BOUNDS ON THE LARGEST KRONECKER AND INDUCED MULTIPLICITIES OF FINITE GROUPS

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Abstract. We give new bounds and asymptotic estimates on the largest Kronecker and induced multiplicities of finite groups. The results apply to large simple groups of Lie type and other groups with few conjugacy classes.

1. Introduction

Given a finite group G, what is the largest dimension $b(G)$ of an irreducible complex representation of G? Which representations attain it? These questions are both fundamental and surprisingly challenging. For large simple groups of Lie type they have been intensely studied especially in the last few years, when asymptotic tools allowed for the general picture to emerge. For S_n and A_n, these questions are classical and have been the subject of intense investigation for decades. Despite some remarkable successes the precise asymptotics is yet to be completely determined. See Section 4 for precise statements and §9.1 for the references.

In recent years, Stanley initiated the study of the largest *Kronecker* and *Littlewood–Richardson coefficients* for the symmetric group (see §9.2). He computed their asymptotics and asked to determine the characters which attain these asymptotics. In our recent paper [PPY] we resolve both problems. Perhaps surprisingly, we show that the answer is always the asymptotically largest degree, suggesting connection with the earlier work.

In this paper we generalize some of our results from S_n to general finite groups with few conjugacy classes. This is a large class which includes quasisimple groups of Lie type of rank ≥ 2, large permutation groups, and even some nilpotent groups of large class.

For a finite group G, the *Kronecker multiplicity* $g(\rho,\varphi,\psi)$, where $\rho,\varphi,\psi \in \operatorname{Irr}(G)$, are defined by the equation:

$$\varphi \cdot \psi \,=\, \sum_{\rho\in\operatorname{Irr}(G)} g(\rho,\varphi,\psi)\,\rho, \tag{1.1}$$

where $\varphi\cdot\psi$ is the usual product of characters: $[\varphi\cdot\psi](x) = \varphi(x)\psi(x)$. Similarly, for every subgroup $H < G$, $\rho \in \operatorname{Irr}(G)$ and $\pi \in \operatorname{Irr}(H)$, we define the *induced multiplicities* $c(\rho,\pi)$ by the equation:

$$\operatorname{Ind}_H^G \pi \,=\, \sum_{\rho\in\operatorname{Irr}(G)} c(\rho,\pi)\,\rho. \tag{1.2}$$

While there is a great deal of literature for determining these coefficients for classical Chevalley groups like $\mathrm{GL}_n(q)$ and $\mathrm{SO}_n(q)$, very little is known about their asymptotics. Even less is

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known for other types and other families of groups. In this paper we obtain bounds on the largest Kronecker and induced multiplicities and illustrate them in many examples.

1.1. **Kronecker multiplicities.** Let G be a finite group and let $k(G) = |\mathrm{Irr}(G)|$ denotes the number of conjugacy classes of G. Define the *largest Kronecker multiplicity* of G:

$$\mathbf{K}(G) := \max_{\rho,\varphi,\psi\in\mathrm{Irr}(G)} g(\rho,\varphi,\psi).$$

Theorem 1.1. *We have:*

$$\frac{b(G)^2}{k(G)^{1/2}|G|^{1/2}} \le \mathbf{K}(G) \le b(G).$$

Since $\sqrt{|G|/k(G)} \le b(G) \le \sqrt{|G|}$, see (3.2), this implies that for $k(G)$ small the bound in the theorem is quite sharp. The next result shows that $\mathbf{K}(G)$ is attained on characters of large degree in that case.

Theorem 1.2. *Let $\varphi, \psi \in \mathrm{Irr}(G)$. Suppose $\varphi(1), \psi(1) \ge b(G)/a$ for some $a \ge 1$. Then there exists $\rho \in \mathrm{Irr}(G)$, such that:*

$$\rho(1) \ge \frac{b(G)}{a\cdot k(G)^{1/2}} \quad \textit{and} \quad g(\rho,\varphi,\psi) \ge \frac{b(G)}{a^2\cdot k(G)}.$$

1.2. **Induced multiplicities.** Let $H < G$. For all $\rho \in \mathrm{Irr}(G)$ and $\pi \in \mathrm{Irr}(H)$, define the *largest induced multiplicity*:

$$\mathbf{C}(G,H) := \max_{\rho\in\mathrm{Irr}(G)} \max_{\pi\in\mathrm{Irr}(H)} c(\rho,\pi).$$

Theorem 1.3. *Let $H < G$. Then:*

$$\frac{1}{k(H)^{1/2}k(G)^{1/2}}[G:H]^{1/2} \le \mathbf{C}(G,H) \le [G:H]^{1/2}.$$

In other words, when $k(H), k(G)$ are small, the largest induced multiplicities are close to $\sqrt{[G:H]}$. The following result again shows that large induced multiplicities are attained at characters of large degree.

