, \nu) + \sum_{\eta \in P_\nu} g(\lambda, \mu, \eta) = \sum_{\alpha \vdash n-s, \beta \vdash s, \gamma \vdash s} c_{\alpha\beta}^\lambda c_{\alpha\gamma}^\mu g(\beta, \pi, \gamma).$$

Since the RHS above is strictly positive, there must be some $\alpha \vdash n - s$ and $\beta \vdash s$, for which both $c_{\alpha\beta}^\lambda, c_{\alpha\gamma}^\mu > 0$. For that to happen we must have $\alpha \subseteq \lambda$ and $\alpha \subseteq \mu$. Therefore, $\alpha \subseteq \lambda \cap \mu$, and in particular, $|\lambda \cap \mu| \geq n - s$. On the other hand,

$$|\lambda \cap \mu| \leq |\lambda| - |\lambda_i - \mu_i| < n - t \leq n - s,$$

a contradiction. This establishes part (i). \square

Proof of part (ii). Suppose now that $|\lambda_i - \mu_i| \leq t$ for all i . Let $\omega = \lambda \cup \mu$ and $\rho = \lambda \cap \mu$. Then we have $\rho \subseteq \omega$ and $\omega_i - \rho_i \leq t$. Fixing these ρ and ω for every partition θ , s.t. $\rho - t^\ell \subseteq \theta$ (here negative parts in partitions are treated as 0), we construct another partition $\varphi(\theta)$ based on ρ and ω as described below. The idea is to shift the connected components in the Young diagram of ω/ρ and then perform the same shifts of parts in θ , so that the resulting partitions are much smaller, but the skew shape $\theta/(\rho - t^\ell)$ is preserved under φ .

Denote by I the set of rows where the skew shape $\omega/(\rho - t^\ell)$ is disconnected, i.e. $I = \{i : \rho_i \geq \omega_{i+1} + t, 1 \leq i \leq \ell\}$. We assume that all partitions have $(\ell + 1)$ st part equal to 0, so $\omega_{\ell+1} = 0$. Note that each connected component in the skew diagram $\omega/(\rho - t^\ell)$ is of size at most $2t$ times its height, since each row has at most $2t$ boxes. For every index j , let $i_j \in I$ be the smallest index in I which is larger than j , i.e. $i_j = \min\{i \in I, i \geq j\}$. In other words, i_j is the index of the lowest row in the connected component containing row j . Set

$$\varphi(\omega)_j = \omega_j - \rho_{i_j} + t(\ell + 1 - i_j) \quad \text{and} \quad \varphi(\rho)_j = \rho_j - \rho_{i_j} + t(\ell + 1 - i_j)$$

We see that $\varphi(\omega)_j \leq t(i_j - j + 1) + t(\ell + 1 - i_j) = t(\ell + 2 - j)$, so $|\varphi(\omega)| < 2t\ell^2$. Moreover, it is easy to see that the skew shape $\varphi(\omega)/\varphi(\rho)$ can be obtained from ω/ρ via horizontal shifts of connected components, since the connected components in both are identical.

For any partition θ , such that $\theta_i \geq \rho_i - t$ for all i , we now define

$$\varphi(\theta) := \theta - \rho + \varphi(\rho).$$

For any integer m we also define $\varphi(m) := m - |\rho| + |\varphi(\rho)|$, so that if $m = |\theta|$ then $\varphi(m) = |\varphi(\theta)|$. Last, we set $\phi(\nu) = (\phi(\nu_1), \nu_2, \dots, \nu_\ell)$. Observe that the connected components of θ/ρ are identical to the connected components of $\varphi(\theta)/\varphi(\rho)$, and similarly for the components of ω/θ and $\varphi(\omega)/\varphi(\theta)$, $\theta/(\rho - t^\ell)$ and $\varphi(\theta)/\varphi(\rho - t^\ell)$, respectively. See Figure 2 and Example 5.2 for an illustration of the map φ .

Let r be as in the preliminaries and consider equation (3), so $r \leq t$. We have $\alpha \vdash n - r$ and $\beta, \gamma \vdash r$. Observe that it suffices to consider only nonzero terms in the rhs of (3). Then, the condition $c_{\alpha\beta}^\lambda c_{\alpha\gamma}^\mu > 0$ implies that $\alpha \subseteq \lambda \cap \mu = \rho$. Since $|\rho| - |\alpha| \leq |\rho| - n + r \leq r \leq t$, we also have $\rho - t^\ell \subseteq \alpha$ and we can construct $\varphi(\alpha)$. Note that $c_{\alpha\beta}^\lambda$ is equal to the number of LR tableaux of shape λ/α of type β . Since the shapes λ/α and $\varphi(\lambda)/\varphi(\alpha)$ have identical connected components, the numbers of these LR tableaux are equal, i.e.

$$c_{\alpha\beta}^\lambda = c_{\varphi(\alpha)\beta}^{\varphi(\lambda)} \quad \text{and} \quad c_{\alpha\gamma}^\mu = c_{\varphi(\alpha)\gamma}^{\varphi(\mu)}.$$

Moreover, for any integer r , such that $r \leq t$, and partitions $\varphi(\alpha)$ and β , such that $\beta \vdash r$ and $\tilde{\alpha} \vdash n - r - |\rho| + |\varphi(\rho)| = \varphi(n - r)$, for which

$$c_{\tilde{\alpha}\beta}^{\varphi(\lambda)} > 0 \quad \text{and} \quad c_{\tilde{\alpha}\gamma}^{\varphi(\mu)} > 0,$$

the operation φ has an inverse. In other words, there is a unique $\alpha = \tilde{\alpha} + \rho - \varphi(\rho)$, such that $\varphi(\alpha) = \tilde{\alpha}$, by definition. We conclude that φ is a bijection on the set of partitions α for which the corresponding LR coefficients are nonzero.

