VANISHING OF SCHUBERT COEFFICIENTS

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Abstract. *Schubert coefficients* are nonnegative integers $c^w_{u,v}$ that arise in Algebraic Geometry and play a central role in Algebraic Combinatorics. It is a major open problem whether they have a combinatorial interpretation, i.e, whether $c^w_{u,v} \in \#\mathsf{P}$.

We study the closely related *vanishing problem of Schubert coefficients*: $\{c^w_{u,v} =^? 0\}$. Until this work it was open whether this problem is in the polynomial hierarchy PH. We prove that $\{c^w_{u,v} =^? 0\}$ in coAM assuming the GRH. In particular, the vanishing problem is in Σ^{p}_2.

Our approach is based on constructions *lifted formulations*, which give polynomial systems of equations for the problem. The result follows from a reduction to *Parametric Hilbert's Nullstellensatz*, recently studied in [A+24]. We extend our results to all classical types. Type D is resolved in the appendix (joint with David Speyer).

1. Introduction

1.1. **Foreword.** This paper occupies an unusual ground, bridging between Schubert Calculus, a subarea of Algebraic Geometry and Algebraic Combinatorics, on the one hand, and Computational Complexity on the other. Since these are far apart, we make an effort to give as much background as we can to make the results comprehensible.

Below is a very lengthy introduction which includes a lot of standard results and relevant background in Algebraic Combinatorics, as well as their meaning in terms of computational complexity classes. Eventually we get to the point when we can state our results and explain their importance. We beg the reader's forgiveness for being introductory or even basic, at times. The experts can simply skip those parts.

Throughout the introduction we assume the reader is familiar with standard notions and results in Computational Complexity. The reader in need of a quick reminder is referred to Section 3, where we give a quick overview of standard complexity classes.

1.2. **Littlewood–Richardson coefficients.** We begin with the Littlewood–Richardson (LR) coefficients $\{c^\lambda_{\mu,\nu}\}$ that are the most important special case of the Schubert coefficients. The are also a major source of inspiration behind many generalizations.

Schur polynomials $s_\lambda = s_\lambda(x_1,\ldots,x_n) \in \mathbb{N}[x_1,\ldots,x_n]$, are symmetric polynomials corresponding to irreducible characters of $\mathrm{GL}_n(\mathbb{C})$ and are given by an explicit determinant formula, see e.g. [Mac95, Sta99]. Here λ is an *integer partition* with at most n parts: $\lambda = (\lambda_1 \geq \ldots \geq \lambda_n)$. We use a shorthand $\boldsymbol{x} := (x_1,\ldots,x_n)$. Schur polynomials can also be defined as

$$s_\lambda(\boldsymbol{x}) := \sum_{A\in \mathrm{SSYT}(\lambda)} \boldsymbol{x}^A \quad \text{where} \quad \boldsymbol{x}^A := \prod_{(i,j)\in\lambda} x_{A(i,j)}\,.$$

Here the summation is over *semistandard Young tableaux* A of shape λ, i.e. integer functions $A:\lambda\to\mathbb{N}$ on a Young diagram which weakly increase in rows and strictly increase in columns, see Figure 1.1.

From here it is easy to see that Schur functions form a linear basis in the ring $\Lambda_n := \mathbb{C}[x_1,\ldots,x_n]^{S_n}$ of symmetric polynomials in n variables. Therefore, their multiplication structure constants in

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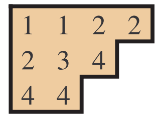

A

FIGURE 1.1. Semistandard Young tableau $A \in \mathrm{SSYT}(4,3,2)$ and the corresponding monomial $\boldsymbol{x}^A = x_1^2\, x_2^3\, x_3\, x_4^3$.

Λ_n are well defined.[1] *Littlewood–Richardson (LR) coefficients* are now defined as:

$$s_\mu \cdot s_\nu \, = \, \sum_\lambda c^\lambda_{\mu,\nu}\, s_\lambda \, .$$

Schur polynomials encode fundamental symmetries in algebra and geometry. They are ubiquitous in Enumerative and Algebraic Combinatorics [Ful97, Sta99], Representation Theory [Sag01], Algebraic Geometry [AF24, Man01], Quantum Information Theory [OW21], Algebraic Complexity [C+20], and Geometric Complexity Theory (GCT) [BI18, IP17]. Unsurprisingly, just about every aspect of Schur polynomials and their generalizations has been studied to great extent.

The LR-coefficients are especially well studied from both combinatorial and computational points of view [Pan23]. There are at least 20 combinatorial interpretations for $\{c^\lambda_{\mu,\nu}\}$, some of which are transparently #P-functions for the unary input and a few others for the binary input [Pak24, §11.4]. Denote by LR the counting problem of computing LR-coefficients. It is known that computing LR-coefficients is #P-complete for the input in binary [Nar06].

Conjecture 1.1 ([Pan23, Conj. 5.14])**.** LR *is* #P*-complete for input in unary.*

Most remarkably, the *vanishing of LR-coefficients* $\{c^\lambda_{\mu,\nu} =^? 0\}$ is in P even for the input in binary. This was shown in [DM06, MNS12] as an easy consequence of the celebrated *saturation theorem* by Knutson and Tao [KT99]. See also [BI13] for a fast algorithm based on network max-flows.

This "vanishing in P" property inspired an important part of Mulmuley's program GCT towards proving that $\mathsf{VP} \neq \mathsf{VNP}$ [Mul09]. Specifically, Mulmuley aimed to modify and extend this property to *Kronecker coefficients* which generalize the LR-coefficients, see §2.1(*iii*). Some of Mulmuley's conjectures were refuted in [BOR09], while his general approach was undermined in [BIP19, IP17]. Most recently, it was shown in [PP20] that even *reduced Kronecker coefficients* do not have the saturation property, adding further doubts to the program.

1.3. **Schubert polynomials.** *Schubert polynomials* $\mathfrak{S}_w \in \mathbb{N}[x_1,\ldots,x_n]$ indexed by permutations, are celebrated generalizations of Schur polynomials, and are not symmetric in general. They were introduced by Lascoux and Schützenberger [LS82, LS85], to represent cohomology classes of Schubert varieties in the complete flag variety (see below).

Schubert polynomials have been intensely studied from algebraic, combinatorial, and (more recently) complexity points of view. See [Mac91, Man01] for classic introductory surveys, [Knu16, Knu22] for overviews of recent results, [AF24, KM05] for geometric aspects, and [Las95] for historical remarks. We refer to [Pak24, §10] for an overview of computational complexity aspects.

We postpone the algebro-geometric background for Schubert polynomials until §4.2. We do not need it at this stage. Instead, below we give two definitions.

Algebraic Definition: For a permutation $w_\circ = (n, n-1, \ldots, 2, 1)$, let

$$\mathfrak{S}_{w_\circ} \, := \, x_1^{n-1} x_2^{n-2} \cdots x_{n-1}\, .$$

[1] Formally, one needs to take multiplication in the inverse limit Λ of Λ_n, so that these structure constants depend only on partitions and not on n.

A permutation $w \in S_n$ is said to have a *descent* at i, if $w(i) > w(i+1)$. Denote by $\operatorname{Des}(w)$ the *set of descents* of w, and by $\operatorname{des}(\sigma) := |\operatorname{Des}(\sigma)|$ the *number of descents*. Define the *divided difference operator*

$$\partial_i F \;:=\; \frac{F - s_i F}{x_i - x_{i+1}}\,,$$

where the transposition $s_i := (i, i+1)$ acts on $F \in \mathbb{C}[x_1, \ldots, x_n]$ by transposing the variables. For all $i \in \operatorname{Des}(w)$, let

$$\mathfrak{S}_{ws_i} \;:=\; \partial_i \mathfrak{S}_w\,,$$

and define all Schubert polynomials recursively.

This definition is due to Lascoux and Schützenberger [LS82]. It follows that $\mathfrak{S}_w \in \mathbb{Z}[\boldsymbol{x}]$ are homogeneous polynomials of degree $\operatorname{inv}(w)$. Here $\operatorname{inv}(w) := \{(i,j) \,:\, i < j,\, w(i) > w(j)\}$ is the number of inversions in w.

This definition can be generalized to other root systems (also called *types*). Indeed, in the corresponding *Weyl group*, the transpositions s_i are replaced by simple reflections, and $w_\circ$ is replaced by the longest element (*long Weyl element*), see [BH95]. We omit the details which are both standard and well explained in [AF24].

The disadvantage of this definition is a nonobvious combinatorial nature of the coefficients. It is known that $[\boldsymbol{x}^\alpha]\mathfrak{S}_w \in \mathbb{N}$. This was established in [BJS93, FS94], and follows from the following combinatorial definition due to Billey and Bergeron [BB93].

<u>Combinatorial Definition:</u> For a permutation $w \in S_n$, denote by $\operatorname{RC}(w)$ the set of *RC-graphs* (also called *pipe dreams*), defined as tilings of a staircase shape with *crosses* and *elbows* as in the figure below, such that:

(i) curves start in row k on the left and end in column $w(k)$ on top, for all $1 \le k \le n$, and

(ii) no two curves intersect twice.

It follows from these conditions that every $H \in \operatorname{RC}(w)$ has exactly $\operatorname{inv}(w)$ crosses.

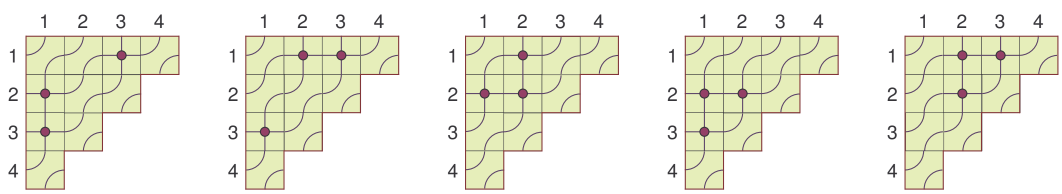

FIGURE 1.2. Graphs in RC(1432) and the corresponding Schubert polynomial $\mathfrak{S}_{1432} = x_1x_2x_3 + x_1^2x_3 + x_1x_2^2 + x_2^2x_3 + x_1^2x_2$ with monomials in this order.

The *Schubert polynomial* $\mathfrak{S}_w \in \mathbb{N}[x_1, x_2, \ldots]$ is defined as

$$\mathfrak{S}_w(\boldsymbol{x}) \;:=\; \sum_{H \in \operatorname{RC}(w)} \boldsymbol{x}^H \quad \text{where} \quad \boldsymbol{x}^H \;:=\; \prod_{(i,j)\,:\,H(i,j)=\boxplus} x_i\,. \tag{1.1}$$

In other words, $\boldsymbol{x}^H$ is the product of x_i's over all crosses $(i,j) \in H$, see Figure 1.2. Note that Schubert polynomials stabilize when fixed points are added at the end, e.g. $\mathfrak{S}_{1432} = \mathfrak{S}_{14325}$. Thus we can pass to the limit $\mathfrak{S}_w$, where $w \in S_\infty$ is a permutation $\mathbb{N} \to \mathbb{N}$ with finitely many nonfixed points.

Permutation $w \in S_n$ is called *Grassmannian* if it has at most one descent. One can show that in this case $\mathfrak{S}_w$ coincides with the Schur function for the partition given by the *Rothe diagram*

$$\operatorname{R}(w) \;:=\; \big\{\big(w(j), i\big) \,:\, i < j,\, w(i) > w(j)\big\} \,\subset \mathbb{N}^2.$$

It follows from this definition that the *Schubert–Kostka numbers* $K_{w,\alpha} := [\boldsymbol{x}^\alpha]\mathfrak{S}_w$ are nonnegative integers, and moreover that they are in $\#\mathsf{P}$ as a counting function. They are not known to be

#P-complete in the natural presentation (this would imply Conjecture 1.2 below). See, however, [ARY21] for other complexity properties of these coefficients.

We refer to [LLS21] for an alternative #P description of the Schubert–Kostka numbers in terms of *bumpless pipe dreams*, and to [GH23] for a poly-time bijection relating two descriptions. Finally, we refer to [ST23] for the generalization of RC-graphs in type B, C, and D. Note that all complexity properties of the Schubert polynomials in type A (defined above), translate verbatim to other types.

1.4. **Schubert coefficients.** It is well known and easy to see that Schubert polynomials $\{\mathfrak{S}_w : w \in S_\infty\}$ form a linear basis in the ring $\mathbb{Z}[x_1, x_2, \ldots]$. *Schubert coefficients* are defined as structure constants:

$$\mathfrak{S}_u \cdot \mathfrak{S}_v \, = \, \sum_{w \in S_\infty} c^w_{u,v} \, \mathfrak{S}_w \, .$$

It is known that $c^w_{u,v} \in \mathbb{N}$ for all $u, v, w \in S_\infty$, as they have both geometric and algebraic meanings [BSY05] which generalize the number of intersection points of lines. Since Schubert polynomials are Schur polynomials for Grassmannian permutations, Schubert coefficients generalize the LR-coefficients.

Denote by SCHUBERT the function which computes Schubert coefficients $\{c^w_{u,v}\}$. The following conjecture remains wide open:

Conjecture 1.2 ([Pak24, §13.4(10)])**.** SCHUBERT *is* #P*-hard.*

This conjecture follow immediately from Conjecture 1.1.[2] It would follow from an even more basic conjecture that counting contingency tables is #P-complete when the input is in unary [Pak24, §13.4(1)].

A major open problem in Algebraic Combinatorics is whether Schubert coefficients have a *combinatorial interpretation* [Sta00, Problem 11]. For various special cases, see [Knu16, Kog01, MPP14] and a detailed discussion in [Pak24, §11.4]. Following [IP22], this can be rephrased as a problem whether SCHUBERT is in #P. We refer to [KZ17, KZ23] for the most general #P-formulas for permutations with at most 3 descents, and for permutations with separated descents (cf. §1.6).

Conjecture 1.3 ([Pak24, Conj. 10.1])**.** SCHUBERT *is not in* #P*.*

The following argument by Alejandro Morales shows that SCHUBERT $\in$ GapP $:=$ #P $-$ #P, see [Pak24, Prop. 10.2]. Let $\sigma \in S_n$ and let $\rho_n := (n-1, \ldots, 1, 0) \in \mathbb{N}^n$. Define

$$\Phi(\sigma) \, := \, \big\{ (\alpha, \beta, \gamma) \in (\mathbb{N}^n)^3 \, : \, \alpha + \beta + \gamma = \sigma\rho_n \big\}.$$

It was shown by Postnikov and Stanley in [PS09, Cor. 17.13], that Schubert coefficients have a *signed combinatorial interpretation*:

$$c^w_{u,v} \, = \, \sum_{\sigma \in S_n} \, \sum_{(\alpha,\beta,\gamma) \, \in \, \Phi(\sigma)} \operatorname{sign}(\sigma) \, K_{u,\alpha} \, K_{v,\beta} \, K_{w,\gamma} \, . \tag{1.2}$$

Since the Schubert–Kostka numbers $\{K_{u,\alpha}\}$ are in #P by (1.1), this implies that Schubert coefficients are in GapP. An alternative proof which generalizes to other root systems, is given by the authors in [PR24a, Thm 1.4].