Theorem 1.4. *Let $H < G$ and $\rho \in \mathrm{Irr}(G)$. Suppose $\rho(1) \ge |G|^{1/2}/a$, for some $a \ge 1$. Then there exists $\pi \in \mathrm{Irr}(H)$, such that:*

$$\pi(1) \ge \frac{|H|^{1/2}}{a\cdot k(H)} \quad \textit{and} \quad c(\rho,\pi) \ge \frac{[G:H]^{1/2}}{a\cdot k(H)}.$$

Remark 1.5. Note that Kronecker multiplicities are a special case of induced multiplicities. To see this, take $G = H \times H$ and a diagonal subgroup $H < G$, and we have $\mathbf{C}(H\times H, H) = \mathbf{K}(H)$. Observe that the bounds for $\mathbf{K}(H)$ which follow from Theorem 1.3 in this case are weaker than the bounds in Theorem 1.1 (see Remark 7.5). This follows from the dependence of $\mathbf{C}(G,H)$ on the embedding $H \hookrightarrow G$. For example, for $G = H \times H$ as above and $(H \times 1) \hookrightarrow G$, we have $\mathbf{C}(H \times H, H) = b(H)$, which can be much larger than $\mathbf{K}(H)$ (see §6).

Remark 1.6. As we mentioned earlier, for the symmetric groups $G = S_n$ and $H = S_k \times S_{n-k}$ the Kronecker and induced multiplicities are called the Kronecker and the Littlewood–Richardson coefficients, respectively. They play a crucial role in Algebraic Combinatorics and its applications, and have been intensely studied from both enumerative, algebraic, geometric, probabilistic and computational point of view (see §9.2 and [PPY] for the references).

Structure of the paper. In sections 2 and 3, we review known bounds on $k(G)$ and $b(G)$, respectively, for various classes of groups. In Section 4 we discuss the symmetric group case and our state of knowledge on $b(S_n)$. Then, in sections 5 and 6, we apply our bounds to various examples of groups and subgroups. We prove theorems 1.1 and 1.2 in Section 7. We then prove our theorems 1.3 and 1.4 in Section 8. We conclude with open problems and final remarks (Section 9).

Notation. Most our notation are standard. Let $\mathrm{Irr}(G)$ denotes the set of *irreducible characters* of G, let $\mathrm{Conj}(G)$ be the set of conjugacy classes, and $k(G) = |\mathrm{Irr}(G)| = |\operatorname{Conj}(G)|$ the *number of conjugacy classes*. By $|C_G(x)|$ we denote the size of the centralizer of element $x \in G$.

2. Number of conjugacy classes

2.1. **General bounds.** There are many general lower and upper bounds for $k(H)$; we will only mention some key results but will not be able to review it. The subject was initiated by E. Landau in 1903 with the first quantitative bound $k(H) = \Omega\big(\log\log|H|\big)$ by Erdős and Turán (1968). Recently, Jaikin-Zapilrain [Jai] showed the first super-log lower bound for nilpotent groups, but for general finite groups there is only a sub-log bound due to Pyber [Pyb], slightly improved in [BMT, Kel]. For a nilpotent group H of bounded class r, Sherman [She] proved:

$$k(H) \,\geq\, r|H|^{1/r} - r + 1. \tag{2.1}$$

In a different direction, for a permutation group $H < S_n$, Kovács and Robinson [KR] showed that $k(H) \leq 5^n$. This was improved to 2^{n-1} in [LP], and further to $k(H) \leq 5^{(n-1)/3}$ for $n \geq 4$, in [GM].

Finally, there are general upper and lower bounds on the number of conjugacy classes, notably:

$$\frac{k(H)}{[G:H]} \,\leq\, k(G) \,\leq\, k(H)\cdot[G:H] \quad \text{for } H < G, \tag{2.2}$$

see [Gal]. Sometimes these bounds are written in terms of the *commuting probability.* Notably, Guralnick and Robinson [GR] prove

$$k(G) \,\leq\, \sqrt{|G|\,k(F)}\,,$$

where F is the Fitting subgroup of G. In particular, $k(G) \leq \sqrt{|G|}$ when $Z(G) = 1$, i.e. when the center G is trivial.

2.2. **Number of conjugacy classes for groups of Lie type.** It was shown by Liebeck and Pyber [LP] that for a completely reducible subgroup $G < \mathrm{GL}_n(q)$, we have $k(G) \leq q^{10n}$. Further, when G is a quasisimple group of Lie type over $\mathbb{F}_q$ of rank r, they show $k(G) \leq (6q)^r$. Fulman and Guralnick [FG] further improve these bounds to

$$q^r \,\leq\, k(G) \,\leq\, 27.2q^r\,, \tag{2.3}$$

with better constants in special cases. In fact, in many cases either sharp asymptotic bounds, or even the exact formulas are known, see examples in Section 6.

3. Largest degree

3.1. **General bounds.** Let $\rho(1)$ denote the *degree* of ρ, and let

$$b(G) \, := \, \max_{\rho\in \mathrm{Irr}(G)} \rho(1)$$

denote the largest degree. Recall the *Burnside identity*:

$$\sum_{\rho\in \mathrm{Irr}(G)} \rho(1)^2 \, = \, |G|. \tag{3.1}$$

This immediately implies that for all finite groups G, we have:

$$\sqrt{|G|/k(G)} \, \le \, b(G) \le \sqrt{|G|}\,. \tag{3.2}$$

There are a few lower and upper bounds for general groups. Notably, if $b(G) < \sqrt{|G|} - \frac{1}{2}$, then

$$b(G) \, \le \, \sqrt{|G|} \, - \, \frac{1}{2} \sqrt[4]{|G|},$$

and this is the best bound of this type [HLS], improving on earlier bounds by Isaacs [Isa2] and others.

