THE NATURE OF PARTITION BIJECTIONS I. INVOLUTIONS.

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Abstract. We analyze involutions which prove several partition identities and describe them in a uniform fashion as projections of "natural" partition involutions along certain bijections. The involutions include those due to Franklin, Sylvester, Andrews, as well as few others. A new involution is constructed for an identity of Ramanujan, and analyzed in the same fashion.

Introduction

Combinatorial methods in Partition Theory have been developing ever since Sylvester's magnum opus [S], where a large body of work by Sylvester and his students were presented. Arguably, *Franklin's involution* published a year earlier [F] played the most important role in convincing field's practitioners in the importance of the approach.

Franklin's involution construction was as beautiful and simple as it was mysterious. Undoubtedly, generations of researchers in the field scratched their head trying to explain its origin.[1] This difficulty led to a general understanding that finding similar constructions in other cases is more of an art than a science, and requires a considerable degree of ingenuity. Still, the quest for a beautiful proof led to a successful emulation of the method in a few notable cases.

Just a year after Franklin's involution, Sylvester found an involutive proof of Jacobi triple product identity [S,P1]. Years later, Schur found an involutive proof of an equivalent version of Rogers–Ramanujan identities [Sc,P1]. In modern times, Andrews found an involutive proof of Gauss identity [A1,P1]. Knuth and Paterson found involutive proofs of extensions of Euler's and Jacobi identities based on a careful analysis of Franklin's involution [KP]. Most recently, Chen, Hou and Lascoux found a combinatorial proof of another Gauss identity by using a geometric argument of a similar type [CHL].

While all these involutions are in the same spirit, until now there seem to be no natural order among them. The aim of this paper is to bring such an order. In

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[1]Andrews explains Franklin's involution from a historical point of view in [A4]. His idea of looking into Sylvester's identity goes along similar lines as we are doing in this paper (see section 9).

fact, we show that in a certain precise sense all these involutions follow automatically from appropriate bijective proofs of more general identities. In other words, obtaining such involutions requires no ingenuity at all modulo appropriate (and, as the reader will see, relatively simple) bijective proofs. We refer the reader to section 9 for philosophical and historical discussion on the merits of the approach.

In the first part of the paper we present separate descriptions of identities and the corresponding involutions in different sections. Each section contains one involutive and one bijective proof, intimately related to each other. In the second part we formally establish the relationships between two proofs of the same identity. We begin by giving a general setup for obtaining involutions from these bijective proofs, and then go over the identities one by one.

Let us mention that in the first part we give several new proofs of identities, as well as some new bijections. In other cases, we simply recall either involutive or bijective proofs available in the literature, so we can use these for our analysis later. To simplify the exposition and for reader's convenience we modify and sometimes simplify their exposition. The identities are ordered somewhat arbitrarily, according to their degree of complexity rather than anything else. We analyze the following identities, one in each section:

1) Euler's Pentagonal Theorem,
2) Gauss product identity,
3) Shanks identity,
4) Jacobi triple product identity,
5) Gauss q-binomial identity,
6) Ramanujan's identity.

Perhaps the most significant 'traditional' contribution of this paper is given in section 6, where we present an explicit involution proving a curious identity of Ramanujan for *"false theta functions"* [A3]. This identity, taken from Ramanujan's "lost" notebook [R], was singled out by Andrews [A3], who proved it analytically and translated it into a language of partitions. *"It would be nice to have a combinatorial proof of this result,"* Andrews writes [A3]. Bijective and involutive proofs we present in this paper are the first such proofs.

We should warn the reader that in the main part of the paper very few references or historical remarks are given. We postpone them till section 11

Notation. Much of the notation follows our recent survey article [P1]. We refer the reader to [A2,P1] for the introduction to the subject, other results and further references.

A Young diagram of a partition λ is denoted by $[\lambda]$ and is represented graphically by means of Young diagrams. By a slight abuse of speech will make no distinction between partitions and Young diagrams. Throughout the paper we use $\ell(\lambda)$ and $s(\lambda)$ to denote the number of parts and the smallest part, respectively. Let λ' denote the conjugate partition to λ. By $\mathcal{P}$ and $\mathcal{D}$ denote the set of all partitions and partitions into distinct parts, respectively. Finally, we use $\mathbb{N} = \{1, 2, \dots\}$, and $\mathbb{Z}_{\geq 0} = \{0, 1, 2, \dots\}$.

1. Euler's Pentagonal Theorem

1.1 Involutive proof. *Franklin's involution* gives a combinatorial proof of *Euler's Pentagonal Theorem*, which is equivalent to the following identity:

$$(*) \qquad \prod_{i=1}^{\infty}(1-t^i) \;=\; \sum_{m=-\infty}^{\infty}(-1)^m t^{m(3m-1)/2}\,.$$

While the proof we present here is standard, we recall it for completeness. Let us first restate $(*)$ as an enumerative result for integer partitions:

Theorem 1. *Let $\mathcal{D}_n^0$ and $\mathcal{D}_n^1$ be the sets of integer partitions of n into distinct parts with even and odd number of parts, respectively. Then*

$$|\mathcal{D}_n^0| \,-\, |\mathcal{D}_n^1| \;=\; \begin{cases} (-1)^m\,, & \text{if} \quad n = m(3m\pm 1)/2\,, \\ \quad 0\;\;, & \text{otherwise.} \end{cases}$$

Denote by $\mathcal{D}_n = \mathcal{D}_n^0 \cup \mathcal{D}_n^1$ the set of all partitions $\lambda \vdash n$ into disjoint parts. Define an involution $\alpha : \mathcal{D}_n \to \mathcal{D}_n$ as follows. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \in \mathcal{D}$. Compare the sizes of *horizontal* and *diagonal* lines of squares in Young diagram $[\lambda]$ (see Figure 1). Let $s = s(\lambda)$ and $g = g(\lambda)$ be the lengths of these lines. If $s > g$, move the diagonal line below the horizontal line. Otherwise, if $s \le g$, move the horizontal line to the right of the diagonal (see Figure 1). If $s = g$ or $s = g + 1$ and the lines have a common square, stay put.

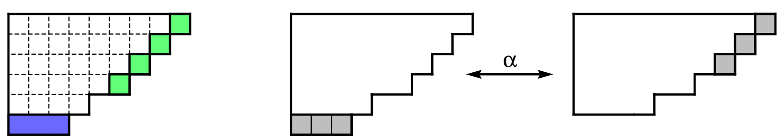

Figure 1. Horizontal and diagonal lines $g(\lambda) = 4$, $s(\lambda) = 3$, where $\lambda = (9,8,7,6,4,3) \in \mathcal{D}_{37}$. Franklin's involution $\alpha : (9,8,7,6,4,3) \to (10,9,8,6,4)$.

Let $\alpha(\lambda)$ denote the resulting partition. The partitions θ_m, $m \in \mathbb{Z}$ of pentagonal shape (see Figure 2) are the only fixed points of α. Denote $\theta_0 = \varnothing$. The fixed points θ_m correspond to the r.h.s. of $(*)$. It is easy to check that α is an involution which changes parity in the number of parts, except for fixed points. This completes the proof of Euler's Theorem.

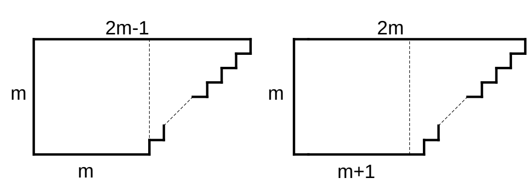

Figure 2. Fixed points θ_{-m} and θ_m, $m \in \mathbb{N}$, of Franklin's involution α.

1.2 Bijective proof. Here is an elegant proof discovered by Sylvester. Start with the following identity:

$$(**)\qquad \prod_{i=1}^{\infty}(1+zt^i) \;=\; \sum_{m=0}^{\infty} z^m\, t^{\frac{m(3m+1)}{2}} \prod_{i=1}^{m} \frac{1+zt^i}{1-t^i} \;+\; \sum_{m=1}^{\infty} z^m\, t^{\frac{m(3m-1)}{2}} \prod_{i=1}^{m-1} \frac{1+zt^i}{1-t^i}\,.$$

Note that when $z=-1$ we obtain $(*)$.

Here is a Durfee square type bijective proof. Start with the r.h.s. of $(**)$ and interpret the power $t^{\frac{m(3m+1)}{2}}$ as the size of pentagonal regions θ_m (see Figure 2; recall that $m \in \mathbb{Z}$). Interpret the nominator and denominator in the product as a pair of partitions μ, ν with at most m parts for $m \geq 0$, at most $(m-1)$ parts for $m < 0$, and such that μ has distinct parts. Now place $[\mu]$ and $[\nu]$ below and to the right of θ_m, as in Figure 3. The resulting Young diagram $\lambda = \varphi(m,\mu,\nu)$ corresponds to the l.h.s. of $(**)$. Note that $\lambda \in \mathcal{D}_n$, where $n = |\mu| + |\nu| + \frac{m(3m+1)}{2}$. It is easy to check that φ is a bijection.

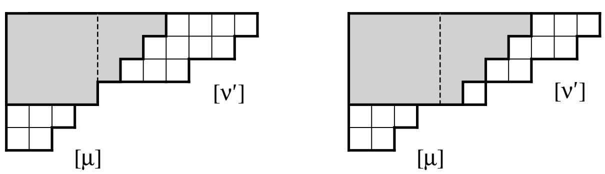

FIGURE 3. Examples of Sylvester's bijection φ, for $m = \pm 4$.

