Acute triangulations of polyhedra and $\mathbb{R}^n$

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Abstract

We study the problem of *acute triangulations* of convex polyhedra and the space $\mathbb{R}^n$. Here an acute triangulation is a triangulation into simplices whose dihedral angles are acute. We prove that acute triangulations of the n–cube do not exist for $n \geq 4$. Further, we prove that acute triangulations of the space $\mathbb{R}^n$ do not exist for $n \geq 5$. In the opposite direction, in $\mathbb{R}^3$ we construct nontrivial acute triangulations of all Platonic solids. We also prove nonexistence of an acute triangulation of $\mathbb{R}^4$ if all dihedral angles are bounded away from $\pi/2$.

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1 Introduction

The problem of finding acute triangulations has a long history going back to the early days of Discrete and Computational Geometry. It has a number of connections to other areas and some real world applications, most notably the finite element method. Until recently, most results dealt with the 2–dimensional case, where the problem has been largely resolved; it is now known that every n–gon in the plane has an acute triangulation into $O(n)$ triangles (a combination of results in [BGR] and [Mae]).

In the last few years, several papers [ESÜ, Kří, KPP, VHZG] broke the dimension barrier in both positive and negative direction (see below). In this paper we continue this exploration, nearly completely (negatively) resolving the problem in dimension 4 and higher, and making further advancement in dimension 3. This extended abstract is based on [KPP].

We must address one important limitation of our paper, essentially highlighting the current poor state of art. The real problem (both in theory and in application) is to find acute triangulations *efficiently* (say, in polynomial time). Instead, we prove that such algorithm is theoretically impossible in dimension $n \geq 5$, and in practice for $n \geq 4$. In a positive direction, we make the first real effort to prove that the general goal can conceivably be accomplished in 3 dimensions.

1.1 Definitions and main results

For a polytope $P \subset \mathbb{R}^n$, an *acute triangulation* is a subdivision of P into finitely many *acute simplices* (i.e. with acute dihedral angles) which form a simplicial complex, so e.g. in the plane, a vertex of one simplex cannot lie in the interior of an edge of another. (See Figure 1, where on the left we have a subdivision of the square which forms a simplicial complex, and on the right we have a subdivision which does not.) A triangulation is *non-trivial* if it has at least one interior vertex. An *acute triangulation* of $\mathbb{R}^n$ is a subdivision of $\mathbb{R}^n$ into acute (infinitely many) simplices which form a simplicial complex.

Fig. 1

Theorem A. *Every Platonic solid in* $\mathbb{R}^3$ *has a non-trivial acute triangulation.*

For an important example of the cube, we roughly do the following. We first triangulate the cube into a regular 3–simplex and four standard 3–simplices. We then subdivide each of these 3–simplices into 543 pieces, to obtain combinatorially what we call the *special subdivision* (based on the 600–cell, see Section 2).

Theorem B. *There are no non-trivial acute triangulations of polytopes in* $\mathbb{R}^n$*, for all* $n \geq 5$*. In particular there are no acute triangulations of* $\mathbb{R}^n$*, for* $n \geq 5$*.*

Theorem C. *The* 4*–cube does not have an acute triangulation. Neither do the* isosceles 4–orthoscheme[1] *and the* Schläfli simplex $\widetilde{D}_4$.[2]

Together, Theorems A, B and C imply the following result which remained open for nearly 50 years:

Corollary D. *The* n*–cube has an acute triangulation if and only if* $n \leq 3$*.*

The next two theorems are successive far-reaching extensions of Theorem C.

Theorem E. *The space* $\mathbb{R}^4$ *has no periodic acute triangulation.*

Theorem F. *The space* $\mathbb{R}^4$ *has no acute triangulation into simplices whose dihedral angles are less than* $\pi/2 - \varepsilon$*, for any given* $\varepsilon > 0$*.*

Theorem F is a *bounded geometry* version of Křížek's conjecture stating that $\mathbb{R}^4$ has no acute triangulation [Kří]. Theorems B, C and F are proved by a topological argument using a (known) advanced extension of Dehn–Sommerville equations. The proof of the strongest result (Theorem F) uses the concept of p–parabilicity of graphs, its recent properties, and our new advanced isoperimetric type inequality.