[2]The argument in [MQ17, p. 885] claiming that $\{c^w_{uv}\}$ is #P-hard via reduction to the LR-coefficients is erroneous as it conflates different input sizes.

1.5. **Main results.** The *Schubert vanishing problem* is a decision problem

$$\textsc{SchubertVanishing} := \{c^w_{u,v} =^? 0\}.$$

This problem is also heavily studied and resolved in special cases, particularly in type A. We postpone an extensive discussion of the prior work until §1.6.

We now proceed to state the main result of the paper. Recall the complexity class $\mathsf{AM} \subseteq \Pi^{\mathrm{p}}_2$ of decision problems that can be decided in polynomial time by an *Arthur–Merlin protocol* with two messages, see e.g. [AB09, Pap94]. See §3.3 for connections to other complexity classes.

Theorem 1.4 (Main theorem). *$\textsc{SchubertVanishing}$ is in coAM assuming* GRH. *The result holds for the vanishing of Schubert coefficients in types A, B, C and D.*

Here the GRH stands for the *Generalized Riemann Hypothesis*, that all nontrivial zeros of L-functions $L(s, \chi_k)$ have real part $\frac{1}{2}$. In fact, tracing back the references shows that a weaker assumption, the *Extended Riemann Hypothesis* (ERH) suffices. We stick with the GRH as better known, and refer to [BCRW08, §6] for definitions and relationships between these hypotheses, and to [Roj07] for discussion of an even weaker number-theoretic assumption.

Main Theorem 1.4 proves the result for all classical types.[3] The theorem has several far-reaching complexity implications, see §2.2. Notably, the theorem shows that $\textsc{SchubertVanishing}$ is in the polynomial hierarchy PH (assuming GRH). This was out of reach until now.

Indeed, until recently, $\textsc{SchubertVanishing} \subseteq \mathsf{PSPACE}$ was the only known upper bound. Morales's observation above gives $\textsc{SchubertVanishing} \in \mathsf{C_{=}P}$. By itself this does not suggest that $\textsc{SchubertVanishing} \in \mathsf{PH}$. Indeed, Tarui's theorem implies that $\mathsf{C_{=}P}$ is not in PH unless PH collapses to a finite level: $\mathsf{PH} = \Sigma^{\mathrm{p}}_m$ for some m, see §3.2.

To contrast the Main Theorem, we make a conjecture in the opposite direction:

Conjecture 1.5. *$\textsc{SchubertVanishing}$ is not in coNP.*

We have mixed feelings about this conjecture. On the one hand, Main Theorem 1.4 suggests that Conjecture 1.5 is false, see §2.2(3). On the other hand, note that if $\textsc{Schubert} \in \#\mathsf{P}$, then $\textsc{SchubertVanishing} \in \mathsf{coNP}$. Thus, Conjecture 1.5 implies Conjecture 1.3.

Conjecture 1.6 (cf. [ARY19, §4]). *$\textsc{SchubertVanishing}$ is coNP-hard.*

This is a decision counterpart of Conjecture 1.2.

1.6. **Prior work on the Schubert vanishing problem.** The literature on the vanishing of Schubert coefficients and its various extensions is too extensive to be fully reviewed. Below are some highlights that are most relevant from the complexity theoretic point of view. Although many of these conditions extend to all types, we restrict our presentation to type A, which is also best studied. We refer to [SY22, §5] for a comparison of some of these conditions from a combinatorial point of view.

1.6.1. *Poly-time conditions.* It follows immediately from the algebraic definition via divided differences, that $c^w_{u,v} = 0$ when $\operatorname{des}(w) > \operatorname{des}(u) + \operatorname{des}(v)$. In [Knu01], Knutson shows that $c^w_{u,v} = 0$ when $\operatorname{Des}(u) \cap \operatorname{Des}(v) \cap \operatorname{Des}(ww_\circ) \neq \varnothing$ (see also [PW24, Cor. 4.15]). The apparent asymmetry disappears in view of Equation (4.2) below. In both cases, the sufficient conditions for Schubert vanishing can be trivially verified in polynomial time.

Another simple sufficient condition for Schubert vanishing is given by $u \not\leqslant w$, where "$\leqslant$" is the *strong Bruhat order*. This condition follows immediately from the algebraic definition (see e.g.

[3] The first version of this paper claimed the result only for types A, B and C, but not for D, see [PR24b]. For non-classical types E_6, E_7, E_8, F_4 and G_2, there is only a finite number of Schubert coefficients, so the problem is computationally uninteresting.

[SY22, §5.1]). Famously, verifying that $u \not\leqslant w$ can be done in polynomial time via *Ehresmann's tableau condition*, see e.g. [Man01, Prop. 2.1.11].

In [SY22, §1.2, §4.3], St. Dizier and Yong gave a necessary condition for nonvanishing of Schubert coefficients in terms of certain fillings of Rothe diagrams of permutations u, v, w with weights given by permutations satisfying additional inequalities. Using [ARY19], the authors then prove [SY22, Thms. A and B], that the existence of such tableaux can be decided in polynomial time, thus giving polynomial time sufficient conditions for the vanishing of Schubert coefficients.

Most recently, [HW24, Cor. 5.12] showed that $\zeta(w) > \zeta(u) + \zeta(v)$ implies that $c^w_{u,v} = 0$, where $\zeta(w)$ is the number of nonzero rows in the Rothe diagram $\mathrm{R}(w)$. This gives yet another simple sufficient condition for vanishing of Schubert coefficients.

1.6.2. NP *conditions.* Every time there is a combinatorial interpretation of Schubert coefficients is given in a special case, this can be viewed as a sufficient condition for non-vanishing. Formally, a $\#\mathsf{P}$ formula for $c^w_{u,v}$ gives an NP sufficient condition for deciding $\{c^w_{u,v} >^? 0\}$, since a single combinatorial object counted by the formula gives a polynomial witness for positivity.

As we mentioned above, Knutson and Zinn-Justin [KZ17] gave a remarkable combinatorial interpretation for Schubert coefficients $c^w_{u,v}$ where $\mathrm{des}(u), \mathrm{des}(v), \mathrm{des}(w) \leq 3$. Similarly, in a followup paper [KZ23], the authors gave a combinatorial interpretation for the case $\mathrm{Des}(u) \cap \mathrm{Des}(v) = \varnothing$, and, more generally, for $\big|\mathrm{Des}(u) \cap \mathrm{Des}(v)\big| \leq 3$. Both cases contain Grassmannian permutations as special case, and thus give a far-reaching generalizations of LR-coefficients. Since computing the set of descents is in P, the Schubert vanishing problem $\{c^w_{u,v} =^? 0\}$ is in NP in these cases.

In [Pur06], Purbhoo presents two sufficient conditions: one for vanishing and one for non-vanishing of Schubert coefficients. These conditions are given combinatorially, in terms of what he calls *root games* which can be viewed as certain sequences of subsets of elements of the poset of positive roots. In type A, the poset is isomorphic to the shifted staircase. The sequence start at *initial position* and end with a *winning position*, with steps given by simple moves.

The sufficient condition for vanishing is given by a *doomed position* where no move can be made [Pur06, Thm 3.6], a property verifiable in polynomial time (cf. [Sea15, Thm 4.1.6]). A sufficient condition for non-vanishing is given by a *winning root game* [Pur06, Thm 3.7]. Since each winning game can be verified in P, this sufficient condition is in NP.

Finally, in [BV08, Thm 5.1], Billey and Vakil study the *permutation arrays* introduced by Eriksson and Linusson [EL00a, EL00b]. These permutation arrays are subsets of $\{1, \ldots, n\}^4$ with certain poly-time checkable conditions. The authors used elimination theory to give a necessary condition for non-vanishing of Schubert coefficients $c^w_{u,v} > 0$ in terms of a unique permutation array with certain properties. Existence of such permutation array can be verified in polynomial time, making this necessary condition in NP (uniqueness is a byproduct of the construction). See also [AB07, Prop. 9.7] by Ardila and Billey, for an extension of this rule.

1.6.3. *Algebraic conditions.* In classical types, a necessary and sufficient condition for vanishing of Schubert coefficients was given by Purbhoo [Pur06] (see also [Bel06, Pur04]). These are linear algebraic conditions, which can be viewed as a polynomial sized system of complex algebraic equations where some variables are assumed to be generic. An important feature of this approach is the dual nature of the system, as it gives an algebraic certificate for vanishing rather than non-vanishing. We give a quick description of this system in Lemma B.7 in the appendix, as we use it to give complexity analysis of SchubertVanishing in the (nonstandard) BSS model of computation.

A more direct description of an algebraic system is given by Billey and Vakil in [BV08, Thm 5.4], which has exactly $c^w_{u,v}$ solutions for a generic values of some variables. They also describe the system of conditions for these variable being generic under assumption that the set of solutions is 0-dimensional [BV08, Cor. 5.5]. The authors do not give a complexity analysis for this system and our approach does not apply; see §8.1 for further details and a complexity discussion.

1.7. **Hilbert's Nullstellensatz.** Let $R = \mathbb{C}[x_1, \ldots, x_s]$ for some $s > 0$. *Hilbert's weak Nullstellensatz* is a fundamental result in Algebra, which states that a polynomial system

$$f_1 \, = \, \ldots \, = \, f_m \, = \, 0 \quad \text{where} \quad f_i \in R \tag{1.3}$$

has no solutions over $\mathbb{C}$ if and only if there exist $(g_1, \ldots, g_m) \in R^m$, such that

$$\sum_{i=1}^{m} f_i \, g_i \, = \, 1 \, .$$

Now let $f_1, \ldots, f_m \in \mathbb{Z}[x_1, \ldots, x_s]$. The decision problem HN (*Hilbert's Nullstellensatz*), asks if the polynomial system (1.3) has a solution over $\mathbb{C}$.[4] Here and everywhere below, the *size* of the polynomial system is the sum of bit-lengths of the coefficients in the polynomials f_i.

Mayr and Meyer [MM82] showed that HN is in $\mathsf{EXPSPACE}$, and is also NP-hard. In a series of major algebraic results [Kol88, KPS01, Jel05], it was shown that one can take g_i of single exponential size. This in turn reduces HN to a single-exponential-sized system of linear equations, and implies that HN is in PSPACE. This is a major unconditional result. It was not known if HN is in the polynomial hierarchy until Koiran's breakthrough result:

Theorem 1.7 ([Koi96, Thm 2])**.** HN *is in* AM *assuming* GRH.

For the proof, Koiran's needs existence of primes in certain intervals and with modular conditions, thus the GRH assumption. For the proof of Theorem 1.4, we need the following strengthening of Theorem 1.7 to finite algebraic extensions. Let

$$f_1, \ldots, f_m \, \in \, \mathbb{Z}(y_1, \ldots, y_k)[x_1, \ldots, x_s] \, .$$

The decision problem HNP (*Parametric Hilbert's Nullstellensatz*) asks if the polynomial system (1.3) has a solution over $\overline{\mathbb{C}(y_1, \ldots, y_k)}$. In a remarkable recent work, Ait El Manssour, Balaji, Nosan, Shirmohammadi, and Worrell extended Theorem 1.7 to HNP :

Theorem 1.8 ([A+24, Thm 1])**.** HNP *is in* AM *assuming* GRH.

The proof of the authors is rather interesting as it substantially simplified and "algebraized" Koiran's original approach in [Koi96, Koi97], avoiding the use of semi-algebraic geometry. The background behind HNP is also very interesting, as are its computational aspects. We refer the reader to a [A+24] for an extensive discussion of this and other related work.

1.8. **Schubert vanishing via Hilbert's Nullstellensatz.** We prove Theorem 1.4 as an application of Theorem 1.8. More precisely, we prove the following:

Lemma 1.9 (Main lemma)**.** $\neg$SchubertVanishing *reduces to* HNP *in types A, B, C and D.*

The lemma, combined with Theorem 1.8 immediately implies Theorem 1.4. Let us emphasize that the vanishing of Schubert coefficients in type D stated in Theorem 1.4, but is not included in the main part of the paper. The reason is that the reduction in proof of the lemma is both delicate and technical, and does not go through in type D, see Remark A.2.

In type D, we obtain the result in Lemma C.1. For the proof, we introduce a different approach in the Appendix C (joint with David Speyer). To clarify the strength of this new approach, we present the construction uniformly for all classical types.

[4] By the Nullstellensatz, this is equivalent to asking if there is a solution over $\overline{\mathbb{Q}}$.

1.9. **Proof outline.** The proof of the Lemma 1.9 is somewhat technical and based on prior constructions. But at the end, we produce an explicit polynomial system (1.3) with polynomially many parameters.

Our starting point is an equivalent algebro-geometric definition of Schubert coefficients, arguably the closest to Schubert's original work. We use *Kleiman's transversality theorem* [Kle74] to interpret Schubert coefficients $\{c^w_{u,v}\}$ as counting the number of points in the intersection of corresponding generically translated Schubert varieties. This is a well-known approach, both in Algebraic Geometry [AF24] and in computational applications [HS17].

From there, the main issue is to formulate the transversality condition by adding new variables (parameters) and explicit polynomial equations. Following the original approach in [HS17] in type A, we obtain and analyze the so called *lifted formulations* for each type. These polynomial systems are relatively clean in type A, see Section 5 and an example in §5.4. In Section 6, we present lifted formulations in types B and C.

A lifted formulation in type D similar to that in other classical types is given in Appendix A. Unfortunately, that formulation does not prove Lemma 1.9 in type D. This is because one of the equations is an $n \times n$ determinant evaluation: $\det(X) = 1$, and which has exponentially many terms. Recall that systems in HN and HNP must have polynomially many equations, each with polynomially many terms; See also §8.2 and Remark A.2. Still, the systems constructed in Appendix A turned out to be useful for results in Appendix B.

In Appendix C, we prove Lemma 1.9 in type D (see Lemma C.1). We present an alternative construction of lifted formulations in all classical types, including type D. This section is joint with David Speyer.

Without going to details, lifted formulations define membership in a Schubert variety as a system of bilinear equations. They are more efficient than the primal–dual formulation in [HHS16] (cf. [L+21]). It is a minor miracle that one can avoid determinantal equations and obtain the desired polynomial size systems of polynomials in the three classical types as in the lemma. Finally, we note that we are using the conventions in [AIJK23] which are different from those in [HS17].

2. Discussion and implications

To clarify the importance of our results, let us recall three somewhat related problems.