One should, of course, expect better upper bounds for large non-solvable groups. For example, if $\rho \in \mathrm{Irr}(G)$ and $\rho(1)^2 \ge |G|/2$, then ρ^2 contains every irreducible character, i.e. $g(\rho,\rho,\chi) > 0$ for all $\chi \in \mathrm{Irr}(G)$. This property is known for all simple groups of Lie type [HSTZ] except for $\mathrm{PSU}_n(q)$, and is a subject of intense study for A_n and S_n, see [Ike, LuS, PPV]. In the opposite direction, for all simple groups one has $b(G) \ge \sqrt[3]{|G|}$, see [KS].

3.2. **Largest degree for groups of Lie type.** For a natural class of reductive linear algebraic groups G of dimension d, rank r over $\mathbb{F}_q$, Kowalski [Kow, Prop. 5.5] uses an argument by J. Michel to prove:

$$b(G) \, \le \, \frac{|G|}{(q-1)^r\,|G|_p} \, \le \, (q+1)^{(d-r)/2},$$

where N_p denotes the largest power of p which divides N. He also proves that the first inequality is sharp when $q = p^a$ is large enough, and obtains explicit bounds for several series, such as $\mathrm{GL}_n(q)$, $\mathrm{Sp}_{2n}(q)$, etc.

More general and sometimes more precise bounds were obtained later by Larsen, Malle and Tiep [LMT]. For all $G(q)$ over $\mathbb{F}_q$, of dimension d, rank r, characteristic p, they prove:

$$A\,(\log_q r)^{\alpha}\,|G|_p \, \le \, b(G) \, \le \, B\,(\log_q r)^{\beta}\,|G|_p\,, \tag{3.3}$$

for some universal constants $A, B > 0$ and $\alpha, \beta \ge 0$. In fact, the log terms disappear for exceptional groups of Lie type. They also obtain sharp explicit bounds in special cases, see Section 6. In full generality, we obtain the following result.

Theorem 3.1. *Let $\mathcal{G}$ be a simple algebraic group of characteristic p, rank r, and a finite group $G(q) := \mathcal{G}^F$ over $\mathbb{F}_q$, corresponding to a Frobenius map $F : \mathcal{G} \to \mathcal{G}$. Then:*

$$C\,\frac{\big(|G|_p\big)^2\,(\log_q r)^{\gamma}}{q^{r/2}\,\sqrt{|G|}} \, \le \, \mathbf{K}\big(G(q)\big) \, \le \, D\,(\log_q r)^{\delta}\,|G|_p$$

where $C, D > 0$ and $\gamma, \delta \ge 0$ are universal constants independent of $\mathcal{G}$ and q.

Proof. In Theorem 1.1, use bounds on $k(G_n)$ and $b(G_n)$ in (2.3) and (3.3), respectively. $\square$

4. Symmetric groups

4.1. **Largest degree.** Recall that $k(S_n) = p(n)$, the number of integer partitions of n. The *Hardy–Ramanujan asymptotic formula* gives:

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}} \quad \text{as} \quad n \to \infty.$$

In 1985, Vershik and Kerov [VK2] proved that for all n large enough:

$$\sqrt{n!}\, e^{-c_1\sqrt{n}(1+o(1))} \le b(S_n) \le \sqrt{n!}\, e^{-c_2\sqrt{n}(1+o(1))}, \tag{4.1}$$

where

$$c_1 = \pi\sqrt{\frac{1}{6}} \approx 1.2825 \quad \text{and} \quad c_2 = \frac{\pi - 2}{\pi^2} \approx 0.1157 \tag{4.2}$$

Note that the lower bound follows from (3.2), but the upper bound is rather remarkable. The following result is an application.

4.2. **Smaller degrees.** Let $\mathrm{W}(G) = \{\rho \in \mathrm{Irr}(G),\, \rho(1) < b(G)\}$, and let $\varepsilon(G)$ be defined as follows:

$$\varepsilon(G) = \frac{\sum_{\rho\in \mathrm{W}(G)} \rho(1)^2}{b(G)^2}.$$

One can think of $\varepsilon(S_n)$ as the ratio of probability of non-largest to largest characters of S_n w.r.t. the Plancherel measure, see [Bia, Rom, VK1]. In [LMT], the authors show that $\varepsilon(S_n) = \Omega(1)$. In fact, they prove that there exist a universal constant $\epsilon > 0$ s.t. $\varepsilon(G) > \epsilon$ for all non-abelian finite simple groups G. The former result was improved in [HHN] to $\varepsilon(S_n) = \Omega(n)$.

Theorem 4.1. *There exist universal constants $a_2 > a_1 > 0$, such that:*

$$e^{a_1\sqrt{n}} \le \varepsilon(S_n) \le e^{a_2\sqrt{n}}$$

Proof. For the upper bound, we have:

$$\varepsilon(S_n) \le \frac{n!}{b(S_n)},$$

and the result follows from the lower bound in (4.1). For the lower bound, let $M(n)$ denote the number of characters of the largest degree, i.e. $M(n) = |\mathrm{Irr}(S_n) \smallsetminus \mathrm{W}(S_n)|$. It was proved in [HHN, Prop. 3.5(1)] that $\varepsilon(S_n) \ge M(n)/16$. On the other hand, from the upper bound in (4.1), we have:

$$\varepsilon(S_n) = \frac{n! - M(n)\, b(S_n)^2}{b(S_n)^2} \ge e^{2c_2\sqrt{n}(1+o(1))} - M(n),$$

and the result follows by combining these two inequalities. □

Remark 4.2. We conjecture that the sequence $M(1), M(2), \ldots$ is bounded (cf. [KP] for some computational evidence). Curiously, for non-largest degrees this is known not to hold. Formally, let

$$f(n) := \max_k \left|\left\{\lambda \vdash n \ : \ \chi^\lambda(1) = k\right\}\right|.$$

For example, $f(13) = 6$ since $d(94) = d(76) = d(10\,21) = d(321^8) = d(2^6 1) = d(2^4 1^5) = 429$, where $d(\lambda) := \chi^\lambda(1)$. Craven [Cra] showed that the sequence $f(1), f(2), \ldots$ is unbounded.

