

# KRONECKER PRODUCTS, CHARACTERS, PARTITIONS, AND THE TENSOR SQUARE CONJECTURES

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**ABSTRACT.** We study the remarkable *Saxl conjecture* which states that tensor squares of certain irreducible representations of the symmetric groups  $S_n$  contain all irreducibles as their constituents. Our main result is that for sufficiently large  $n$  they contain representations corresponding to Young diagrams of hooks, two row and diagrams obtained from hooks and two rows by adding a finite number of squares. For that, we develop a new sufficient condition for the *positivity of Kronecker coefficients* in terms of characters, and apply this tool using combinatorics of rim hook tableaux in combination with known results on unimodality of certain partition functions. We also present connections and speculations on random characters of  $S_n$ .

## 1. INTRODUCTION AND MAIN RESULTS

Different fields have different goals and different open problems. Most of the time, fields peacefully coexist enriching each other and the rest of mathematics. But occasionally, a conjecture from one field arises to present a difficult challenge in another, thus exposing its technical strengths and weaknesses. The story of this paper is our effort in the face of one such challenge.

Motivated by *John Thompson's conjecture* and *Passman's problem* (see §10.8), Heide, Saxl, Tiep and Zalesski recently proved that with a few known exceptions, every irreducible character of a simple group of Lie type is a constituent of the tensor square of the Steinberg character [HSTZ]. They conjecture that for every  $n \geq 5$ , there is an irreducible character  $\chi$  of  $A_n$  whose tensor square  $\chi \otimes \chi$  contains every irreducible character as a constituent.<sup>1</sup> Here is the symmetric group analogue of this conjecture:

**Conjecture 1.1** (Tensor square conjecture). *For every  $n \geq 3$ ,  $n \neq 4, 9$ , there is a partition  $\mu \vdash n$ , such that tensor square of the irreducible character  $\chi^\mu$  of  $S_n$  contains every irreducible character as a constituent.*

The *Kronecker product problem* is a problem of computing multiplicities

$$g(\lambda, \mu, \nu) = \langle \chi^\lambda, \chi^\mu \otimes \chi^\nu \rangle$$

of an irreducible character of  $S_n$  in the tensor product of two others. It is often referred as “classic”, and “one of the last major open problems” in algebraic combinatorics [BWZ, Reg]. Part of the problem is its imprecise statement: we are talking about finding an explicit combinatorial interpretation here rather than computational complexity (see Subsection 10.9).

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<sup>1</sup>Authors of [HSTZ] report that this conjecture was checked by Eamonn O'Brien for  $n \leq 17$ .

Despite a large body of work on the Kronecker coefficients, both classical and very recent (see e.g. [BO2, Bla, BOR, Ike, Reg, Rem, RW, Val1, Val2] and references therein), it is universally agreed that “frustratingly little is known about them” [Bür]. Unfortunately, most results are limited to partitions of very specific shape (hooks, two rows, etc.), and the available tools are much too weak to resolve the tensor square conjecture.

During a talk at UCLA, Jan Saxl made the following conjecture, somewhat refining the tensor square conjecture.<sup>2</sup>

**Conjecture 1.2** (Saxl conjecture). *Denote by  $\rho_k = (k, k-1, \dots, 2, 1) \vdash n$ , where  $n = \binom{k+1}{2}$ . Then for every  $k \geq 1$ , the tensor square  $\chi^{\rho_k} \otimes \chi^{\rho_k}$  contains every irreducible character of  $S_n$  as a constituent.*

Andrew Soffer checked the validity of this conjecture for  $k \leq 8$ .<sup>3</sup> While we believe the conjecture, we also realize that it is beyond the reach of current technology. In Section 4, we briefly survey the implications of known tools towards the tensor product and Saxl conjectures. More importantly, we then develop a new tool, aimed specifically at the Saxl conjecture:

**Lemma 1.3** (Main Lemma). *Let  $\mu = \mu'$  be a self-conjugate partition of  $n$ , and let  $\nu = (2\mu_1 - 1, 2\mu_2 - 3, 2\mu_3 - 5, \dots) \vdash n$  be the partition whose parts are lengths of the principal hooks of  $\mu$ . Suppose  $\chi^\lambda[\nu] \neq 0$  for some  $\lambda \vdash n$ . Then  $\chi^\lambda$  is a constituent of  $\chi^\mu \otimes \chi^\mu$ .*

Curiously, the proof uses representation theory of  $A_n$  and is based on the idea of the proof of [BB, Thm 3.1] (see Section 9). We use this theorem to obtain the following technical results (among others).