2. Gauss product identity

2.1 Involutive proof. The following identity due to Gauss is one of the most beautiful results in Partition Theory:

$$(\circ)\qquad \prod_{i=1}^{\infty} \frac{1-t^i}{1+t^i} \;=\; \sum_{k=-\infty}^{\infty} (-1)^k\, t^{k^2}\,.$$

Define a *MacMahon diagram* $\langle\lambda\rangle$ to be a Young diagram $[\lambda]$ with a *marked* subset of corners of $[\lambda]$. Denote by $\mathcal{M}$ the set of all MacMahon diagrams. Observe that

$$(\circ')\qquad \prod_{i=1}^{\infty} \frac{1-t^i}{1+t^i} \;=\; \sum_{\langle\lambda\rangle \in \mathcal{M}} (-1)^{\ell(\lambda)}\, t^{|\lambda|}\,.$$

Indeed, consider two partitions $\mu \in \mathcal{D}$ and $\nu \in \mathcal{P}$, corresponding to nominator and denominator, respectively. Mark the rightmost square in each row of $[\mu]$. Then take a union of rows of $[\mu]$ and $[\nu]$, by arranging them in decreasing order, such

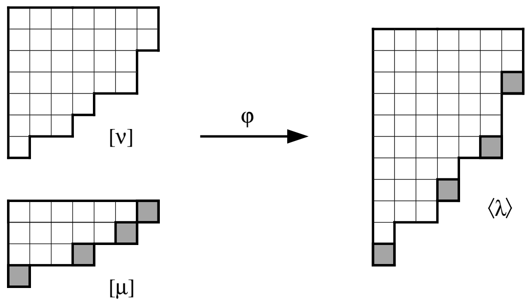

FIGURE 4. Bijective proof of ($\circ'$).

that the rows of $[\mu]$ are always below rows of $[\nu]$ of the same length. This gives a MacMahon diagram $\langle\lambda\rangle = \varphi(\mu,\nu) \in \mathcal{M}$ (see Figure 4).

The following involution $\beta : \mathcal{M} \to \mathcal{M}$ is equivalent to that due to Andrews. Define $v = v(\lambda) = s(\lambda')$ to be the length of the *vertical line* of squares in $[\lambda]$, a conjugate notion to horizontal line. Similarly, let $u = u\langle\lambda\rangle$ be the number of unmarked squares in a vertical line. By the definition of a MacMahon diagram, $v = u + 1$ if the vertical line contains a marked square, and $v = u$ otherwise. Finally, let $s = s(\lambda)$.

Now, if $v < s$, or $u < v = s$, attach a row of length v to the horizontal line; mark the last square if the vertical line does not contain a marked square, or vice versa (see Figure 5). Conversely, if $s < v$, or $s = u = v$, attach a column of length s to the vertical line and make it marked if the horizontal line was unmarked, or vice versa. Denote by β the involution we obtain.

There are four exceptional cases when β is undefined: when $\langle\lambda\rangle$ is an $r \times (r+1)$ rectangle with no marked squares, an $(r+1) \times r$ rectangle with one marked square, and an $r \times r$ rectangle with or without a marked square. Let these be the first points of β. The first two cases cancel each other, while the last two give the terms on the r.h.s. of ($\circ$).

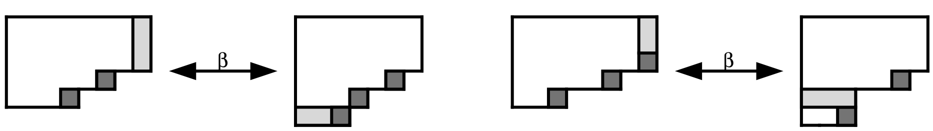

FIGURE 5. Examples of Andrews's involution.

2.2 Bijective proof. We prove the following extension of Gauss identity

$$(\circ\circ)\qquad \begin{aligned}\prod_{i=1}^{\infty} \frac{1+azt^i}{1-zt^i} &= \sum_{k=0}^{\infty} z^k\, t^{k^2} \left[\prod_{i=1}^{k} \frac{1+azt^i}{1-zt^i}\right] \left[\prod_{i=1}^{k} \frac{1+at^i}{1-t^i}\right] \\ &\quad + a \sum_{k=1}^{\infty} z^k\, t^{k^2} \left[\prod_{i=1}^{k-1} \frac{1+azt^i}{1-zt^i}\right] \left[\prod_{i=1}^{k-1} \frac{1+at^i}{1-t^i}\right]\end{aligned}$$

Observe that when $a = 1$ and $z = -1$, the products in square brackets cancel each other and we get $(\circ)$. The proof of $(\circ\circ)$ follows easily from the Durfee square considerations for MacMahon diagrams.

Let $m\langle\lambda\rangle$ denote the number of marked squares. Observe that the map $\varphi :$ $(\mu, \nu) \to \langle\lambda\rangle$ described as above proves also the following extension of $(\circ')$:

$$(\circ\circ') \qquad \prod_{i=1}^{\infty} \frac{1 + a\, z\, t^i}{1 - zt^i} \;=\; \sum_{\langle\lambda\rangle \in \mathcal{M}} a^{m\langle\lambda\rangle}\, z^{\ell(\lambda)}\, t^{|\lambda|}\,.$$

Thus it remains to show that the r.h.s. of $(\circ\circ)$ is equal to the r.h.s. of $(\circ\circ')$.

Consider a *Durfee square* in $\langle\lambda\rangle$, defined as the largest size square which fits $\langle\lambda\rangle$. Denote such $k \times k$ square by $\langle\delta_k\rangle$. Suppose the lower right corner of the Durfee square $\langle\delta_k\rangle$ is unmarked (see Figure 6). Then removing $\langle\delta_k\rangle$ from the MacMahon diagram $\langle\lambda\rangle$ breaks it into two MacMahon diagrams with at most k rows. These two diagrams correspond to two products of in the first summation on the r.h.s. of $(\circ\circ)$. Similarly, if the lower right corner of $\langle\delta_k\rangle$ is marked, these two MacMahon diagrams have at most $(k-1)$ rows, and correspond to two products of in the second summation on the r.h.s. of $(\circ\circ)$. This completes the bijective proof.

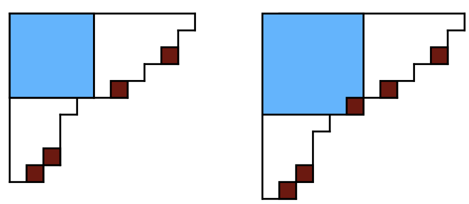

FIGURE 6. Examples of Durfee square construction.

3. Shanks identity

3.1 Involutive proof. The following identity due to Shanks is a 'finite analogue' of Euler's Pentagonal Theorem:

$$(\circledast) \qquad \prod_{i=1}^{m} (1 - t^i) \sum_{k=0}^{m} \frac{(-1)^k\, t^{mk + k(k+1)/2}}{(1-t)(1-t^2)\cdots(1-t^k)} \;=\; \sum_{r=-m}^{m} (-1)^r\, t^{r(3r-1)/2}\,,$$

for all $m \in \mathbb{N}$. Note that when $m \to \infty$ we obtain Euler's identity $(*)$. Knuth and Paterson showed that $(\circledast)$ can be proved by extending Franklin's involution α. Below we briefly outline the construction.

Let $\mathcal{D}$ denote the set of partitions λ with distinct parts, and let $r(\lambda)$ denotes the size of the corresponding Durfee square. We claim that the l.h.s. of $(\circledast)$ can be expressed as follows:

$$(\circledast') \qquad \prod_{i=1}^{m} (1 - t^i) \sum_{k=0}^{m} \frac{(-1)^m\, t^{mk + k(k+1)/2}}{(1-t)(1-t^2)\cdots(1-t^k)} \;=\; \sum_{\lambda \in \mathcal{D}\,:\, r(\lambda) \le m} (-1)^{\ell(\lambda)}\, t^{|\lambda|}\,.$$

Indeed, given $\lambda \in \mathcal{D}$ such that $r(\lambda) \le m$, let μ denote the set of parts $\le m$. Clearly, $\mu \in \mathcal{D}$ and corresponds to the product in $(\circledast')$. Now a trapezoid shape region of size $rm + r(r+1)/2$ and a Young diagram to its right correspond to the sum on the l.h.s. of $(\circledast')$, which proves the claim (see first Young diagram in Figure 7).

It remains to prove that the r.h.s. of $(\circledast')$ coincides with the r.h.s. of $(\circledast)$. Indeed, note that Franklin's involution α preserves the Durfee square size. Applying α to the set of all $\lambda \in \mathcal{D}$ with $r(\lambda) \le m$ cancels all terms on the r.h.s. of $(\circledast')$ except for partitions θ_r, with $-m \le r \le m$. This gives the r.h.s. of $(\circledast)$ and completes the proof of Shanks identity.

3.2 Bijective proof. The following identity is a common generalization of Sylvester's identity $(**)$ and Shanks identity $(\circledast)$:

$$(\circledast\circledast)\qquad \prod_{i=1}^{m}(1+zt^i)\sum_{k=0}^{m}\frac{z^k\,t^{mk+k(k+1)/2}}{(1-t)(1-t^2)\cdots(1-t^k)} = \sum_{r=0}^{m} z^r\,t^{r(3r+1)/2}\prod_{i=1}^{r}\frac{1+zt^i}{1-t^i} + \sum_{r=1}^{m} z^r\,t^{r(3r-1)/2}\prod_{i=1}^{r-1}\frac{1+zt^i}{1-t^i}\,.$$

When $z=-1$ we obtain Shanks identity $(\circledast)$. Sylvester's identity $(**)$ follows in the limit $m\to\infty$. The identity follows from a simple bijection. As before, we show that both sides of $(\circledast\circledast)$ are equal to

$$\sum_{\lambda\in\mathcal{D}\,:\,r(\lambda)\le m} z^{\ell(\lambda)}\,t^{|\lambda|}\,.$$

For the l.h.s. this was established in the previous section. For the r.h.s. the proof is exactly the same as in case of Sylvester's identity $(**)$. An example is shown in Figure 7. We omit the details.

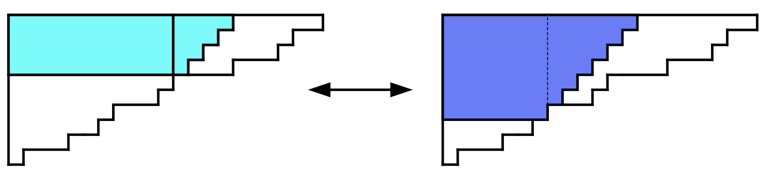

FIGURE 7. Bijective proof of $(\circledast\circledast)$.

