Partition congruences by involutions

Christine Bessenrodt

Fachbereich Mathematik
Universität Hannover
D-30167 Hannover, Germany
bessen@math.uni-hannover.de

Igor Pak

Department of Mathematics, MIT
Cambridge, MA 02139
pak@math.mit.edu

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Abstract

We present a general construction of involutions on integer partitions which enable us to prove a number of modulo 2 partition congruences.

Introduction

The theory of partitions is a beautiful subject introduced by Euler over 250 years ago and is still under intense development [2]. Arguably, a turning point in its history was the invention of the "constructive partition theory" symbolized by Franklin's involution [10] and commemorated in Sylvester's magnum opus [17]. Based on explicit constructions of bijections and involutions, this approach was taken to a new high by Schur's proof of Rogers-Ramanujan's identities and led to numerous new proofs and identities. We refer to [14] for an extensive survey of history and recent developments of the subject.

By themselves, partition congruences became a subject of intense interest ever since Ramanujan's celebrated discovery of the congruence $p(5n-1) \equiv 0 \mod 5$. Despite various proofs, extensions and even Dyson's 'rank' combinatorial interpretation [7], there is still no bijective proof of Ramanujan's congruences. In fact, the few partition congruences which are known to have

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combinatorial proofs are mod 2 congruences, all proved by explicit involutions. The idea of this paper is to present a certain new class of involutions which prove a wide range of modulo 2 partition congruences and identities.

Let us start with Euler's classical Pentagonal Theorem, which is equivalent to the following identity:

$$(*) \qquad \prod_{i=1}^{\infty}(1-t^i) \,=\, \sum_{m=-\infty}^{\infty}(-1)^m t^{m(3m-1)/2}\,.$$

One way to prove $(*)$ is to show that the number of partitions of n into distinct parts with an odd number of parts is equal to the number of partitions of n into distinct parts with an even number of parts, unless n is a pentagonal number. This is exactly the approach used by Franklin [10]; his classical involution proves $(*)$ by switching the parity of the number of parts in a partition. The proof was soon recognized as of great importance by Cayley and other contemporaries and became a key result in Sylvester's program of studying partitions [17]. Hardy described the proof as *"beautiful"* [11], and Rademacher called it *"the first American theorem"*.[1] In his historical investigation [3], Andrews showed that Franklin's involution easily follows from an easy Durfee square type proof of Sylvester's identity. This even led to speculations that this was in fact how Franklin's proof was obtained, a speculation later disproved[2]. Most recently, this approach was formalized in [16].

In recent years, Franklin's proof had a new life with several more general identities proved by means of the very same involution (see e.g. [6, 12, 13]). Just last year, a note [15] by the second author showed that one of Fine's partition results follows easily from Franklin's involution. We refer to [15] for the full story, but let us mention here that Fine published a note [8] where, in Andrews' words "[Fine] *announced several elegant and intriguing partition theorems. These results were marked by their simplicity of statement and* [...] *by the depth of their proof."* The paper [15] presents combinatorial proofs of all of Fine's results except for the following:

Fine's Theorem *The number of partitions of n into distinct parts and with odd smallest part is odd if and only if n is a perfect square.*

This result remained elusive until now. In this paper we present an explicit involutive proof of this Fine's theorem, together with a number

[1] This quote was communicated to us by George Andrews, who attended Rademacher's lectures while at UPenn.

[2] George Andrews, personal communication.

of extensions and generalizations. It turns out that there is a common general underlying principle behind these involutions as well as Franklin's involution. As the reader shall see, the proof we present is really a "proof from the book", and after reading this paper will wonder why it took so long to find this connection.

The structure of the paper is as follows. After basic definitions (Section 1), we start with Vahlen's classical involution and its restricted version (Sections 2 and 3). Then follows Section 4 on Sylvester's transformation and main results are given in Section 5. In Section 6 we present a number of examples and special cases, which include extensions of Fine's Theorem above. We suggest the reader check our calculations as this may prove helpful for a better grasp of the material. The connection to Franklin's involution is described in Section 7. We conclude with final remarks and questions for further study.

1 Basic definitions

A *partition* of n is an integer sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ and $|\lambda| := \lambda_1 + \lambda_2 + \cdots = n$. We refer to the λ_i as the *parts* of the partition λ. Let $\mathcal{D}$ and $\mathcal{P}$ denote the set of partitions with distinct parts and the set of all partitions, respectively. Denote by $\ell(\lambda)$ and $s(\lambda)$ the number of parts and the smallest part in λ, respectively; for convenience, set $s(\varnothing) = \infty$. For partitions λ and μ denote by $\lambda \cup \mu$ the partition obtained by taking the (multiset) union of the parts.