1.2 Previous work

The problem of finding acute triangulations has a long history in classical geometry, and is elegantly surveyed in [BKKS], which argues that it goes back to Aristotle. In recent decades, it was further motivated by the *finite element method* which requires "good" meshes (triangulations of surfaces) for the numerical algorithms to run. Although the requirements for meshes largely depend on the algorithm, the sharp angle conditions seem to be a common feature, and especially important in this context. We refer to [SF] for the introduction to the subject, and to [She] for the state of art.[3]

It is important to emphasize that for a given convex polygon $P \subset \mathbb{R}^2$, the real difficulty in the acute triangulation problem is finding the positions of Steiner points in the interior and on the boundary of P. Once such set of points is fixed, the unique triangulation on these vertices that can possibly be acute is the corresponding *Delaunay triangulation* (see [Ede]), and the latter can be constructed efficiently (see [Aur, For]). This makes this problem different from most problems in [DRS], where typically an optimal triangulation is sought on a fixed set of vertices.

Let us mention also a motivation coming from the recreational literature, where the subject of dissections has been popular in general (see [Lin]), and of acute triangulations in particular [CL, Man]. In this context, the problem of acute triangulations of a square, cube, and hypercubes seem to be of special interest [Epp]. It is crucial in this context that the simplices must be acute rather than non-obtuse (when the angle $\pi/2$ is allowed), since in the latter case every n–cube can be triangulated into $n!$ non-obtuse (path-) simplices.

[1] This is a path-simplex in $\mathbb{R}^4$ with intervals of unit lengths, which can be defined by the inequalities $0 \leq x_1 \leq x_2 \leq x_3 \leq x_4 \leq 1$.

[2] This 4–simplex is sometimes called *cube corner* and can be defined by $x_1 + x_2 + x_3 + x_4 \leq 1$, and $x_1, x_2, x_3, x_4 \geq 0$.

[3] Proceedings of recent *International Meshing Roundtable* conferences, specifically their *tetrahedral meshes* sections, provide further up-to-date results on both theoretical and practical aspects of this important problem.

In one of the first papers on the subject, Burago and Zalgaller proved in [BZ] that any non-convex polygon (possibly, with holes) has an acute triangulation. Unfortunately, their argument was largely forgotten as it was inexplicit and did not give a bound on the number of triangles required. In a long series of papers [BE, BGR, BMR, Mae, Sar, Yuan] first polynomial, and then linear bounds were obtained for non-obtuse, and, eventually, for acute triangulations. We refer to [Zam] for the historical outline, a short survey, and further references.

In higher dimensions, several results have been recently obtained. First, Eppstein, Sullivan and Üngör [ESÜ] showed that the space $\mathbb{R}^3$ has a periodic acute triangulation. Then, Křížek [Kří] showed that no vertex in $\mathbb{R}^n$ for $n \geq 5$ can be surrounded by a finite number of acute simplices (as in Theorem B),[4] and conjectured that the space $\mathbb{R}^4$ also cannot be triangulated into acute tetrahedra. Finally, most recently (and independently form our [KPP]), VanderZee, Hirani, Zharnitsky and Guoy [VHZG] used an advanced numerical simulation technique to find an acute triangulation for the (usual) cube in $\mathbb{R}^3$ (as in Theorem A). Their construction is ad hoc, but better suitable for applications (it uses 1370 tetrahedra as opposed to 2715 tetrahedra in our construction).

The main approach in our paper is topological, both in motivation of the positive results, and in the tools for negative results. A variation on the use of 600–cell to obtain a topological version of Theorem A was used previously by Świątkowski and the third named author in [PŚ], to construct the so called *flag-no-square* subdivisions in dimension 3, used originally to construct Gromov hyperbolic groups with prescribed boundaries. A variation on Theorem B was previously obtained by Kalai in [Kal] (in fact, both results are extension of the same special case).