2.1. **Comparison to other problems.** To better explain how surprising the results are in this paper, let us recall complexity results on related problems.

(i) ***Characters of the symmetric group.*** Let χ^λ denote irreducible characters of the symmetric group S_n, where $\lambda \vdash n$ is a partition of n. They play the same role that Schur functions play in the representation theory of $\mathrm{GL}_n(\mathbb{C})$. The character values $\chi^\lambda(\mu)$, where $\lambda, \mu \vdash n$, are integers. As a function, χ is GapP-complete even when the input is in unary, see [Hep94].

In [IPP24], Ikenmeyer, Pak and Panova showed that squared characters $\big(\chi^\lambda(\mu)\big)^2$ are not in $\#\mathsf{P}$ unless $\mathsf{PH} = \Sigma_2^{\mathsf{p}}$. This is the analogue of Conjecture 1.3. This was done by proving that the *character vanishing problem* $\{\chi^\lambda(\mu) =^? 0\}$ is $\mathsf{C_=P}$-complete. By Tarui's theorem (see §3.2), this shows that the character vanishing problem is not in PH, unless PH collapses to a finite level.

(ii) ***Defect of Stanley's poset inequality.*** Let $P = (X, \prec)$ be a finite poset on n elements, and let $x, z_1, \ldots, z_k \in X$ be fixed distinct elements. Fix distinct integers $c_1, \ldots, c_k \in \{1, \ldots, n\}$. *Linear extensions* of P are order-preserving bijections $f : X \to \{1, \ldots, n\}$. They generalize *standard Young tableaux* of both straight and skew shape.

Let $N(a)$ denote the number of linear extensions f of P, s.t. $f(z_i) = c_i$ and $f(x) = a$. Famously, Stanley's poset inequality gives log-concavity of $\{N(a)\}$:

$$N(a)^2 \,\geq\, N(a+1) \cdot N(a-1).$$

Denote $\delta(a) := N(a)^2 - N(a+1)\cdot N(a-1)$ the defect of this inequality. It follows from [BW91] that δ is GapP-complete.

Similarly to the example above, for $k \geq 2$, it was shown by Chan and Pak [CP24a], that δ is not in $\#\mathsf{P}$ unless $\mathsf{PH} = \Sigma_2^{\mathsf{P}}$. This was done by proving that the *Stanley Vanishing problem* $\{\delta(a) =^? 0\}$ is not in PH unless PH collapses to the finite level. In particular, the authors showed that the Stanley Vanishing problem is not in coNP unless $\mathsf{PH} = \Sigma_2^{\mathsf{P}}$, similar to the claim in Conjecture 1.5. See also [CP24b, CP24c] for a similar approach to a other *coincidence problems*.

Rather remarkably, it remains open if the Stanley Vanishing problem is $\mathsf{C_=P}$-complete for $k \geq 2$, as the proof bypasses this problem. Curiously, and in the opposite direction, for $k \in \{0,1\}$ it was shown in [SvH23] and [CP24a] that the Stanley vanishing problem is in P. *Pak's Conjecture* states that the defect δ is not in $\#\mathsf{P}$ in these cases as well [Pak24, Conj. 6.3]. At the moment, there are no known tools to establish this result.

(*iii*) ***Kronecker coefficients.*** Let $\lambda, \mu, \nu \vdash n$ be partitions of n. Denote by $g(\lambda,\mu,\nu)$ the *Kronecker coefficients* defined as follows:

$$g(\lambda,\mu,\nu) \,:=\, \langle \chi^\lambda \chi^\mu, \chi^\nu \rangle \,=\, \frac{1}{n!} \sum_{\sigma\in S_n} \chi^\lambda(\sigma)\, \chi^\mu(\sigma)\, \chi^\nu(\sigma)\,.$$

By definition, $g(\lambda,\mu,\nu) \in \mathbb{N}$. Kronecker coefficients are the structure constants in the ring of S_n characters, and thus play the same role that LR-coefficients play for $\mathrm{GL}_n(\mathbb{C})$ characters.

Kronecker coefficients are closely related to Quantum Computing and appear naturally in Geometric Complexity Theory, see e.g. [Aar16, Mul09]. They can also be defined for all types and are known to be hard to compute and to analyze [CDW12, PP17].

Finding a combinatorial interpretation for Kronecker coefficients remains a major open problem in Algebraic Combinatorics first posed by Murnaghan in 1938, see [Sta00, Problem 10]. It was conjectured by the first author [Pak24, Conj. 9.1], that g is not in $\#\mathsf{P}$, a conjecture that remains wide open.

It was shown by Ikenmeyer, Mulmuley and Walter in [IMW17], that g is $\#\mathsf{P}$-hard, while the Kronecker Vanishing Problem $\{g(\lambda,\mu,\nu) =^? 0\}$ is coNP-hard, even when the input is in unary. In a different direction, it was shown by Ikenmeyer and Subramanian [IS23], that g is in $\#\mathsf{BQP}$. Similarly, the vanishing problem was shown in coQMA by Bravyi et al. [B+24]. As pointed out in [IS23, Rem. 1.2], these results suggest that proving the Kronecker Vanishing problem is $\mathsf{C_=P}$-complete (as was done in [IPP24] for characters), would be difficult. Indeed, the authors note that this would imply $\mathsf{NQP} \subseteq \mathsf{QMA}$, an open problem in Quantum Computing [KMY09], cf. §3.3.

2.2. Connections and implications.

There are several ways to think of results in the paper.

(0) Most straightforwardly, we showed that the vanishing of Schubert coefficients is in coAM, i.e. relatively low in the polynomial hierarchy (Theorem 1.4). If one assumes Conjecture 1.5, this upper bound is close to optimal.

(1) Looking at the complicated sign-reversing structure of (1.2) and assuming that the Schubert–Kostka numbers are $\#\mathsf{P}$-complete (cf. Conjecture 1.2), one would naturally assume positive and negative terms are "independent" $\#\mathsf{P}$-functions, cf. §2.1(*i*). This would imply that the Schubert vanishing problem is $\mathsf{C_=P}$-complete, and prove Conjecture 1.6. By Tarui's theorem (see §3.2), this would imply that SchubertVanishing $\notin \mathsf{PH}$ unless PH collapses to a finite level. Main Theorem 1.4 implies that this independence assumption is flawed and that in fact SchubertVanishing $\in \mathsf{PH}$ (assuming GRH).

(2) Another way to think about our results is an effort to show that Conjecture 1.3 *cannot be proved* by showing that the Schubert vanishing problem is not in PH. This would be a natural approach to the problem, similar to that in (*i*) and (*ii*) discussed above. All of this is assuming both the GRH and a non-collapse of PH, of course. This is a much stronger argument than the implication of an open problem in quantum computing discussed at the end of §2.1(*iii*).

Note that the *relativization argument* employed in [IP22] to prove that some functions are not in #P is inapplicable in this setting. Thus, Theorem 1.4 implies that there are currently no other tools to attack Conjecture 1.3. The same applies to Pak's conjecture in §2.1(*ii*), that the defect of Stanley's inequality for $k = 0$, is not in #P.

(3) There is a plausible argument in favor of the *derandomization assumption* $\mathsf{BPP} = \mathsf{P}$, see [Wig23] partly motivated by [IW97].[5] Several versions of this assumption imply that $\mathsf{AM} = \mathsf{NP}$ [KvM02, MV05], and disprove Conjecture 1.5.[6] As we mentioned above, this by itself does not disprove Conjecture 1.3; cf. discussion of Pak's conjecture in §2.1(*ii*). In the opposite direction, there are also some important indications against derandomization assumptions, see e.g. discussion in [SY09, §4.3]. Notably, it is unlikely that $\mathrm{HN} \in \mathsf{NP}$ as discussed in [Koi96, §6] (preprint version).

(4) Although the underlying algebraic properties of Kronecker and Schubert coefficients are unrelated, they both generalize LR–coefficients whose vanishing problem is in P. There are other connections between complexity classes of best upper bounds QMA and AM for the vanishing problems of Kronecker and Schubert coefficients, respectively (see Bravyi et al. [B+24] and our Theorem 1.4). For example, if $\mathsf{BQP} \subseteq \mathsf{AM}$ then $\mathsf{PH} = \Sigma_2^{\mathrm{p}}$ [Aar10]. Of course, $\mathsf{BQP} \subseteq \mathsf{QMA}$ are not believed to be in PH, see e.g. [RT22].

(5) Main Theorem 1.4 also shows that SchubertVanishing is not NP-complete (assuming GRH), as this would imply a collapse in the polynomial hierarchy to the second level: $\mathsf{PH} = \Sigma_2^{\mathrm{p}}$ [BHZ87, Thm 2.3].

3. Background in computational complexity

We refer to [AB09, Pap94] for the definitions and standard results, and to [Aar16, Wig19] for further background. For the BSS computational model and basic results on the corresponding polynomial and counting hierarchies, see [BCSS98, BC13]. Below we give a somewhat informal and incomplete reminder of standard computational complexity classes.

3.1. **Basic notions and the polynomial hierarchy.** The notion of a *combinatorial object* can be viewed as follows. A *word* is a binary sequence $x \in \{0,1\}^*$. The *size* of x is the length $|x|$. In other words, combinatorial objects of size N are encoded by words of length N, which in turn correspond to integers $0 \leq a < 2^N$.

The *language* as a subset of words: $L \subseteq \{0,1\}^*$. For example, the language of words which encode all Hamiltonian graphs can be defined as collection of pairs (G, C) where $G = (V, E)$ is a simple graph, $E \subseteq \binom{E}{2}$, and $C \subseteq E$ is a Hamiltonian cycle in G. Language L is called a *polynomial* if the membership problem $x \in^? L$ can be decided in time polynomial in the size of x (*poly-time*).

Class P is a class of decision problems which can be decided in polynomial time. For example, deciding if a graph is connected is in P. Class PSPACE is a class decision problems which can be decided in polynomial space.

Fix a polynomial language L. Class NP is a class of problems of the type: given prefix y, does there exist suffix z, s.t. $yz \in L$? For example, given a simple graph $G = (V, E)$, the problem whether G has a Hamiltonian cycle is in NP. Class coNP is a class of problems opposite to NP, i.e. given prefix y, is it true that for all z, we have $yz \notin L$? For example, deciding if graph G is non-Hamiltonian is in coNP. To see this, rephrase the problem in the positive: decide if for every subset of edges $C \subseteq E$, we have C is *not* a Hamiltonian cycle in G.

Polynomial hierarchy PH is a union of classes Σ_m^{p} and Π_m^{p} of the form

$$\exists x_1 \, \forall x_2 \, \exists x_3 \, \ldots \, \exists(\forall) x_m \ : \ x_1 x_2 x_3 \cdots x_m \in^? L, \quad \text{and}$$
$$\forall x_1 \, \exists x_2 \, \forall x_3 \, \ldots \, \forall(\exists) x_m \ : \ x_1 x_2 x_3 \cdots x_m \in^? L,$$

[5]The poll in [Gas19] reports that about 98% of "experts" and about 60% of "non-experts" believe that $\mathsf{P} = \mathsf{BPP}$. Since the problem has been open for over 60 years, it is worth considering all possibilities.

[6]We explore this direction in a short followup paper [PR24c] written from a combinatorial point of view.

respectively. It is important here that m is fixed. Note that $\Sigma_1^{\rm p} = \mathsf{NP}$, $\Pi_1^{\rm p} = \mathsf{coNP}$, and $\mathsf{PH} \subseteq \mathsf{PSPACE}$. It is known that $\mathsf{BPP} \subseteq \Sigma_2^{\rm p} \cap \Pi_2^{\rm p}$ and thus very low in the polynomial hierarchy.

For a class X, we write X-complete for the class of *complete problems* in X, i.e. problems A $\in \mathsf{X}$ such that every problem B $\in \mathsf{X}$ reduces to A. Not all classes are known to have complete problems. Famously, 3SAT and HAMILTONICITY are NP-complete. For two classes X and Y, we write X^{Y} for a class of problems which can be solved by a solver in X with an oracle Y.

3.2. **Counting problems.** Fix a polynomial language L. Class $\#\mathsf{P}$ is a class of problems of the type: compute the number of suffixes z with a given prefix y, such that $yz \in L$. For example, compute the number of Hamiltonian cycles C in a given simple graph G. Class PP is the corresponding class of decision problems, whether the number above is greater than a given integer k.

Class FP is a subclass of $\#\mathsf{P}$ of counting problems which can be solved in poly-time. Class $\mathsf{FPSPACE}$ is a class of counting problem which can be solved in polynomial space. Clearly, $\mathsf{FP} \subseteq \#\mathsf{P} \subseteq \mathsf{FPSPACE}$. Counting hierarchy CH is a union of classes of decision problems corresponding to classes $\Sigma^{\rm p}$, whether the number of solutions is $\geq k$. Clearly, $\mathsf{PP} \subseteq \mathsf{CH} \subseteq \mathsf{PSPACE}$.

A $\#\mathsf{P}$ *function* $f : \{0,1\}^* \to \mathbb{N}$ is a counting function corresponding to a problem in $\#\mathsf{P}$. We write $f \in \#\mathsf{P}$ in this case. Let $\mathsf{GapP} := \#\mathsf{P} - \#\mathsf{P}$ denote set of differences of such functions, and let $\mathsf{GapP}_{\geq 0}$ denote the set of nonnegative functions in GapP. For example, Schubert coefficients, Kronecker coefficients and the defect of Stanley's inequality (see §2.1), are all in $\mathsf{GapP}_{\geq 0}$.

Toda's theorem, see e.g. [AB09, Gol08, Pap94], says that $\mathsf{PH} \subseteq \mathsf{P}^{\#\mathsf{P}}$. It implies that a $\#\mathsf{P}$-complete function is not in PH unless $\mathsf{PH} = \Sigma_m^{\rm p}$ for some m, i.e. the polynomial hierarchy *collapses*. *Tarui's theorem* [Tar91] states that $\mathsf{PH} \subseteq \mathsf{NP}^{\mathsf{C}_=\mathsf{P}}$. The proof is an easy argument based on Toda's theorem, see e.g. [IP22].

Complexity class $\mathsf{C}_=\mathsf{P}$ is the class of decision problems whether two $\#\mathsf{P}$-functions have equal values on two given inputs. Clearly, $\mathsf{coNP} \subseteq \mathsf{C}_=\mathsf{P} \subseteq \mathsf{PSPACE}$ where the first inclusion follows since one can take $g \equiv 0$. For a function $f \in \#\mathsf{P}$, the corresponding *coincidence problem* is defined as

$$\mathrm{C}_f \ := \ \big\{ f(x) =^? f(y) \ : \ x, y \in \{0,1\}^* \big\}.$$

Clearly, $\mathrm{C}_f \in \mathsf{C}_=\mathsf{P}$. It follows from Tarui's theorem, that if C_f is $\mathsf{C}_=\mathsf{P}$-complete, then we have $\big(f(x) - f(y)\big)^2 \notin \#\mathsf{P}$ unless $\mathsf{PH} = \Sigma_m^{\rm p}$ for some m, see [CP24b, IP22].