In fact, Moretó showed that Craven's result implies unbounded maximal multiplicity for *all* large finite groups [Mor].

5. General linear groups

5.1. **Bounds on Kronecker multiplicities.** For $G_n = \mathrm{GL}_n(q)$, there are sharp bounds on all parameters we need. We have:

$$\left(1 - \frac{1}{q} - \frac{1}{q^2}\right) q^{n^2} \,\le\, |G_n| \,\le\, q^{n^2},$$

where the first inequality is given in [Pak]. Similarly,

$$q^n \,-\, q^{n-1} \,\le\, k(G_n) \,\le\, q^n,$$

where the lower bound follows from (2.3) and upper bound is given in [MR, Lemma 5.9(ii)]. Finally,

$$\frac{1}{4}\bigl(1+\log_q(n+7)/2\bigr)^{3/4} q^{n(n-1)/2} \,\le\, b(G_n) \,\le\, 13\bigl(1+\log_q(n+1)\bigr)^{2.54} q^{n(n-1)/2},$$

see [LMT, Thm. 5.1].[1] Theorem 1.1 then gives the upper and lower bounds:

$$\frac{1}{16}\bigl(1+\log_q(n+7)/2\bigr)^{3/2} q^{n(n-3)/2} \,\le\, \mathbf{K}(G_n) \,\le\, 13\bigl(1+\log_q(n+1)\bigr)^{2.54} q^{n(n-1)/2}.$$

5.2. **Induced multiplicities from a block subgroup.** Let q be fixed, $n = 2m$, $n \to \infty$, and let $G_n = \mathrm{GL}_n(q)$ be as above. Consider a subgroup $H_n := (G_m \times G_m)$ of G_n of index $[G_n : H_n] = q^{\frac{n^2}{2}+O(n)}$. Clearly, $k(H_n) = k(G_m)^2$. Theorem 1.3 then gives:

$$\mathbf{C}(G_n, H_n) \,=\, q^{\frac{n^2}{4}+O(n)}.$$

5.3. **Induced multiplicities from a parabolic subgroup.** Similarly, let $n = 2m$, $G_n = \mathrm{GL}_n(q)$, and let $B_n < G_n$ be a subgroup of matrices $(x_{ij}) \in G_n$ with $x_{ij} = 0$ for all $i > m$, $j \le m$. Thus $[G_n : B_n] = q^{\frac{n^2}{4}+O(n)}$. We also have $k(B_n) = q^{O(n)}$. To prove this, take a normal subgroup A_n of upper right $m \times m$ matrices, $B_n/A_n \simeq H_n$ as above, and consider the action of B_n on A_n. Then use the exact formula in [FF] and estimates in [Pak] (we omit the details). Now Theorem 1.3 gives:

$$\mathbf{C}(G_n, B_n) \,=\, q^{\frac{n^2}{8}+O(n)}.$$

6. Further examples

6.1. **Linear groups of rank 1.** Let $G_p := \mathrm{SL}_2(p)$, where p is a prime. Then:

$$|G_p| = p^3 - p, \quad k(G_p) = p+4, \quad b(G_p) = p+1.$$

In this case the whole character table can be computed by hand, so the lower bound in Theorem 1.1 is neither sharp nor useful.

[1]There does not seem to be a closed formula for $b(\mathrm{GL}_n(q))$, however the *Steinberg character* St is asymptotically the largest unipotent irreducible character; see discussion in [LMT, §5] (cf. [HSTZ]).

6.2. **Suzuki groups.** Let $G_n := \mathrm{Suz}(q)$, where $q = 2^{2n+1}$ and $n \to \infty$. Then:

$$|G_n| = q^2(q^2+1)(q-1), \ \ k(G) = q+3, \quad b(G_n) = q^2 + O(q^{3/2}).$$

By Theorem 1.1 we have a sharp bound: $\mathbf{K}(G_n) = q^2 + O(q^{3/2})$.

6.3. **Unitriangular groups.** Let $H_n = \mathrm{U}_n(q)$ be the group of upper triangular matrices with ones on the diagonal. Let q be fixed and $n \to \infty$. We have:

$$|H_n| = q^{\binom{n}{2}}, \quad q^{\frac{n^2}{12}+O(n)} \le k(H_n) \le q^{\frac{7n^2}{44}+O(n)}, \quad b(H_n) = q^{\mu(n)},$$

where $\mu(n) = \lfloor (n-1)^2/4 \rfloor$, see [Isa1]. Here the lower bound on $k(H_n)$ is by Higman [Hig], and the upper bound on $k(H_n)$ is by Soffer [Sof]. Note that Sherman's bound (2.1) is quite weak in this case. Similarly, the lower bound in (3.2) is very weak in this case, while the upper bound is quite sharp. Theorem 1.1 then gives:

$$q^{\frac{n^2}{11}+O(n)} \le \mathbf{K}(H_n) \le q^{\frac{7n^2}{44}+O(n)}.$$

It would be interesting to see if this bound can be improved, perhaps, by using the supercharacter theory, see [DI, Yan].