Applying φ and the above observations, we can now rewrite (3) with $\tau \leftarrow \mu$, $\sigma \leftarrow \lambda$, $\zeta \leftarrow \eta$ and then apply it again for $\tau \leftarrow \varphi(\mu)$, $\sigma \leftarrow \varphi(\lambda)$. We have

$$\begin{aligned} (5) \quad [s_\mu] (s_\lambda * (s_{n-r} s_\eta)) &= \sum_{\alpha \vdash n-r, \beta \vdash r, \gamma \vdash r} c_{\alpha\beta}^\lambda c_{\alpha\gamma}^\mu g(\beta, \eta, \gamma) \\ &= \sum_{\alpha \vdash n-r, \beta \vdash r, \gamma \vdash r} c_{\varphi(\alpha)\beta}^{\varphi(\lambda)} c_{\varphi(\alpha)\gamma}^{\varphi(\mu)} g(\beta, \eta, \gamma) \\ &= \sum_{\tilde{\alpha} \vdash \varphi(n-r), \beta \vdash r, \gamma \vdash r} c_{\tilde{\alpha}\beta}^{\varphi(\lambda)} c_{\tilde{\alpha}\gamma}^{\varphi(\mu)} g(\beta, \eta, \gamma) \\ &= [s_{\varphi(\mu)}] (s_{\varphi(\lambda)} * (s_{\varphi(n-r)} s_\eta)). \end{aligned}$$

We now show that this leads to part (ii) in the lemma. The idea is that s_ν , for $\nu = (n - s, \pi)$, can be expressed as a linear combination of terms of the form $s_{n-r} s_\eta$ with $r \leq s \leq t$. We then apply (5) to obtain the desired equality for the Kronecker coefficients.

By the Jacobi-Trudi identity, for any partition ξ , we have

$$\begin{aligned} (6) \quad s_\xi &= \det [h_{\xi_i - i + j}]_{i,j=1}^\ell = \sum_{j=1}^\ell (-1)^{j-1} h_{\xi_1 - 1 + j} \det [h_{\xi_i - i + l}]_{i=2, k=1, l \neq j}^\ell \\ &= \sum_{j=0}^{\ell-1} (-1)^j h_{\xi_1 + j} \left(\sum_{\eta \vdash s-j} c_\eta s_\eta \right) = \sum_{j=0}^{\ell-1} \sum_{\eta \vdash n-j} (-1)^j c_\eta s_{\xi_1 + j} s_\eta. \end{aligned}$$

Here we expand the $(\ell - 1) \times (\ell - 1)$ determinants of homogenous symmetric functions as sums of Schur functions, so c_η are the coefficients with which they appear in the sum.

Finally, applying (5) with $\nu = (n - s, \pi)$ and (6) with $\xi \leftarrow \nu$, we have

$$\begin{aligned}
 g(\lambda, \mu, \nu) &= [s_\mu] (s_\lambda * s_\nu) = [s_\mu] \left\{ s_\lambda * \left(\sum_{j=0}^{\ell-1} \sum_{\eta \vdash n-j} (-1)^j c_\eta s_{n-s+j} s_\eta \right) \right\} \\
 &= \sum_{j=0}^{\ell-1} \sum_{\eta \vdash n-j} (-1)^j c_\eta [s_\mu] (s_\lambda * (s_{n-s+j} s_\eta)) \\
 &= \sum_{j=0}^{\ell-1} \sum_{\eta \vdash n-j} (-1)^j c_\eta [s_{\varphi(\mu)}] (s_{\varphi(\lambda)} * (s_{\varphi(n-s+j)} s_\eta)) \\
 &= [s_{\varphi(\mu)}] (s_{\varphi(\lambda)} * \sum_{j=0}^{\ell-1} \sum_{\eta \vdash n-j} (-1)^j c_\eta s_{\varphi(n-s)+j} s_\eta) \\
 &= [s_{\varphi(\mu)}] (s_{\varphi(\lambda)} * s_{(\varphi(n-s), \pi)}) = g(\varphi(\mu), \varphi(\lambda), \varphi(\nu)).
 \end{aligned}$$

Here the penultimate equality comes from application of (6) with $\xi_1 = \varphi(n - s)$. The last equality follows from the fact that $\varphi(\nu) = (\varphi(n - s), \pi)$, since $|\pi| \leq t$. This completes the proof of the lemma. \square

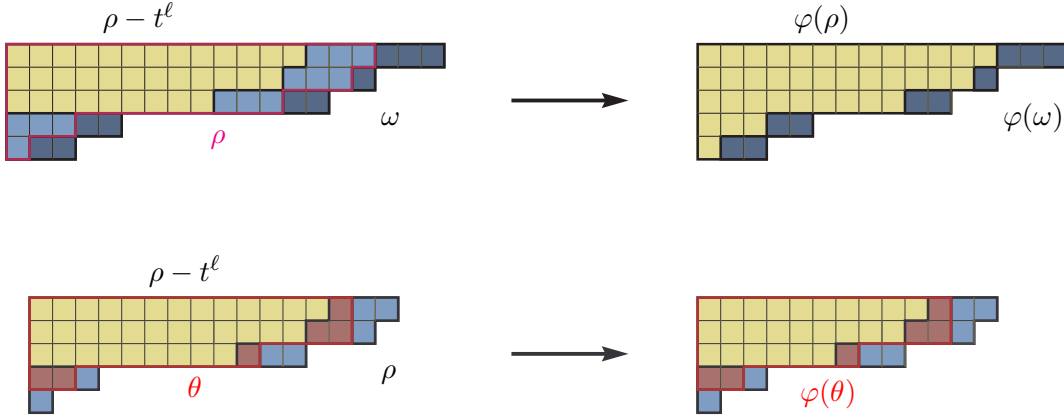


FIGURE 2. Example of the map φ .