3.3. **More involved complexity classes.** Class BPP is defined as a class of decision problems which can be decided in *probabilistic polynomial time*. In other words, in poly-time the problem $\{x \in^? L\}$ can be decided with probability of error at most $\frac{1}{3}$. This probability can be amplified by a repeated application of the algorithm if necessary.

If BPP is a probabilistic version of P, then $\exists{\cdot}\mathsf{BPP}$ is defined as probabilistic version of NP. In this case the decision problem asks if there is a word in the BPP *language*, i.e. a language whose membership problem $\{x \in^? L\}$ is in BPP.

One can view complexity problems as *interactive proofs*, with quantifiers describing communications between prover and verifier. When we have a BPP verifier, i.e. when the verifier (Arthur) has powers to flip coins and the prover (Merlin) has unlimited computational powers. such communication is called the *Arthur–Merlin protocol*. The number of quantifiers becomes the number of messages between the prover and the verifier.

The complexity class AM is a class of decision problems that can be decided in polynomial time by an Arthur–Merlin protocol with two messages, see e.g. [AB09, §8.2]. Recall the inclusions

$$\mathsf{NP} \ \subseteq \ \exists{\cdot}\mathsf{BPP} \ \subseteq \ \mathsf{MA} \ \subseteq \ \mathsf{AM} \ \subseteq \ \Pi_2^{\rm p} \ \subseteq \ \mathsf{PH}.$$

Here MA is a standard but slightly different Arthur–Merlin protocol with two messages that we will not need. Famously, *graph isomorphism* is in coAM, since graph non-isomorphism can be established by a simple interactive protocol (see e.g. [AB09, Thm 8.13]).

Finally, in Quantum Computing, one considers *quantum Turing machines*. Class BQP is the class of decision problems which can be solved in polynomial time on such machines with probability of error at most $\frac{1}{3}$. This class should be viewed as analogous to BPP. Similarly, class QMA is defined as analogous to class MA. Finally, class NQP is a quantum analogue of NP, and it is known that $\mathsf{NQP} = \mathsf{coC_{=}P}$ [FGHP98].

4. Definitions, notation, and geometric setup

4.1. **Standard notation.** We use $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $[n] = \{1, \ldots, n\}$. We use $[a, b]$ to denote the interval $\{a, a + 1, \ldots, b\} \subset \mathbb{Z}$. From this point on, unless stated otherwise, the underlying field is always $\mathbb{C}$.

We use $e_1, \ldots, e_n$ to denote the standard basis in $\mathbb{C}^n$, and $\mathbf{0}$ to denote zero vector. We use bold symbols such as $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ to denote sets and vectors of variables, and bars such as $\overrightarrow{x}$ and $\overrightarrow{y}$, to denote complex vectors. We say that $A = (a_{ij})$ is *symmetric* if $a_{ij} = a_{ji}$ for all $i, j \in [n]$. Similarly, we say that $A = (a_{ij})$ is *skew-symmetric* if $a_{ij} = -a_{ji}$ for all $i, j \in [n]$.

Recall the following standard notation for *complex reductive simple Lie groups*. We have the *general linear group* $\mathsf{GL}_n(\mathbb{C})$, the *odd special orthogonal group* $\mathsf{SO}_{2n+1}(\mathbb{C})$, the *symplectic group* $\mathsf{Sp}_{2n}(\mathbb{C})$ and the *even special orthogonal group* $\mathsf{SO}_{2n}(\mathbb{C})$. These groups correspond to *root systems* A_n, B_n, C_n and D_n, respectively, and are called *groups of type* A, B, C and D, respectively.

To distinguish the types, we use parentheses in Schubert coefficients, e.g. $c^w_{u,v}(A)$, and superscript in other cases, e.g. X^A_u. We omit the dependence on the type when it is clear from the context. We use bullets (both in subscript and superscript) to denote flags, e.g. $F_\bullet$ and $E^\bullet$.

4.2. **Geometric and Algebraic Combinatorics notation.** Throughout the paper we follow conventions of [AF24, AIJK23]. We assume the reader is familiar with some basic notions and results in Algebraic Geometry, Lie theory and specifically Schubert theory. Below we recall some key notation that we use throughout the paper.

Let G be a *complex reductive Lie group*. Take $\mathsf{B} \subset \mathsf{G}$ and $\mathsf{B}_- \subset \mathsf{G}$ to be the *Borel subgroup* and *opposite Borel subgroup*, respectively. The *torus subgroup* is defined as $\mathsf{T} = \mathsf{B} \cap \mathsf{B}_-$. The *Weyl group* is defined as the normalizer $\mathcal{W} \cong N_{\mathsf{G}}(\mathsf{T})/\mathsf{T}$. Finally, the celebrated *Bruhat decomposition* says that

$$\mathsf{G} \ = \ \bigsqcup_{w \in \mathcal{W}} \mathsf{B}_- \dot{w} \mathsf{B},$$

where $\dot{w}$ is the preimage of w in the normalizer $N_{\mathsf{G}}(\mathsf{T})$.

The *generalized flag variety* is defined as G/B. Recall that G/B has finitely many orbits under the left action of B_-. These are called *Schubert cells* and denoted Ω_w. Schubert cells are indexed by the Weyl group elements $w \in \mathcal{W}$. See [MT11] for additional background.

The *Schubert varieties* X_w are the Zariski closures of Schubert cells Ω_w. The *Schubert classes* $\{\sigma_w\}_{w \in \mathcal{W}}$ are the Poincaré duals of Schubert varieties. These form a $\mathbb{Z}$–linear basis of the cohomology ring $H^*(\mathsf{G}/\mathsf{B})$. The *Schubert coefficients* $c^w_{u,v}$ are defined as structure constants:

$$\sigma_u \smile \sigma_v \ = \ \sum_{w \in \mathcal{W}} c^w_{u,v} \sigma_w \, . \tag{4.1}$$

The fact that in type A these constants coincide with the definition in §1.4, is proved in [LS82]. This definition easily implies the S_3-symmetries of Schubert coefficients:

$$c^{w_\circ w}_{u,v} \ = \ c^{w_\circ w}_{v,u} \ = \ c^{w_\circ v}_{u,w} \, . \tag{4.2}$$

By the *Kleiman transversality* [Kle74], coefficients $c^w_{u,v}$ count the number of points in the intersection of generically translated Schubert varieties:

$$c^w_{u,v} \ = \ \#\bigl\{X_u(E_\bullet) \cap X_v(F_\bullet) \cap X_{w_\circ w}(G_\bullet)\bigr\}, \tag{4.3}$$

where $E_\bullet$, $F_\bullet$ and $G_\bullet$ are generic flags. Here $w_\circ$ is the *long word* in $\mathcal{W}$, see [AF24, Section 15.1] for details.

5. Lifted formulation in type A

5.1. **Definitions.** In type A, we have $\mathsf{G} := \mathsf{GL}_n(\mathbb{C})$. The Weyl group $\mathcal{W} = S_n$ is the symmetric group on n letters. We also have $\mathsf{G}/\mathsf{B} = \mathcal{F}l_n$ is the *complete flag variety*. Here $\mathcal{F}l_n$ consists of flags

$$F_\bullet \,:=\, \mathbf{0} \subset F_1 \subset \cdots \subset F_n \,=\, \mathbb{C}^n \quad \text{where} \quad \dim F_i = i\,.$$

Fix the *coordinate flag* $E^\bullet$ given by $E^i := \langle e_n, \ldots, e_{i+1}\rangle$. We explicitly define Schubert cells as

$$\Omega_w^A(E^\bullet) \,=\, \big\{F_\bullet \,:\, \dim(F_i \cap E^j) = k_w(i,j) \text{ for } i,j \in [n]\big\},$$

where

$$k_w(i,j) \,:=\, \#\big\{r \in [i] \,:\, w_r > j\big\} \quad \text{for all} \quad 1 \le i,j \le n\,.$$

Then the Schubert variety in type A is an orbit closure which can also be defined as

$$X_w^A(E^\bullet) \,=\, \overline{\Omega_w^A}(E^\bullet).$$

Take $F^\bullet := \pi E^\bullet$ and $G^\bullet := \rho E^\bullet$, where $\pi = (y_{ij})$ and $\rho = (z_{ij})$ are matrices of indeterminates. Let $\boldsymbol{y} = \{y_{11}, \ldots, y_{nn}\}$ and $\boldsymbol{z} = \{z_{11}, \ldots, z_{nn}\}$ be the sets of variables in these matrices. Fix three permutations $u, v, w \in S_n$.

We use the system for solving

$$X_u^A(F^\bullet) \,\cap\, X_v^A(G^\bullet) \,\cap\, X_w^A(E^\bullet)$$

as formulated in [HS17, Rem. 2.7]. Since our conventions are opposite those in [HS17], we include the necessary translations here.

Using the classical determinantal formulation of the Schubert problem or an abridged version as in Billey–Vakil [BV08], one can similarly define a system of equations to solve the Schubert problem. However, these may involve exponentially many equations. See [HS17] for further discussion as well as comparisons with other formulations of Schubert systems.

Let $d := \max \mathrm{Des}(w)$ denote the maximal descent of w. The *Stiefel coordinates* $\mathcal{X}_w^A$ for the Schubert cell Ω_w^A are the collection of $n \times d$ matrices (m_{ij}) such that

$$m_{ij} = \begin{cases} 1 & \text{if } i = w_j \\ 0 & \text{if } i < w_j \text{ or } w_i^{-1} < j \\ x_{ij} & \text{otherwise.} \end{cases} \tag{5.1}$$

For convenience, we relabel matrix entries $\{x_{ij}\}$ as $\boldsymbol{x} = \{x_1, \ldots, x_s\}$, where $s = \binom{n}{2} - \mathrm{inv}(w)$, reading across rows, top to bottom. To every $\omega \in \mathcal{X}_w^A$ with entries $\boldsymbol{x}$, we associate the flag given by the span of the column vectors $e_i(\boldsymbol{x})$ of ω. Take

$$\begin{aligned} \boldsymbol{\alpha} \,&:=\, \big\{\alpha_{ij} : j \in [i], \text{ where } u_j < u_i\big\}, \\ \boldsymbol{\beta} \,&:=\, \big\{\beta_{ij} : j \in [i], \text{ where } v_j < v_i\big\}. \end{aligned}$$

For $i \in [d]$, define the n-vectors $g_i(\boldsymbol{x}, \boldsymbol{\alpha})$ and $h_i(\boldsymbol{x}, \boldsymbol{\beta})$ with entries in $\mathbb{Z}[\boldsymbol{x}, \boldsymbol{\alpha}, \boldsymbol{\beta}]$ such that

$$\begin{aligned} g_i(\boldsymbol{x}, \boldsymbol{\alpha}) \,&:=\, e_i(\boldsymbol{x}) \,+\, \sum_{\substack{j \in [i] \\ u_j < u_i}} \alpha_{ij}\, e_j(\boldsymbol{x}), \\ h_i(\boldsymbol{x}, \boldsymbol{\beta}) \,&:=\, e_i(\boldsymbol{x}) \,+\, \sum_{\substack{j \in [i] \\ v_j < v_i}} \beta_{ij}\, e_j(\boldsymbol{x}). \end{aligned}$$

5.2. **Characterization of Schubert varieties.** In this section we outline a key construction of [HS17]. Using the following fact, one may introduce auxiliary variables to characterize membership in the Schubert cell in terms of bilinear equations rather than determinants.

Proposition 5.1 ([HS17, Lemma 2.1]). *Let $\Phi_\bullet \in \mathcal{F}l_n$ be in general position with $E^\bullet$, where $\Phi_i = \langle f_1, \ldots, f_i\rangle$ for all $i \in [d]$. Take $w \in S_n$ with $d = \max \mathrm{Des}(w)$. Then $\Phi_\bullet \in \Omega^A_w(E^\bullet)$ <u>if and only if</u> for each $k \in [d]$, there exist unique $\{\beta_{jk}\}$ such that*

$$g_k \, = \, f_k \, + \sum_{\substack{j\in[k] \\ w_j<w_k}} \beta_{jk}\, f_j \, \in \, E^{w_k-1} \, - \, E^{w_k}\,.$$

Note that the statement of the proposition is translated into our conventions to make it amenable to generalization for other types. We include their proof for completeness.

Proof. Suppose $\Phi_\bullet \in \Omega^A_w(E^\bullet)$. We proceed by induction on k. For the base case, this translates to $g_1 = f_1 \in E^{w_1-1} - E^{w_1}$, which follows since $\Phi_\bullet \in \Omega^A_w(E^\bullet)$. Suppose the result holds for all $k' < k$. Note that $\Phi_{k-1} + (E^{w_k-1} \cap \Phi_k)$ must be 0 or 1-dimensional modulo Φ_{k-1}. Thus there must exist some $f \in \Phi_{k-1}$ such that $f_k + f \in E^{w_k-1}$. Since $\Phi_\bullet \in \Omega^A_w(E^\bullet)$, we further have $f_k + f \in E^{w_k-1} - E^{w_k}$.

Since $f \in \Phi_{k-1}$ and $f = \sum_{j<k} \alpha_j f_j$, by the inductive assumption we have $f = \sum_{j<k} \alpha'_j g_j$. Note that if $w_j > w_k$ and $E^{w_j-1} \subset E^{w_k-1}$, we may assume that

$$f \, = \, \sum_{\substack{j<k \\ w_j<w_k}} \alpha'_j\, g_j\,.$$

Since $g_j \in E^{w_j-1} - E^{w_j}$, we obtain uniqueness of α'_j. Using the inductive assumption, we can now rewrite each such g_j in terms of f_i's, giving

$$g_k \, := \, f_k + f \, = \, f_k \, + \sum_{\substack{j\in[k] \\ w_j<w_k}} \beta_{jk}\, f_j \, \in \, E^{w_k-1} - E^{w_k}\,.$$

This completes the proof. □

5.3. **The construction.** Let d denote the maximal descent among u, v, w :

$$d \, := \, \max\big(\mathrm{Des}(u) \cup \mathrm{Des}(v) \cup \mathrm{Des}(w)\big)\,.$$

We consider $\omega \in \mathcal{X}^A_w$ with column vectors $e_j(\boldsymbol{x})$. By construction, the corresponding flag $\omega^\bullet$ lies in $\Omega^A_w(E^\bullet)$. Then construct $g_i(\boldsymbol{x}, \boldsymbol{\alpha})$ and $h_i(\boldsymbol{x}, \boldsymbol{\beta})$ in terms of $e_j(\boldsymbol{x})$.