6.4. **Unitriangular subgroup.** Let q be fixed. In notation above, note that $H_n = \mathrm{U}_n(q)$ is a subgroup of $G_n = \mathrm{GL}_n(q)$ of index $[G_n : H_n] = q^{\frac{n^2}{4}+O(n)}$, as $n \to \infty$. Theorem 1.3 then gives:

$$q^{\frac{15n^2}{88}+O(n)} \le \mathbf{C}(G_n, H_n) \le q^{\frac{n^2}{4}+O(n)}.$$

6.5. **The Monster group.** Let M be the *Monster group*. We have:

$$|M| \approx 8.08 \cdot 10^{53}, \qquad k(M) = 194, \qquad b(M) \approx 2.59 \cdot 10^{26}.$$

In notation of §4.2, we have $\varepsilon(M) \approx 11.02$, which follows from many "large but not largest" irreducible characters. On the other hand, equation (3.2) gives:

$$\sqrt{|M|/k(M)} \approx 6.45 \cdot 10^{25} \le b(M) \approx 2.59 \cdot 10^{26} \le \sqrt{|M|} \approx 8.99 \cdot 10^{26}.$$

The large gap in the first inequality can be explained by a large number of characters with very small degree.

We compare the exact value of $\mathbf{K}(M)$ computed directly from the character table [C+], with estimates in Theorem 1.1:

$$\frac{b(M)^2}{\sqrt{k(M)|M|}} \approx 5.35 \cdot 10^{24} \le \mathbf{K}(M) \approx 2.15 \cdot 10^{25} \le b(M) \approx 2.59 \cdot 10^{26}.$$

Again both gaps can be similarly explained by the presence of many "relatively small" irreducible characters which allow the isotypical components to be relatively evenly distributed (cf. the proof of Theorem 1.1 in §7.2). In fact, $\mathbf{K}(M)$ is much larger than the *average Kronecker multiplicity*:

$$\frac{1}{k(M)^3} \sum_{\rho,\varphi,\psi \in \mathrm{Irr}(M)} g(\rho,\varphi,\psi) \approx 3.38 \cdot 10^{22},$$

which can be explained by the fact that if even one of the three characters has small degree, then so does $g(\rho,\varphi,\psi)$, see (7.2).

In fact, the bound $\mathbf{A}(G) \geq |G|$ (see next section) is unusually tight in this case:[2]

$$\begin{aligned} \mathbf{A}(M) &= 808017424794512875894769468067441075690144312450960558 \\ |M| &= 808017424794512875886459904961710757005754368000000000 \end{aligned}$$

There is a simple explanation, of course: the centralizer sizes z_α rapidly decrease as we go down the list. Here are the first three of them other than $z_1 = |M|$, corresponding to the three largest maximal subgroups of M:

$$2|B| \approx 8.31 \cdot 10^{33}, \qquad 2^{25}|Co_1| \approx 1.40 \cdot 10^{26}, \qquad 3|Fi_{24}| \approx 3.77 \cdot 10^{24}.$$

When these are subtracted from $\mathbf{A}(M)$ we obtain a relatively small remainder:

$$\mathbf{A}(M) - |M| - 2|B| - 2^{25}|Co_1| - 3|Fi_{24}| \approx 1.00 \cdot 10^{19}.$$

7. Kronecker multiplicities

7.1. General inequalities. First, note:

$$g(\rho, \varphi, \psi) = \langle \rho, \varphi \cdot \psi \rangle = \langle \overline{\rho} \cdot \varphi \cdot \psi, 1 \rangle.$$

This implies the symmetries

$$g(\rho, \varphi, \psi) = g(\overline{\varphi}, \overline{\rho}, \psi) = g(\overline{\varphi}, \psi, \overline{\rho}) = \ldots \tag{7.1}$$

In particular, we have a general upper bound:

$$g(\rho, \varphi, \psi) \leq \rho(1) \cdot \min\{\varphi(1)/\psi(1), \psi(1)/\varphi(1)\} \leq \rho(1). \tag{7.2}$$

Proposition 7.1. *Let $\rho, \varphi, \psi \in \mathrm{Irr}(G)$. Suppose $g(\rho, \varphi, \psi) \geq b(G)/a$, for some $a \geq 1$. Then: $\rho(1), \varphi(1), \psi(1) \geq b(G)/a$.*

Proof. This follows immediately from (7.2) and the symmetries (7.1). □

7.2. Largest Kronecker multiplicity. Recall the definition of $\mathbf{K}(G)$ given in the introduction. Let

$$\mathbf{A}(G) := \sum_{\rho, \varphi, \psi \in \mathrm{Irr}(G)} g(\rho, \varphi, \psi)^2. \tag{7.3}$$

Lemma 7.2. *We have:*

$$\mathbf{A}(G) = \sum_{\alpha \in \mathrm{Conj}(G)} z_\alpha, \tag{7.4}$$

where $z_\alpha = |C(\alpha)|$ is the size of the centralizer of an element $x \in \alpha$.

