**Richard Stanley: D-finiteness of certain series associated with group algebras**

Let $G$ be a group and $\mathbb{Z}G$ its integral group algebra. For $u \in \mathbb{Z}G$ let $f_u(n) = [1]u^n$, the coefficient of the identity element of $G$ when $u^n$ is expanded in terms of the basis $G$. Set $F_u(x) = \sum_{n \geq 1} f_u(n)x^n$. If $F = F_d$, the free group on $d$ generators, then it is known that $F_u(x)$ is algebraic. This goes back to Chomsky and Schützenberger and seems first to have been explicitly stated by Haiman. If $G = \mathbb{Z}^d$ then it follows from standard facts about $D$-finite series that $F_u(x)$ is $D$-finite, though it need not be algebraic.

Maxim Kontsevich asked whether $F_u(x)$ is always $D$-finite when $G = \text{GL}(d, \mathbb{Z})$. This remains open, though it is known that the question of whether $F_u(x) = 0$ is undecidable. More generally, we can ask for which groups $G$ is $F_u(x)$ algebraic for all $u \in \mathbb{Z}G$, and for which groups is $F_u(x)$ $D$-finite for all $u \in \mathbb{Z}G$.

**Nick Wormald: Reduction of degree in the coefficients of a generating function**

**Problem:** Let $[y]_k$ denote $y(y-1)\cdots(y-k+1)$ and $A_t(y, z)$ the coefficient of $x^t$ in

$$\log \sum_{k \geq 0} \frac{[y]_k[z]_k}{k!} x^k.$$  

Clearly $A_t$ has total degree at most $2t$. Show that $A_t$ has total degree at most $t+1$ for $t \geq 1$.

**Notes:**

1. A short solution was quickly found by Ira Gessel, Gilles Schaeffer and Richard Stanley, each independently.
2. A similar question: show that for $t \geq 1$ the coefficient of $x^t$ in

$$\log \sum_{k \geq 0} \frac{[y]_{2m}}{m!} x^m$$

has degree $t+1$. This was also solved by Ira Gessel, using the solution to the main problem.
3. Ira Gessel has obtained the leading coefficients in both questions.