By Proposition 5.1, $\omega^\bullet \in \Omega^A_u(F^\bullet)$ if and only if for every $i \in [d]$, $g_i(\boldsymbol{x}, \boldsymbol{\alpha}) \in F^{u_i-1} - F^{u_i}$ for $\boldsymbol{\alpha}$ unique. To impose this condition, we require

$$(y_{j1} \;\; y_{j2} \;\; \cdots \;\; y_{jn}) \cdot g_i(\boldsymbol{x}, \boldsymbol{\alpha}) \, = \, 0, \text{ for each } j < u_i\,. \tag{5.2}$$

Similarly, $\omega^\bullet \in \Omega^A_v(G^\bullet)$ if and only if for every $i \in [d]$, $h_i(\boldsymbol{x}, \boldsymbol{\beta}) \in G^{v_i-1} - G^{v_i}$ for $\boldsymbol{\beta}$ unique. For this last condition, we need

$$(z_{j1} \;\; z_{j2} \;\; \cdots \;\; z_{jn}) \cdot h_i(\boldsymbol{x}, \boldsymbol{\beta}) \, = \, 0, \text{ for each } j < v_i\,. \tag{5.3}$$

Let $\mathcal{S}^A(u, v, w)$ denote the system given by Equations (5.2) and (5.3). Note that the system $\mathcal{S}^A(u, v, w)$ consists of at most $3\binom{n}{2} = O(n^2)$ bilinear equations and variables with coefficients in $\mathbb{Z}[\boldsymbol{y}, \boldsymbol{z}]$. Moreover, all these coefficients have monomials in $\{0, 1\}$.

Define $w_\circ := (n, n-1, \ldots, 1) \in S_n$ to be the long word. Let $c^w_{u,v}(A)$ denote the Schubert coefficients in type A defined as in §1.4 or by (4.1).

Proposition 5.2. *The system $\mathcal{S}^A(u, v, w_\circ w)$ has $c^w_{u,v}(A)$ solutions over $\overline{\mathbb{C}(\boldsymbol{y}, \boldsymbol{z})}$.*

Proof. We first note

$$\begin{aligned} c^w_{u,v}(A) &= \#\{X^A_u(F^\bullet) \cap X^A_v(G^\bullet) \cap X^A_{w_\circ w}(E^\bullet)\}. \\ &= \#\{\Omega_u(F^\bullet) \cap \Omega_v(G^\bullet) \cap \Omega_{w_\circ w}(E^\bullet)\}. \end{aligned}$$

The first equality follows by definition. The next follows since by Kleiman transversality [Kle74], all points in the first intersection of Schubert varieties must lie in the interior.

By Proposition 5.1, for generic choices of $\boldsymbol{y}, \boldsymbol{z}$ solutions to $\mathcal{S}^A(u, v, w_\circ w)$ are biject with flags $\omega^\bullet \in \Omega_u(F^\bullet) \cap \Omega_v(G^\bullet) \cap \Omega_w(E^\bullet)$. Thus for generic evaluations $\overrightarrow{y}, \overrightarrow{z}$ of $\boldsymbol{y}, \boldsymbol{z}$, respectively, the system $\mathcal{S}^A(u, v, w_\circ w)$ has $c^w_{u,v}(A)$ solutions over $\mathbb{C}$.

Take $I \subseteq \mathbb{Z}(\overrightarrow{y}, \overrightarrow{z})[\boldsymbol{x}, \boldsymbol{\alpha}, \boldsymbol{\beta}]$ to be the ideal generated by the vanishing expressions in the system $\mathcal{S}^A(u, v, w_\circ w)$.[7] A generic choice of evaluation has the ideal I be zero–dimensional and of degree $c^w_{u,v}(A)$. Since variables $\boldsymbol{y}, \boldsymbol{z}$ are independent, their evaluations $\overrightarrow{y}$ and $\overrightarrow{z}$ are algebraically independent, and the result follows. □

5.4. **An example.** For the purposes of illustration, we formulate the system of equations for the vanishing problem in the case $n = 4$.

Let $E^\bullet$ be the standard flag. Take $F^\bullet := \pi E^\bullet$ and $G^\bullet := \rho E^\bullet$ for $\pi = (y_{ij})$ and $\rho = (z_{ij})$ matrices of indeterminates. Take $u = 2143$, $v = 3124$, and $w = 4132$ in S_4. Then $w_\circ w = 1423$. Note $d = 3$ in this case.

Then the Stiefel coordinates are given by

$$\mathcal{X}^A_{w_\circ w} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ x_1 & 0 & 1 \\ x_2 & 0 & x_4 \\ x_3 & 1 & 0 \end{pmatrix} : x_1, x_2, x_3, x_4 \in \mathbb{C} \right\}.$$

Let $e_i(\boldsymbol{x})$ denote the i-th column of $\mathcal{X}^A_{w_\circ w}$ for $i \in \{1, 2, 3\}$. To restrict solutions to the intersection $X^A_{w_\circ w}(E^\bullet) \cap X^A_u(F^\bullet)$, we require

$$\begin{aligned} g_1(\boldsymbol{x}, \boldsymbol{\alpha}) &= c_1(\boldsymbol{x}) \in F^1 - F^2, \\ g_2(\boldsymbol{x}, \boldsymbol{\alpha}) &= c_2(\boldsymbol{x}) \in F^0 - F^1, \text{ and} \\ g_3(\boldsymbol{x}, \boldsymbol{\alpha}) &= \alpha_{3,1} c_1(\boldsymbol{x}) + \alpha_{3,2} c_2(\boldsymbol{x}) + c_3(\boldsymbol{x}) \in F^3 - F^4. \end{aligned}$$

To restrict solutions to the intersection $X^A_{w_\circ w}(E^\bullet) \cap X^A_u(F^\bullet) \cap X^A_v(G^\bullet)$, we require

$$\begin{aligned} h_1(\boldsymbol{x}, \boldsymbol{\beta}) &= c_1(\boldsymbol{x}) \in G^2 - G^3, \\ h_2(\boldsymbol{x}, \boldsymbol{\beta}) &= c_2(\boldsymbol{x}) \in G^0 - G^1, \text{ and} \\ h_3(\boldsymbol{x}, \boldsymbol{\beta}) &= \beta_{3,2} c_2(\boldsymbol{x}) + c_3(\boldsymbol{x}) \in G^1 - G^2. \end{aligned}$$

These inclusions determine the following system $\mathcal{S}^A(u, v, w_\circ w)$:

$$\begin{cases} y_{k1}(\alpha_{3,1}) + y_{k2}(\alpha_{3,1}x_1 + 1) + y_{k3}(\alpha_{3,1}x_2 + x_4) + y_{k4}(\alpha_{3,1}x_3 + \alpha_{3,2}) = 0 \text{ for } k \in \{1, 2, 3\}, \\ y_{11} + y_{12}(x_1) + y_{13}(x_2) + y_{14}(x_3) = 0, \\ z_{k1} + z_{k2}(x_1) + z_{k3}(x_2) + z_{k4}(x_3) = 0 \text{ for } k \in \{1, 2\}, \\ z_{12} + z_{13}(x_4) + z_{14}(\beta_{3,2}) = 0. \end{cases}$$

Note that this is a square system in $\mathbb{K}[x_1, x_2, x_3, x_4, \alpha_{31}, \alpha_{32}, \beta_{32}]$, with 7 equations in 7 variables, and coefficients in $\mathbb{K} = \mathbb{Z}[y_{11}, y_{12}, \ldots, y_{34}, z_{11}, z_{12}, \ldots, z_{24}]$.

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[7] Here $I \subseteq \mathbb{K}[\boldsymbol{x}, \boldsymbol{\alpha}, \boldsymbol{\beta}]$ where $\mathbb{K} = \mathbb{Z}[\overrightarrow{y}, \overrightarrow{z}]$. We write $\mathbb{Z}(\overrightarrow{y}, \overrightarrow{z})[\boldsymbol{x}, \boldsymbol{\alpha}, \boldsymbol{\beta}]$ to make presentation uniform in all types and clarify the connection to HNP as presented in [A+24].

6. Lifted formulations in types B and C

6.1. **Definitions.** Let $\{e_{\overline{n}}, \ldots, e_{\overline{1}}, e_1, \ldots, e_n\}$ denote a basis in $\mathbb{C}^{2n}$. Define the *skew symmetric* bilinear form on $\mathbb{C}^{2n}$, such that $\langle e_i, e_j\rangle = \langle e_{\overline{i}}, e_{\overline{j}}\rangle = 0$ and $-\langle e_i, e_{\overline{j}}\rangle = \langle e_{\overline{j}}, e_i\rangle = \delta_{ij}$ for all $i, j \in [n]$. Here and throughout this section we use the notation $\overline{k} := -k$ for all $k \in [n]$.

In type C, we have $\mathsf{G} = \mathsf{Sp}_{2n}(\mathbb{C})$ the symplectic group, defined with respect to the bilinear form $\langle\cdot,\cdot\rangle$. The corresponding Weyl group is the *hyperoctahedral group* $\mathcal{W}_n = S_n \ltimes \{\pm 1\}^n$, which can be viewed as the set of *signed permutations* of n letters. It will be convenient to realize $\mathcal{W}_n$ as a subgroup of S_{2n} where we impose $w_{\overline{i}} = \overline{w_i}$ for each $i \in [n]$.

An *isotropic flag* $F_\bullet$ is a flag of spaces

$$\mathbf{0} \subset F_n \subset F_{n-1} \subset \cdots \subset F_1 \subset \mathbb{C}^{2n},$$

where $\dim F_i = n + 1 - i$ for each $i \in [n]$ and each F_i is *isotropic*, i.e. satisfies $F_i \subseteq F_i^{\perp}$. We identify the flag variety G/B with the set of such isotropic flags.

Recall that *torus subgroup* T in this case consists of diagonal matrices T given by $\mathsf{diag}(T) = (a_1, \ldots, a_{2n})$ with $a_{n+i} = a_i^{-1}$ for all $i \in [n]$. Define the *antidiagonal matrix* $\mathrm{D}_n = (d_{ij})_{i,j\in[n]}$ is defined by $d_{ij} = 1$ if $i + j = n + 1$, and $d_{ij} = 0$ otherwise. Finally, denote

$$\mathrm{J} := \begin{pmatrix} 0 & \mathrm{D}_n \\ -\mathrm{D}_n & 0 \end{pmatrix}. \tag{6.1}$$

Representing isotropic flags $F_\bullet$ with matrices ω, the isotropic condition translates to

$$\left\{\omega \cdot \mathrm{J} \cdot \omega^T = \mathrm{J}\right\}. \tag{6.2}$$

Consider spaces $E_i := \langle e_n, \ldots, e_i\rangle$ for $i \in [n]$. We extend this chain to a complete flag $E_\bullet$ by setting

$$E_{\overline{i}} := E_{i+1}^{\perp} = \langle e_n, \ldots, e_1, e_{\overline{1}}, \ldots, e_{\overline{i}}\rangle$$

for each $i \in [n]$. For all $w \in \mathcal{W}_n$, we can explicitly define Schubert cells in type C as follows:

$$\Omega_w^C(E_\bullet) = \left\{F_\bullet \ : \ \dim(F_i \cap E_j) = k_w(\overline{i}, j) \ \text{ for } \ i \in [n],\, j \in [\overline{n}, n],\, j \neq 0\right\},$$

where $k_w(\overline{i}, j) := \#\{r \leq \overline{i} \ : \ w_r \geq j\}$. Then the Schubert variety in type C is an orbit closure which can be defined as the follows:

$$X_w^C(E_\bullet) = \overline{\Omega_w^C}(E_\bullet).$$

To construct a generic element $g \in \mathsf{G}$, we use a generalization of the *Cayley transform*, see [Weyl39, §10].[8] Let $M = (m_{ij})$ be a $2n \times 2n$ matrix such that $M\mathrm{J} = -\mathrm{J}M^T$, i.e., $M \in \mathfrak{g}$. These are precisely block matrices

$$M = \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix}, \ \text{where } B = B^T \text{ and } C = C^T. \tag{6.3}$$

Equivalently, $m_{ij} = m_{(2n+1-j)(2n+1-i)}$ if $i \leq n < j$ or $j \leq n < i$ and $m_{ij} = -m_{(2n+1-j)(2n+1-i)}$ otherwise. Thus M can be expressed in terms of those entries weakly above the antidiagonal.

Consider $g = (\mathrm{I}_{2n} + M)^{-1}(\mathrm{I}_{2n} - M)$, where $g = (g_{ij})$. It is straightforward to check g satisfies Equation (6.2) since $M\mathrm{J} = -\mathrm{J}M^T$ and that such elements g are dense in G. Thus we may represent a generic element of G as a $2n \times 2n$ matrix $g = (g_{ij})$, where $g = (\mathrm{I}_{2n} + M)^{-1}(\mathrm{I}_{2n} - M)$ constructed as above. Here we treat the entries m_{ij} for $i + j \leq 2n + 1$ as parameters. We treat g_{ij} for $i, j \in [2n]$ as variables constrained by $g \cdot (\mathrm{I}_{2n} + M) = (\mathrm{I}_{2n} - M)$.

Using this construction, we obtain generic elements $\pi = (\pi_{ij})$ and $\rho = (\rho_{ij})$ of G, where $\pi \cdot (\mathrm{I}_{2n} + Y) = (\mathrm{I}_{2n} - Y)$ and $\rho \cdot (\mathrm{I}_{2n} + Z) = (\mathrm{I}_{2n} - Z)$ for Y, Z matrices of the form of Equation (6.3) with entries in parameters $\boldsymbol{y} = \{y_1, \ldots, y_t\}$ and $\boldsymbol{z} = \{z_1, \ldots, z_t\}$, respectively. Here $t = 2n^2 + n$. Let $\boldsymbol{\pi} = \{\pi_{ij}\}_{i,j\in[2n]}$ $\boldsymbol{\rho} = \{\rho_{ij}\}_{i,j\in[2n]}$ be sets of variables. Take $F_\bullet := \pi E_\bullet$ and $G_\bullet := \rho E_\bullet$.

[8]This construction was suggested to us by David Speyer (personal communication).

Fix three elements $u, v, w \in \mathcal{W}_n$. We consider the system for solving

$$X_u^C(F_\bullet) \cap X_v^C(G_\bullet) \cap X_w^C(E_\bullet).$$

The *Stiefel coordinates* $\mathcal{X}_w^C$ for the Schubert cell Ω_w^C, are the collection of $2n \times 2n$ matrices (m_{ij}) such that

$$m_{ij} = \begin{cases} 1 & \text{if } i = w_j \\ 0 & \text{if } i < w_j \text{ or } w_i^{-1} < j \\ x_{ij} & \text{otherwise,} \end{cases} \tag{6.4}$$

and such that Equation (6.2) is satisfied.