Proof of Lemma 7.2. By definition, we have:

$$g(\rho, \varphi, \psi) = \frac{1}{|G|} \sum_{x \in G} \overline{\rho(w)} \varphi(x) \psi(x) = \frac{1}{|G|} \sum_{x \in G} \rho(w) \overline{\varphi(x)} \overline{\psi(x)},$$

[2] See A. Hulpke's answer in `https://math.stackexchange.com/questions/2668042`

noting that $\chi(g^{-1}) = \overline{\chi(g)}$ for finite groups. Hence, we can write the sum of squares as

$$\mathbf{A}(G) = \sum_{\rho,\varphi,\psi\in\mathrm{Irr}(G)} g(\rho,\varphi,\psi)^2 = \frac{1}{|G|^2}\sum_{x,y\in G}\sum_{\rho}\overline{\rho(x)}\rho(y)\sum_{\varphi}\varphi(x)\overline{\varphi(y)}\sum_{\psi}\psi(x)\overline{\psi(y)}$$
$$= \frac{1}{|G|^2}\sum_{x,y\in G}\left(\sum_{\rho}\rho(x)\overline{\rho(y)}\right)^3 = \frac{1}{|G|^2}\sum_{\alpha\in\mathrm{Conj}(G)}\left(\frac{|G|}{z_\alpha}\right)^2 (z_\alpha)^3 = \sum_{\alpha\in\mathrm{Conj}(G)} z_\alpha\,.$$

Here the last equality follows from orthogonality of the columns in the character table. □

Proposition 7.3. *We have:*

$$\frac{|G|^{1/2}}{k(G)^{3/2}} \le \mathbf{K}(G) \le b(G).$$

Proof. In (7.4), we have $\mathbf{A}(G) \ge z_1 = |G|$. This gives the lower bound. The upper bound follows from (7.2). □

Theorem 1.2 can be viewed as a converse of Proposition 7.1.

Proof of Theorem 1.2. Let ρ be the character in the largest term in the RHS of

$$\frac{b(G)^2}{a^2} \le \varphi(1)\cdot\psi(1) = \sum_{\rho\in\mathrm{Irr}(G)} g(\rho,\varphi,\psi)\,\rho(1).$$

On the one hand,

$$g(\rho,\varphi,\psi) \ge \frac{1}{k(G)\cdot b(G)}\cdot\frac{b(G)^2}{a^2} = \frac{b(G)}{a^2\,k(G)}\,.$$

On the other hand,

$$\rho(1)^2 \ge g(\rho,\varphi,\psi)\,\rho(1) \ge \frac{1}{k(G)}\cdot\frac{b(G)^2}{a^2}\,,$$

which implies the result. □

7.3. **Refined Kronecker multiplicities.** Fix $\rho,\varphi\in\mathrm{Irr}(G)$. Define the *largest refined Kronecker multiplicity*

$$\mathbf{K}(G;\rho,\varphi) := \max_{\psi\in\mathrm{Irr}(G)} g(\rho,\varphi,\psi)$$

Clearly, $\mathbf{K}(G;\rho,\varphi)\le\mathbf{K}(G)$.

Proposition 7.4. *For all $\rho,\varphi\in\mathrm{Irr}(G)$, we have:*

$$\frac{\rho(1)\,\varphi(1)}{k(G)^{1/2}|G|^{1/2}} \le \mathbf{K}(G;\rho,\varphi) \le \min\{\rho(1),\varphi(1)\}.$$

Proof. Let

$$A(\rho,\varphi) := \sum_{\psi} g(\rho,\varphi,\psi)^2.$$

Recall Burnside's identity (3.1) and

$$\sum_{\psi} g(\rho,\varphi,\psi)\psi(1) = \varphi(1)\rho(1).$$

Now apply the Cauchy–Schwarz inequality to vectors $\big(\psi(1)\big), \big(g(\rho,\varphi,\psi)\big) \in \mathbb{R}^{k(G)}$, both indexed by $\psi \in \mathrm{Irr}(G)$. We obtain:

$$A(\rho,\varphi) \,\ge\, \frac{\rho(1)^2\varphi(1)^2}{|G|}\,.$$

Therefore, for the maximal term in the summation $A(\rho,\varphi)$, we have:

$$\max_{\psi} g(\rho,\varphi,\psi) \,\ge\, \frac{\rho(1)\varphi(1)}{k(G)^{1/2}|G|^{1/2}}\,.$$

This implies the lower bound. The upper bound follows from (7.2). □

Proof of Theorem 1.1. In Proposition 7.4, take $\rho,\varphi \in \mathrm{Irr}(G)$ s.t. $\rho(1)=\varphi(1)=b(G)$. □

Remark 7.5. Note that Theorem 1.1 and equation (3.2) imply the lower bound in Proposition 7.3. In fact, the latter lower bound is same bound mentioned in Remark 1.5 for the diagonal subgroup $\mathbf{K}(H) = \mathbf{C}(H\times H, H)$.