4. JACOBI TRIPLE PRODUCT IDENTITY

4.1 Involutive proof. *Jacobi triple product identity* is one of the most famous and the most useful results in Partition Theory. Here is an equivalent version:

$$(\diamond)\qquad \prod_{i=1}^{\infty}(1+st^i)\prod_{j=0}^{\infty}(1+s^{-1}t^j)\prod_{r=1}^{\infty}(1-t^r) = \sum_{k=-\infty}^{\infty} s^k t^{\frac{k(k+1)}{2}}\,.$$

The following involutive proof is equivalent to that given by Sylvester. First, interpret the l.h.s. of $(\diamond)$ as

$$(\diamond') \qquad \sum_{\lambda,\,\nu\,\in\mathcal{D},\,\mu\in\mathcal{D}'} (-1)^{\ell(\nu)}\, z^{\ell(\lambda)-\ell(\mu)}\, t^{|\lambda|+|\mu|+|\nu|}\,,$$

where $\mathcal{D}$ is a set of partitions into disjoint positive parts $\lambda_1 > \lambda_2 > \ldots > 0$, and $\mathcal{D}'$ is a set of partitions into disjoint *nonnegative* parts $\mu_1 > \mu_2 > \ldots \geq 0$. The possibility of a zero part in μ accounts to starting the second product from $j = 0$ on the l.h.s. of $(\diamond)$.

We present an explicit involution γ which cancels the terms in $(\diamond')$. As in section 1.1, let $g(\lambda)$ and $s(\lambda)$ be the lengths of the diagonal and horizontal lines in a diagram $[\lambda]$. We say that λ has *triangular shape* if $\lambda = \rho_k := (k, k-1, \ldots, 1)$, and $k \geq 1$. An involution γ is defined on $(\lambda, \mu, \nu) \in \mathcal{D} \times \mathcal{D}' \times \mathcal{D}$, so that it preserves $|\lambda| + |\mu + |\mu|$, $\ell(\lambda) - \ell(\mu)$, and changes parity of ν except on the fixed points $(\rho_k, \varnothing, \varnothing)$ and $(\varnothing, \rho_k, \varnothing)$. Everywhere below we assume that $\ell(\lambda) \geq \ell(\mu)$. The case $\ell(\lambda) < \ell(\mu)$ can be treated similarly, by switching the role of λ and μ.

If $g(\lambda) \geq s(\nu)$, move the horizontal line in $[\nu]$ to the right of the diagonal line in $[\lambda]$. If $g(\lambda) < s(\nu)$ and $\lambda \neq \rho_k$, move the diagonal line in $[\lambda]$ to below the horizontal line in $[\nu]$. In both cases leave μ unchanged (see Figure 8).

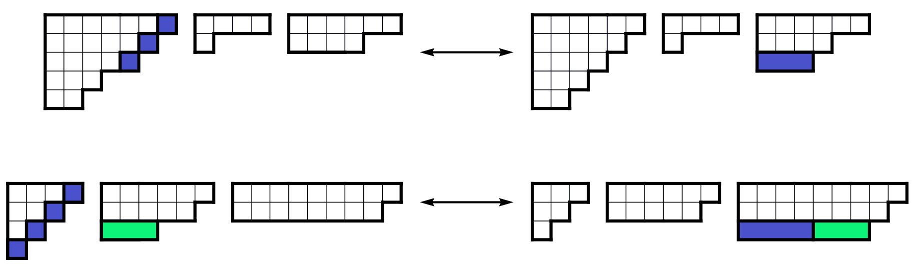

FIGURE 8. Examples of involution γ on triples $(\lambda, \mu, \nu) \in \mathcal{D} \times \mathcal{D}' \times \mathcal{D}$.

If $\lambda = \rho_k$ and $g(\lambda) + s(\mu) < s(\nu)$, move the diagonal line in $[\lambda]$ and the horizontal line in $[\mu]$ to join them below the horizontal line in $[\nu]$. Note that λ becomes ρ_{k-1} in this case. Conversely, if $\lambda = \rho_k$ and $g(\lambda) + s(\mu) \geq s(\nu)$, remove the horizontal line in $[\nu]$ and split it: add $(k+1)$ to $[\lambda]$ to make ρ_{k+1} and attach $s(\nu) - (k+1)$ below the horizontal line in $[\mu]$ (see Figure 8).

It is easy to check that the above construction defines an involution γ on triples of partitions (λ, μ, ν), and satisfies all the properties described above. Since the fixed points of γ correspond to summands on the r.h.s. of $(\diamond)$, we obtain the result.

4.2 Bijective proof. First, let us rewrite the identity as follows:

$$(\diamond\diamond) \qquad \prod_{i=1}^{\infty}(1 + st^i) \prod_{j=0}^{\infty}(1 + s^{-1}t^j) \;=\; \sum_{k=-\infty}^{\infty} s^k t^{\frac{k(k+1)}{2}} \prod_{i=1}^{\infty} \frac{1}{1-t^i}$$

The following proof is due to Sylvester and Hathaway. Interpret the l.h.s. and the r.h.s of $(\diamond\diamond)$ as follows:

$$(\diamond\diamond') \qquad \sum_{\lambda\in\mathcal{D},\mu\in\mathcal{D}'} z^{\ell(\lambda)-\ell(\mu)}\, t^{|\lambda|+|\mu|} \;=\; \sum_{k=-\infty}^{\infty} z^k\, t^{|\rho_{|k|}|} \sum_{\tau\in\mathcal{P}} t^{|\tau|}\,.$$

We define a bijection $\psi : (\rho_k, \tau) \to (\lambda, \mu)$ as in Figure 9. Basically, we start by attaching $\rho_{|k|}$ to the side of $[\tau]$ (depending whether k greater that 0 or not), and then split the resulting diagram along the line $i - j = k$. reading parts to the right and below the line gives λ and μ, respectively. We leave the details to the reader.

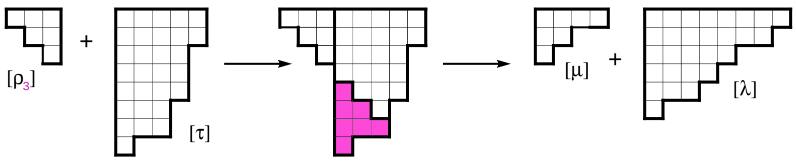

FIGURE 9. Example of bijection ψ.

5. Gauss q-binomial identity

5.1 Involutive proof. Define *q-binomial coefficients* as follows:

$$\binom{m}{k}_q = \frac{(m)!_q}{(k)!_q\,(m-k)!_q}\,, \quad \text{where } (r)!_q = \prod_{i=1}^{r} \frac{(1-q^i)}{(1-q)}\,.$$

The following is another identity due to Gauss:

$$(\lozenge) \qquad \sum_{k=1}^{m} (-1)^k \binom{m}{k}_q = \begin{cases} 0, & \text{if } m \text{ is odd,} \\ (1-q)(1-q^2)\cdots(1-q^{m-1}), & \text{if } m \text{ is even.} \end{cases}$$

The following involutive proof is due to Chen, Hou and Lascoux [CHL]. Start by rewriting $(\lozenge)$ in the following equivalent form:

$$(\lozenge') \qquad \sum_{k=1}^{m} (-1)^k \frac{q^k}{(1-q)\cdots(1-q^k)} \cdot \frac{q^{m-k}}{(1-q)\cdots(1-q^{m-k})} = \begin{cases} 0, & \text{if } m \text{ is odd,} \\ \dfrac{q^m}{(1-q^2)(1-q^4)\cdots(1-q^{m/2})}, & \text{if } m \text{ is even.} \end{cases}$$

Denote by $\mathcal{W} = \mathcal{W}(m)$ the set of triples (λ, μ, k) such that $\ell(\lambda) \le k$ and $\ell(\mu) \le (m-k)$. Define

$$(\lozenge'') \qquad P_m(q) = \sum_{(\lambda,\mu,k)\in\mathcal{W}} (-1)^k\, t^{|\lambda|+|\mu|+m}$$

and observe that $P_m(q)$ equals the l.h.s. of $(\lozenge')$. We define an involution $\sigma : \mathcal{W} \to \mathcal{W}$ as follows.

For a triple (λ, μ, k), if $\lambda_1 < \mu_1$, move the first part μ_1 from μ to λ and set $k \leftarrow k+1$. If $\lambda_1 \ge \mu_1$, and not all parts in λ which are $> \mu_1$ appear even number of times, find the largest such part $c > \mu_1$ and move it from λ to μ; set $k \leftarrow k-1$. If all parts in λ which are $> \mu_1$ appear even number of times, and $\mu \ne \varnothing$, there

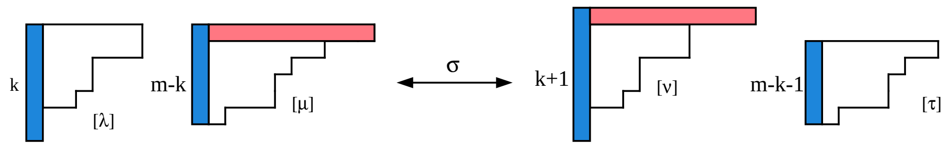

FIGURE 10. Involution $\sigma : (\lambda, \mu, k) \to (\nu, \tau, k \pm 1)$.

are two cases. First, if part μ_1 appears in λ an odd number of times, move part μ_1 from μ to λ and set $k \leftarrow k-1$. Similarly, if part μ_1 appears in λ an even number of times, move part μ_1 from λ to μ and set $k \leftarrow k+1$. Finally, if $\mu = \varnothing$ and λ has every part appear an even number of times, stay put. Denote the resulting triple by $\sigma(\lambda, \mu, k)$.

It is easy to check that σ is an involution on $\mathcal{W}$. By construction, involution σ changes the parity of k which implies ($\lozenge'$).

5.2 Bijective proof. Start with the following simple identity:

$$(\triangle) \qquad \sum_{k=0}^{\infty} \frac{a\, z^k q^k}{(1-aq)(1-aq^2)\cdots(1-aq^k)} \;=\; \prod_{i=1}^{\infty} \frac{1}{(1-azq^i)}$$

Here is a quick bijective proof. Interpret the r.h.s. of ($\triangle$) as $\sum_\lambda z^\ell(\lambda)\, q^{|\lambda|}$, where summation is over all partitions λ. Note that the term insider summation on the l.h.s. of ($\triangle$) corresponds to partitions with the largest part k. This implies ($\triangle$).