For convenience, we relabel matrix entries $\{x_{ij}\}$ as $\boldsymbol{x} = \{x_1, \ldots, x_s\}$, reading across rows, top to bottom. To every $\omega \in \mathcal{X}_w^C$ with entries $\boldsymbol{x}$, we associate the flag given by the span of the column vectors $e_i(\boldsymbol{x})$ of ω. Take

$$\begin{aligned} \boldsymbol{\alpha} &:= \{\alpha_{ij} : j < i, \quad \text{where} \quad u_j < u_i\}, \\ \boldsymbol{\beta} &:= \{\beta_{ij} : j < i, \ \text{where} \ \ v_j < v_i\}. \end{aligned}$$

For $i \in [\overline{n}, n]$, $i \neq 0$, define the $2n$-vectors $g_i(\boldsymbol{x}, \boldsymbol{\alpha})$ and $h_i(\boldsymbol{x}, \boldsymbol{\beta})$ with entries in $\mathbb{Z}[\boldsymbol{x}, \boldsymbol{\alpha}, \boldsymbol{\beta}]$ such that

$$\begin{aligned} g_i(\boldsymbol{x}, \boldsymbol{\alpha}) &:= e_i(\boldsymbol{x}) + \sum_{\substack{j<i \\ u_j<u_i}} \alpha_{ij}\, e_j(\boldsymbol{x}), \\ h_i(\boldsymbol{x}, \boldsymbol{\beta}) &:= e_i(\boldsymbol{x}) + \sum_{\substack{j<i \\ v_j<v_i}} \beta_{ij}\, e_j(\boldsymbol{x}). \end{aligned}$$

Proposition 6.1. *Let $\Phi_\bullet \in \mathsf{G}/\mathsf{B}$ be in general position with $E_\bullet$, where $\Phi_i = \langle f_1, \ldots, f_i \rangle$ for $i \in [\overline{n}, n]$, $i \neq 0$. Take $w \in \mathcal{W}$. Then $\Phi_\bullet \in \Omega_w^C(E_\bullet)$ <u>if and only if</u> for each for $k \in [\overline{n}, n]$, $k \neq 0$, there exist unique $\{\beta_{jk}\}$ such that*

$$g_k = f_k + \sum_{\substack{j<i \\ w_j<w_i}} \beta_{jk}\, f_j \in E_{w_k} - E_{w_k+1}.$$

Proof. This follows from precisely the same argument as in Proposition 5.1. $\square$

6.2. **The construction in type C.** We consider $\omega \in \mathcal{X}_w^C$ with column vectors $e_j(\boldsymbol{x})$. By construction, the corresponding flag $\omega_\bullet$ lies in $\Omega_w^C(E_\bullet)$ after imposing Equation (6.2). Then construct $g_i(\boldsymbol{x}, \boldsymbol{\alpha})$ and $h_i(\boldsymbol{x}, \boldsymbol{\beta})$ in terms of $e_j(\boldsymbol{x})$.

By Proposition 6.1, $\omega_\bullet \in \Omega_u^C(F_\bullet)$ if and only if $i \in [\overline{n}, n]$, $i \neq 0$, we have $g_i(\boldsymbol{x}, \boldsymbol{\alpha}) \in F_{u_i} - F_{u_i+1}$ for $\boldsymbol{\alpha}$ unique. To impose this condition, we require

$$(\pi_{j1} \ \ \pi_{j2} \ \ \cdots \ \ \pi_{j,2n}) \cdot g_i(\boldsymbol{x}, \boldsymbol{\alpha}) = 0, \ \text{for each } j < u_i. \tag{6.5}$$

Similarly, $\omega_\bullet \in \Omega_v^C(G_\bullet)$ if and only if for every $i \in [\overline{n}, n]$, $i \neq 0$, we have $h_i(\boldsymbol{x}, \boldsymbol{\beta}) \in G_{v_i} - G_{v_i+1}$ for $\boldsymbol{\beta}$ unique. For this last condition, we need

$$(\rho_{j1} \ \ \rho_{j2} \ \ \cdots \ \ \rho_{j,2n}) \cdot h_i(\boldsymbol{x}, \boldsymbol{\beta}) = 0, \text{ for each } j < v_i. \tag{6.6}$$

Let $\mathcal{S}^C(u, v, w)$ denote the system given by Equations (6.2), (6.5), and (6.6) along with the constraints $\pi = (1+Y)^{-1}(1-Y)$ and $\rho = (1+Z)^{-1}(1-Z)$. Equation (6.2) introduces $12n^2$ quadratic equations in fewer than $4n^2$ variables. Equations (6.5) and (6.6) add at most $2\binom{2n}{2}$ trilinear equations and $2\binom{2n}{2} + 8n^2$ variables with $0, 1$ coefficients. The remaining constraints on π and ρ each introduce $4n^2$ linear equations with coefficients in $\mathbb{Z}[\boldsymbol{y}, \boldsymbol{z}]$.

Let $c^w_{u,v}(C)$ denote the Schubert coefficients in type C, defined as structure constants of Schubert polynomials given by the Algebraic Definition in §1.3 or a combinatorial construction in [ST23]. Alternatively, Schubert coefficients in type C can be defined by Equation (4.1).

Proposition 6.2. *The system $\mathcal{S}^C(u, v, w_\circ w)$ has $c^w_{u,v}(C)$ solutions over $\overline{\mathbb{C}(\boldsymbol{y}, \boldsymbol{z})}$.*

The proof follows verbatim the proof of Proposition 5.2.

6.3. **The construction in type B.** In type B, we have $\mathsf{G} = \mathsf{SO}_{2n+1}(\mathbb{C})$ defined analogously. Let $c^w_{u,v}(B)$ denote the Schubert coefficients in type B, defined as above. For $w \in \mathcal{W}_n$ let

$$s(w) := \{i \in [n] : w_i < 0\}.$$

Finally, define

$$\mathcal{S}^B(u, v, w) := \mathcal{S}^C(u, v, w).$$

Proposition 6.3. *The system $\mathcal{S}^B(u, v, w_\circ w)$ has $2^{s(u)+s(v)-s(w)} c^w_{u,v}(B)$ solutions over $\overline{\mathbb{C}(\boldsymbol{y}, \boldsymbol{z})}$.*

Proof. By [BH95], we have

$$c^w_{u,v}(B) = 2^{s(w)-s(u)-s(v)}\, c^w_{u,v}(C). \tag{6.7}$$

Thus the result follows by Proposition 6.2. □

7. Proof of Main Lemma 1.9

Fix type $Y \in \{A, B, C\}$. Let $u, v, w \in \mathcal{W}$ be elements in the corresponding Weyl group. Consider the system $\mathcal{S}^Y(u, v, w)$ defined as in Sections 5 and 6. By Propositions 5.2, 6.2, and 6.3, the number of solutions of the system $\mathcal{S}^Y(u, v, w)$ is equal to the corresponding Schubert coefficient $c^w_{u,v}(Y)$.

Observe that the size of $\mathcal{S}^Y(u, v, w)$ is $O(n^2)$ in each case. Thus, these systems are instances of HNP by construction, with sets of variables $(\boldsymbol{x}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\pi}, \boldsymbol{\rho})$ and parameters $(\boldsymbol{y}, \boldsymbol{z})$. Therefore, $\neg$SchubertVanishing is in HNP in each case, as desired. □

8. Final remarks

8.1. Since the Billey–Vakil algebraic system discussed in §1.6.3 is the most advanced effort in the direction of this work, it is worth elaborating as to why it cannot be used to make a reduction to HNP. There are three major issues, in fact. First, note that system of constraints [BV08, Eq. (8)] on the variables to be generic has exponentially many equations. Second, these equations are of the form $\det(X) = 0$, thus have exponentially many terms. Finally, the additional condition of the set of solutions being 0-dimensional is also quite delicate. It has also been studied by Koiran [Koi97] in the context of HN, but the previous two issues again apply in this case.

8.2. Recall Purbhoo's algebraic system discussed in §1.6.3, and given explicitly in Lemma B.7. This system also has exponentially many constraints of the form $\det(X) \neq 0$, at least one of which has to be satisfied. Each of these can be easily converted into a polynomial equation of the form $\det(X) \cdot y = 1$ with an auxiliary variable (cf. §C.3). Unfortunately, these equations cannot be used to make a reduction to HNP for the same reason as above, since they have exponentially many terms. This issue also appears in our original approach to type D that is given in Appendix A, see Remark A.2.

8.3. Note that Conjecture 1.6 would follow if the Knutson and Zinn-Justin sufficient condition for nonvanishing was shown NP-complete. Similarly, it would be interesting to see if the Purbhoo sufficient conditions for nonvanishing and the Billey–Vakil sufficient conditions for vanishing given in §1.6.2 are NP-complete and coNP-complete, respectively. Note these results by themselves would not imply Conjecture 1.6.

8.4. Compared to the original draft of this paper [PR24b], this version includes a new Appendix C.[9] The result in type D and for the BSS model are moved outside of the main body of the paper to Appendix A and B, respectively, while the example that used to be in the appendix is now moved to §5.4. In a short companion paper [PR24c], we have a discussion of combinatorial and philosophical implications of Main Theorem 1.4 in the context of various number theoretic and complexity theoretic assumptions.

8.5. Note that Main Theorem 1.4 can be generalized in several directions. Notably, the theorem can be extended to the vanishing problem of generic k-fold intersection

$$X_{u_1}\big(E_\bullet^{(1)}\big) \cap X_{u_2}\big(E_\bullet^{(2)}\big) \cap \ldots \cap X_{u_k}\big(E_\bullet^{(k)}\big),$$

for every fixed k. In a different direction, Main Theorem 1.4 can be extended to enriched cohomology theories.[10]

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[9]The extended abstract of [PR24b] is to appear in *Proc. 57-th ACM STOC* (2025), 12 pp.

[10]This work is in preparation.

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Appendix A. Lifted formulation in type D

Recall that our original approach to Main Theorem 1.4 does not extend to type D. An approach that does is given in Appendix C. Nevertheless we decided to include a lifted formulation in type D similar to our lifted formulations in types A, B and C. There are several reasons for doing that.

First, we wanted to highlight the technical issue that failed the proof in this case. Second, we use this construction in the BSS model given in Appendix B. Even if the new construction Appendix C would also work, this lifted formulation is more concise and is of independent interest. Finally, it is worth expounding on a construction in each type before presenting a uniform construction in all types.

A.1. **Definitions.** As in type C, let $e_{\overline{n}}, \ldots, e_{\overline{1}}, e_1, \ldots, e_n$ denote a basis in $\mathbb{C}^{2n}$. Define the *symmetric* bilinear form on $\mathbb{C}^{2n}$ such that $\langle e_i, e_j\rangle = \langle e_{\overline{i}}, e_{\overline{j}}\rangle = 0$ and $\langle e_i, e_{\overline{j}}\rangle = \langle e_{\overline{j}}, e_i\rangle = \delta_{ij}$ for $i, j \in [n]$. As before, we use $\overline{k} := -k$ for $k \in [n]$.

In type D, we have $\mathsf{G} = \mathsf{SO}_{2n}(\mathbb{C})$ is the special orthogonal group with respect to the bilinear form $\langle\cdot,\cdot\rangle$. The corresponding Weyl group $\mathcal{W}_n^+$ is a subgroup of index two in the hyperoctahedral group $\mathcal{W}_n$ It can be viewed the group of signed permutations of n letters with an even number of sign changes.

An *isotropic flag* $F_\bullet$ is a flag of spaces

$$\mathbf{0} \subset F_{n-1} \subset F_{n-1} \subset \cdots \subset F_1 \subset F_0 \subset \mathbb{C}^{2n},$$

where $\dim F_i = n - i$ for all $i \in [n]$, and each F_i is *isotropic*, i.e. $F_i \subseteq F_i^{\perp}$.

In type D, we identify the flag variety G/B with the set of those isotropic flags $F_\bullet$ such that $E_0 \cap F_0$ is even dimensional.

Recall that *torus subgroup* T in this case consists of diagonal matrices T given by $\mathsf{diag}(T) = (a_1, \ldots, a_{2n})$ with $a_{n+i} = a_i^{-1}$ for all $i \in [n]$. Now set $\mathrm{J} = \mathrm{D}_{2n}$, the antidiagonal $2n \times 2n$ matrix.

Representing flags $F_\bullet$ with matrices ω, the isotropic condition translates to[11]

$$\text{(A.1)} \qquad \begin{cases} \omega \cdot \mathrm{J} \cdot \omega^T \,=\, \mathrm{J} \\ \quad \det(\omega) \,=\, 1. \end{cases}$$

Consider spaces $E_i := \langle e_n, \ldots, e_i\rangle$ for $i \in [n]$. We extend this chain to a complete flag $E_\bullet$ by setting

$$E_{\overline{i}} \,:=\, E_{i+1}^{\perp} \,=\, \langle e_n, \ldots, e_1, e_{\overline{1}}, \ldots, e_{\overline{i}}\rangle$$

for each $i \in [n]$. We take $E_n := \mathbf{0}$. Taking $w \in \mathcal{W}_n^+$, we can explicitly define Schubert cells in type D :

$$\Omega_w^D(E_\bullet) \,=\, \bigl\{F_\bullet \,:\, \dim(F_i \cap E_j) = k_w(\overline{i}, j) \text{ for } \ 0 \le i \le n-1,\, j \in [\overline{n}, n] \cup \{\overline{0}, 0\}\bigr\},$$

where $k_w(\overline{i}, j) := \#\{r < \overline{i} \,:\, w_r > j\}$. Then the Schubert variety in type D is the orbit closure which can be defined as follows:

$$X_w^D(E_\bullet) \,=\, \overline{\Omega_w^D}(E_\bullet).$$

To construct a generic $g \in \mathsf{G}$, we again use a generalization of the Cayley transform, see [Weyl39, §10], just as in Section 6. Let $M = (m_{ij})$ be a $2n \times 2n$ matrix such that $M\mathrm{J} = -\mathrm{J}M^T$, i.e., $M \in \mathfrak{g}$. These are precisely the matrices M such that

$$\text{(A.2)} \qquad m_{ij} = -m_{(2n+1-j)(2n+1-i)}.$$

Now M can be expressed in terms of those entries strictly above the antidiagonal.

Consider $g = (\mathrm{I}_{2n} + M)^{-1}(\mathrm{I}_{2n} - M)$, where $g = (g_{ij})$. It is straightforward to check g satisfies Equation (6.2) since $M\mathrm{J} = -\mathrm{J}M^T$ and that such elements g are dense in G. Thus we may represent a generic element of G as a $2n \times 2n$ matrix $g = (g_{ij})$, where $(\mathrm{I}_{2n} + M) \cdot g = (\mathrm{I}_{2n} - M)$

[11] Equation $\omega \cdot \mathrm{J} \cdot \omega^T = \mathrm{J}$ gives $\det(\omega) \in \{\pm 1\}$. Ensuring that $\det(\omega) = 1$ is a non-local parity condition that is unavoidable in this setup.

constructed as above. Here we treat m_{ij} where $i+j \leq 2n+1$ as parameters. We treat g_{ij} as variables constrained by $(\mathrm{I}_{2n}+M)\cdot g = (\mathrm{I}_{2n}-M)$.