8. Induced multiplicities

8.1. **General inequalities.** Let $H < G$ be a subgroup of a finite group G of index $[G:H] = |G|/|H|$. For all $\rho \in \mathrm{Irr}(G)$ and $\pi \in \mathrm{Irr}(H)$, define the *induced multiplicities* $c(\rho,\pi)$ as follows:

$$c(\rho,\pi) \,:=\, \big\langle \rho, \pi\!\uparrow_H^G \big\rangle \,=\, \big\langle \rho\!\downarrow_H^G, \pi \big\rangle.$$

We have:

$$\sum_{\rho\in\mathrm{Irr}(G)} c(\rho,\pi)\,\rho(1) \,=\, [G:H]\cdot\pi(1) \qquad \text{and} \qquad \sum_{\pi\in\mathrm{Irr}(H)} c(\rho,\pi)\,\pi(1) \,=\, \rho(1). \tag{8.1}$$

Lemma 8.1. *For every $H < G$, we have:*

$$\sum_{\rho\in\mathrm{Irr}(G)}\sum_{\pi\in\mathrm{Irr}(H)} c(\rho,\pi)^2 \,=\, \sum_{\alpha\in\mathrm{Conj}(H)} \frac{z_\alpha(G)}{z_\alpha(H)}, \tag{8.2}$$

where $z_\alpha(H) = |C_H(x)|$ denotes the size of the centralizer of $x\in\alpha$ within H, and $z_\alpha(G) = |C_G(x)|$ is the size of the centralizer within G.

Proof. Denote by $\xi = \rho|_H$ the restriction of the character ρ to H. We have:

$$c(\rho,\pi) \,=\, \sum_{\alpha\in\mathrm{Conj}(H)} z_\alpha^{-1}\,\xi(\alpha)\,\overline{\pi(\alpha)},$$

Then:

$$\begin{aligned}\sum_{\rho\in\mathrm{Irr}(G)}\sum_{\pi\in\mathrm{Irr}(H)} c(\rho,\pi)^2 \,&=\, \sum_{\alpha,\gamma\in\mathrm{Conj}(H)} z_\alpha^{-1} z_\gamma^{-1} \sum_{\rho\in\mathrm{Irr}(G)}\sum_{\pi\in\mathrm{Irr}(H)} \xi(\alpha)\overline{\xi(\gamma)}\,\overline{\pi(\alpha)}\pi(\gamma)\\ &=\, \sum_{\alpha\in\mathrm{Conj}(H)} z_\alpha(H)^{-2}\big(z_\alpha(H)\cdot z_\alpha(G)\big) \,=\, \sum_{\alpha\in\mathrm{Conj}(H)}\frac{z_\alpha(G)}{z_\alpha(H)}\,,\end{aligned}$$

as desired. □

Corollary 8.2. *For every $H < G$, we have:*

$$\sum_{\rho\in\mathrm{Irr}(G)}\sum_{\pi\in\mathrm{Irr}(H)} c(\rho,\pi)^2 \,\ge\, [G:H].$$

Proof. Since in (8.2) the RHS $\geq z_1(G)/z_1(H) = [G:H]$, we obtain the inequality. □

Remark 8.3. Note that Lemma 7.2 easily follows from Lemma 8.1 by taking the diagonal subgroup in $G \times G$ as in Remark 1.5. The details are straightforward. We chose to keep both proofs for clarity of exposition.

Lemma 8.4. *For every $H < G$, we have:*

$$\sum_{\rho\in\mathrm{Irr}(G)} c(\rho,\pi)^2 \,\leq\, [G:H] \quad \text{and} \quad \sum_{\pi\in\mathrm{Irr}(H)} c(\rho,\pi)^2 \,\leq\, [G:H].$$

Proof. We have:

$$\sum_{\rho\in\mathrm{Irr}(G)} c(\rho,\pi)^2 \,\leq\, \sum_{\rho\in\mathrm{Irr}(G)} c(\rho,\pi)\,\frac{\rho(1)}{\pi(1)} \,=\, \frac{1}{\pi(1)}\cdot\pi(1)\,[G:H] \,=\, [G:H],$$

$$\sum_{\pi\in\mathrm{Irr}(H)} c(\rho,\pi)^2 \,\leq\, \sum_{\pi\in\mathrm{Irr}(G)} c(\rho,\pi)\,\frac{\pi(1)\cdot[G:H]}{\rho(1)} \,=\, \frac{1}{\rho(1)}\cdot\rho(1)\,[G:H] \,=\, [G:H],$$

where we repeatedly use both equations in (8.1). □

Corollary 8.5. *For every $H < G$, we have:*

$$[G:H] \,\leq\, \sum_{\rho\in\mathrm{Irr}(G)}\ \sum_{\pi\in\mathrm{Irr}(H)} c(\rho,\pi)^2 \,\leq\, [G:H]\,\min\bigl\{k(G),k(H)\bigr\}.$$

Note that $k(H)$ can be much larger that $k(G)$. For example, take $H = \mathbb{Z}_2^{n/2}$ and $G = S_n$. Then $k(H) = 2^{n/2}$, while $k(S_n) = e^{\Theta(\sqrt{n})}$.