Finally, the idea behind both the statement and the proof of Theorem F is based on large body of work on tessellations of the space by convex polyhedra with bounded geometry. Perhaps the earliest, is the result of Alexandrov that in every triangulation of the plane into bounded triangles the average degree of vertices (when defined) must be at least 6 [Ale]. Another is a classical result by Niven that convex n–gons of bounded geometry cannot tile the plane for $n \geq 7$ [Niv] (see also [Ful]). The idea is always to use the isoperimetric inequalities compared with direct counting estimates, an approach which works in higher dimensions as well (see e.g. [KS, LM]). Our proof of Theorem F is a variation on the same line of argument. We refer to [GS] for historical background and further references.

1.3 Implications of our work for the finite element method

This paper has both good and bad news for the finite element method practitioners. We start with the good. Until now, a typical (far from optimal) practical approach to getting an acute triangulation of a given (non-convex) polyhedron, would be to start with the triangulation of the interior along the fundamental tetrahedron in the BCC lattice, and then improve and refine the triangulation near the boundary.[5] This tetrahedron, one of the Sommerville's tetrahedra, tiles the space, but although not acute, it is non-obtuse and highly symmetric. The acute triangulation in [ESÜ] suggests that at a cost of creating less symmetric triangles, one can ensure that the same approach can make all but "boundary" tetrahedra to be acute. Our work (Theorem A) suggests that perhaps one can abandon this periodicity approach altogether and create an acute triangulation of the whole polyhedron.

[4]There is a crucial error in this proof, see Křížek's correction in *Disc. Comp. Geom.* (to appear).
[5]This makes such triangulations useful for some but not all applications.

Main Conjecture. *Every (possibly, non-convex) polyhedron $P \subset \mathbb{R}^3$ has an acute triangulation. Moreover, such triangulation will have a polynomial number of tetrahedra (in the number of vertices of P), and can be computed in polynomial time.*

Given how little is known (basically, nearly nothing until recently) this might sound too speculative. We refer to [KPP] for a reasoning behind (based on this work).

Now for the bad news. In many practical applications, mostly when time is a dimension, the finite element method is applied in $\mathbb{R}^4$. Our Theorem E shows that the periodicity approach (as in [ESÜ]) cannot be implemented. Moreover, Theorem F proves that the very large dihedral angles are unavoidable in any large simplicial acute triangulation in $\mathbb{R}^4$, which suggests that some numerical methods might be slow in four dimensions.

Convention. In the entire article we adopt a convention that simplicial complexes and triangulations of (homology) manifolds are not allowed to have edges connecting a vertex to itself, and they are also not allowed to have multiple simplices spanned on the same set of vertices.

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2 Acute triangulations in $\mathbb{R}^3$

In this section we describe acute triangulations of the 3–cube and the regular octahedron. The starting point is the following observation:

Observation 2.1. The link of an interior edge of an acute triangulation of a polyhedron in $\mathbb{R}^3$ is a simplicial loop of length at least 5.

In view of this observation, let us make the following definition.

Definition 2.2. A triangulation of an n–dimensional homology manifold is *rich* if the links of all interior $(n-2)$–simplices are loops of length at least 5.

Note that being rich is a purely combinatorial (i.e. non-metric) property. Observation 2.1 states that an acute triangulation of the 3–cube (or a regular octahedron) must be rich.

Consider the 600–cell, the convex regular 4–polytope with Schläfli symbol $\{3; 3; 5\}$ (see, e.g., [Cox]). Denote by X_{600} the boundary of the 600–cell, a 3–dimensional simplicial polyhedron homeomorphic to the 3–dimensional sphere. It consists of 600 3–simplices[6] and

[6] To streamline and simplify the presentation, we refer to triangles as 2–simplices, to tetrahedra as 3–simplices, etc.

has 120 vertices. Its vertex links are icosahedra and its edge links are pentagons. We first focus on the combinatorial simplicial structure of X_{600}. Denote by X_{543} the subcomplex of X_{600} which we obtain by removing from X_{600} the interiors of all simplices intersecting a fixed 3–simplex. (The number 543 in the subscript refers to the number of 3–simplices in X_{543}.)