Example 5.2. Let $\lambda = (19, 15, 12, 5, 1)$, $\mu = (16, 16, 14, 3,)$ and $t = 3$. These partitions give $\omega = (19, 16, 14, 5, 3)$ and $\rho = (16, 15, 12, 3, 1)$. The relevant partitions are displayed in the top picture of Figure 2: the dark blue area is the skew shape ω/ρ , whereas all blue areas represent the connected components of $\omega/(\rho - 3^5)$. We have that $I = \{3, 6\}$ and see that the map φ shifts the top 3 parts of any partition 3 boxes to the left and leaves the bottom parts intact, giving $\varphi(\omega) = (16, 13, 11, 5, 3)$ and $\varphi(\rho) = (13, 12, 9, 3, 1)$. Now, for $\theta = (14, 14, 10, 2)$, we have $\varphi(\theta) = (11, 11, 7, 2)$, as in the figure.

5.2. Proofs of complexity results. The following result (Main Lemma) gives a bound on the computational complexity of Kronecker coefficients in the general case. Together with the Reduction Lemma it gives Theorem 3.5. Incidentally, its proof can be used to derive Theorem 3.2, as we explain.

Before we proceed, we need the following technical result. For integer vectors $a, b, c \in \mathbb{Z}_{\geq 0}^\ell$, denote by $C(a, b, c)$ the number of *three dimensional contingency arrays* (3-way statistical tables) with 2-way marginals a, b, c .

Lemma 5.3. *In the notation above, let $R = \max\{a_i, b_i, c_i \mid i = 1, \dots, \ell\}$. Then the number $C(a, b, c)$ can be computed in time $(\ell \log R)^{O(\ell^3 \log \ell)}$.*

Proof. Note that these contingency arrays are just the integer points in a polytope in $\mathbb{R}^{\ell^3} = \{(\dots, x_{ijk}, \dots) \mid i, j, k = 1, \dots, \ell\}$, defined by the ℓ^3 inequalities

$$x_{ijk} \geq 0, \text{ for } 1 \leq i, j, k \leq \ell$$

and the 3ℓ equations

$$\sum_{j,k} x_{ijk} = a_i, \quad \sum_{i,k} x_{ijk} = b_j, \quad \sum_{i,j} x_{ijk} = c_k.$$

Recall Barvinok's algorithm (Theorem 2.3), which computes the number of integer points in a polytope of dimension d and input size L in time $L^{O(d \log d)}$. Here we have $L = O(\ell^3 \log R)$ and $d = O(\ell^3)$, which gives the result. \square

Lemma 5.4 (Main Lemma). *Let $\alpha, \beta, \gamma \vdash n$ be partitions of the same size, such that $\alpha_1, \beta_1, \gamma_1 \leq m$ and $\ell(\alpha), \ell(\beta), \ell(\gamma) \leq \ell$. Then $g(\alpha, \beta, \gamma)$ can be computed in time $(\ell \log m)^{O(\ell^3 \log \ell)}$.*

Proof. Consider three sets of variables, $x = (x_1, \dots, x_\ell)$, $y = (y_1, \dots, y_\ell)$ and $z = (z_1, \dots, z_\ell)$. We use the determinantal formula for Schur functions:

$$s_\theta(x_1, \dots, x_\ell) = \frac{\det \left[x_i^{\theta_i + \ell - j} \right]_{i,j=1}^\ell}{\Delta(x)},$$

where Δ is the Vandermonde determinant.

Thus, for a symmetric function $F(x_1, \dots, x_\ell)$, we have

$$[s_\theta] F = \left[x_1^{\theta_1 + \ell - 1} \dots x_\ell^{\theta_\ell} \right] \Delta(x) F(x),$$

i.e. the coefficient of s_θ in the expansion of F as a linear combination of Schur functions is equal to the coefficient of the respective monomial in the expansion of the polynomial $\Delta(x)F(x)$ as a sum of monomials. Applying this to identity (2) and using the staircase partition $\delta = (\ell - 1, \dots, 1, 0)$, gives

$$(7) \quad \begin{aligned} g(\alpha, \beta, \gamma) &= [s_\alpha(x)s_\beta(y)s_\gamma(z)] \prod_{i,j,k} \frac{1}{1 - x_i y_j z_k} = [x^{\alpha+\delta} y^{\beta+\delta} z^{\gamma+\delta}] \Delta(x)\Delta(y)\Delta(z) \frac{1}{1 - x_i y_j z_k} \\ &= [x^{\alpha+\delta} y^{\beta+\delta} z^{\gamma+\delta}] \prod_{1 \leq i < j \leq \ell} (x_i - x_j)(y_i - y_j)(z_i - z_j) \prod_{i,j,k=1}^\ell (1 + x_i y_j z_k + \dots + (x_i y_j z_k)^{m+\ell}). \end{aligned}$$

Here we truncated the infinite sums $(1 - x_i y_j z_k)^{-1}$ to the maximal powers which could be involved in computing the necessary coefficients.

Using the notation from Lemma 5.3 we have that

$$\prod_{i,j,k=1}^\ell (1 + x_i y_j z_k + \dots + (x_i y_j z_k)^{m+\ell}) = \sum_{a,b,c \in \{0,1,\dots,m+\ell\}^\ell} C(a, b, c).$$