A *joint partition* of n is a pair of partitions (λ, μ) such that $\lambda \in \mathcal{P}$, $\mu \in \mathcal{D}$, and $n = |\lambda| + |\mu|$. Denote by $\mathcal{J} = \mathcal{P} \times \mathcal{D}$ the set of joint partitions. Clearly,

$$\sum_{(\lambda,\mu)\in\mathcal{J}} a^{\ell(\lambda)}\, b^{\ell(\mu)}\, t^{|\lambda|}\, z^{|\mu|} \, = \, \prod_{i=1}^{\infty} \frac{1 + b\, z^i}{1 - a\, t^i} \, .$$

Graphically, one can present partitions and joint partitions by using Young diagrams and MacMahon's diagrams as in Figure 1. Here, a MacMahon diagram corresponding to $(\lambda, \mu) \in \mathcal{J}$ is presented by a Young diagram of shape $\nu = \lambda \cup \mu$ with *marked* squares in the corners, so that rows with marked squares correspond to parts of the partition μ (see [14]).

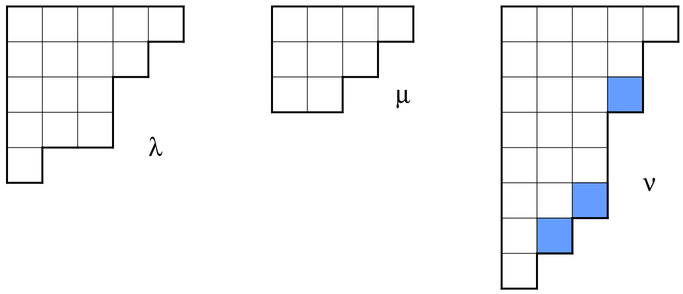

Figure 1. Young diagrams corresponding to partitions $\lambda = (5,4,3,3,1) \in \mathcal{P}$, $\mu = (4,3,2) \in \mathcal{D}$, and MacMahon's diagram of shape $\nu = \lambda \cup \mu$ corresponding to $(\lambda, \mu) \in \mathcal{J}$.

2 Vahlen's involution

Consider the following trivial identity:

$$(\star) \qquad \prod_{i=1}^{\infty} (1-t^i) \cdot \prod_{i=1}^{\infty} \frac{1}{1-t^i} \, = \, 1\,.$$

The left hand side can be viewed as a weighted sum of $(-1)^{\ell(\mu)}$ over all joint partitions $(\lambda, \mu) \in \mathcal{J}$. Let us prove identity $(\star)$ by constructing an involution $\phi : \mathcal{J} \to \mathcal{J}$, defined as follows. If $s(\lambda) < s(\mu)$, move the smallest part from λ to μ. Otherwise, if $s(\lambda) \geq s(\mu)$, move the smallest part from μ to λ. It is easy to see that the involution ϕ has exactly one fixed point: an empty joint partition, which represents the r.h.s. of $(\star)$. The involution ϕ is called *Vahlen's involution* [14].

It is easy to generalize $(\star)$ to any subset of integers $I \in \mathbb{N}$:

$$(\star\star) \qquad \prod_{i \in I} (1-t^i) \cdot \prod_{i \in I} \frac{1}{(1-t^i)} \, = \, 1\,,$$

with the proof given again by Vahlen's involution ϕ.

3 Restriction of Vahlen's involution

Consider a subset $\mathcal{R}_{\leq k}$ of joint partitions $(\lambda, \mu) \in \mathcal{J}$ with $\ell(\lambda) \leq k$, and such that $s(\lambda) < s(\mu)$ whenever $\ell(\lambda) = k$. Let us prove that

$$(\circ) \qquad \sum_{(\lambda,\mu) \in \mathcal{R}_{\leq k}} (-1)^{\ell(\lambda)} \, t^{|\lambda|+|\mu|} \, = \, 1\,.$$

Use Vahlen's involution ϕ again. Observe that when $\ell(\lambda) < k$, we can always apply ϕ. In case $\ell(\lambda) = k$ there is no room to add the smallest part from μ. But that is unnecessary due to the condition $s(\lambda) < s(\mu)$ in this case. This implies ($\circ$).

There is another way to define a restriction of ϕ which will not be used later in the paper. Define $\mathcal{S}_{\leq k} \subset \mathcal{J}$ to be the subset of joint partitions (λ, μ) with $\ell(\mu) \leq k$, such that $s(\mu) \leq s(\lambda)$ whenever $\ell(\mu) = k$. Then

$$(\circ\circ) \qquad \sum_{(\lambda,\mu)\in\mathcal{S}_{\leq k}} (-1)^{\ell(\lambda)}\, t^{|\lambda|+|\mu|} \, = \, 1\,.$$

The proof again follows from Vahlen's involution.