Using this construction, we obtain generic elements $\pi = (\pi_{ij})$ and $\rho = (\rho_{ij})$ of G, where $(\mathrm{I}_{2n}+Y)\cdot\pi = (\mathrm{I}_{2n}-Y)$ and $(\mathrm{I}_{2n}+Z)\cdot\rho = (\mathrm{I}_{2n}-Z)$ for Y, Z matrices in parameters $\boldsymbol{y} = \{y_1, \ldots, y_t\}$ and $\boldsymbol{z} = \{z_1, \ldots, z_t\}$, respectively, satisfying Equation (A.2) Here $t = 2n^2+n$. Let $\boldsymbol{\pi} = \{\pi_{ij}\}_{i,j\in[2n]}$ $\boldsymbol{\rho} = \{\rho_{ij}\}_{i,j\in[2n]}$ be sets of variables. Take $F_\bullet := \pi E_\bullet$ and $G_\bullet := \rho E_\bullet$.

Pick $u, v, w \in \mathcal{W}_n^+$. We consider the system for solving

$$X_u^D(F_\bullet) \cap X_v^D(G_\bullet) \cap X_w^D(E_\bullet).$$

The *Stiefel coordinates* $\mathcal{X}_w^D$ for the Schubert cell Ω_w^D, are those $2n \times 2n$ matrices (m_{ij}) such that

$$m_{ij} = \begin{cases} 1 & \text{if } i = w_j \\ 0 & \text{if } i < w_j \text{ or } w_i^{-1} < j \\ x_{ij} & \text{otherwise,} \end{cases} \tag{A.3}$$

and such that Equation (A.1) is satisfied.

We relabel matrix entries $\{x_{ij}\}$ as $\boldsymbol{x} = \{x_1, \ldots, x_s\}$, $s \leq \binom{2n}{2}$, reading across rows, top to bottom. To every $\omega \in \mathcal{X}_w^D$ with entries $\boldsymbol{x}$, we associate the flag given by the span of the column vectors $e_i(\boldsymbol{x})$ of ω. Take

$$\begin{aligned} \boldsymbol{\alpha} &:= \{\alpha_{ij} \,:\, j < i, \quad \text{where} \quad u_j < u_i\}, \\ \boldsymbol{\beta} &:= \{\beta_{ij} \,:\, j < i, \quad \text{where} \quad v_j < v_i\}. \end{aligned}$$

For $i \in [\overline{n}, n],\ i \neq 0$, define the $2n$-vectors $g_i(\boldsymbol{x}, \boldsymbol{\alpha})$ and $h_i(\boldsymbol{x}, \boldsymbol{\beta})$ with entries in $\mathbb{Z}[\boldsymbol{x}, \boldsymbol{\alpha}, \boldsymbol{\beta}]$ such that

$$\begin{aligned} g_i(\boldsymbol{x}, \boldsymbol{\alpha}) &:= e_i(\boldsymbol{x}) + \sum_{\substack{j<i \\ u_j<u_i}} \alpha_{ij}\, e_j(\boldsymbol{x}), \\ h_i(\boldsymbol{x}, \boldsymbol{\beta}) &:= e_i(\boldsymbol{x}) + \sum_{\substack{j<i \\ v_j<v_i}} \beta_{ij}\, e_j(\boldsymbol{x}). \end{aligned}$$

A.2. **The construction in type D.** We consider $\omega \in \mathcal{X}_w^D$ with column vectors $e_i(\boldsymbol{x})$. By construction, the corresponding flag $\omega_\bullet$ lies in $\Omega_w^D(E_\bullet)$ after imposing Equation (A.1). Then construct $g_i(\boldsymbol{x}, \boldsymbol{\alpha})$ and $h_i(\boldsymbol{x}, \boldsymbol{\beta})$.

Extending Proposition 6.1, $\omega_\bullet \in \Omega_u^D(F_\bullet)$ if and only if $i \in [\overline{n}, n],\ i \neq 0$, we have $g_i(\boldsymbol{x}, \boldsymbol{\alpha}) \in F_{u_i} - F_{u_i+1}$ for $\boldsymbol{\alpha}$ unique. To impose this condition, we require

$$(\pi_{j1}\ \ \pi_{j2}\ \ \cdots\ \ \pi_{j,2n}) \cdot g_i(\boldsymbol{x}, \boldsymbol{\alpha}) \;=\; 0, \text{ for each } j < u_i. \tag{A.4}$$

Similarly, $\omega_\bullet \in \Omega_v^D(G_\bullet)$ if and only if for every $i \in [\overline{n}, n],\ i \neq 0$, we have $h_i(\boldsymbol{x}, \boldsymbol{\beta}) \in G_{v_i} - G_{v_i+1}$ for $\boldsymbol{\beta}$ unique. For this last condition, we need

$$(\rho_{j1}\ \ \rho_{j2}\ \ \cdots\ \ \rho_{j,2n}) \cdot h_i(\boldsymbol{x}, \boldsymbol{\beta}) \;=\; 0, \text{ for each } j < v_i. \tag{A.5}$$

Let $\mathcal{S}^D(u, v, w)$ denote the system given by Equations (A.1), (A.4), and (A.5) along with the constraints $(1+Y)\cdot\pi = (1-Y)$ and $(1+Z)\cdot\rho = (1-Z)$. Equation (A.1) introduces $12n^2$ quadratic equations and one degree $2n$ equation in fewer than $4n^2$ variables. Equations (A.4) and (A.5) add at most $2\binom{2n}{2}$ cubic equations and $2\binom{2n}{2}$ variables with $0, 1$ coefficients. The remaining constraints on π and ρ each introduce $4n^2$ linear equations with coefficients in $\mathbb{Z}[\boldsymbol{y}, \boldsymbol{z}]$.

Let $c_{u,v}^w(D)$ denote the Schubert coefficients in type D, defined as above or as structure coefficients in Equation (4.1) for $\mathsf{G} = \mathsf{SO}_{2n}$.

Proposition A.1. *The system $\mathcal{S}^D(u, v, w_\circ w)$ has $c_{u,v}^w(D)$ solutions over $\overline{\mathbb{C}(\boldsymbol{y}, \boldsymbol{z})}$.*

The proof follows verbatim the proof of Proposition 5.2.

Remark A.2. Let us emphasize that the determinantal equation in (A.1) gives the system $\mathcal{S}^D(u,v,w)$ of exponential size. Thus the proof of Lemma 1.9 for types A, B and C does not immediately extend to the type D setting. We do this in Appendix C.

Appendix B. Blum–Shub–Smale model

In this section we study complexity of the Schubert vanishing problem in a nonstandard model of computation pioneered by Blum, Shub and Smale. We removed this section from the main body of the paper to avoid the destruction from the Main Theorem 1.4. We believe these results are of independent interest, as they illustrate the power of computation over fields when it comes to problems in Algebraic Combinatorics.

B.1. **Computation over fields.** The *Blum–Shub–Smale* (BSS) *model of computation* [BSS89, BCSS98] was introduced to describe computations over general fields $\mathbb{K}$. It can be viewed as analogous to (Boolean) Turing machine, but in this case the registers that can store arbitrary numbers from $\mathbb{K}$, and rational functions over $\mathbb{K}$ can be computed in a single time step.

One can then similarly define complexity classes $\mathsf{P}_{\mathbb{K}}$, $\mathsf{NP}_{\mathbb{K}}$, etc. Although the original paper works largely over $\mathbb{R}$, many results extend to general fields. Theorem B.1 is one of the key results in the area. The counting complexity classes for computations over $\mathbb{K}$ were introduced by Bürgisser and Cucker [BC06]. Notably, denote by $\#\mathsf{P}_{\mathbb{K}}$ the class of counting functions for the number of accepting paths of problems in $\mathsf{NP}_{\mathbb{K}}$.

The relationships between these classes in many ways resemble the classical model. Notably, Bürgisser and Cucker prove the following analogue of Toda's theorem (assuming GRH):

$$\#\mathsf{P}_{\mathbb{C}} \subseteq \mathsf{FP}_{\mathbb{C}}^{\mathsf{NP}_{\mathbb{C}}} \implies \mathsf{PH} = \Sigma_4^{\mathrm{p}},$$

i.e., the *classical* polynomial hierarchy collapses to the fourth level [BC06, Cor. 8.7].

B.2. **Schubert vanishing in BSS model.** Motivated by the algebro-geometric considerations, we consider the problem $\mathrm{HN}_{\mathbb{C}}$ which asks if the polynomial system (1.3) with coefficients in $\mathbb{C}$ has a solution over $\mathbb{C}$. This is a natural problem in the BSS model of computation described above. One of the key insights of the Blum–Shub–Smale theory is the following remarkable theorem:

Theorem B.1 ([BCSS98, §5.4])**.** $\mathrm{HN}_{\mathbb{C}}$ *is in* $\mathsf{NP}_{\mathbb{C}}$*. Moreover,* $\mathrm{HN}_{\mathbb{C}}$ *is* $\mathsf{NP}_{\mathbb{C}}$*-complete.*

This inclusion combined with the system of polynomial equations constructed in the proof of Lemma 1.9 gives the following result:

Theorem B.2. SchubertVanishing *is in* $\mathsf{NP}_{\mathbb{C}}$ *in types A, B, C and D.*

While the proof in types A, B and C is straightforward from the proof of Lemma 1.9, there is a technical issue in type D. As we mentioned above, we follow the approach in the lemma, one of the equations in type D has exponential size (Remark A.2). Nevertheless, it can be *evaluated* in poly-time over $\mathbb{C}$, and thus can be used to derive the result.

Our next result is a different kind of inclusion and also holds in all classical types.

Theorem B.3. SchubertVanishing *is in* $\mathsf{P}_{\mathbb{R}}$ *in types A, B, C and D.*

The proof of the theorem uses a completely different approach from that in Lemma 1.9. Instead, we use deep results in [Bel06, Pur06] to give a linear algebraic formulation of the Schubert vanishing problem which quickly implies the result. Curiously, this approach cannot be used to give an independent proof of Lemma 1.9.

Remark B.4. Note that $\mathsf{NP}_\mathbb{R}$ is likely much more powerful than $\mathsf{NP}_\mathbb{C}$ since the former is best known as the *existential theory of the reals*: $\exists\mathbb{R} = \mathsf{NP}_\mathbb{R}$ and likely not in PH. In fact, $\mathsf{NP} \subseteq \exists\mathbb{R} \subseteq \mathsf{PSPACE}$ are the best known lower and upper bounds [Sch10]. Thus, classes $\mathsf{NP}_\mathbb{C}$ and $\mathsf{P}_\mathbb{R}$ are not easily comparable.

B.3. **Back to Schubert coefficients.** Let $\#\mathsf{P}_\mathbb{C}$ denote the class of counting functions for the number of accepting paths of problems in $\mathsf{NP}_\mathbb{C}$. Denote by $\#\mathrm{HN}$ and $\#\mathrm{HN}_\mathbb{C}$ the problems of counting solutions of problems in HN and $\mathrm{HN}_\mathbb{C}$, respectively. By analogy with the Blum–Shub–Smale Theorem B.1, Bürgisser and Cucker prove the following:

Theorem B.5 **([BC06, Thm 3.4(i)]).** *$\#\mathrm{HN}_\mathbb{C}$ is in $\#\mathsf{P}_\mathbb{C}$. Moreover, $\#\mathrm{HN}_\mathbb{C}$ is $\#\mathsf{P}_\mathbb{C}$-complete.*

We refer to [BC13] for a thorough treatment of this theory. Our last result in this model is the following inclusion.

Theorem B.6. Schubert *is in $\#\mathsf{P}_\mathbb{C}$ in types A, B, C and D.*

The proof of this theorem uses the inclusion in Theorem B.5 and follows the approach in the proof of Theorem B.2. Conjecture 1.2 suggests that this result is likely optimal in the BSS counting hierarchy $\mathsf{CH}_\mathbb{C}$.

B.4. **Proof of Theorem B.2 and Theorem B.6.** Fix type $Y \in \{A, C, D\}$ and consider the corresponding system $\mathcal{S}^Y(u,v,w)$. Take an evaluation of this system at generic values $\overrightarrow{y}$ and $\overrightarrow{z}$. The size of the system is $O(n^2)$. Note that in type D we do not expand the determinant and take an evaluation directly.

By Propositions 5.2, 6.2, and A.1, deciding if a given flag $\widetilde{F}_\bullet$ is a solution to $\mathcal{S}^Y(u,v,w)$ is in $\mathsf{NP}_\mathbb{C}$. Now Theorem B.5 implies the result for all $Y \in \{A, C, D\}$. In type B, the result follows from type C and (6.7). This completes the proof of Theorem B.2.

Finally, Theorem B.6 follows verbatim the argument above since the number of solutions of $\mathcal{S}^Y(u,v,w)$ is *equal* to the corresponding Schubert coefficient $c^w_{u,v}(Y)$ for $Y \in \{A, C, D\}$. In type B, the result follows from type C and (6.7). $\square$

B.5. **Proof of Theorem B.3.** Fix type $Y \in \{A, B, C, D\}$ and let $\mathsf{G} = \mathsf{G}_Y$ be the corresponding reductive group. In each case, G is a matrix group lying in an ambient vector space V. Let N denote the subgroup of unipotent matrices, so we have

$$\mathsf{N} \subset \mathsf{B} \subset \mathsf{G} \subset V.$$

Let $\mathfrak{n}$ denote the Lie algebra of N. We think of $\mathfrak{n}$ as a subspace of V.

For $w \in \mathcal{W}$, define $Z_w := \mathfrak{n} \cap (w\mathsf{B}_- w^{-1})$. Equivalently, Z_w is the subspace of $\mathfrak{n}$ generated by basis vectors e_α for α a positive root in the root system for G.

Lemma B.7. [Pur04, Lemma 2.2.1] *Let $Y \in \{A, B, C, D\}$. For generic $\rho, \omega, \tau \in \mathsf{N}$, we have:*

$$c^w_{u,v}(Y) = 0 \quad \textit{if and only if} \quad \rho Z_u \rho^{-1} + \omega Z_v \omega^{-1} + \tau Z_{w_\circ w} \tau^{-1} = \mathfrak{n}.$$

Here the sum is the usual sum of vector subspaces of V.