Using ($\triangle$) for $a = 1$ and $a = -1$, we obtain:

$$(\lozenge\lozenge) \qquad \sum_{m=0}^{\infty} P_m(q) z^m \;=\; \prod_{i=1}^{\infty} \frac{1}{(1+zq^i)} \prod_{i=1}^{\infty} \frac{1}{(1-zq^i)} \;=\; \prod_{i=1}^{\infty} \frac{1}{(1-z^2q^{2i})}\,.$$

Using ($\triangle$) once again, we conclude that the r.h.s. of ($\lozenge\lozenge$) is equal to the generating function for the r.h.s. of ($\lozenge'$). Since $P_m(q)$ is equals to the l.h.s. of ($\lozenge'$) (see ($\lozenge''$) above), we obtain the result.

6. Ramanujan's identity

6.1 Involutive proof. The following identity was 'found' by Andrews in Ramanujan's "Lost" Notebook:

$$(\heartsuit) \qquad \sum_{r=0}^{\infty} \frac{t^r}{(1+t)(1+t^3)\cdots(1+t^{2r+1})} \;=\; \sum_{k=0}^{\infty} (-1)^k\, t^{6k^2+4k}(1+t^{4k+2})\,.$$

To make the result look more like Theorem 1, Andrews translated the identity in the language of partitions:

Theorem 2. *Let $\mathcal{Q}_n^1$ and $\mathcal{Q}_n^3$ be the sets of partitions λ into odd parts, such that the largest part is repeated an odd number of times while other parts are repeated an even number of times, and such that $\ell(\lambda)$ is 1 and 3 modulo 4, respectively. Then:*

$$|\mathcal{Q}_n^1| \,-\, |\mathcal{Q}_n^3| \;=\; \begin{cases} (-1)^k\,, & \textit{if} \quad n = 12k^2+8k+1 \quad \textit{or} \quad n = 12k^2+16k+5\,, \\ \quad 0\;, & \textit{otherwise.} \end{cases}$$

It was noted in [P1] that this result, in fact, is very symmetric: set $\mathcal{Q}_n = \mathcal{Q}_n^1 \cup \mathcal{Q}_n^3$ consists of all partitions $\lambda \vdash n$, such that both λ and a conjugate partition λ' have only odd parts. We present an involutive proof of ($\heartsuit$) in three steps, two of which are involutive.

Step one. Let $\mathcal{B}_n \subset \mathcal{Q}_n$ be the set of partitions $\lambda \in \mathcal{Q}_n$ such that all parts of λ are greater or equal to the number of parts, and and all parts of λ are congruent to the number of parts modulo 4. In other words, we require $s(\lambda) \geq \ell(\lambda)$ and $4 \mid \lambda_i - \ell(\lambda)$, for all $i = 1 \ldots \ell(\lambda)$.

Our first involution $\varrho : \mathcal{Q}_n \to \mathcal{Q}_n$ has the set of fixed points $\mathcal{B}_n$ and proves that

$$(\star) \qquad \sum_{\lambda \,\in\, \mathcal{Q}_n} (-1)^{(\ell(\lambda)-1)/2} \; = \; \sum_{\lambda \,\in\, \mathcal{B}_n} (-1)^{(\ell(\lambda)-1)/2}\,.$$

The idea is based on a Durfee square type construction (see section 2). Start with $\lambda \in \mathcal{Q}_n$. Remove Durfee square $[\delta_r]$ from $[\lambda]$ and obtain two diagrams $[\mu']$ and $[\nu]$. By definition of $\mathcal{Q}_n$, the size r of the Durfee square is odd and both μ and ν' have only even parts, while μ' and ν have only odd parts. In other words, partitions μ' and ν have only odd parts, which are repeated even number of times. Denote by s the smallest of the parts in both partitions: $s = \min\{s(\mu), s(\nu)\}$. Let $2a$ and $2b$ be the number of times part s appears in μ' and ν, respectively.

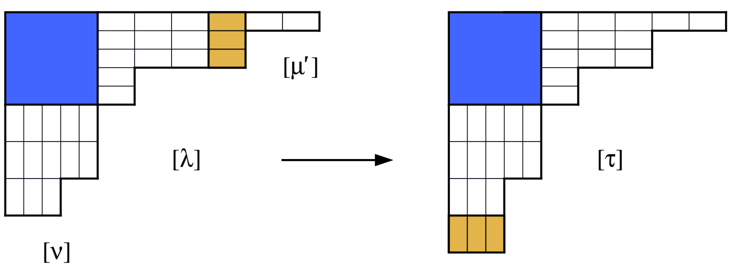

FIGURE 11. An example of involution $\varrho : \lambda \to \tau$.

Now, if a is odd, move two parts of size s from μ' to ν. If a is even and $b > 0$, move two parts of size s from ν to μ'. If a is even and $b = 0$, let $s \leftarrow s + 2$ and repeat the choices. The only pairs (μ, ν) that remain satisfy $\nu = \varnothing$ and $4 \,|\, \mu_i$ for all $i = 1, \ldots, \ell(\mu)$. Now add the Durfee square back to obtain Young diagram of a partition $[\tau] = \varrho(\lambda)$. Note that $\ell(\sigma) = \ell(\lambda) \pm 2$, and the fixed points of ϱ are partitions $\tau \in \mathcal{B}_n$. This implies ($\star$).

Step two. Let $\mathcal{C}_n$ be the set of partitions $\lambda \in \mathcal{D}_n$, such that $\lambda_{2i} = \lambda_{2i+1} + 2$ and $\lambda_{2i-1} = 1 \mod 4$, for all $i = 1 \ldots (\ell(\lambda) - 1)/2$. We prove that $|\mathcal{C}_n| = |\mathcal{B}_n|$ by a direct bijection $\eta : \mathcal{B}_n \to \mathcal{C}_n$. Simply transform the Durfee square $[\delta_k]$ into $[2k-1, 2k-3, \ldots, 3, 1]$ as in Figure 12. Note that $\ell(\mu) = \ell(\lambda)$ for all $\mu = \eta(\lambda)$.

Step three. After steps 1 and 2 it remains to show that

$$(\star') \qquad \sum_{n=1}^{\infty} \sum_{\lambda \in \mathcal{C}_n} (-1)^{(\ell(\lambda)-1)/2}\, t^n \; = \; \sum_{k=-\infty}^{\infty} (-1)^k\, t^{12k^2+8k+1}\,.$$

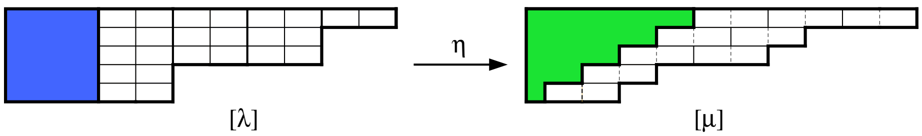

FIGURE 12. An example of bijection $\eta : \mathcal{B}_n \to \mathcal{C}_n$.

Define partitions $\xi_k = (8k+1, 8k-1, \ldots, 4k+3, 4k+1)$, for $k \geq 0$, and $\xi_{-k} = (8k-3, 8k-5, \ldots, 4k-1, 4k-3)$, for $k > 0$ (see Figure 13). Note that $|\xi_k| = 12k^2 + 8k + 1$, for all $k \in \mathbb{Z}$. Below we present an involution χ on $\mathcal{C}_n$ which changes parity of $(\ell(\lambda) - 1)/2$ for all $\lambda \in \mathcal{C}_n$, and has only partitions ξ_k as fixed points.

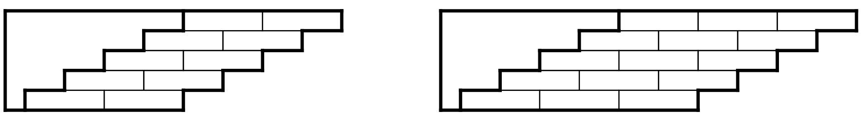

FIGURE 13. Fixed points ξ_2 and ξ_{-3} of the involution χ.

Start with $\lambda \in \mathcal{C}_n$, and let $\ell = \ell(\lambda)$. Denote by $\widetilde{g} = \widetilde{g}(\lambda)$ the smallest r such that $\lambda_{2r-1} > \lambda_{2r} + 2$; let $\widetilde{g} = (\ell(\lambda) + 1)/2$ if no such r exist. Similarly, let $\widetilde{s} = \widetilde{s}(\lambda) = (s(\lambda) + 3)/4$.

If $\widetilde{s} \leq \widetilde{g}$, remove the last two parts $\lambda_{\ell-1}$, λ_ℓ and add 4 to the first $2\widetilde{s} - 1$ parts. Conversely, if $\widetilde{s} > \widetilde{g}$, subtract 4 from the first $2\widetilde{g} - 1$ parts, and add parts $(4\widetilde{g} - 1)$ and $(4\widetilde{g} - 3)$ to λ (see Figure 14).

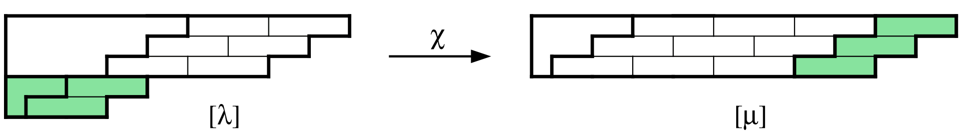

FIGURE 14. An example of involution $\chi : \lambda \to \mu$ on $\mathcal{C}_{57}$. Here $\widetilde{s}(\lambda) = \widetilde{g}(\lambda) = 2$, $\widetilde{s}(\mu) = 5$ and $\widetilde{g}(\mu) = 2$.

It is easy to check that the involution ξ is well defined for all $\lambda \neq \xi_k$. This implies $(\star')$ and therefore $(\heartsuit)$.