4 Sylvester's transformation

Let $\mathcal{P}_{\leq k}$ denote the set of partitions $\lambda \in \mathcal{P}$ with $\ell(\lambda) \leq k$. Similarly, let $\mathcal{D}_k$ denote the set of partitions $\mu \in \mathcal{D}$ with $\ell(\mu) = k$. Consider a map $\pi_k : \mathcal{P}_{\leq k} \to \mathcal{D}_k$ defined by $\pi_k(\lambda_1, \lambda_2, \lambda_3 \ldots) = (\lambda_1 + k, \lambda_2 + (k-1), \lambda_3 + (k-2), \ldots)$. It is easy to see that π_k is a bijection. Pictorially, it can be presented by adding a triangular shape region (see Figure 2 below). This transformation was first introduced by Sylvester [17] (see also [14]). Observe that $|\pi_k(\lambda)| = |\lambda| + \binom{k+1}{2}$, which immediately implies

$$(\lozenge) \qquad \sum_{\mu\in\mathcal{D}_k} t^{|\mu|} \, = \, t^{\binom{k+1}{2}} \sum_{\lambda\in\mathcal{P}_{\leq k}} t^{|\lambda|}\,.$$

Summing ($\lozenge$) over all $k = 0, 1, 2, \ldots$ we obtain one of the classical Euler identities:

$$(\blacklozenge) \qquad \prod_{i=1}^{\infty} (1 + t^i) \, = \, 1 \, + \, \sum_{k=1}^{\infty} \frac{t^{\binom{k+1}{2}}}{(1-t)(1-t^2)\cdots(1-t^k)}\,.$$

Let us present a generalization of Sylvester's transformation. Fix an infinite integer sequence $A = (a_0, a_1, a_2, a_3, \ldots)$, where $a_0 > 0$, and define $\mathcal{P}(A)$ to be the set of partitions $\lambda \in \mathcal{P}$ which satisfy:

$\lambda_{\ell-i} - \lambda_{\ell-i+1} \geq a_i$, for all $i = 0, \ldots, \ell - 1$, where $\ell = \ell(\lambda)$, $\lambda_{\ell+1} = 0$.

For example, when $A = (1, 0, 0 \ldots)$ we have $\mathcal{P}(A) = \mathcal{P}$. Similarly, when $A = (1, 1, 1, \ldots)$ we have $\mathcal{P}(A) = \mathcal{D}$.

Denote by $\mathcal{P}_k(A)$ the set of partitions $\lambda \in \mathcal{P}(A)$ with $\ell(\lambda) = k$. Finally, consider a map $\pi_{k,A} : \mathcal{P}_{\le k} \to \mathcal{P}_k(A)$ defined by

$$\pi_{k,A}(\lambda_1, \ldots, \lambda_{k-1}, \lambda_k) \ = \ (\lambda_1 + a_0 + a_1 + \ldots + a_{k-1}, \ldots, \lambda_{k-1} + a_1 + a_0, \lambda_k + a_0)$$

It is easy to see that $\pi_{k,A}$ is a bijection generalizing bijection π_k defined above. Define $h_k(A) = a_{k-1} + 2\, a_{k-2} + \ldots + k\, a_0$, and observe that $|\pi_{k,A}(\lambda)| = h_k(A) + |\lambda|$. We conclude:

$$(\blacklozenge\blacklozenge) \qquad \sum_{\lambda \in \mathcal{P}(A)} t^{|\lambda|} \ = \ 1 \ + \ \sum_{k=1}^{\infty} \frac{t^{h_k(A)}}{(1-t)(1-t^2)\cdots(1-t^k)} \, .$$

5 Main results

Fix $A = (1, a_1, a_2, \ldots)$ as above. Define $\mathcal{R}(A)$ to be the set of joint partitions $(\lambda, \mu) \in \mathcal{J}$ such that $\lambda \in \mathcal{P}(A)$, $\mu \in \mathcal{D}$, and $s(\lambda) \le s(\mu)$. In $\mathcal{R}(A)$, let $\mathcal{R}(A; n)$ be the subset of joint partitions (λ, μ) of n, i.e., $|\lambda| + |\mu| = n$. Let $\mathcal{R}_k(A)$ denote the set of joint partitions $(\lambda, \mu) \in \mathcal{R}(A)$ with $\ell(\lambda) = k$, and let $\mathcal{R}^{\pm}(A)$ be the set of joint partitions $(\lambda, \mu) \in \mathcal{R}(A)$ with $(-1)^{\ell(\mu)} = \pm 1$. We tacitly use the corresponding notation for subsets of joint partitions.