Equation (7) then gives

$$(8) \quad g(\alpha, \beta, \gamma) = \sum_{\sigma^1, \sigma^2, \sigma^3 \in S_\ell} \text{sgn}(\sigma^1 \sigma^2 \sigma^3) C(\alpha + 1 - \sigma^1, \beta + 1 - \sigma^2, \gamma + 1 - \sigma^3),$$

where the sum goes over triples of permutations on $[1, \dots, \ell]$ and

$$\alpha + 1 - \sigma^1 = (\alpha_1 + 1 - \sigma_1^1, \dots, \alpha_\ell + 1 - \sigma_\ell^1),$$

and similarly for the other terms. Applying Lemma 5.3 for each of the summands in the above sum, we get that the Kronecker coefficient is computed in time

$$(\ell^2 \log(m + \ell))^{O(\ell^3 \log \ell)} (\ell!)^3 = (\ell \log m)^{O(\ell^3 \log \ell)},$$

as desired. \square

Proof of Theorem 3.2. Notice that (8) can be presented as the following difference: the number of contingency 3d arrays with marginals of the form $\alpha + 1 - \sigma^1, \beta + 1 - \sigma^2, \gamma + 1 - \sigma^3$, where $\text{sgn}(\sigma^1 \sigma^2 \sigma^3) = 1$, minus the number of contingency 3d arrays with marginals of the form $\alpha + 1 - \sigma^1, \beta + 1 - \sigma^2, \gamma + 1 - \sigma^3$, where $\text{sgn}(\sigma^1 \sigma^2 \sigma^3) = -1$. Since each of these two numbers counts polynomially verifiable objects, and there are $(\ell!)^3/2$ many of them, we conclude the numbers are in $\#\text{P}$. This implies the result. \square

Proof of Theorem 3.5. Let $n = |\nu|$ and $t = n - \nu_1$. Then we have $t \leq \ell M$. In at most ℓ steps we can check whether $|\lambda_i - \mu_i| > t$ for some i , in which case part (i) of Lemma 5.1 immediately implies that $g(\lambda, \mu, \nu) = 0$. Otherwise, part (ii) of the Reduction Lemma 5.1 implies that $g(\lambda, \mu, \nu) = g(\varphi(\lambda), \varphi(\mu), \varphi(\nu))$. Note that, by the definition of ϕ , we can compute $\varphi(\theta)$ in at most 3ℓ steps.

In the Main Lemma 5.4, let $\alpha = \varphi(\lambda), \beta = \varphi(\mu), \gamma = \varphi(\nu)$ and $m \leq 2t\ell^2 \leq 2M\ell^3$. Then the lemma implies that the Kronecker coefficients can be found in time

$$O(\ell \log N) + (\ell^2 \log M)^{O(\ell^3 \log \ell)},$$

where the term $O(\ell \log N)$ comes from the initial comparison of the parts of the partitions and the application of ϕ . \square

6. PARTITIONS OF FIXED LENGTHS

Here we consider the Kronecker Positivity problem KP when ℓ is fixed. While the Main Lemma 5.4 already gives that $\text{KP} \in \text{P}$ in this case, we present a different proof and a new complexity bound for this result.

Theorem 6.1. *When ℓ is fixed, the problem $\text{KP} \in \text{P}$. Moreover, for $\lambda_1, \mu_1, \nu_1 \leq N$ and $\ell(\lambda), \ell(\mu), \ell(\nu) \leq \ell$, there is an algorithm which decide whether $g(\lambda, \mu, \nu) > 0$ in time $O(\log N)$.*

Proof of Theorem 6.1. Let $v_1, \dots, v_k \in \mathbb{Z}_+^{3\ell}$ be the basis of generators of the Kronecker semigroup \mathcal{K}_ℓ (see Section 2.4). Let A be the $(3\ell) \times k$ matrix whose columns are the vectors v_1, \dots, v_k , i.e. $A = [v_1 \dots v_k]$. Deciding whether $g(\lambda, \mu, \nu) > 0$ is equivalent to deciding whether the vector $X = (\lambda, \mu, \nu)^T$ is a nonnegative integral combination of the vectors v_1, \dots, v_k . In other words, we need to decide whether $AY = X$ has a solution $Y \in \mathbb{Z}_+^{3\ell}$.

Since ℓ and k are fixed constants, the matrix A is also fixed. The first part of the theorem follows from Lenstra's theorem stating that this integer linear program (ILP) can be solved in polynomial time in the size of the input (i.e. X) for every fixed dimension (see e.g. [Sch]). The second part follows from more recent results on the complexity of feasibility of ILP (see [Eis]). \square

Remark 6.2. Observe that the dependence on $(\log N)$ in the theorem is linear, rather than polynomial of degree (ℓ^3) in the Main Lemma 5.4, making this algorithm more efficient. Note also that the proof of Corollary 2.2 of the finite generation of \mathcal{K}_ℓ given in [CHM] is inexplicit. Thus we have no control over the size $k = k(\ell)$ as in the proof, not even whether this is a computable function of ℓ . Getting any bounds in this direction would be interesting, as it would make effective the constant implied by the $O(\cdot)$ notation.

7. COMPLEXITY OF DECIDING WHETHER A CHARACTER IS ZERO

We now consider an analogue of the KP problem, which we call CHARP. Note that complexity of characters has been studied before, see [Hep] for a treatment of the problem when the input is in unary and a conclusion that the decision problem is then PP-complete.

IS THE CHARACTER $\chi^\lambda[\nu] = 0$ (CHARP):

Input: Integers N, ℓ , partitions $\lambda = (\lambda_1, \dots, \lambda_\ell)$, $\nu = (\nu_1, \dots, \nu_\ell)$, where $0 \leq \lambda_i, \nu_i \leq N$, and $|\lambda| = |\nu|$.

Decide: whether $\chi^\lambda[\nu] = 0$.

The main result of this section is the following theorem.

Theorem 7.1. *The problem CHARP is NP-hard.*

The proof follows from the following observation, implying that CHARP is at least as hard as the Knapsack problem.