8.2. **Largest induced multiplicity.** Recall the definition of $\mathbf{C}(G,H)$ from the introduction. We have:

Proof of Theorem 1.3. The lower bound follows immediately from Corollary 8.2, while the upper bound follows from Lemma 8.4. □

Proof of Theorem 1.4. Let π be the character in the largest term in the RHS of

$$|G|^{1/2}/a \,\leq\, \rho(1) \,=\, \sum_{\pi\in\mathrm{Irr}(H)} c(\rho,\pi)\,\pi(1).$$

On the one hand, by the upper bound in Theorem 1.3 we have:

$$\pi(1) \,\geq\, \frac{\rho(1)}{k(H)\cdot\mathbf{C}(G,H)} \,\geq\, \frac{|G|^{1/2}/a}{k(H)\cdot[G:H]^{1/2}} \,=\, \frac{|H|^{1/2}}{a\,k(H)}.$$

On the other hand,

$$c(\rho,\pi) \,\geq\, \frac{\rho(1)}{k(H)\cdot b(H)} \,\geq\, \frac{|G|^{1/2}/a}{k(H)\cdot|H|^{1/2}} \,=\, \frac{[G:H]^{1/2}}{a\,k(H)},$$

as desired. □

Remark 8.6. Theorem 1.4 above is patterned after Theorem 1.2. Note, however, that we do not have an analogue of a much simpler Proposition 7.1.

9. Final remarks

9.1. The study of $b(S_n)$ was initiated back in 1954, in one of the earliest uses of computer calculations in combinatorics and algebra [BMSW]. The study continued in a long series of papers [BDJ, Mc, LoS, VK1, VK2, VP], in part due to connections to random unitary matrices and longest increasing subsequences in random permutations. Let us single out papers [LoS, VK1] which determined the *limit shape* of the partition λ corresponding to the largest character χ^λ, and [BDJ] which determined the exact distribution of shapes λ. Stanley's questions are best viewed as part of this research direction. We refer to [Rom] for a comprehensive overview of the area.

For general groups, parameter Snyder in [Sny] introduced parameter $e(G)$ defined by $b(G)\big(b(G)+e(G)\big) = |G|$. Parameter $e(G)$ is closely related to $\varepsilon(G)$, and was the motivation for a series of recent papers improving bound on both [HLS, HHN, Isa2, LMT].

9.2. The literature on Kronecker and Littlewood–Richardson coefficients is so vast, there is no single source that would give it justice. We refer to [Sta2] for a comprehensive introduction to the subject and to [Ful, vL] for connections to Algebra and Geometry, and to [PPY] for further references.

We should mention that from the point of view of Schur duality, one can describe the classical Littlewood–Richardson coefficients $c^\lambda_{\mu\nu}$ of S_n as a special case of Kronecker multiplicities for $\mathrm{GL}_N(q)$. Indeed, by taking $N \geq 2\ell(\lambda)$ where $\ell(\lambda)$ is the number of parts in λ, and taking q large enough, the Kronecker multiplicities $g(\chi^\lambda, \chi^\mu, \chi^\nu)$ of the corresponding $\mathrm{GL}_N(q)$-reps become polynomial in q. Letting $q \to 1$ in these polynomials recovers $c^\lambda_{\mu\nu}$. Thus, estimating the Kronecker multiplicities for $\mathrm{GL}_N(q)$ is likely to be difficult.

9.3. It was shown by Bufetov [Buf] that w.r.t. the Plancherel measure there is a concentration of

$$\frac{1}{\sqrt{n}} \log \frac{\chi^\lambda(1)^2}{n!} \quad \text{as } n \to \infty$$

at some $h \in [-2c_1, -2c_2]$, where c_1, c_2 are given in (4.2). If such h was determined, this would further improve the asymptotic bounds on $\varepsilon(S_n)$ given in the proof of Theorem 4.1. Numerical experiments in [VP] suggest that there is a limit

$$\eta = \lim_{n\to\infty} \frac{1}{\sqrt{n}} \log \frac{b(S_n)^2}{n!}$$

and that $h < \eta$.

9.4. It was noted by McKay [Mc] and Kowalski [Kow, p. 80] that for some families of groups a nice interpretation for the sum of degrees are known:

$$f(G) = \sum_{\chi\in\mathrm{Irr}(G)} \chi(1).$$

Namely, $f(S_n)$ is the number of involutions, $f\big(\mathrm{GL}_n(q)\big)$ is the number of symmetric matrices, etc. We refer to [Vin] for the unified view of these results and review of prior work by Gow, Klyachko, and others. We should mention that for our applications, these formulas give weaker bounds compared to (3.2). For S_n, this was pointed out in [VK2], who improved upon McKay's lower bound.

9.5. It was pointed out to us by the referee that the lower bound in Theorem 1.1 can be strengthened to

$$\frac{b(G)^2}{f(G)} \leq \mathbf{K}(G),$$

since $f(G) \leq k(G)^{1/2} |G|^{1/2}$ by the Cauchy–Schwarz inequality.

Similarly, by analogy with $f(G)$ and $\mathbf{A}(G)$, one can also bound the *average Kronecker multiplicity* via:

$$k(G)^2 \leq \sum_{\rho,\varphi,\psi \in \mathrm{Irr}(G)} g(\rho,\varphi,\psi) \leq k(G)\,|G|.$$

(cf. §6.5). Both sides are tight for abelian groups. Thus, abelian groups have the smallest average Kronecker multiplicity among groups of the same order.

9.6. It would be interesting to see if we always have $\mathbf{C}(G,H) \leq \sqrt{b(G)/b(H)}$, which would be sharper in some cases and match the upper bound $\mathbf{K}(G) \leq b(G)$ in the diagonal embedding case, see Remark 1.5. We have not checked this speculation on a computer.

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