Theorem 1. *For any $A = (1, a_1, a_2, \ldots)$ and any $k, n \in \mathbb{N}$, we have*

$$|\mathcal{R}_k^+(A; n)| - |\mathcal{R}_k^-(A; n)| = \delta_{n, h_k(A)}.$$

Thus we have the identity

$$\sum_{(\lambda,\mu) \in \mathcal{R}(A)} (-1)^{\ell(\mu)} q^{\ell(\lambda)} t^{|\lambda| + |\mu|} \ = \ \sum_{k=0}^{\infty} q^k t^{h_k(A)} \, .$$

Proof. We construct an involution $\varkappa : \mathcal{R}(A) \to \mathcal{R}(A)$ which keeps $\ell(\lambda)$ fixed, and which changes the parity of $\ell(\mu)$ unless $(\lambda, \mu) \in \mathcal{R}(A)$ is a fixed point. Fixed points of $\varkappa$ are joint partitions $(\lambda^{(k)}, \varnothing)$, where $\lambda^{(k)} = \pi_{k,A}(\varnothing)$. Since $|\lambda^{(k)}| = h_k(A)$, this implies the result.

The involution $\varkappa$ is defined as follows. Start with $(\lambda, \mu) \in \mathcal{R}(A)$ and let $k = \ell(\lambda)$. Define $\nu = \pi_{k,A}^{-1}(\lambda)$ and $(\nu', \mu') = \phi(\nu, \mu)$. Finally, let $\lambda' = \pi_{k,A}(\nu')$ and set $\varkappa(\lambda, \mu) = (\lambda', \mu')$.

Note that if $\ell(\lambda) = \ell(\nu)$, then the condition $s(\lambda) \le s(\mu)$ translates into $s(\nu) = s(\lambda) - 1 < s(\mu)$, so the restriction of ϕ is applicable in this case. From here and $\ell(\lambda') = \ell(\lambda)$, we conclude that $\varkappa$ is an involution, which restricts to an involution on $\mathcal{R}_k(A)$. The fixed points of $\varkappa$ are the joint partitions $(\lambda, \mu) = (\lambda^{(k)}, \varnothing)$ which correspond to the fixed points $(\nu, \mu) = (\varnothing, \varnothing)$ of the involution ϕ. Moreover, by the construction of Vahlen's involution ϕ, the parity of μ' differs from the parity of μ unless (ν, μ) is a fixed point of ϕ. This completes the proof. □

The following result is a natural generalization of Theorem 1 to *modular diagrams* (see e.g. [14]). Rather than define the latter, we state the result in terms of joint partitions.

Fix an integer m and an infinite residue pattern $r = (r_1, r_2, \ldots)$, $1 \le r_i < m$ for all i. Let $B = (r; ma_1, ma_2, \ldots)$. Define $\mathcal{R}(B, m)$ to be the set of joint partitions $(\lambda, \mu) \in \mathcal{J}$ such that $\lambda \in \mathcal{P}(A)$, for $A = (r_1, ma_1, ma_2, \ldots)$, and with $\lambda_i \equiv r_{k+1-i} \mod m$ (for $i = 1, \ldots, k = \ell(\lambda)$), $\mu \in \mathcal{D}$, $\mu_i \equiv 0 \mod m$ (for all i), and $s(\lambda) < s(\mu)$. Define $\mathcal{R}_k^{\pm}(B, m; n)$ similarly as before. For $k \in \mathbb{N}$, $h_k(B)$ is the smallest number n with $\mathcal{R}_k(B, m; n) \ne \varnothing$ (the set then contains a unique partition $(\lambda^{(k)}, \varnothing)$).

Theorem 2. *For any $B = (r; ma_1, ma_2, \ldots)$ as above, we have*

$$|\mathcal{R}_k^{+}(B, m; n)| - |\mathcal{R}_k^{-}(B, m; n)| = \delta_{n, h_k(B)}.$$

The proof follows verbatim the proof of Theorem 1. One should replace Vahlen's involution with its generalization as in (⋆⋆). Similarly, one should use the partition $\lambda^{(k)}$ and proceed as above. We omit the details.

6 Examples and special cases

Suppose $A = (1, 0, 0, \ldots)$. Then $\mathcal{P}(A) = \mathcal{P}$, $\lambda^{(k)} = (1^k)$, and $h_k(A) = k$ for all $k \ge 1$. Theorem 1 in this case gives:

Corollary 1. *Let $\mathcal{Q}$ be the set of joint partitions (λ, μ) such that $s(\lambda) \le s(\mu)$. Then*

$$\sum_{(\lambda,\mu) \in \mathcal{Q}} (-1)^{\ell(\mu)} q^{\ell(\lambda)} t^{|\lambda| + |\mu|} = \frac{1}{1 - q\,t}$$

In particular, the set $\mathcal{Q}(n)$ of joint partitions $(\lambda, \mu) \in \mathcal{Q}$ of n is of odd order for all n.