Proposition 7.2. *Suppose that $\ell(\lambda) = 2$ and $\nu_i \equiv 0 \pmod{2}$. Then CHARP is NP-hard.*

Proof. Let $\lambda = (n - k, k)$. Frobenius' formula (Jacobi-Trudi identity) gives

$$\chi^\lambda = \chi^{(n-k)} \circ \chi^{(k)} - \chi^{(n-k+1)} \circ \chi^{(k-1)} = \chi^{(n,k)/(k)} - \chi^{(n,k-1)/(k-1)}.$$

We evaluate the characters for skew shapes by the usual Murnaghan–Nakayama rule. In this case the height of each border strip has to be 1 to fit into skew tableaux consisting of disconnected rows. For any multiset of positive integers $R = \{r_1, \dots, r_q\}$ and any integer m , denote by $P_R(m)$ the number of ways to write m as a sum of entries from R . In other words

$$P_R(m) = \#\{(i_1, i_2, \dots) : 1 \leq i_1 < i_2 < \dots \leq q, \text{ s.t. } r_{i_1} + r_{i_2} + \dots = m\}.$$

For any a and b the Murnaghan–Nakayama rule gives

$$\chi^{(a+b,a)/(a)}[\nu] = P_{\mathcal{X}}(a),$$

where $\mathcal{X} = \{\nu_1, \dots, \nu_\ell\}$. Hence

$$\chi^\lambda[\nu] = P_{\mathcal{X}}(k) - P_{\mathcal{X}}(k - 1).$$

Suppose that $k \equiv 0 \pmod{2}$, then $P_{\mathcal{X}}(k-1) = 0$, so $\chi^\lambda[\nu] = 0$ if and only if $P_{\mathcal{X}}(k) = 0$. The problem to determine whether $P_{\mathcal{X}}(k) = 0$ is the classical *Knapsack problem*:

Given the input $(k, \nu_1, \dots, \nu_\ell)$, determine whether there are $\epsilon_i \in \{0, 1\}$ for $i = 1, \dots, \ell$, such that

$$k = \sum_{i=1}^{\ell} \epsilon_i \nu_i.$$

This decision problem is well known to be in NP. Hence in this particular case we have $\text{CHARP} \in \text{NP}$. Analogously, the case $k \equiv 1 \pmod{2}$ is equivalent to deciding whether $\pi_{\mathcal{X}}(k-1) = 0$, which is again in NP. \square

8. FINAL REMARKS

8.1. As mentioned in the introduction, the Main Lemma 5.4 implies that one can compute Kronecker coefficients in time polynomial in the size of the parts, but exponential in the number of parts. This phenomenon is similar to the well known distinction between *weak* and *strong* NP-completeness (see e.g. [GJ, Pap, Vaz]), corresponding to the input given in *binary* and in *unary*. It applies to other counting problems as well. For example, counting the number of perfect matchings in graphs with large multiple edges and fixed number n of vertices is easily polynomial, while for large n the problem is classically #P-complete even when edge multiplicities are 0 or 1. More generally, the number of BIN PACKING solutions is strongly #P-complete, via a standard reduction to #TRIPARTITE MATCHINGS problem (see e.g. [Pap]). Of course, for other counting problems this phenomenon fails. For example, the counting of KNAPSACK solutions is polynomial when the input is in unary.

From this point of view, we believe that the bounds in the Main Lemma cannot be substantially improved.

Conjecture 8.1. *The Kronecker coefficients $g(\lambda, \mu, \nu)$ and the LR coefficients $c_{\beta\gamma}^\alpha$ are strongly #P-hard.*

Of course, the second part of the conjecture implies the first part, via Murnaghan's reduction (see below). The final reduction in [Nar] proving #P-completeness of LR coefficients, is from *contingency tables* (see [DG] for an introduction). It is easy to see that the decision problem for existence of contingency tables with given marginals is polynomial when the input is in unary, and NP-complete when the input is in binary. However, despite the large body of literature, it seems open whether the counting problem is strongly #P-complete. Note, however, that De Loera and Onn proved that for the three-way statistical tables (see §5.2), the counting problem is strongly #P-complete, even when one dimension $m \geq 2$ is bounded, see [DO].

8.2. The Littlewood–Richardson coefficients are much better understood than the Kronecker coefficients. In fact, the LR coefficients are actually a special case of the Kronecker coefficients:

$$(9) \quad c_{\mu\nu}^\lambda = g((n - |\lambda|, \lambda), (n - |\mu|, \mu), (n - |\nu|, \nu)),$$

for any partitions λ, μ, ν , such that $|\lambda| = |\mu| + |\nu|$, and any sufficiently large n . This equality is due to Murnaghan and Littlewood [L2, Mu2, Mu3]. Suppose that $\nu_1 \leq M$. Then the Reduction Lemma applied to the partitions in (9), gives the following alternative for the

LR coefficients. When $|\lambda_i - \mu_i| > |\nu|$ for some i , we have $c_{\mu\nu}^\lambda = 0$; otherwise, there exist partitions $\psi(\lambda), \psi(\mu)$ of sizes at most $\ell^3 M$, such that $c_{\mu\nu}^\lambda = c_{\psi(\mu)\nu}^{\psi(\lambda)}$. Note that these results can also be obtained directly from the Littlewood–Richardson rule, as in the proof of the Reduction Lemma. Let us mention also that for $i = 1$, part (i) of the Reduction Lemma is proved in [Kle] (see also [JK, §2.9]).

The analogue of Theorem 3.5 for LR coefficients is the following result.

Corollary 8.2. *When $\nu_1 \leq M$, $\lambda_1, \mu_1 \leq N$ and $\ell(\mu), \ell(\lambda), \ell(\nu) \leq \ell$, the Littlewood–Richardson coefficient $c_{\mu\nu}^\lambda$ can be computed in time $O(\ell \log N) + (\ell \log M)^{O(\ell^3 \log \ell)}$.*

This result seems already nontrivial and hard to establish directly. It would be nice to improve the complexity in the corollary (cf. §8.4 below). In a different direction, it would be interesting to extend the Main Theorem 3.5 to *plethystic constants* $a_{\lambda\mu}^\pi$ (see e.g. [JK, M1, Sta]).

8.3. An analogous to KP, yet much simpler, is the positivity problem for the Littlewood–Richardson coefficients:

LRP PROBLEM: Given the input (λ, μ, ν) as in KP, decide whether $c_{\mu\nu}^\lambda > 0$.