Lemma 2.3 ([PŚ, Lemmas 2.5 and 2.7])**.**

(1) X_{543} *is topologically a* 3*–ball. It is rich.*

(2) Its boundary is a 2*–sphere simplicially isomorphic to the simplicial complex which we obtain from the boundary of a* 3*–simplex by subdividing each face as in Figure 2.*

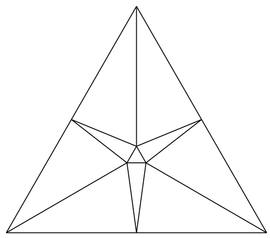

Fig. 2

We recall the following definition:

Definition 2.4 ([PŚ, Definitions 2.2 and 2.8])**.** Given a simplicial complex of dimension at most 3, its *special subdivision* is the simplicial complex obtained by:

(i) subdividing each edge into two (by adding an extra vertex in the interior of the edge),

(ii) subdividing each 2–simplex as in Figure 2,

(iii) subdividing each 3–simplex so that it becomes isomorphic to X_{543}.

We are ready to describe the combinatorial structure of our triangulation of the 3–cube and the regular octahedron. Assume that the cube lies in $\mathbb{R}^3$ with the vertices at points $(\pm 1, \pm 1, \pm 1)$. Consider the triangulation W of the cube into five 3–simplices so that one of them (denote it by T_0) has vertices $(1,1,1), (-1,-1,1), (-1,1,-1)$ and $(1,-1,-1)$, while the remaining four 3–simplices (denote them $T_1, \ldots, T_4$) are the components of the complement to T_0 in the cube. Note that $T_1, \ldots, T_4$ are congruent (equal up to a rigid motion); we call such 3–simplices *standard* (see e.g. [Pak]).[7] Let W^* be the special subdivision of W defined as above.

Similarly, let Y be the triangulation of the regular octahedron into eight standard 3–simplices obtained as cones from the center over the faces. Let Y^* be the special subdivision of Y.

By [PŚ, Proposition 2.13], subdivisions W^* and Y^* are both rich and have a potential of giving an acute realization. This is true indeed, and proves Theorem A for the cube and the regular octahedron: In fact, we provide acute triangulations of all 3–simplices of W and Y, combinatorially equivalent to X_{543}, and matching on common part of the boundary. In other words, we prove the following intermediate result:

Theorem 2.5. *There is a (non-trivial) acute triangulation, combinatorially equivalent to* X_{543}*, of (i) the regular* 3*–simplex, and (ii) the standard* 3*–simplex.*

[7]This tetrahedron is also called the *cube-corner*.

Below we describe the construction for the 3–cube. At some points we use a computer program. We provide the exact position of all vertices of both triangulations from Theorem 2.5 in Appendix A. There are three steps of the construction. First, we construct an acute triangulation of T_0. Then we "flatten" it to obtain an acute triangulation of T_1. Then we construct another acute triangulation of T_0 so that it matches the one of T_1 on the common part of the boundary.

Step 1. Note that the vertices of the 600–cell, whose boundary we called X_{600}, lie on a sphere in $\mathbb{R}^4$. Moreover all 3–simplices in this realization of X_{600} are regular, hence acute. Let now $\widetilde{X}_{543}$ be the realization of X_{543} in $\mathbb{R}^3$, whose vertices are obtained by stereographic projection of the $\mathbb{R}^4$ realization. We choose the center of the projection to be the center of the (spherical) 3–simplex in X_{600} disjoint from X_{543}. It turns out that this mapping does not disturb the angles significantly.

We move the vertices of $\partial\widetilde{X}_{543}$ radially so that they arrange on the boundary of a regular 3–simplex which we identify with T_0. If we scale the size of ∂T_0 correctly, this triangulation of T_0 is already acute, i.e. it satisfies Theorem 2.5(i). However, it is not the one listed in Appendix A, we will modify it later in Step 3.

Step 2. The subdivision of the standard 3–simplex, say T_1, is more difficult. Our computer program uses the following algorithm to find the position of the vertices. We "flatten" the acute triangulation of T_0 obtained in Step 1 in order to obtain an acute triangulation of T_1. We gradually move one of the boundary vertices (marked A on Figure 4) towards the center, keeping the three vertices marked C in the points where the circles inscribed into triangles AB_iB_j meet the edges AB_i. We also keep the nine points marked D on their faces, and scale and translate together all the interior vertices.