When $n = 3$, we have $\mathcal{Q}(3) = \{(3, \varnothing), (21, \varnothing), (1^3, \varnothing), (1^2, 1), (1, 2)\}$, and therefore $|\mathcal{Q}(3)| = 5$. The involution $\varkappa$ defined in the proof works as follows:

$$(3, \varnothing) \longleftrightarrow (1, 2)\,, \qquad (21, \varnothing) \longleftrightarrow (1^2, 1)\,, \qquad (1^3, \varnothing) \ \circlearrowleft .$$

When $n = 4$, we have $\mathcal{Q}(4) = \{(4, \varnothing), (31, \varnothing), (2^2, \varnothing), (21^2, \varnothing), (1^4, \varnothing), (21, 1), (1^3, 1), (2, 2), (1^2, 2), (1, 3), (1, 21)\}$, and $|\mathcal{Q}(4)| = 11$.

It is instructive to compare Corollary 1 with the following Gauss identity:

$$(\circledast) \qquad \prod_{i=1}^{\infty} \frac{1 - t^i}{1 + t^i} = 1 \, + \, 2 \sum_{k=1}^{\infty} (-1)^k t^{k^2} .$$

This shows that the total number of joint partitions (λ, μ) of n is even for all $n \geq 1$.

Now suppose $A = (1, 1, 1, \dots)$. Then $\mathcal{P}(A) = \mathcal{D}$, $\lambda^{(k)} = (k, k-1, \dots, 2, 1)$, and $h_k(A) = \binom{k+1}{2}$ for all $k \geq 1$. Theorem 1 in this case gives:

Corollary 2. *Let $\mathcal{Q}$ be the set of joint partitions (λ, μ) such that $\lambda \in \mathcal{D}$ and $s(\lambda) \leq s(\mu)$. Then*

$$\sum_{(\lambda,\mu) \in \mathcal{Q}} (-1)^{\ell(\mu)} q^{\ell(\lambda)} t^{|\lambda| + |\mu|} \, = \, \sum_{k=0}^{\infty} q^k t^{\binom{k+1}{2}} .$$

In particular, $|\mathcal{Q}(n)|$ is odd if and only if n is a triangular number.

When $n = 5$, we have $\mathcal{Q}(5) = \{(5, \varnothing), (41, \varnothing), (32, \varnothing), (31, 1), (21, 2), (2, 3), (1, 4), (1, 31)\}$, $|\mathcal{Q}(5)| = 8$. Similarly, when $n = 6$, we have $\mathcal{Q}(6) = \{(6, \varnothing), (51, \varnothing), (42, \varnothing), (321, \varnothing), (41, 1), (31, 2), (21, 21), (21, 3), (3, 3), (2, 4), (1, 5), (1, 41), (1, 32)\}$, and $|\mathcal{Q}(6)| = 13$. An example of the involution $\varkappa$ in this case is given in Figure 2.

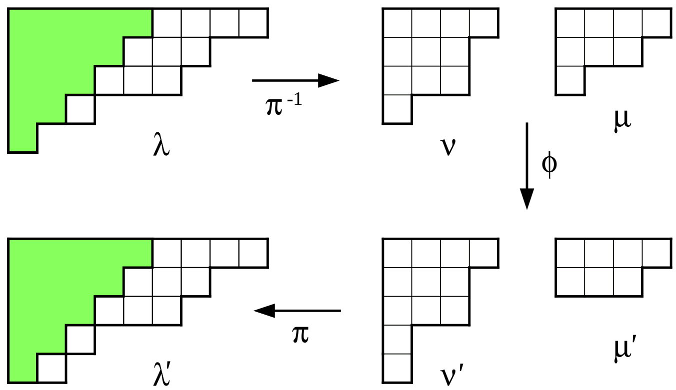

Figure 2. An example of the involution $\varkappa : (97631, 431) \to (97632, 43)$.