Following [BCSS98, §17.5], define complexity class $\mathsf{BPP}^U_\mathbb{R}$ to be the class of problems which can be decided in probabilistic polynomial time by a BSS machine over $\mathbb{R}$. Here the machine is allowed to pick uniform random numbers in $[0,1]$. It is known that $\mathsf{BPP}^U_\mathbb{R} = \mathsf{P}_\mathbb{R}$, see [BCSS98, §17.6]. Thus, it suffices to show that SchubertVanishing $\in \mathsf{BPP}^U_\mathbb{R}$.

In all types, generate upper-unitriangular matrices ρ, ω, τ whose entries are uniform random complex numbers $a + i\,b$, where $a, b \in [-1, 1]$ and $i = \sqrt{-1}$. In type A, this gives a *full measure* on an ε-ball in N around $\mathbf{0}_\mathsf{N}$. With probability 1, all generated numbers will be algebraically independent. Note that although these are complex matrices, we will treat them as arrays where entries are two real numbers, and with multiplication defined as for complex numbers (in the usual way).

For types B, C, and D, we then compute the resulting entries constrained by the isotropic condition in $O(n^3)$ time. Computing the corresponding inverse matrices using Cramer's rule, can be done in $O(n^3)$ time in the BSS model.

Proceed to pick standard bases in Z_u, Z_v, and $Z_{w_\circ w}$, again minding relevant isotropy conditions. Compute subspaces as in Lemma B.7. Check that the sum $\rho Z_u \rho^{-1} + \omega Z_v \omega^{-1} + \tau Z_{w_\circ w} \tau^{-1}$ of these subspaces spans the whole $\mathfrak{n}$. This again can be determined in $O(n^3)$ time. By the lemma, this gives a $\mathsf{BPP}_{\mathbb{R}}^{U}$ algorithm for SchubertVanishing. □

Appendix C. A new HN system for all classical types (joint with David E Speyer[12,13])

In this section we prove the following result, thus completing the proof of Lemma 1.9:

Lemma C.1 (cf. Lemma 1.9)**.** $\neg$SchubertVanishing *reduces to* HNP *in type* D.

For the proof, we give an alternative construction of Schubert systems in all classical types. This gives a new proof of Lemma 1.9 in types A, B and C.

C.1. **Setup.** Consider G a complex Lie group of classical type: $\mathsf{GL}_n(\mathbb{C})$, $\mathsf{SO}_{2n+1}(\mathbb{C})$, $\mathsf{Sp}_{2n}(\mathbb{C})$, or $\mathsf{SO}_{2n}(\mathbb{C})$. We can equivalently express Equation (4.3) as the following:

$$\begin{aligned} c^w_{u,v} &= \#\big\{X_u(E_\bullet) \cap X_v(F_\bullet) \cap X_{w_\circ w}(G_\bullet)\big\} \\ &= \#\big\{X_u(E_\bullet) \cap \pi X_v(E_\bullet) \cap \rho X_{w_\circ w}(E_\bullet)\big\} \\ &= \#\big\{\Omega_u(E_\bullet) \cap \pi \Omega_v(E_\bullet) \cap \rho \Omega_{w_\circ w}(E_\bullet)\big\} \end{aligned}$$

The first equality follows since can associate generic reference flags $E_\bullet, F_\bullet, G_\bullet$ in G/B with generic π, $\rho \in \mathsf{G}$. The second equality follows by Kleiman transversality. Using the definition of Schubert cells, we see

$$c^w_{u,v} > 0 \iff \mathsf{B}_-\dot{u}\mathsf{B} \cap \pi\mathsf{B}_-\dot{v}\mathsf{B} \cap \rho\mathsf{B}_-(\dot{w_\circ w})\mathsf{B} \neq \varnothing \tag{C.1}$$

for generic $\pi, \rho \in \mathsf{G}$. Thus we can consider the intersection

$$\Xi := \mathsf{B}_-\dot{u}\mathsf{B} \cap \pi\mathsf{B}_-\dot{v}\mathsf{B} \cap \rho\mathsf{B}_-(\dot{w_\circ w})\mathsf{B}. \tag{C.2}$$

Remark C.2. Equation (4.3) defines Schubert coefficients as the number of points in a 0-dimensional intersection. The intersection in Ξ is positive dimensional, so the number of points in Ξ is infinite. If desired, one can add additional intersections to this expression in order to cut Ξ down to a finite set of cardinality c^w_{uv}. For our purposes, however, Ξ will suffice as is.

C.2. **Describing generic group elements.** For equations defining the intersection Ξ in Equation (C.2), we construct a generic element $g \in \mathsf{G}$.

For $\mathsf{GL}_n(\mathbb{C})$, we generate g as in Section 5, where $g = (y_{ij})$ a matrix of parameters. For $\mathsf{Sp}_{2n}(\mathbb{C})$, we generate g as in Section 6. In particular we create block skew-antisymmetric matrix of parameters M, i.e. satisfying Equation (6.3). Then we create a matrix of variables g and set $g = (\mathrm{I}_{2n} + M)^{-1}(\mathrm{I}_{2n} - M)$.

Similarly, for $\mathsf{SO}_{2n}(\mathbb{C})$, we generate g as in Section A. In particular we create skew-antisymmetric matrix of parameters M, i.e. satisfying Equation (A.2). Then we create a matrix of variables g and set $g = (\mathrm{I}_{2n+1} + M)^{-1}(\mathrm{I}_{2n+1} - M)$. For $\mathsf{SO}_{2n+1}(\mathbb{C})$, we may generate g similarly to $\mathsf{SO}_{2n}(\mathbb{C})$,

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using the same argument as in Section A. In particular we create skew-antisymmetric matrix of parameters M. Then we create a matrix of variables g and set $g = (\mathrm{I}_{2n+1} + M)^{-1}(\mathrm{I}_{2n+1} - M)$.

We summarize these descriptions in Table 1 below. Here for each G, we describe the set of parameters $\mathbf{Par}(g)$, the set of variables $\mathbf{Var}(g)$, and the set of equations $\mathbf{Eq}(g)$ needed to ensure $g \in \mathsf{G}$. Note in each case that the sizes of $\mathbf{Par}(g)$, $\mathbf{Var}(g)$, and $\mathbf{Eq}(g)$ are in $O(n^2)$.

G	$\mathbf{Par}(g)$	$\mathbf{Var}(g)$	$\mathbf{Eq}(g)$
GL_n	$\{m_{ij}\}_{i,j\in[n]}$	$\varnothing$	$\varnothing$
SO_{2n+1}	$\{m_{ij}\}_{i+j\leq 2n+2,\ i,j\in[2n+1]}$	$\{g_{ij}\}_{i,j\in[2n+1]}$	$(\mathrm{I}_{2n+1}+M)\cdot g = (\mathrm{I}_{2n+1}-M)$
Sp_{2n}	$\{m_{ij}\}_{i+j\leq 2n+1,\ i,j\in[2n]}$	$\{g_{ij}\}_{i,j\in[2n]}$	$(\mathrm{I}_{2n}+M)\cdot g = (\mathrm{I}_{2n}-M)$
SO_{2n}	$\{m_{ij}\}_{i+j\leq 2n+1,\ i,j\in[2n]}$	$\{g_{ij}\}_{i,j\in[2n]}$	$(\mathrm{I}_{2n}+M)\cdot g = (\mathrm{I}_{2n}-M)$

TABLE 1. Generating a matrix $g \in \mathsf{G}$.

C.3. **Describing generic Borel subgroup elements.** To construct equations defining the intersection in Equation (C.2), we must describe the equations ensuring that an upper triangular matrix $B = (b_{ij})$ lies $\mathsf{B} \subset \mathsf{G}$.

For $\mathsf{GL}_n(\mathbb{C})$, we have $B = (b_{ij})_{i,j\in[n]}$ lies in B if B is invertible. Thus, for b a variable, we have $B \in \mathsf{B}$ precisely when

$$b \cdot \prod_{i=1}^{n} b_{ii} \;=\; 1 \tag{C.3}$$

is satisfiable.

For $\mathsf{SO}_{2n+1}(\mathbb{C})$, let $\mathrm{J} := \mathrm{D}_{2n+1}$ be the antidiagonal matrix. Since we require $B \in \mathsf{G}$, matrix B must satisfy $B^T \cdot \mathrm{J} \cdot B = \mathrm{J}$ and $\det(B) = 1$, cf. Equation (6.2). Imposing $B^T \cdot \mathrm{J} \cdot B = \mathrm{J}$ implies that $b_{ii}b_{(2n+2-i)(2n+2-i)} = 1$ for each $i \in [2n+1]$. Since B is upper triangular, this implies $\det(B) = b_{nn} = \pm 1$. So, in order to have B in SO_{2n+1}, we simply impose that $B^T J B = J$ and $b_{nn} = 1$.

For $\mathsf{Sp}_{2n}(\mathbb{C})$ set

$$\mathrm{J} \;:=\; \begin{pmatrix} 0 & \mathrm{D}_n \\ -\mathrm{D}_n & 0 \end{pmatrix},$$

cf. Equation (6.1). To have $B \in \mathsf{G}$, then B must satisfy Equation (6.2). That is, we impose $B^T \cdot \mathrm{J} \cdot B = \mathrm{J}$.

For $\mathsf{SO}_{2n}(\mathbb{C})$, set $\mathrm{J} := \mathrm{D}_{2n}$. To have $B \in \mathsf{G}$, we have matrix B must satisfy Equation (A.1). That is, we have $B \cdot \mathrm{J} \cdot B^T = \mathrm{J}$ and $\det(B) = 1$. Imposing $B^T \cdot \mathrm{J} \cdot B = \mathrm{J}$ implies that we have $b_{ii}b_{(2n+1-i)(2n+1-i)} = 1$ for each $i \in [2n]$. Thus, since B is upper triangular, we have $B \cdot \mathrm{J} \cdot B^T = \mathrm{J}$ implies $\det(B) = 1$. Therefore, it suffices to impose $B^T \cdot \mathrm{J} \cdot B = \mathrm{J}$.

We summarize in Table 2 the description of the set of variables $\mathbf{Var}(B)$ and set of equations $\mathbf{Eq}(B)$ needed to ensure that $B \in \mathsf{B} \subset \mathsf{G}$. Note that in each case that the sizes of $\mathbf{Var}(B)$ and $\mathbf{Eq}(B)$ are $O(n^2)$. Of course, this construction similarly generates $B \in \mathsf{B}_- \subset \mathsf{G}$ after transposing.

C.4. **Describing the Equations.** Now we describe the equations defining Ξ in (C.2). Observe that $\Xi \neq \varnothing$ if and only if there exist $P_1, P_2, P_3 \in \mathsf{B}_-$ and $Q_1, Q_2, Q_3 \in \mathsf{B}$ such that

$$P_1\dot{u}Q_1 \;=\; \pi P_2\dot{v}Q_2 \quad\text{ and }\quad P_1\dot{u}Q_1 \;=\; \rho P_3(\dot{\overline{w_\circ w}})Q_3\,.$$

Let $\mathcal{E}^Y(u,v,w)$, $Y \in \{A,B,C,D\}$, denote the system given by the following equations:

(i) $\mathbf{Eq}(\pi) \cup \mathbf{Eq}(\rho)$ in Table 1,

G	$\mathbf{Var}(B)$	$\mathbf{Eq}(B)$
GL_n	$\{b_{ij}\}_{i\le j\in[n]}\cup\{b\}$	$\{b\cdot\prod_{i\in[n]}b_{ii}=1\}$
SO_{2n+1}	$\{b_{ij}\}_{i\le j\in[2n+1]}$	$\{b_{ii}\,b_{(2n+2-i)(2n+2-i)}=1\}_{i\in[2n+1]}\cup\{b_{nn}=1\}$
Sp_{2n}	$\{b_{ij}\}_{i\le j\in[2n]}$	$\{b_{ii}\,b_{(2n+2-i)(2n+2-i)}=1\}_{i\in[2n]}$
SO_{2n}	$\{b_{ij}\}_{i\le j\in[2n]}$	$\{b_{ii}\,b_{(2n+2-i)(2n+2-i)}=1\}_{i\in[2n]}$

TABLE 2. Generating a matrix $B\in\mathsf{B}$.

(ii) $\mathbf{Eq}(P_1)\cup\mathbf{Eq}(P_2)\cup\mathbf{Eq}(P_3)\cup\mathbf{Eq}(Q_1)\cup\mathbf{Eq}(Q_2)\cup\mathbf{Eq}(Q_3)$ in Table 2,
(iii) $P_1\dot{u}Q_1=\pi P_2\dot{v}Q_2$ and
(iv) $P_1\dot{u}Q_1=\rho P_3(\dot{w_\circ w})Q_3$.

Here Y is the type corresponding to G. The system $\mathcal{E}^Y(u,v,w)$ has the following variables:
(i) $\boldsymbol{\alpha}:=\mathbf{Var}(\pi)\cup\mathbf{Var}(\rho)$ in Table 1, and
(ii) $\boldsymbol{\beta}:=\mathbf{Var}(P_1)\cup\mathbf{Var}(P_2)\cup\mathbf{Var}(P_3)\cup\mathbf{Var}(Q_1)\cup\mathbf{Var}(Q_2)\cup\mathbf{Var}(Q_3)$ in Table 2.

We solve $\mathcal{E}^Y(u,v,w)$ over $\mathbb{C}(\boldsymbol{y},\boldsymbol{z})$, where $\boldsymbol{y}=\mathbf{Par}(\pi)$ and $\boldsymbol{z}=\mathbf{Par}(\rho)$ from Table 1.

Remark C.3. In types B and C, one should be careful to choose a representative $\dot{\omega}$ for $\omega\in\mathcal{W}$ with determinant 1. In type C, we may use a signed permutation matrix where $\omega_{ij}\in\{0,-1\}$ if $1\le i\le n$ and $n+1\le j\le 2n$, and $\omega_{ij}\in\{0,1\}$ otherwise. In type B, one may do this, or instead set the central value of $\omega_{(n+1)(n+1)}$ to be ±1 as needed. Note that in type D this property holds for any representative.

Proof of Lemma C.1 and Lemma 1.9. Fix type $Y\in\{A,B,C,D\}$. Let $u,v,w\in\mathcal{W}$ be elements in the corresponding Weyl group. By Equation (C.1), it follows that $\mathcal{E}^Y(u,v,w)$ is satisfiable over $\mathbb{C}(\boldsymbol{y},\boldsymbol{z})$ if and only if $c^w_{u,v}>0$. It is straightforward to check $\mathcal{E}^Y(u,v,w)$ has size $O(n^2)$. Thus, these systems are instances of HNP by construction, with sets of variables $(\boldsymbol{\alpha},\boldsymbol{\beta})$ and parameters $(\boldsymbol{y},\boldsymbol{z})$. Therefore, $\neg$SCHUBERTVANISHING is in HNP in each case, as desired. □