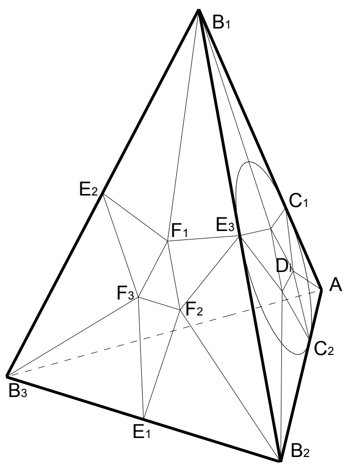

Fig. 3

Whenever some angle stops being acute during this operation, we suspend the flattening process to correct the angles. This is done by slightly moving the responsible vertices so that the angle becomes smaller. Vertices are moved only in a way that does not disturb the combinatorial structure, i.e. points A, B_i, C_i, E_i are not moved at all; movement of D_i and F_i is restricted to their faces; all the interior vertices except the two outermost layers (of 12 and 16 vertices, respectively) are moved together so that the structure is not disrupted.

When all the angles are corrected, we resume the flattening, until we obtain the standard 3–simplex T_1. This completes the description of the triangulation in Theorem 2.5, part (ii) (see also Appendix A).

Step 3. The position of the vertices F_i on the equilateral face of T_1 is now different from their position on the face of T_0, because we had to move F_i during the correcting process in Step 2. So in the triangulation of T_0 constructed in Step 1 we move all 12 vertices corresponding to F_i to the position matching with the standard T_i. It turns out that it is then enough to scale the interior structure to obtain an acute triangulation (see Appendix A).

Now we attach acute triangulations of all T_i, constructed in Steps 2 and 3, to obtain an acute triangulation of the cube (see Remark 2.6 below). Similarly, by attaching eight copies of the standard 3–simplex triangulated as in Step 2, we obtain an acute triangulation of the regular octahedron.

For the last two Platonic solids, note that the regular icosahedron has a straightforward acute triangulation as a union of cones from the center to the facets. Finally, for the regular dodecahedron, first subdivide it into a union of 120 path tetrahedra. Then take their special subdivision and realize it metrically in a similar manner as with the standard 3–simplex. We omit the details. □

Remark 2.6. The animation of our triangulation of the 3–cube is available at

`http://www.mimuw.edu.pl/~erykk/papers/acute.html`.

For the details and the exact values of all the parameters which have been guessed, see the implementation of above algorithm, available with the animation.

3 Dehn–Sommerville equations in dimension 4

In this short section we present some known results in geometric combinatorics.

Denote by $f_i(M)$, $f_i(\partial M)$ (we later abbreviate this to f_i, f_i^∂), the number of i–dimensional simplices of a triangulation of a compact m–dimensional homology manifold M and its boundary ∂M. Recall the following Dehn–Sommerville type equations (see Appendix **??** for the history of this generalization).

Theorem 3.1 ([Kla, Theorem 1.1] and [NS]). *Let M be a compact m–dimensional triangulated homology manifold with boundary. For $k = 0, \ldots, m$ we have*

$$f_k(M) - f_k(\partial M) = \sum_{i=k}^{m} (-1)^{i+m} \binom{i+1}{k+1} f_i(M).$$

If $m = 4$, then for $k = 1, 2$ we obtain the following.

Corollary 3.2. *If M is 4–dimensional, and $f_i = f_i(M), f_i^\partial = f_i(\partial M)$, then:*

(i) $2f_1 - f_1^\partial = 3f_2 - 6f_3 + 10f_4$,

(ii) $-f_2^\partial = -4f_3 + 10f_4$.

These equalities will be used repeatedly in the next two sections.

4 Rich triangulations of 4–manifolds

In this section we prove the following combinatorial result on rich triangulations (see Definition 2.2) of 4–dimensional homology manifolds. We keep the notation f_i, f_i^∂ from Section 3.