In Theorem 2, suppose $m = 2$, $r = \mathbf{1} = (1, 1, 1, \ldots)$, $B = (\mathbf{1}; 2, 2, \ldots)$. Then $\mathcal{R}(B, 2)$ is the set of joint partitions (λ, μ), where λ is a partition into distinct odd parts, μ is a partition into distinct even parts, and $s(\lambda) \le s(\mu)$. Taking the union $\lambda \cup \mu$ of the parts of λ, μ gives a bijection $\iota : \mathcal{R}(B, 2) \to \mathcal{Q} \subset \mathcal{D}$ into the set of partitions τ into distinct parts with the smallest part $s(\tau)$ odd. For $k \in \mathbb{N}$, let $\mathcal{Q}_k$ denote the set of partitions in $\mathcal{Q}$ with k odd parts. Note that here $\lambda^{(k)} = (2k-1, \ldots, 3, 1)$, and $h_k(A) = k^2$. Theorem 2 in this case gives:

Corollary 3. *Let $\mathcal{Q}(n)$ be the set of partitions of n into distinct parts, with odd smallest part. For $k \in \mathbb{N}$, let $\mathcal{Q}_k^{\pm}(n)$ denote the partitions in $\mathcal{Q}(n)$ with k odd parts, and with an even and odd number of even parts, respectively. Then*

$$|\mathcal{Q}_k^+(n)| - |\mathcal{Q}_k^-(n)| = \delta_{n,k^2}\,.$$

In particular, $|\mathcal{Q}^+(n)| - |\mathcal{Q}^-(n)| = 1$ if n is a perfect square, and it is 0 otherwise.

Clearly, Corollary 3 extends Fine's Theorem (see the introduction). When $n = 9$ we have $\mathcal{Q}(9) = \{9, 81, 63, 621, 531\}$, $|\mathcal{Q}(9)| = 5$, and the involution works as follows:

$$9 \longleftrightarrow 81\,, \qquad 63 \longleftrightarrow 621\,, \qquad 531 \;\circlearrowleft\,.$$

To see how the involution $\eta = \iota\,\varkappa\,\iota^{-1}$ works in general, see Figure 3.

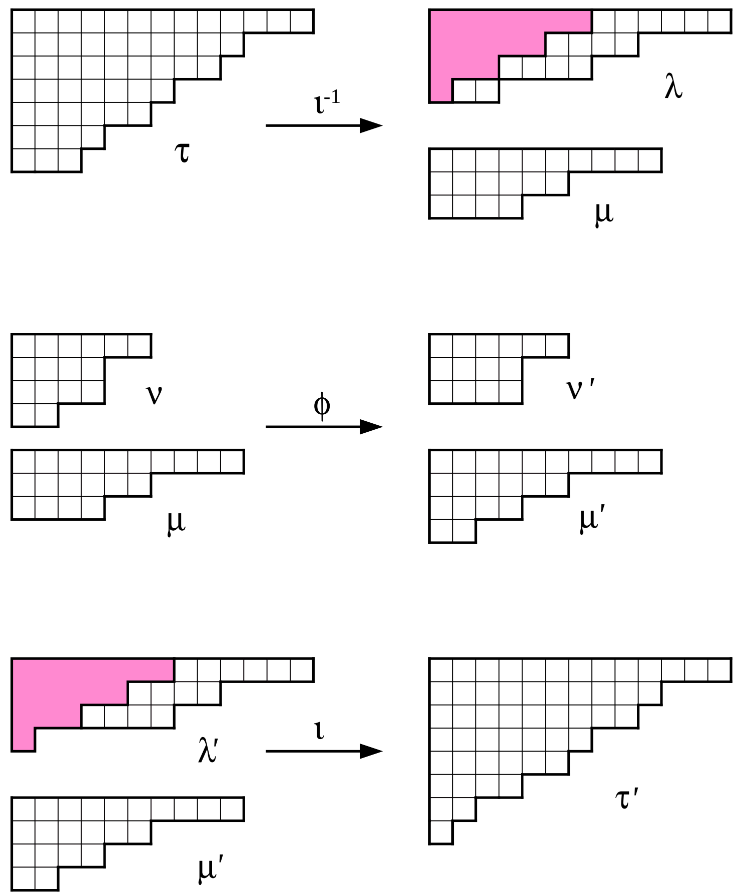

Figure 3. An example of steps of the involution $\eta : \mathcal{Q}(52) \to \mathcal{Q}(52)$. Here $\eta(13, 10, 9, 7, 6, 4, 3) = (13, 10, 9, 7, 6, 4, 2, 1)$.

In Theorem 2, suppose $m = 2$, $r = \mathbf{1}$, $B = (\mathbf{1}; 0, 0, \dots)$. For a partition τ let $\ell_1(\tau)$ and $\ell_0(\tau)$ denote the number of odd and even parts of τ, respectively. Similarly as for Corollary 3, we obtain the following result.