Knutson and Tao’s proof of the *Saturation Conjecture* [KT] implies that this decision problem is in P, since it reduces to a feasibility problem of a linear program with $O(\ell^2)$ inequalities and constraints of size $O(N)$, see [MNS, BI2]. Such problems can be solved in polynomial time in the size s of the input, $s = \Theta(\ell \log N)$.

Unfortunately, the saturation theorem does not hold for Kronecker coefficients (see e.g. [Kir, §2.5]). Mulmuley’s original approach to KP was via Conjecture 3.4 and a weak version of the Saturation Conjecture. The original version of the latter was disproved in [BOR1], and modified by Mulmuley in the appendix to [BOR1] (see also [M1]).

While the decision problem for the positivity of LR-coefficients is in NP even without the Knutson–Tao theorem, conjectures 3.1 and 3.3 remain out of reach. As of now, there are no combinatorial interpretation for Kronecker coefficients $g(\lambda, \mu, \nu)$ except for a few special cases (see the references in [PP3, PPV]).

8.4. For the LR coefficients, one can apply Barvinok’s algorithm for counting integer points in polytopes of BZ triangles, see [DM]. In notation of Corollary 8.2, these polytopes have dimension $d = \theta(\ell^2)$ and input size $L = O(\ell \log N)$. By Theorem 2.3, the resulting algorithm has cost

$$L^{O(d \log d)} = (\ell \log N)^{O(\ell^2 \log \ell)}.$$

This is roughly comparable with the result of Corollary 8.2, larger in some cases and smaller in other.

Note also that in light of Barvinok’s algorithm, one can view the Main Lemma 5.4 as an evidence in support of Mulmuley’s Conjecture 3.4.

8.5. A positive combinatorial interpretation for the Kronecker coefficients, analogous to the LR-rule, would likely show that the decision problem is in NP and the counting problem in #P. Such interpretation would also imply a combinatorial interpretation for the difference between the number of partitions of k and the number of partitions of $k - 1$, which fit into an $\ell \times m$ rectangle (see [PP1]). Formally, this difference is equal to $g(m^\ell, m^\ell, (n - k, k))$; in full generality its combinatorial interpretation is already highly nontrivial and will appear in [PP3] (see also [BO]).

8.6. The known results so far do not prove Mulmuley’s Conjecture 3.1, even when the input is in unary. As the current results suggest, the computational complexity comes from two sources – the length ℓ of the partitions, and the size N of their parts. While it is often possible to reduce the problem to one where the size of the parts is $O(\log N)$, the exponential dependence on ℓ cannot be reduced with current methods. As suggested by the proof of Lemma 5.4 and the equivalent formulas through inverse Kostka coefficients (see [V1]), the Kronecker coefficients are given by alternating sums over all permutations in S_ℓ , whose number is $O(\ell^\ell)$.

8.7. Although KRON is in GapP (Theorem 3.2), it is not immediately obvious that KP is in PSPACE, or even in EXPTIME (see however [M1]). Indeed, the brute force computations with character values of the symmetric group S_n , where $n = O(\ell N)$. Such computations are doubly exponential in time and exponential in space (in the input size). Of course, any such method implies that KP is in EXPSPACE. Note that our proof of Theorem 3.2 via Lemma 5.4 uses only contingency arrays, bypassing Kostka and LR coefficients.

8.8. The semigroup property (Theorem 2.1) was conjectured by Klyachko in 2004, and recently proved in [CHM] (see also [Man]). It is the analogue of the semigroup property of LR coefficients proved by Brion and Knop in 1989 (see [Zel] for the history and the related results).

8.9. Throughout the paper, we are rather relaxed in our treatment of the algorithm timing. Our time complexity is in the cost of arithmetic operations with integers as in the input.

8.10. The *reduced Kronecker coefficients*, see e.g. [BDO, BOR2], are defined as

$$(10) \quad \bar{g}(\lambda, \mu, \nu) = g((n - |\lambda|, \lambda), (n - |\mu|, \mu), (n - |\nu|, \nu)) \quad \text{for } n \text{ large enough.}$$

Note that equation (10) generalizes equation (9) when there is no constraint $|\lambda| = |\mu| + |\nu|$. The fact that $\bar{g}(\lambda, \mu, \nu)$ are well defined has been established by Murnaghan [Mu1] (see also [Bri, V1]), but determining effective bounds on n for which the sequence stabilizes is still an active area. The Reduction Lemma 5.1 immediately implies Murnaghan’s result that they stabilize. Its proof, and more specifically the map φ , also gives the following upper bound.

Corollary 8.3. *Equation (10) holds for all triples of partitions (λ, μ, ν) , such that $|\nu| \leq |\mu| \leq |\lambda|$ and $n \leq \max\{\lambda_1, \mu_1\} + |\nu| + |\lambda|$.*

This result matches the bound in [V1] in the cases when $\lambda_1 \geq \mu_1$, but is slightly weaker otherwise.

Sketch of proof. Apply the map φ from Lemma 5.1 with $t = |\nu|$ to the partitions $(n - |\lambda|, \lambda)$ and $(n - |\mu|, \mu)$. When $n \geq \max\{\lambda_1, \mu_1\} + |\nu| + |\lambda|$, we have $1 \in I$. Then, for any partition θ as in the proof, the first part $\varphi(\theta)_1$ does not depend on n anymore. Thus $\varphi(n - |\lambda|, \lambda)$, $\varphi(n - |\mu|, \mu)$, $\varphi(n - |\nu|, \nu)$ are also independent of n . The rest follows from the proof of Lemma 5.1. \square

It would be interesting to see if the Reduction Lemma can be further extended to imply better bounds, or whether there is a plethystic generalization.

8.11. Note that the elementary construction in the proof of Theorem 7.1 also implies that computing characters is $\#P$ -hard, a result obtained earlier in [Hep]. In [PPV], the authors prove that positivity of certain Kronecker coefficients is a consequence of nonzero character values, which are easier to establish via the Murnaghan–Nakayama rule in certain large cases. Unfortunately, the fact that CHARP is NP-hard (Theorem 7.1), implies that this approach is unlikely to have complexity implications. Similarly, using characters to efficiently compute the Kronecker coefficients via the formula in §2.2, is most likely going to be futile (cf. [BCS, §13.5]).