Theorem 4.1. *Every rich triangulation of a compact 4–dimensional homology manifold M with Euler characteristic χ satisfies*

$$2f_0 \leq 2\chi + f_1^\partial.$$

In particular, if M is closed, then $f_0 \leq \chi$.

Before we present the proof of the theorem, let us give the following four corollaries in the case when the homology manifold M is closed.

Proof of theorems C and E. A periodic triangulation τ of $\mathbb{R}^4$ descends to a triangulation τ' of a 4–torus. Since the Euler characteristic of a 4–torus equals 0, by Theorem 4.1, triangulation τ' is not rich. Hence, by Observation 2.1, τ is not acute. This proves Theorem E. For Theorem C, observe that an acute triangulation of the 4–cube could be promoted, by reflecting, to a periodic acute triangulation of $\mathbb{R}^4$. For the isosceles orthoscheme, observe that it tiles the cube and thus (periodically) the whole space $\mathbb{R}^4$ by reflections. Similarly, albeit less obviously, the cube corner 4–simplex tiles the space $\mathbb{R}^4$ according to reflections of the affine Lie group $\widetilde{D}_4$. □

Proof of Theorem B. Let v be an interior vertex and let ρ be a codimension 5 simplex (a vertex for $\mathbb{R}^5$, an edge for $\mathbb{R}^6$ etc) containing v. The link L of ρ is a 4–dimensional homology sphere, hence its Euler characteristic equals 2. Since L is 4–dimensional, it must have at least 6 vertices. Hence, by Theorem 4.1, L is not rich. Thus the link of one of the codimension 2 simplices containing ρ is a cycle of length shorter than 5. Hence one of the dihedral angles adjacent to v is not acute. □

Proof of Theorem 4.1. Let τ be a rich triangulation of M. We compute the number N of flags ($\rho_2 \subset \rho_4$) of a 2–simplex ρ_2 contained in a 4–simplex ρ_4. On one hand, it is equal to $10f_4$, since each 4–simplex has ten 2–dimensional faces. On the other hand, by richness, each interior 2–simplex (there are $f_2 - f_2^\partial$ of those) is contained in at least five 4–simplices. Thus, we have:

$$N \;=\; 10f_4 \;\geq\; 5(f_2 - f_2^\partial). \tag{1}$$

By the definition of the Euler characteristic, we have:

$$\chi - f_0 = -f_1 + f_2 - f_3 + f_4. \tag{2}$$

Applying consecutively formula (2), then Corollary 3.2 parts (i) and (ii), and formula (1), we obtain:

$$\begin{aligned} 2(\chi - f_0) + f_1^\partial &= -2f_1 + 2(f_2 - f_3 + f_4) + f_1^\partial = \\ &= -(f_1^\partial + 3f_2 - 6f_3 + 10f_4) + 2(f_2 - f_3 + f_4) + f_1^\partial = \\ &= -f_2 + 4f_3 - 8f_4 = -f_2 + (f_2^\partial + 10f_4) - 8f_4 = \\ &= 2f_4 - (f_2 - f_2^\partial) \geq 0, \end{aligned}$$

as desired. □

5 Acute triangulations of $\mathbb{R}^4$

In this section we address the problem whether there is an acute triangulation of $\mathbb{R}^4$. We know already that every such acute triangulation of $\mathbb{R}^4$ cannot be periodic (Theorem E).

We say that a triangulation of $\mathbb{R}^p$ has *bounded geometry* if there is a global upper bound on the ratio of edge lengths in every p–simplex.

Theorem 5.1. *There is no acute triangulation of $\mathbb{R}^4$ with bounded geometry.*

This result easily implies Theorem F, as the following argument shows.

Proof of Theorem F modulo Theorem 5.1. If the dihedral angles are bounded away from $\frac{\pi}{2}$, then the angles of 2–simplices are bounded away from $\frac{\pi}{2}$ (see e.g. [Kří]). Hence the angles of 2–simplices are also bounded away from 0. By the sine law, this gives a bound on the ratio of lengths of edges in each 2–simplex, which results in a bound of the ratio of lengths of edges in each 4–simplex. $\square$

Now Theorem 5.1 follows from the following intermediate results which are proved in Appendix B.