Corollary 4. *Let $\mathcal{Q}$ be the set of partitions with distinct even parts and odd smallest part. Then*

$$\sum_{\tau \in \mathcal{Q}} (-1)^{\ell_0(\tau)} q^{\ell_1(\tau)} t^{|\tau|} = \frac{1}{1 - q\,t}$$

Let $\mathcal{Q}^{\pm}(n)$ denote the set of partitions of n in $\mathcal{Q}$ with an even and odd number of even parts, respectively, then in particular $|\mathcal{Q}^{+}(n)| - |\mathcal{Q}^{-}(n)| = 1$ for all $n \geq 1$.

When $n = 7$ we have $\mathcal{Q}(7) = \{7, 61, 51^2, 43, 421, 41^3, 3^21, 321^2, 31^4, 21^5, 1^7\}$, $|\mathcal{Q}(7)| = 11$. In particular, Corollary 4 says that $|\mathcal{Q}(n)|$ is always odd. As in Corollary 1, it is instructive to compare this result with another Gauss identity:

$$(\ast\!\ast) \qquad \prod_{i=1}^{\infty} \frac{1 - t^{2i}}{1 + t^{2i-1}} \, = \, \sum_{k=1}^{\infty} (-1)^{k-1} \, t^{\binom{k}{2}} \, .$$

This shows that the *total* number of partitions of n with no repeated even parts is even unless n is a triangular number. From here we immediately obtain the following result.

Corollary 4′. *Let $\mathcal{Q}(n)$ be the set of partitions of n with distinct even parts and even smallest part. Then $|\mathcal{Q}(n)|$ is even if and only if n is a triangular number.*

Consider the following generalization of the previous situation. Let $j \in \{1, \ldots, m-1\}$, $r = (j, j, j, \ldots) = \mathbf{j}$, $B = (\mathbf{j}; 0, 0, \ldots)$. Then we obtain:

Corollary 5. *Let $\mathcal{Q}(n)$ be the set of partitions of n into parts $\equiv 0, j \mod m$, with the smallest part $\equiv j \mod m$ and no repeated parts divisible by m. Let $\ell_0(\tau)$ and $\ell_j(\tau)$ denote the number of parts of τ congruent to 0 and $j \mod m$, respectively. Then*

$$\sum_{\tau \in \mathcal{Q}} (-1)^{\ell_0(\tau)} q^{\ell_j(\tau)} t^{|\tau|} \, = \, \frac{1}{1 - q\, t^j}$$

In particular, the number of partitions in $\mathcal{Q}(n)$ with an even number of parts divisible by m minus the number of partitions in $\mathcal{Q}(n)$ with an odd number of parts divisible by m is equal to 1, whenever n is a multiple of j, and 0 otherwise.

The proof of the corollary follows verbatim the proof of Corollary 4. We skip the details.

7 Variations on the theme

Rather than state general theorems, let us indicate in special cases a few directions in which our results can be generalized.

Proposition 1. *Let $\mathcal{Q}(n)$ be the number of partitions τ of n with no repetitions of odd parts ≥ 3, and with odd largest part τ_1 or smallest* *part* $s(\tau) = 1$. *Let* $\ell_0(\tau)$ *and* $\ell_1(\tau)$ *denote the number of even and odd parts in* τ. *Then*

$$\sum_{\tau \in \mathcal{Q}} (-1)^{\ell_0(\tau)} q^{\ell_1(\tau)} t^{|\tau|} \, = \, \frac{1}{1 - q\, t}$$

In particular, $|\mathcal{Q}(n)|$ *is odd, for all* $n \geq 1$.

In other words, partitions $\tau \in \mathcal{Q}(n)$ satisfy the following conditions:

- $|\tau| = n$,
- no part $3, 5, 7, \ldots$ is repeated,
- $s(\tau) = 1$ or τ_1 is odd.

For example, $\mathcal{Q}(6) = \{51, 41^2, 321, 31^3, 21^4, 2^2 1^2, 1^6\}$ and $|\mathcal{Q}(6)| = 7$.

The proof of Proposition 1 follows along the same lines as the proof of Corollary 4. Here the crucial difference is in the use of Vahlen's involution: instead of ϕ one should use its sister map ψ where the *largest part* is moved in place of the *smallest part*.

Formally, define an involution $\zeta : \mathcal{Q} \to \mathcal{Q}$ as follows. For $\tau \in \mathcal{Q}$, let $2a+1$ be the largest odd part, and let $2b$ be the largest even part. If $a \geq b$ and $a > 0$, i.e., when $\tau_1 \geq 3$ is odd, define $\tau' = \zeta(\tau)$ to be the partition obtained from τ by replacing the part $2a + 1$ by the parts 1, $2a$. Note that $\tau' \in \mathcal{Q}$ since $s(\tau') = 1$. If $b > a$, i.e., when τ_1 is even and $s(\tau) = 1$, remove parts 1 and $2b$ from τ, and add part $2b + 1$. Then $\tau' \in \mathcal{Q}$ since $\tau_1' = 2b + 1$ is odd. Finally, if $\tau_1 = 1$, i.e., $\tau = 1^n$, stay put.

