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A direct bijective proof of the hook-length formula. (English. English summary)

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This is probably the most important recent contribution to bijective combinatorics. The hook-length formula states that the number of standard Young tableaux of a given shape λ, where λ is a partition of n (viewed as a Ferrers diagram), is given by $n!/\prod_\rho h_\rho$, where the product ranges over all cells ρ of λ and where h_ρ is the hook-length corresponding to ρ. In view of such a nice, combinatorial formula, it is natural to ask for a "nice", i.e., bijective, proof that would explain the form of the formula. That is, one is looking for a bijection which would explain the formula

$$(*) \qquad \#(\text{SYT of shape } \lambda)\prod_\rho h_\rho = n!.$$

There do already exist such "hook bijections", by D. S. Franzblau and D. Zeilberger [J. Algorithms 3 (1982), no. 4, 317–343; MR 84b:05016], J. B. Remmel [Linear and Multilinear Algebra 11 (1982), no. 1, 45–100; MR 83h:05010], Zeilberger [Discrete Math. 51 (1984), no. 1, 101–108; MR 86b:05008a], and the reviewer [Electron. J. Combin. 2 (1995), Research Paper 13, approx. 9 pp. (electronic); MR 96h:05209]. However, none of them is regarded as really "satisfactory". In contrast, the bijection presented in the paper under review (an announcement of the bijection, but without proof, is contained in [I. M. Pak and A. V. Stoyanovskii, Funktsional. Anal. i Prilozhen. 26 (1992), no. 3, 80–82; MR 93h:20014]) must be considered as the natural hook bijection. It fulfills the dream that one would start with an arbitrary filling of the shape λ with $1, 2, \cdots, n$ (the number of such fillings being counted by the right-hand side of $(*)$), sort the entries step by step by jeu de taquin moves in order to obtain a standard Young tableau in the end, and, simultaneously, record the moves by objects that are counted by the hook product (the second term on the left-hand side of $(*)$). The recording, which of course is the non-obvious part in such a construction, is tricky, but simple. The inverse of the algorithm also has a simple, but highly nontrivial description.

{Reviewer's remark: The reviewer combined the recording idea of this paper with a modified jeu de taquin from his earlier paper [Discrete Math. Theor. Comput. Sci. 3 (1998), 11–32 (electronic)] to construct in ["Another involution principle-free bijective proof of Stanley's hook-content formula", Preprint, http://xxx.lanl.gov/abs/math.CO/9807068] an equally natural bijective proof of the hook-content formula for the generating function for semistandard tableaux of a given shape with bounded entries, which, as a by-product, can also be used to generate such tableaux at random. A slight modification of this algorithm gives an algorithm for the random generation of plane partitions inside a given box.} *Christian Krattenthaler* (A-VUC)