Definition 5.2. Let $G = (V, E)$ be a simple connected (locally finite) infinite graph, and let $\Omega \subset V$ be a finite subset of vertices. Denote by $\partial\Omega$ the *vertex-boundary* of Ω, defined as the subset of $V \setminus \Omega$ consisting of vertices adjacent to vertices in Ω.

We say that $I\colon \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ is an *isoperimetric function* for G, if the inequality $I(|\Omega|) \leq |\partial\Omega|$ holds for every finite $\Omega \subset V$.

Proposition 5.3. *The* 1*–skeleton of any acute triangulation of $\mathbb{R}^4$ with bounded geometry has linear isoperimetric function.*

The following result gives a direct contradiction to Proposition 5.3. Following Benjamini–Curien [BC, Section 2.2], we recall the following definition:

Definition 5.4. Let $G = (V, E)$ be a locally finite connected graph and let $\Gamma(v)$ be the set of all semi-infinite self avoiding simplicial paths in G starting from $v \in V$. For any $m\colon V \to \mathbb{R}_+$ (so called *metric*), the *length* of a path γ in G is defined by

$$\mathrm{Length}_m(\gamma) = \sum_{v\in\gamma} m(v).$$

If $m \in L^p(V)$, we denote by $||m||_p$ the usual L^p norm. The graph G is *p–parabolic* if the *p–extremal length* of $\Gamma(v)$,

$$\sup_{m\in L^p(V)} \inf_{\gamma\in\Gamma(v)} \frac{\mathrm{Length}_m(\gamma)^p}{||m||_p^p}$$

is infinite. This definition does not depend on the choice of $v \in V$.

Lemma 5.5. *Let G be the* 1*–skeleton of a triangulation of $\mathbb{R}^p$ with bounded geometry, where $p \geq 2$. Then G is p–parabolic.*

This lemma can be obtained from the Bonk and Kleiner result [BK, Corollary 8.8]. To make the proof complete and self-contained, we include a concise proof in Appendix B.

To finish the proof, we need the following recent result by Benjamini and Curien:

Proposition 5.6 ([BC, Proposition 4.1(1)]). *Let $G = (V, E)$ be an infinite locally finite connected graph. If G is p–parabolic and I is an isoperimetric function, then for $D > p$ we have*

$$\sum_{k=1}^{\infty} \frac{1}{I(k)^{\frac{p}{p-1}}} = \infty.$$

Proof of Theorem 5.1. Assume that there is is an acute triangulation τ of $\mathbb{R}^4$ with bounded geometry. Let G be the 1–skeleton of τ. By Lemma 5.5 we have that the graph G is 4–parabolic. By Proposition 5.6, we have that $k \to Ck$ is not an isoperimetric function for G. This contradicts Proposition 5.3. □

6 Conclusion

We presented both positive and negative results on acute triangulations of polytopes and acute triangulations of the space $\mathbb{R}^n$. The results suggest that it is often possible to obtain acute triangulation in $\mathbb{R}^3$. We do this explicitly for all Platonic solids. Independently with [VHZG], this is is the first ever construction of a non-trivial acute triangulation. Our construction is based on geometry of the 600–cell, a regular polytope in $\mathbb{R}^4$ and topological considerations. In addition, we resolved an old folklore conjecture of Gardner, et al. finding that the n–cube has an acute triangulation if and only if $n \leq 3$.

In the negative direction, we show that the standard finite element idea cannot possibly work in dimensions higher than 4, since there are no periodic triangulations in $\mathbb{R}^4$ and any non-trivial triangulations in $\mathbb{R}^n$, $n \geq 5$. We proved that several examples of convex polytopes in $\mathbb{R}^4$ (the 4–cube, the 4–dimensional cube corner, and the 4–dimensional isosceles orthoscheme) do not have an acute triangulation. These are the first (correct) negative results in higher dimension. Finally, we give a partial solution of Křížek's conjecture (Theorem F).[8]

[8] In fact, we do not believe in the conjecture in full generality (see [KPP] for the reasoning).

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