6.2 Bijective proof. We start with an equivalent version of Ramanujan's identity $(\heartsuit)$:

$$(\spadesuit) \qquad \sum_{r=0}^{\infty} \frac{t^{2r+1}}{(1+t^2)(1+t^6)\cdots(1+t^{4r+2})} = \sum_{k=-\infty}^{\infty} (-1)^k \, t^{12k^2+8k+1}$$

The involutive proof we presented above follows these analytic steps:

$$\begin{aligned}
(\spadesuit') \quad & \sum_{r=0}^{\infty} \frac{t^{2r+1}}{(1+t^2)(1+t^6)\cdots(1+t^{4r+2})} \;\mathring{=}\; \sum_{n=1}^{\infty} \sum_{\lambda\in\mathcal{Q}_n} (-1)^{(\ell(\lambda)-1)/2}\, t^n \\
&= \sum_{k=1}^{\infty} \frac{(-1)^k\, t^{(2k+1)^2}}{(1-t^2)(1-t^6)\cdots(1-t^{4k+2})\,(1+t^2)(1+t^6)\cdots(1-t^{4k+2})} \\
&= \sum_{k=1}^{\infty} \frac{(-1)^k\, t^{(2k+1)^2}}{(1-t^4)(1-t^{12})\cdots(1-t^{8k+4})} \;\mathring{=}\; \sum_{n=1}^{\infty} \sum_{\lambda\in\mathcal{B}_n} (-1)^{(\ell(\lambda)-1)/2}\, t^n \\
&=_\eta \sum_{n=1}^{\infty} \sum_{\lambda\in\mathcal{C}_n} (-1)^{(\ell(\lambda)-1)/2}\, t^n \;=_\chi\; \sum_{k=-\infty}^{\infty} (-1)^k\, t^{|\xi_k|} \\
&\mathring{=} \sum_{k=-\infty}^{\infty} (-1)^k\, t^{12k^2+8k+1}\,.
\end{aligned}$$

Here the equalities $\mathring{=}$ correspond to combinatorial interpretations of power series, while $=_\eta$ and $=_\chi$ denote equalities which follow from bijection η and involution χ, respectively. Note that bijection η gives a tautological power series identity, while involution ϱ dissolves into a Durfee square argument and a simple power series identity (second and third equalities). We return to involution ϱ in section 8.6 To summarize, the only inherently involutive step in the proof is the Franklin style involution χ. Here is a bijective proof of an extension of this result.

We start with the following identity:

$$(\clubsuit) \quad \sum_{k=0}^{\infty} \frac{z^k\, t^{k(k+1)/2}}{(1-at)(1-at^2)\cdots(1-at^k)} = \sum_{m=0}^{\infty} z^m a^m t^{\frac{m(3m+1)}{2}} \prod_{i=1}^{m} \frac{1+za^{-1}t^i}{1-at^i} + \sum_{m=1}^{\infty} z^m a^{m-1} t^{\frac{m(3m-1)}{2}} \prod_{i=1}^{m-1} \frac{1+za^{-1}t^i}{1-at^i}\,.$$

Setting $t \leftarrow t^4$, $a \leftarrow 1/t^2$, $z \leftarrow (-1/t^4)$ gives an identity proved in the last four steps of $(\spadesuit')$. The proof of $(\clubsuit)$ is based on the following combinatorial interpretations of both sides:

$$(\clubsuit') \qquad \sum_{\lambda\in\mathcal{D}} z^{\ell(\lambda)}\, a^{\mathrm{rk}(\lambda)}\, t^{|\lambda|}\,,$$

where $\mathrm{rk}(\lambda) = \lambda_1 - \ell(\lambda)$ is a *rank* of a partition $\lambda \in \mathcal{D}$. The l.h.s. of $(\clubsuit)$ easily equals to $(\clubsuit')$ by breaking a sum over all $\lambda \in \mathcal{D}$ into sums over all $\lambda \in \mathcal{D}$ with $\ell(\lambda) = k$. Now, observe that the r.h.s. of $(\clubsuit)$ coincides with the r.h.s. of $(**)$ for $a = 1$. Thus for the r.h.s. use the same bijection φ as in the proof of Sylvester's identity $(**)$. We omit the details.

7. Involutions from bijections: background

7.1 General setup. Let us start by giving the most general description of the way involutions can be obtained from bijections.

Suppose $\mathcal{A}$ and $\mathcal{B}$ are two sets of combinatorial objects, and let $\varphi : \mathcal{A} \to \mathcal{B}$ be a bijection. Further, suppose we have an involution α on $\mathcal{A}$ with a subset $\widehat{\mathcal{A}} \subset \mathcal{A}$

of fixed points. Then we can define an involution β on $\mathcal{B}$ with a set of fixed points $\widehat{\mathcal{B}} = \varphi(\widehat{\mathcal{A}})$ by projecting α on $\mathcal{B}$:

$$\beta(x) \,:=\, \varphi(\alpha(\varphi^{-1}(x))), \quad \text{for all} \;\; x \in \mathcal{B}\,.$$

In this case we say that involution β is a *projection of* α *along* φ.

The involutions we consider in this paper are always sign-reversing in the following sense. Suppose $\mathcal{A} = \mathcal{A}_+ \sqcup \mathcal{A}_-$ and $\mathcal{B} = \mathcal{B}_+ \sqcup \mathcal{B}_-$ are disjoint unions of sets. If bijection φ satisfies $\varphi(\mathcal{A}_\pm) = \mathcal{B}_\pm$, it is called *sign-preserving*.

Define $\widehat{\mathcal{A}}_\pm = \mathcal{A}_\pm \cap \widehat{\mathcal{A}}$ and $\widehat{\mathcal{B}}_\pm = \mathcal{B}_\pm \cap \widehat{\mathcal{B}}$. Involution α is called *sign-reversing* if $\alpha(\mathcal{A}_+ \setminus \widehat{\mathcal{A}}_+) = \mathcal{A}_- \setminus \widehat{\mathcal{A}}_-$. This immediately implies that

$$|\mathcal{A}_+| \,-\, |\mathcal{A}_-| \,=\, |\widehat{\mathcal{A}}_+| \,-\, |\widehat{\mathcal{A}}_-|\,.$$

By construction, involution β is also sign-reversing and we have

$$|\mathcal{B}_+| \,-\, |\mathcal{B}_-| \,=\, |\widehat{\mathcal{B}}_+| \,-\, |\widehat{\mathcal{B}}_-|\,.$$

Finally, suppose statistics $\pi : \mathcal{A} \to \mathbb{Z}$ and $\pi' : \mathcal{B} \to \mathbb{Z}$ is defined on both sets such that $\pi'(\varphi(x)) = \pi(x)$ and $\pi(\alpha(x)) = \pi(x)$, for all $x \in \mathcal{A}$. Then

$$(\otimes) \qquad \sum_{x\in\mathcal{A}_+} t^{\pi(x)} - \sum_{x\in\mathcal{A}_-} t^{\pi(x)} \,=\, \sum_{x\in\widehat{\mathcal{A}}_+} t^{\pi(x)} - \sum_{x\in\widehat{\mathcal{A}}_-} t^{\pi(x)}\,,$$

and the same holds when one replaces $\mathcal{A}$ with $\mathcal{B}$. Typically, our sets will be infinite sets of partitions λ, and $\pi(\lambda) = |\lambda|$. Extension to more than one statistics is straightforward.

The main result of this paper can be summarized in the following claim:

Meta Theorem *All involutions defined in sections 1–6 are projections of* natural *involutions along the corresponding bijections.*

Here by a "corresponding bijection" we refer to a bijection arising the proof of the same identity as the one proved by an involution. The result may seem surprising given a different, seemingly ad hoc nature of involutions in each case. This can be explained by varying nature of the corresponding bijections and, to a lesser extend by the difference in "natural involutions". Let us introduce the latter before we proceed to formal version of the Meta Theorem.

7.2 Vahlen's involution. Let us start with the following trivial identity:

$$(\oplus) \qquad \frac{1-t^k}{1-t^k} \,=\, 1\,,$$

which can be interpreted as a summation of $(-1)^a t^{(a+b)k}$ over all pairs (k^a, k^b), $a \in \{0,1\}$ and $b \in \mathbb{Z}_{\geq 0}$. The following involution cancels the terms:

$$\alpha : (k^1, k^{b-1}) \longleftrightarrow (k^0, k^b)\,.$$

Consider now the following no less trivial identity:

$$(\oplus\oplus)\qquad \frac{\prod_{i=1}^{k}(1-t^i)}{\prod_{i=1}^{k}(1-t^i)} \,=\, 1\,.$$

The l.h.s. of $(\oplus\oplus)$ can be interpreted as $\sum_{\lambda,\mu}(-1)^{\ell(\lambda)}t^{|\lambda|+|\mu|}$, where the summation goes over all $\lambda\in\mathcal{D}$, $\mu\in\mathcal{P}$ such that $\lambda_1,\mu_1\le k$. *Vahlen's involution* υ is defined as follows. Compare the smallest parts $s(\lambda)$ and $s(\mu)$. If $s(\lambda)\le s(\mu)$, move part $s(\lambda)$ from λ to μ. Otherwise, if $s(\lambda)>s(\mu)$, move part $s(\mu)$ from μ to λ.

Note that Vahlen's involution υ coincides with α when $k=1$. In fact, to obtain Vahlen's involution from α we need to define how to break ties. Vahlen's involution favors smaller parts, but one can also favor larger parts, or, in fact, use any linear order on $\{1,2,\ldots,k\}$.

Vahlen's involution is our first example of a "natural involution". An extension of Vahlen's involution to any set $I=\{r_1,\ldots,r_k\}\subset\mathbb{N}$ is straightforward:

$$(\oplus\oplus\oplus)\qquad \frac{\prod_{i=1}^{k}(1-t^{r_i})}{\prod_{i=1}^{k}(1-t^{r_i})} \,=\, 1\,.$$

7.3 Parity involution. Again, let us start with the following trivial identity:

$$(\ominus)\qquad \frac{1}{(1-t^k)(1+t^k)} \,=\, \frac{1}{1-t^{2k}}\,.$$

The l.h.s. of $(\ominus)$ can be interpreted as a sum of $(-1)^a t^{(a+b)k}$ over a set of pairs of partitions (k^a,k^b) with $a,b\in\mathbb{Z}_{\ge 0}$. The following involution cancels the terms:

$$\alpha:\quad (k^a,k^0)\leftrightarrow(k^{a-1},k^1)\,,\quad (k^{a-2},k^2)\leftrightarrow(k^{a-3},k^3)\,,\quad \ldots$$

The only remaining terms correspond to pairs (k^0,k^{2c}), which give a combinatorial interpretations for the r.h.s. of $(\ominus)$. We call α defined above a *parity involution.*

Here is a more general version of $(\ominus)$:

$$(\ominus\ominus)\qquad \prod_{i=1}^{k}\frac{1}{1-t^i}\,\prod_{i=1}^{k}\frac{1}{1+t^i} \,=\, \prod_{i=1}^{k}\frac{1}{1-t^{2k}}\,.$$

The l.h.s. of $(\ominus\ominus)$ can be interpreted as a sum

$$\sum_{\lambda,\,\mu\,:\,\lambda_1,\,\mu_1\le k}(-1)^{\ell(\lambda)}\,t^{|\lambda|+|\mu|}\,.$$

By analogy with Vahlen's involution, the corresponding parity involution in this case can be described as follows. Let r be the largest (or, say, the smallest—the order does not matter again) part size which either appears in λ or appears in μ odd number of times. If part k appears in μ odd number of times, move it to λ. Otherwise, if part k appears in μ even number of times, and appears in λ at least once, move it from λ to μ. By construction, the only remaining terms correspond to pairs of partitions (λ,μ), where $\lambda=\varnothing$ and each part in μ appear even number of times. This proves $(\ominus\ominus)$.