Acknowledgements. We are grateful to Abdelmalek Abdesselam, Jonah Blasiak, Emmanuel Briand, Jin-Yi Cai, Jesús De Loera, Allen Knutson, Joseph Landsberg, László Lovász, Rosa Orellana and Bernd Sturmfels for interesting conversations and helpful remarks. We are especially thankful to Christian Ikenmeyer and Ernesto Vallejo for reading a draft of this manuscript and help with the references. The first author gives special thanks to Alexander Barvinok for the generosity in explaining his eponymous algorithm many years ago and clarifying its complexity issues. The first author was partially supported by the NSF grant, the second by the Simons Postdoctoral Fellowship.

REFERENCES

- [BO] C. M. Ballantine and R. C. Orellana, A combinatorial interpretation for the coefficients in the Kronecker product $s_{(n-p,p)} * s_\lambda$, *Sém. Lothar. Combin.* **54A** (2006), Art. B54Af, 29 pp.
- [Bar] A. Barvinok, *Integer points in polyhedra*, European Mathematical Society, Zürich, 2008.
- [BP] A. Barvinok and J. E. Pommersheim, An algorithmic theory of lattice points in polyhedra, in *New perspectives in algebraic combinatorics*, Cambridge Univ. Press, Cambridge, UK, 1999, 91–147.
- [BR] A. Berele and A. Regev, Hook Young diagrams with applications to combinatorics and representations of Lie superalgebras, *Adv. Math.* **64** (1987), 118–175.
- [Bla] J. Blasiak, Kronecker coefficients for one hook shape, [arXiv:1209.2018](https://arxiv.org/abs/1209.2018).
- [BSS] G. Benkart, F. Sottile and J. Stroomer, Tableau switching: algorithms and applications, *J. Combin. Theory Ser. A* **76** (1996), 11–43.
- [BDO] C. Bowman, M. De Visscher and R. Orellana, The partition algebra and the Kronecker coefficients, to appear in *Trans. AMS*; [arXiv:1210.5579](https://arxiv.org/abs/1210.5579).
- [BOR1] E. Briand, R. Orellana and M. Rosas, Reduced Kronecker coefficients and counter-examples to Mulmuley’s strong saturation conjecture SH. With an appendix by Ketan Mulmuley, *Comput. Complexity* **18** (2009), 577–600.
- [BOR2] E. Briand, R. Orellana and M. Rosas, The stability of the Kronecker product of Schur functions, *J. Algebra* **331** (2011), 11–27.
- [Bri] M. Brion, Stable properties of plethysm: on two conjectures of Foulkes, *Manuscripta Math.* **80** (1993), 347–371.
- [Bür] P. Bürgisser, Review of [MS], [MR2421083 \(2009j:68067\)](https://arxiv.org/abs/2009j.68067).
- [BCS] P. Bürgisser, M. Clausen and M. A. Shokrollahi, *Algebraic complexity theory*, Springer, Berlin, 1997.
- [BI1] P. Bürgisser and C. Ikenmeyer, The complexity of computing Kronecker coefficients, in *Discrete Math. Theor. Comput. Sci. Proc.*, Assoc. DMTCS, Nancy, 2008, 357–368.
- [BI2] P. Bürgisser and C. Ikenmeyer, Deciding Positivity of Littlewood–Richardson Coefficients, *SIAM J. Discrete Math.* **27** (2013), 1639–1681.
- [CDW] M. Christandl, B. Doran and M. Walter, Computing Multiplicities of Lie Group Representations, in *Proc. 53rd FOCS*, 2012, 639–648; [arXiv:1204.4379](https://arxiv.org/abs/1204.4379).
- [CHM] M. Christandl, A. W. Harrow and G. Mitchison, Nonzero Kronecker coefficients and what they tell us about spectra, *Comm. Math. Phys.* **270** (2007), 575–585.
- [DHTY] J. De Loera, R. Hemmecke, J. Tauzer and R. Yoshida, Effective lattice point counting in rational convex polytopes, *J. Symbolic Comput.* **38** (2004), 1273–1302.