For example, when $n = 6$, the involution ζ acts on $\mathcal{Q}(6)$ as follows:

$$51 \longleftrightarrow 41^2\,, \qquad 321 \longleftrightarrow 2^2 1^2\,, \qquad 31^3 \longleftrightarrow 21^4\,, \qquad 1^7 \circlearrowleft .$$

Note that the number of odd parts is unchanged under ζ, while the parity of the number of even parts changes unless $\tau = 1^n$. This implies Proposition 1.

Let $\mathcal{D}(n)$ be the set of partitions of n into distinct parts, and let $\mathcal{D}^{\pm}(n)$ denote the subsets of partitions with an even and odd number of parts, respectively.

Proposition 2. (Euler) *Let* $n \in \mathbb{N}$. *Then* $|D^+(n)| - |D^-(n)| = (-1)^k$ *if* $n = k(3k \pm 1)/2$ *is a pentagonal number, and 0 otherwise.*

Let $\mathcal{D}_1^{\pm}(n)$ denote the subsets of partitions of $\mathcal{D}(n)$ with an even and odd largest part, respectively.

Proposition 3. (Fine) *Let* $n \in \mathbb{N}$. *Then* $|D_1^+(n)| - |D_1^-(n)| = (-1)^k$ *if* $n = k(3k \pm 1)/2$ *is a pentagonal number, and 0 otherwise.*

To prove these results, remove a pentagonal shape region of area $k(3k \pm 1)/2$ as in Figure 4 to obtain a joint partition (μ, ν), with $\mu_1, \nu_1 \le k$. Now use Vahlen's involution ϕ to these partitions. Check that ϕ changes parity of λ_1 and $\ell(\lambda)$ unless λ is a fixed point. This implies the result.

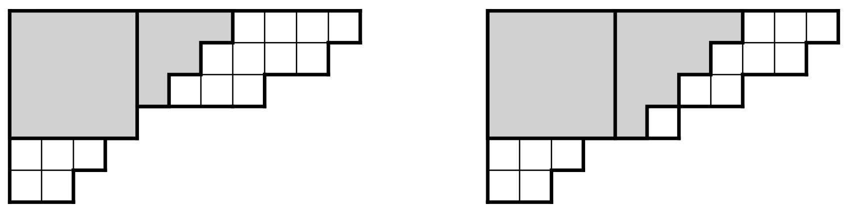

Figure 4. A pentagonal shape in partitions with distinct parts.

Now, of course, Proposition 2 is exactly Euler's Pentagonal Theorem (see introduction). Proposition 3 is one of Fine's theorems (see [15]). The resulting involution in this case coincides with Franklin's involution as discovered by Andrews [3]. We leave the details to the reader.

8 Final remarks

It is well known that the number $p(n)$ of partitions of n take infinitely many even and odd values. Can one use the kind of involution that we describe to give a combinatorial proof of this result? We should point out that even modulo 3 the distribution of $p(n)$ remains open [5].

There is very little hope that known methods can lead to a combinatorial proof of Ramanujan's congruences $p(5n-1) \equiv 0 \mod 5$, even in view of Dyson's rank interpretation. In Oliver Atkin's words[3], *"it is probably bad advice to a young man to look for a true combinatorial proof* [of Ramanujan's congruences]*."*

The form of the curious identities (1.5) and (1.6) from [4] suggests that they should have an involutive proof in a similar manner. Despite several attempts such a proof eluded the authors. We hope the reader give it a try.

The name "joint partitions" was coined recently by Don Knuth as a better alternative to a term "overpartitions" existing in the literature (cf. [14]) The notion of MacMahon diagrams was rediscovered on many occasions, especially in connection with "m-modular diagrams" (see [14]).

Fine's Theorem was announced in [8]. Its proof first appeared in print forty years later in Fine's book [9]. See [15] for a history of these results as

[3]This quote is taken from a letter of Atkin to the second author.

well as some “missed opportunities.”

Both Gauss identities ($\circledast$) and ($\circledast\circledast$) have involutive proofs [1]. Thus one can use an involution principle to prove Corollary $4'$ bijectively (see e.g. [14]). Can one find an “involution principle free” bijective proof? In a different direction, can one start with these involutions and refine them to obtain new partition congruences? What about Schur’s celebrated involution? (see [14])

Finally, can one use the second version of restricted Vahlen’s involution on $\mathcal{S}_{\leq k} \subset \mathcal{J}$ (see Section 3) to construct further partition congruences?

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