It is straightforward to extend of the parity involution of the following generalization of $(\ominus\ominus)$ to any any set $I=\{r_1,\ldots,r_k\}\subset\mathbb{N}$:

$$(\ominus\ominus\ominus)\qquad \prod_{i=i}^{k}\frac{1}{1-t^{r_i}}\,\prod_{i=1}^{k}\frac{1}{1+t^{r_i}} \,=\, \prod_{i=1}^{k}\frac{1}{1-t^{2r_i}}\,.$$

8. Involutions from bijections: case by case

8.1 Franklin's involution. Consider the both sides of Euler's identity $(*)$ and Sylvester's identity $(**)$ with $z = -1$. The l.h.s. in both identities coincide and are equal to

$$\sum_{\lambda \in \mathcal{D}} (-1)^{\ell}(\lambda)\, t^{|\lambda|}\,.$$

On the other hand, the r.h.s. in $(**)$ differs from $(*)$ by a factor $(\oplus\oplus)$ (see section 7.2). We can now bring this case to a general setup as in section 7.1.

Let $\mathcal{D}_+$ and $\mathcal{D}_-$ be the set of all partitions $\lambda \in \mathcal{D}$ with $\ell(\lambda)$ even and odd, respectively. Define $\mathcal{R}$ to be a sets of triples (θ_m, μ, ν) where $\mu \in \mathcal{D}$, $m \in \mathbb{Z}$, and such that $\mu_1, \nu_1 \leq m$ for $m \geq 0$, and $\mu_1, \nu_1 \leq -(m+1)$ for $m < 0$. Let $\mathcal{R}_+$ and $\mathcal{R}_-$ be subsets of triples $(\theta_m, \mu, \nu) \subset \mathcal{R}$, such that and $m + \ell(\mu)$ is even and odd, respectively. Recall Sylvester's bijection $\varphi : \mathcal{R}_\pm \to \mathcal{D}_\pm$ defined in section 1.2. Finally, consider Vahlen's sign-reversing involution υ on triples $(\theta_m, \mu, \nu) \in \mathcal{R}$ as above with $(\theta_m, \varnothing, \varnothing)$ as fixed points. Clearly, Euler's identity follows from $(\otimes)$ in this case.

Proposition 1. *Franklin's sign-reversing involution α on $\mathcal{D}_\pm$ is a projection of Vahlen's involution υ along Sylvester's bijection φ.*

Proof. Start by checking that Sylvester's map is sign-preserving. Now, the smallest parts in μ and ν correspond to the lengths of the horizontal line $s(\lambda)$ and diagonal line $g(\lambda)$ in Young diagram $[\lambda]$. These lines are moved exactly the same way as under Vahlen's involution υ. From above, the fixed points of υ correspond to $\lambda = \theta_m$, $m \in \mathbb{Z}$. This completes the proof. $\square$

8.2 Andrews's involution. Recall Andrews's involution β on the set $\mathcal{M}$ of all MacMahon diagrams (see section 2.1). Let $\langle\lambda\rangle$ be in $\mathcal{M}_+$ and $\mathcal{M}_-$ if $\ell(\lambda)$ is even and odd, respectively. By definition, involution β is sign-reversing. Denote by $\mathcal{R}$ the set of all triples of MacMahon diagrams $(\langle\delta_k\rangle, \langle\mu\rangle, \langle\nu\rangle)$ with $\ell(\mu), \ell(\nu) \leq k$ if $m\langle\delta_k\rangle = 0$, and $\ell(\mu), \ell(\nu) \leq k-1$ if $m\langle\delta_k\rangle = 1$. Let a triple $(\langle\delta_k\rangle, \langle\mu\rangle, \langle\nu\rangle)$ be in $\mathcal{R}_+$ and in $\mathcal{R}_-$ if $k + \ell(\mu)$ is even and odd, respectively. Consider a sign-preserving bijection $\psi : \mathcal{R} \to \mathcal{M}$ defined in section 2.2. Finally, two products on the r.h.s. of $(\circ\circ)$ cancel when $z = -1$. This cancellation can be done by Vahlen's involution, which defines a sign-reversing involution υ on $\mathcal{R}$. Define an order of cancellation by always moving the smallest part, and when both part sizes are the same – the vertical line with a marked square rather than without.

Proposition 2. *Andrews's sign-reversing involution β on $\mathcal{M}_\pm$ is a projection of Vahlen's involution υ along bijection ψ.*

Proof. Start by checking that ψ is sign-preserving and the fixed points of υ are triples $(\langle\delta_m\rangle, \varnothing, \varnothing)$ which are mapped onto fixed points $\langle\delta_m\rangle$ of β. Observe that the smallest parts $s(\mu)$ and $s(\nu)$ correspond to sizes $v(\lambda)$ and $s(\lambda)$ of the vertical and horizontal lines, respectively. The projection of υ along ψ compares these rows and moves smaller one next to the other, changes marking of the corner. Now recall the description of β to see that it coincides with the projection. $\square$

8.3 Knuth-Paterson involution. The case of Shanks identity is essentially a straightforward extension of 8.1. Use Vahlen's involution υ to cancel terms of the products in (⁂) in section 3.2. Now recall Knuth-Paterson restriction of Franklin's involution α. Denote by ϕ the bijection defined in section 3.2. We have

Proposition 3. *Knuth-Paterson sign-reversing involution α is a projection of Vahlen's involution υ along bijection ϕ.*

The proof is follows verbatim the proof of Proposition 1.

8.4 Sylvester's involution. Consider the following version of Jacobi triple product identity:

$$(\diamond\diamond\diamond) \qquad \prod_{i=1}^{\infty}(1+st^i)\prod_{j=0}^{\infty}(1+s^{-1}t^j)\prod_{r=1}^{\infty}(1-t^r) = \sum_{k=-\infty}^{\infty} s^k t^{\frac{k(k+1)}{2}} \prod_{i=1}^{\infty}(1-t^i)\prod_{i=1}^{\infty}\frac{1}{1-t^i}\,.$$

In notation of section 4, write $(\diamond\diamond\diamond)$ as follows:

$$(\diamond\diamond\diamond') \qquad \begin{aligned} &\sum_{\lambda\in\mathcal{D}} z^{\ell(\lambda)}\, t^{|\lambda|} \sum_{\mu\in\mathcal{D}'} z^{-\ell(\mu)}\, t^{|\mu|} \sum_{\nu\in\mathcal{D}} (-1)^{\ell(\nu)}\, t^{|\nu|} \\ &= \sum_{k=-\infty}^{\infty} z^k\, t^{|\rho_{|k|}|} \sum_{\tau\in\mathcal{P}} t^{|\tau|} \sum_{\omega\in\mathcal{D}} (-1)^{\ell(\omega)}\, t^{|\omega|} \end{aligned}$$

Recall a Hathaway-Sylvester's bijection $\psi : (\rho_k, \tau) \to (\lambda, \mu)$ defined in section 4.2, which cancels the product of the first two sums on the r.h.s. of $(\diamond\diamond\diamond')$ with the product of the first two sums on the l.h.s. Extend this bijection to the sets of triples: $\psi : (\rho_k, \tau, \omega) \to (\lambda, \mu, \nu)$, by setting $\omega = \nu \in \mathcal{D}$. Consider Vahlen's involution υ on $(\rho_k, \tau, \omega) \in \mathcal{P}\times\mathcal{D}$ which cancels last two products on the r.h.s. of $(\diamond\diamond\diamond)$, with the only difference that we take a conjugate partition $\tau' \in \mathcal{P}$ instead of $\tau \in \mathcal{P}$ when $k \geq 0$. Finally, recall Sylvester's sign-reversing involution γ on $(\lambda, \mu, \nu) \in \mathcal{D}\times\mathcal{D}'\times\mathcal{D}$, defined in section 4.1.

Proposition 4. *Sylvester's sign-reversing involution γ is a projection of Vahlen's involution υ along bijection ψ.*

Proof. First, define a sign of a triple (ρ_k, τ, ω) to be $(-1)^{\ell(\omega)}$ as in $(\diamond\diamond\diamond)'$, and observe that ψ is sign-preserving. The fixed points of υ are triples $(\rho_k, \varnothing, \varnothing)$ which are mapped onto fixed points of γ, which consists of triple $(\rho_k, \varnothing, \varnothing)$ when $k \geq 0$, and $(\varnothing, \rho_{-k}, \varnothing)$ when $k < 0$.

As in construction of γ and ψ, we consider only the case $\ell(\lambda) \geq \ell(\mu)$; the other case is similar. There are four subcases to consider:

- $g(\lambda) \geq s(\nu)$, $\lambda \neq \rho_k$,
- $g(\lambda) < s(\nu)$, $\lambda \neq \rho_k$,
- $\lambda = \rho_k$, $g(\lambda) + s(\mu) < s(\nu)$,
- $\lambda = \rho_k$, $g(\lambda) + s(\mu) \geq s(\nu)$,

where $k \geq 0$. Arrange Young diagrams $[\lambda]$ and $[\mu]$ by shifting the rows as in bijection φ (see section 4.2). Two examples are shown in Figure 15 : here $\lambda =$

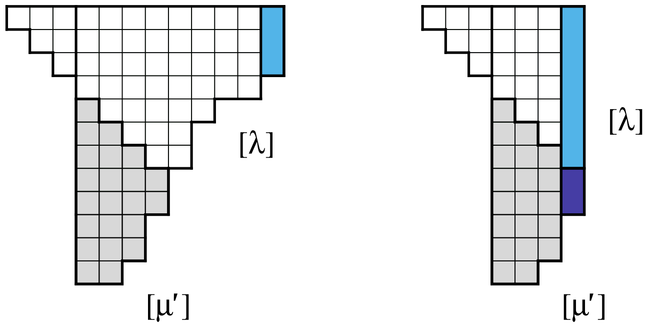

FIGURE 15. Moving parts in Sylvester's involution.