- [DM] J. De Loera and T. B. McAllister, On the computation of Clebsch-Gordan coefficients and the dilation effect, *Experiment. Math.* **15** (2006), 7–19.
- [DO] J. De Loera and S. Onn, The complexity of three-way statistical tables, *SIAM J. Comput.* **33** (2004), 819–836.
- [DG] P. Diaconis and A. Gangolli, Rectangular arrays with fixed margins, in *Discrete probability and algorithms*, Springer, New York, 1995, 15–41.
- [Dvir] Y. Dvir, On the Kronecker product of S_n characters, *J. Algebra* **154** (1993), 125–140.
- [DK] M. Dyer and R. Kannan, On Barvinok’s algorithm for counting lattice points in fixed dimension, *Math. Oper. Res.* **22** (1997), 545–549.
- [Eis] F. Eisenbrand, Fast Integer Programming in Fixed Dimension, in *Lecture Notes in Computer Science*, Vol. 2832 (Proc. 11th ESA), 2003, 196–207.
- [F1] L. Fortnow, Counting complexity, in *Complexity theory retrospective, II*, Springer, New York, 1997, 81–107.
- [F2] L. Fortnow, The Status of the P Versus NP Problem, *Comm. ACM* **52** (2009), No. 9, 78–86.
- [GJ] M. Garey and D. S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-completeness*, Freeman, San Francisco, CA, 1979.
- [Hep] C. T. Hepler, *On The Complexity of Computing Characters of Finite Groups*, thesis, University of Calgary, 1994; available at <https://dspace.ucalgary.ca/handle/1880/45530>
- [Ike] C. Ikenmeyer, *Geometric Complexity Theory, Tensor Rank, and Littlewood-Richardson Coefficients*, thesis, University of Paderborn, 2012; available at <http://bit.ly/SVIR0M>
- [JK] G. James and A. Kerber, *The representation theory of the symmetric group*, Addison-Wesley, Reading, MA, 1981.
- [Kir] A. N. Kirillov, An invitation to the generalized saturation conjecture, *Publ. RIMS* **40** (2004), 1147–1239.
- [Kle] M. Klemm, Tensorprodukte von Charakteren der symmetrischen Gruppe (in German), *Arch. Math. (Basel)* **28** (1977), 455–459.
- [KT] A. Knutson and T. Tao, The honeycomb model of $GL_n(\mathbb{C})$ tensor products. I. Proof of the saturation conjecture, *J. AMS* **12** (1999), 1055–1090.
- [Las] A. Lascoux, Produit de Kronecker des représentations du groupe symétrique, in *Lecture Notes in Math.* **795**, Springer, Berlin, 1980, 319–329.
- [L1] D. E. Littlewood, The Kronecker product of symmetric group representations, *J. London Math. Soc.* **31** (1956), 89–93.
- [L2] D. E. Littlewood, Products and plethysms of characters with orthogonal, symplectic and symmetric groups, *Canad. J. Math.* **10** (1958), 17–32.
- [Mac] I. G. Macdonald, *Symmetric functions and Hall polynomials* (Second ed.), Oxford U. Press, New York, 1995.
- [Man] L. Manivel, On rectangular Kronecker coefficients, *J. Algebraic Combin.* **33** (2011), 153–162.
- [M1] K. D. Mulmuley, Geometric complexity theory VI: The flip via positivity, [arXiv:0704.0229v4](https://arxiv.org/abs/0704.0229v4)
- [M2] K. D. Mulmuley, The GCT program toward the P vs. NP problem, *Comm. ACM* **55** (2012), No. 6, 98–107.
- [MNS] K. D. Mulmuley, H. Narayanan and M. Sohoni, Geometric complexity theory III. On deciding nonvanishing of a Littlewood-Richardson coefficient, *J. Algebraic Combin.* **36** (2012), 103–110.
- [MS] K. D. Mulmuley and M. Sohoni, Geometric complexity theory II. Towards explicit obstructions for embeddings among class varieties, *SIAM J. Comput.* **38** (2008), 1175–1206.
- [Mu1] F. D. Murnaghan, The Analysis of the Direct Product of Irreducible Representations of the Symmetric Groups, *Amer. J. Math.* **60** (1938), 44–65.
- [Mu2] F. D. Murnaghan, On the analysis of the Kronecker product of irreducible representations of S_n , *Proc. Nat. Acad. Sci. USA* **41** (1955), 515–518.
- [Mu3] F. D. Murnaghan, On the Kronecker product of irreducible representations of the symmetric group, *Proc. Nat. Acad. Sci. USA* **42** (1956), 95–98.
- [Nar] H. Narayanan, On the complexity of computing Kostka numbers and Littlewood-Richardson coefficients, *J. Algebraic Combin.* **24** (2006), 347–354.
- [PP1] I. Pak and G. Panova, Unimodality via Kronecker product, to appear in *J. Algebraic Combin.*, [arXiv:1304.5044](https://arxiv.org/abs/1304.5044).
- [PP2] I. Pak and G. Panova, Strict unimodality of q -binomial coefficients, *C.R. Math. Acad. Sci. Paris* **351** (2013), 415–418.

- [PP3] I. Pak and G. Panova, Combinatorics and complexity of Kronecker coefficients, in preparation.
- [PPV] I. Pak, G. Panova and E. Vallejo, Kronecker products, characters, partitions, and the tensor square conjectures, [arXiv:1304.0738](https://arxiv.org/abs/1304.0738).
- [PV1] I. Pak and E. Vallejo, Combinatorics and geometry of Littlewood-Richardson cones, *European J. Combin.* **26** (2005), 995–1008.
- [PV2] I. Pak and E. Vallejo, Reductions of Young tableau bijections, *SIAM J. Discrete Math.* **24** (2010), 113–145.
- [Pap] C. H. Papadimitriou, *Computational complexity*, Addison-Wesley, Reading, MA, 1994.
- [R1] K. W. Regan, Understanding the Mulmuley-Sohoni approach to P vs. NP, *Bull. Eur. Assoc. Theor. Comput. Sci.* **78** (2002), 86–99.
- [R2] A. Regev, Kronecker multiplicities in the (k, l) hook are polynomially bounded, [arXiv:1011.1636](https://arxiv.org/abs/1011.1636).
- [Rem] J. B. Remmel, A formula for the Kronecker products of Schur functions of hook shapes, *J. Algebra* **120** (1989), 100–118.
- [Ros] M. H. Rosas, The Kronecker product of Schur functions indexed by two row shapes of hook shapes, *J. Algebraic Combin.* **14** (2001), 153–173.
- [Sag] B. E. Sagan, *The symmetric group* (Second ed.), Springer, New York, 2001.
- [Sch] A. Schrijver, *Theory of linear and integer programming*, John Wiley, Chichester, 1986.
- [Sta] R. P. Stanley, *Enumerative combinatorics*, Vol. 2, Cambridge U. Press, Cambridge, 1999.
- [V1] E. Vallejo, Stability of Kronecker products of irreducible characters of the symmetric group, *Electron. J. Combin.* **6** (1999), RP 39, 7 pp.
- [V2] E. Vallejo, Multiple hooks in Kronecker products of S_n characters and a conjecture of Berele and Imbo, *J. Algebra* **259** (2003), 300–309.
- [Vaz] V. V. Vazirani, *Approximation algorithms*, Springer, Berlin, 2001.
- [Zel] A. Zelevinsky, Littlewood-Richardson semigroups, in *New perspectives in algebraic combinatorics*, Cambridge Univ. Press, Cambridge, 1999, 337–345.