$(12, 11, 10, 8, 5, 3, 2)$, $\mu = (8, 7, 5, 2)$ in the first example, and $\lambda = (7, 6, 5, 4, 3, 2, 1)$, $\mu(8, 7, 5, 2)$ in the second example.

In the four subcases as above, the first two correspond to the case when $g(\lambda)$ coincides with the smallest part of the conjugate partition τ' formed by attaching $[\lambda]$ and $[\mu]$ as above and removing $[\rho_k]$. Since $g(\lambda)$ is compared with $s(\nu)$, we conclude that Sylvester's involution is a projection of involution υ along φ in these subcases. Similarly, if $\lambda = \rho_k$ as in the second two subcases, we have $g(\lambda) + s(\mu)$ is equal to $s(\tau')$ (see Figure 15). Since now $g(\lambda) + s(\mu)$ is compared with $s(\nu)$, we conclude that Sylvester's involution is a projection of involution υ along φ in these subcases as well. $\square$

8.5 Chen-Hou-Lascoux involution. The case of Gauss q-binomial identity relies on parity involution ($\ominus\ominus$) defined in section 7.3. Recall Chen-Hou-Lascoux sign reversing involution σ acting on

$$\mathcal{W} \, = \, \{(\lambda, \mu, k) \mid \ell(\lambda), \ell(\mu) \le k, \lambda, \mu \in \mathcal{P}, k \in \mathbb{Z}_{\ge 0}\} \, .$$

Let us rewrite ($\Diamond\Diamond$) and ($\Diamond''$) (see section 5) as follows:

$$(\Diamond\Diamond') \qquad \sum_{(\lambda,\mu,k)\in\mathcal{W}} (-1)^k \, t^{|\lambda|+|\mu|} \; = \; \prod_{i=1}^{\infty} \frac{1}{1+q^i} \; \prod_{i=1}^{\infty} \frac{1}{1-q^i} \, .$$

Let $\mathcal{A} = \mathcal{P} \times \mathcal{P}$. Define a sign of $(\lambda, \mu) \in \mathcal{A}$ to be $(-1)^{\ell(\lambda)}$. Consider a sign-reversing parity involution π defined on $\mathcal{A}$ as in section 7.3, with fixed points $\widehat{\mathcal{A}} = \{(\varnothing, \mu) \mid \mu_i \text{ is even for all } 1 \le i \le \ell(\mu)\}$. This corresponds to combinatorial interpretation of the r.h.s. of ($\Diamond\Diamond'$), since the latter is equal to $\prod_{i=1}^{\infty} (1 - z^2 q^{2i})^{-1}$. In the definition of π, choose an order to be the largest of the parts to be compared. Finally, consider a straightforward bijection $\varphi : \mathcal{A} \to \mathcal{W}$ defined in the proof of ($\triangle$) in section 5.2.

Proposition 5. *Chen-Hou-Lascoux sign-reversing involution σ is a projection of parity involution π along bijection φ.*

The proof is a straightforward check of the definitions and is left to the reader.

8.6 Involution for Ramanujan's identity. The involutive proof in section 6.1 is defined in three steps, which consist of involution ϱ, bijection η, and involution χ. We need to consider only the first and the third steps, corresponding to two involutions.

Step one. Start with involution ϱ, which proves the first three equalities in ($\spadesuit'$) in section 6.2. Let $\mathcal{A}_+ = \cup_n \mathcal{Q}_n^1$, $\mathcal{A}_- = \cup_n \mathcal{Q}_n^3$, $\mathcal{A} = \mathcal{A}_+ \cup \mathcal{A}_-$, and let $\widehat{\mathcal{A}} := \cup \mathcal{B}_n \subset \mathcal{A}$ be as in section 6.1. In other words, we define a sign on $\lambda \in \mathcal{A}$ to be $(-1)^{(\ell(\lambda)-1)/2}$. Recall that involution ϱ is sign-reversing with $\mathcal{B}$ as fixed points. Let $\mathcal{U}$ be the set of triples (k, λ, μ), such that $\ell(\lambda), \ell(\mu) \le 2k+1$, $k \ge 0$, all parts in λ, μ are odd, while all parts in λ', μ' are even.

The Durfee square bijection $\varphi : \mathcal{U} \to \mathcal{A}$ maps a triple (k, λ, μ) into a partition λ, such that $[\lambda]$ is obtained by joining $[\delta_{2k+1}]$, $[\mu']$ and $[\nu]$ as in Figure 11. We need a *substitution bijection* $\iota : (k, \mu, \nu) \to (k, \widetilde{\mu}, \widetilde{\nu})$, where $\widetilde{\mu}$ is defined to contain parts $(2\mu_1', 2\mu_3', \dots)$. Note that $\widetilde{\mu}_i = 2 \mod 4$, $\mu_1 \le 4k+2$ and the same condition holds for ν. Define a sign of $(k, \widetilde{\mu}, \widetilde{\nu})$ to be $(-1)^{\ell(\nu)}$

Finally, consider a parity involution π acting on triples $(k, \widetilde{\mu}, \widetilde{\nu})$ by its action on pairs $(\widetilde{\nu}, \widetilde{\mu})$ and leaving k unchanged.

Proposition 6. *Sign-reversing involution ϱ defined in section 6.1 is a projection of parity involution π along composition of bijections $\varphi \circ \iota^{-1}$.*

Proof. First, consider a projection of sign-reversing involution π along bijection ι^{-1}. The effect is an action of pairs of rows in $[\mu]$ and $[\nu]$, whose lengths are compared. Further projection along bijection φ maps these pairs of rows to pairs of vertical and horizontal lines of squares in $[\lambda]$ as shown in Figure 11. It remains to check that $\varphi \circ \iota^{-1}$ is sign–preserving, which is straightforward. □

Step three. In notation of section 6.1, let $\mathcal{C} = \cup_n \mathcal{C}_n$, and define a sign on $\lambda \in \mathcal{C}$ to be $(-1)^{(\ell(\lambda)-1)/2}$. Let $\mathcal{D} = \mathcal{D}_+ \cup \mathcal{D}_-$ be as in section 8.1. We define a *substitution bijection* $\zeta : \mathcal{D} \to \mathcal{C}$ as follows:

$$\begin{aligned}\zeta(\tau_1, \tau_2, \dots, \tau_{\ell-1}, \tau_\ell) &= 4\,(\tau_1 - \ell, \tau_2 - (\ell-1), \dots, \tau_{\ell-1} - 2, \tau_\ell - 1) + \\ &\qquad + (2\ell-1, 2\ell-3, \dots, 3, 1) \\ &= (4\tau_1 - (2\ell+1), 4\tau_2 - (2\ell-1), \dots, 4\tau_{\ell-1} - 5, 4\tau_\ell - 3).\end{aligned}$$

Recall Franklin's sign-reversing involution α on $\mathcal{D}$ (see section 1.1). It is easy to check that ζ is sign-preserving and maps its fixed points ξ_k into fixed points θ_k of α.

Proposition 7. *Sign-reversing involution χ on $\mathcal{C}$ defined in section 6.1 is a projection of Franklin's involution α along bijection ζ.*

Proof. The result is clear after comparing Figure 1 and Figure 14. Observe that if $\zeta(\tau) = \lambda$, then $g(\tau) = \widetilde{g}(\lambda)$ and $s(\tau) = \widetilde{s}(\lambda)$, which are compared accordingly. Moreover, removing a diagonal line of length g in τ corresponds to removing of four squares in the first g rows. Similarly, removing a horizontal line in τ corresponds to removing last two rows in λ. This implies the result. □

As suggested by the r.h.s. of ($\clubsuit$), consider Vahlen's involution υ on $\mathcal{R}$, defined in section 8.1. Finally, let $\varphi : \mathcal{R} \to \mathcal{D}$ be Sylvester's bijection defined in section 1.2.

Corollary 1. *Sign-reversing involution χ on $\mathcal{C}$ defined in section 6.1, is a projection of Vahlen's involution υ along composition of bijections $\zeta \circ \varphi$.*

Proof. The result follows immediately from Propositions 1 and 7. □

9. Involutions from bijections: discussion

This idea behind Meta Theorem is a philosophical claim that partition involutions are not ad hoc arguments, as a casual reader may assume by reading their description, but rather projection of "natural" involutions along the corresponding bijections. As a rule, these bijections prove stronger arguments, have a simpler structure, and are easier to discover and explain. Thus, is a sense, the bijective arguments are the "primal" results and, when found, involutive proofs should be ignored since arguments follow "automatically" from our setup. In a different direction, the existence of a partition involution proof strongly suggests possibility of a bijective argument which would "automatically" imply the involution.

To give it a name, we call this the *automaticity of involutions idea*. Clearly, is is in sharp contrast with the commonly accepted 'truth', so let us spend some time discussing pro and contra arguments[2] We elaborate on the traditional *"not important"*, *"not true"* and *"not new"* lines, and at the same time cover a history of the problem and give some references to the literature.

Of course, being philosophical, the idea cannot be proved or disproved. On the other hand, we do establish a connection in several important examples which seem to exhibit a pattern. In fact, the very first case we consider—of Franklin's involution—is not new. After the results of this paper were obtained, we discovered that Andrews already discovered a connection between Franklin's involution and Sylvester's identity in a somewhat forgotten publication [A4]. Andrews even suggested that this is exactly how it might have happened historically[3], but cautioned that *"intuition and insight are not famous for proceeding in an orderly manner; hence the chronology of events may have been different from that described here."* [A4]. Unfortunately, Andrews never formalized or generalized his observation, nor connected it with Vahlen's involution, but rather presented them as an informal historical speculation.

The results in Propositions 2–7 seem to be new. The case of Jacobi triple product identity is especially exciting since both bijective and involutive proof go back to Sylvester's article [S]. Both Hathaway and Sylvester discovered versions of bijection ψ which was later rediscovered by Wright and a score of others [H,S,W] (see [P1] for further references). The involutive proof, written rather obscurely in [S], was rediscovered in [Zo] and used in [KPP] to obtain extensions of Jacobi identity. Therefore, Proposition 4 establishes a formal connection between two proofs which were considered *different* for over a century (cf. [P1]).

On a smaller historical scale, a similar situation appeared to happen with Gauss product identity (∘). In [A1], Andrews proves this and another similar identity by an explicit involution we present in section 2.1. In the very same paper Andrews proves bijectively what he calls Rogers–Fine identity (see section 10.2) which in fact implies (∘∘). Despite showing that Rogers–Fine identity is an extension Gauss identity, Andrews never made a substitution to conclude that his bijective proof in fact projects onto the involutive proof. Several subsequent investigators who

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[2]A similarly controversial idea in a context of binomial identities was recently presented by Zeilberger [Ze].

[3]This was later refuted [A5].

found equivalent proofs of the Rogers–Fine identity did not realize this fact either (cf. [P1]).

Finally, the case of Gauss q-binomial identity appeared in a recent paper [CHL]. The authors present both proofs (and their extensions to other roots of unity) but never realized that their involution is a simple projection of a parity involution. It seems, had they started with smaller rather than larger parts and conjugated Young diagrams in their figures, they would undoubtedly notice a connection. Or, perhaps, the problem is a lack of a formal setup to make a statement of this kind. Either way, we hope this paper will serve as a guidance in future investigations.

To summarize, we are saying that the involutions we obtain as projections are well studied in the literature along with algebraic and bijective proofs. The fact that they are all projections of bijections is thus of importance and give an a posteriori support to the automaticity of involutions idea.

Now, one can counter this with other notable involutions, such as that of Schur and Bressoud–Zeilberger [Sc,BZ2] (see also [P1]), which cannot be described by these means. We argue in this case that these involutions have a different "nature", whatever that means. As was shown in [P2], the Bressoud–Zeilberger involution follows "automatically" from Dyson's analytic proof with Dyson's adjoint map applied accordingly. Dyson's proof is recursive and thus has a very different nature compared to the Durfee type arguments. As for the Schur's involution, the jury is still out. The construction seems to combine several different elements. Although there is no "automatic proof" in sight, it is quite possible that it fits into a slightly more general framework. We plan to return to Schur's involution in the future.

A skeptical reader may object to automaticity of involutions idea by saying that we know too few partition involutions to make a judgement, that there is no reason to assume that *all* partition involutions are of a certain type just because this seems to work for the involutions we know. We have two counter points for this. First, it seems to us that the relative lack of available partition involutions is an *indication* in favor of the automaticity, We believe that for a partition involution to exist it must be a projection of a bijection, and since very few bijections have projections, this implies the scarcity of partition involution.

Second, we decided to test our idea on a "random" partition identity, by choosing an important Ramanujan's identity (♡) as a example. The fact that the involutive proof we found is a composition of projections suggests that our automaticity of involutions idea may be somewhat overreaching, but not overly so.

The reader may also suggest that conceptually little in this paper is new in view of the celebrated Garsia–Milne involution principle, which routinely employs Vahlen's involution in the proof of partition identities [GM]. The truth is that the involution principle never cares for what kind of bijections it produces. It inputs several involutions (some of which are complicated) and trivial bijections and produces a bijection which in the cases of interest (such as Rogers–Ramanujan identities) does not seem to possess any nice combinatorial properties. Our approach is the opposite: we employ complicated bijections and trivial involutions to produce well known and quite nontrivial involutions.

This last observation only underscores that finding a "good" combinatorial proof still requires the ingenuity and cannot be completely mechanized as suggested in [Ze]. The main goal of this work is a shift in emphasis from involutive to bijective proofs.

10. Connections to other identities

10.1 Euler–Sylvester identity. Before we conclude, let us establish connections between partitions identities we use and those known in the literature. First, recall classical Euler's identity

$$\sum_{k=0}^{\infty} \frac{z^k\, t^{k(k+1)/2}}{(1-t)(1-t^2)\cdots(1-t^k)} \;=\; \prod_{i=1}^{\infty}(1+zt^i)\,.$$

Combined with Sylvester's identity $(**)$ we obtain:

$$(ES)\qquad \sum_{k=0}^{\infty} \frac{z^k\, t^{k(k+1)/2}}{(1-t)(1-t^2)\cdots(1-t^k)} \;=\; \sum_{m=0}^{\infty} z^m\, t^{\frac{m(3m+1)}{2}} \prod_{i=1}^{m} \frac{(1+zt^i)}{(1-t^i)} + \sum_{m=1}^{\infty} z^m\, t^{\frac{m(3m-1)}{2}} \prod_{i=1}^{m-1} \frac{(1+zt^i)}{(1-t^i)}\,.$$

Thus, identity $(\clubsuit)$ is an extension of this combined *Euler–Sylvester identity.* In fact, there is a third power series $F(a,z,t)$ equal to both sides of $(\clubsuit)$:

$$F(a,z,t) := \sum_{k=0}^{\infty} z\, a^{k-1}\, t^k \prod_{i=1}^{k} (1 + z\, a^{-1}\, t^i)\,.$$

This is a "distinct" version of identity $(\triangle)$; the proof is analogous.

10.2 Cauchy and Rogers–Fine identities. Setting $a = 0$ in the *extended Shanks identity* $(\circledast\circledast)$ gives a classical *Cauchy identity* :

$$\prod_{i=1}^{\infty} \frac{1}{1-zt^i} \;=\; 1 + \sum_{k=1}^{\infty} \frac{z^k\, t^{k^2}}{(1-t)(1-t^2)\cdots(1-t^k)\,(1-zt)(1-zt^2)\cdots(1-zt^k)}\,.$$

Combination of the first two equalities in $(\spadesuit')$ is a special case of Cauchy identity.

Consider now the following general four variable *Rogers–Fine identity*:

$$(RF)\qquad 1 + \sum_{n=1}^{\infty} \frac{(1+at)(1+at^3)\cdots(1+at^{2n-1})\, z^n t^{2n}}{(1-bt^2)(1-bt^4)\cdots(1-bt^{2n})} = \sum_{r=0}^{\infty} \frac{(1+azt^{4r+3}) z^r t^{2r(r+1)}}{(1-zt^{2(r+1)})} \prod_{i=1}^{r} \frac{(1+at^{2i-1})(b+azt^{2i+1})}{(1-bt^{2i})(1-zt^{2i})}\,.$$

One can deduce both $(\circ\circ)$ and $(\spadesuit)$ by letting $b=1$ or $b=0$ and making a change of variables. We leave the details to the reader.

10.3 Finite analogues. Just like Shanks identity $(\circledast)$ is a "finite analogue" of Euler's identity, identity $(\circledast\circledast)$ is a finite analogue of Sylvester's identity $(**)$. In the literature there are finite analogues of many other partition identities. For example, MacMahon's identity is a finite analogue of Jacobi triple product identity:

$$\prod_{i=1}^{m}(1+zq^{2i-1}) \prod_{j=1}^{n}(1+z^{-1}q^{2j-1}) \;=\; \sum_{k=-n}^{m} z^k\, q^{k^2} \binom{m+n}{k+n}_{q^2}\,.$$

11. Final remarks

1. As we mentioned earlier, Franklin's involution was published in [F]. Sylvester's identity ($**$) and its bijective proof is given in [S]. Andrews' involutive proof of Gauss identity ($\circ$) and a Durfee square type bijective proof of the Rogers–Fine identity is given in [A1]. Shanks identity ($\circledast$) was introduced in [Sh] to derive a simple proof of the Jacobi triple product identity, and was shown to follows from Franklin's involution in [KP].

Our presentation of the bijective proof of Jacobi identity ($\diamond$) follows [W], and the involutive proof is a modified version of that in [Zo] (see also [P1]). The involutive and bijective proofs of Gauss identity ($\Diamond$) follows [CHL], and Vahlen's involution is given in [V]. Both proofs of Ramanujan's identity ($\heartsuit$) seem to be new. See [A3] for combinatorial proofs of related Ramanujan's identities. Further partitions bijections and references can be found in [A2,P1].

2. An involutive proof of Euler's Pentagonal Theorem of Bressoud and Zeilberger [BZ2] follows automatically from a recursive argument of Dyson proving a somewhat stronger result about the ranks of partitions [P2]. It would be nice to extend he philosophy of this paper to such recursive proofs taking into account that the simplest proofs of Rogers–Ramanujan identities in fact are using a recursive argument. Then compare the resulting involution with Schur's involution.

In a different but not unrelated direction, Bressoud's simple proof of Rogers–Ramanujan identities is based on a finite analogue of these identities [B]. This proof was later extended to a bijective proof of the identities by means of the involution principle [BZ1,BZ3]. Can one explain the underlying nontrivial involution in the same manner as we do in this paper?

There are other variations on Franklin's theme in the recent literature [C,Li]. Can one apply automaticity idea to obtain these results?

Finally, Bessenrodt and the author found an involutive proof of Fine's identity [BP]. While we do show that it is a restriction of Franklin's involution, this involution does not seem to fit into our framework. We challenge the reader to make this connection more formal.

3. By restricting $m \times 2m$ Durfee rectangles to $m \leq k$, one can obtain a finite version of the Rogers–Fine identity together with a bijective proof. This most general bijection for different values of the parameters can then be projected onto Franklin's, Knuth–Paterson and Andrews's involution, and well as on Sylvester's bijection. We leave the details to the reader.

4. Most paper in constructive partition theory are written in one of three styles which roughly correspond to characters in a classical work by Leone [Le]. The problem is the need to show some novel results results in the subject, and bijection in and by itself is not considered to be sufficient; thus the need for generalizations was invented. Papers of the first type have classical versions written upfront with bijections clearly described, and the generalizations sketched for reader's convenience. Papers of the second type describe bijections only for generalizations, leaving the reader out in the cold struggling to understand the bijection constructions even in the simplest case. Finally, papers of the third type combine known involutions and bijections into new bijections by means of the involution principle; these bijections are not analyzed and are never used in those or subsequent publications.

We hope this paper introduces a new style of writing, where the nature of bijections and involutions is explained on a certain level. The first attempt of this kind was made by the author in [P3]. As the title suggests, we plan to continue writing in this style.

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