**Theorem 1.4.** *There is a universal constant  $L$ , such that for every  $k \geq L$ , the tensor square  $\chi^{\rho_k} \otimes \chi^{\rho_k}$  contains characters  $\chi^\lambda$  as constituents, for all*

$$\begin{aligned} \lambda &= (n - \ell, \ell), \quad 0 \leq \ell \leq n/2, \quad \lambda = (n - r, 1^r), \quad 0 \leq r \leq n - 1, \\ \lambda &= (n - \ell - m, \ell, m), \quad m \in \{1, 3, 5, 7, 8, 9\}, \quad L \leq \ell + m \leq n/2, \\ \text{or } \lambda &= (n - r - m, m, 1^r), \quad 1 \leq m \leq 10, \quad L \leq r < n/2 - 5. \end{aligned}$$

Of course, this is only a first step towards proving the Saxl conjecture. While the Main Lemma is a powerful tool, proving that the characters are nonzero is also rather difficult in general, due to the alternating signs in the Murnaghan–Nakayama rule. We use a few known combinatorial interpretations (for small values of  $\ell$ ), and rather technical known results on monotonicity of the number of certain partitions (for larger  $\ell$ ), to obtain the above theorem and various related results. We should emphasize that the constant  $L$  in the theorem can be perhaps found explicitly, but that would entail extending analysis of partition asymptotics, which goes beyond the scope of this paper (see §10.5).

The rest of the paper is structured as follows. We begin with a discussion of combinatorics and asymptotics of the number of integer partitions in Section 2. We then turn to characters of  $S_n$  and basic formulas for their computation in Section 3. There, we introduce two more shape sequences (*chopped square* and *caret*), which will appear throughout the paper. In the next section (Section 4), we present several known results on the Kronecker product, and find easy applications to our problem. In the following three sections we present a large number of increasingly technical calculations evaluating the characters in terms of certain

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<sup>2</sup>UCLA Combinatorics Seminar, Los Angeles, March 20, 2012.

<sup>3</sup>Personal communication.

partition functions, and using Main Lemma and known partition inequalities to derive the results above. In a short Section 8, we discuss and largely speculate what happens for random characters. We prove the Main Lemma in Section 9, and conclude with final remarks.

## 2. INTEGER PARTITIONS

**2.1. Basic definitions.** Let  $\lambda \vdash n$  be a partition of  $n$ , and let  $P_n$  denote the set of partitions of  $n$ .

**2.2. Asymptotics.** Let  $\pi(n) = |P_n|$  be the number of partitions of  $n$ . Then

$$1 + \sum_{n=1}^{\infty} \pi(n)t^n = \prod_{i=1}^{\infty} \frac{1}{1-t^i},$$

and

$$\pi(n) \sim \frac{e^{c\sqrt{n}}}{4\sqrt{3n}}, \quad \text{where } c = \pi\sqrt{\frac{2}{3}}.$$

Denote by  $\pi_k(n)$  the number of partitions  $\lambda \vdash kn$ , such that their  $k$ -core is empty [Mac], i.e. there exists a rim hook tableau of shape  $\lambda$  and weight  $(k^n)$ . Note that  $\pi_1(n) = \pi(n)$ . Then by [Sta2, Exc.7.59e]

$$1 + \sum_{n=1}^{\infty} \pi_k(n)t^n = \prod_{i=1}^{\infty} \frac{1}{(1-t^i)^k},$$

and Lemma 4 in [LP] gives

$$\pi_k(n) \sim \left[ \frac{k^{k+1}}{2^{3k+5} 3^{k+1}} \right]^{1/4} \frac{e^{c\sqrt{kn}}}{n^{(k+3)/4}}, \quad \text{where } c = \pi\sqrt{\frac{2}{3}}.$$

**2.3. Limit shapes.** Let  $\lambda \vdash n$  be a *random partition on  $n$* , i.e. chosen uniformly at random from  $P_n$ . Scale by  $1/\sqrt{n}$  the Young diagram  $[\lambda]$  of a random partition. It is known that for every  $\varepsilon > 0$ , the scaled random shape is w.h.p. within  $\varepsilon$ -distance from the curve

$$e^{-cx/2} + e^{-cy/2} = 1,$$

where  $c$  is as above [DVZ, Ver]. Somewhat loosely, we call such  $\lambda$  the *limit shape*. Note that the limit shape is symmetric and has two infinite tails, so the longest part and the number of parts we have  $\lambda_1, \ell(\lambda) = \omega(\sqrt{n})$ , i.e.  $\ell(\lambda)$  dominates  $\sqrt{N}$  asymptotically. In fact, it is known that for random  $\lambda$ , we have  $\lambda_1, \ell(\lambda) = c^{-1}\sqrt{n}(\log n + O(1))$  w.h.p., as  $n \rightarrow \infty$  (see [Fri]).

**2.4. Partitions into infinite arithmetic progressions.** Fix  $a, m \geq 1$ , such that  $\gcd(a, m) = 1$ . Define integers  $\pi'_{a,m}(n)$  by

$$\sum_{n=0}^{\infty} \pi'_{a,m}(n) t^n = \prod_{r=0}^{\infty} (1 + t^{a+rm}).$$

In other words,  $\pi'_{a,m}(n)$  is the number of partitions of  $n$  into distinct parts in the arithmetic progression  $R = \{a, a+m, a+2m, \dots\}$ . It is known [RS] that

$$\pi'_{a,m}(n+1) > \pi'_{a,m}(n) > 0,$$

for all  $n$  large enough. Below we present a stronger result.

**2.5. Partitions into finite arithmetic progressions.** Denote by  $R = R(a, m, k) = \{a, a+m, a+2m, \dots, a+km\}$  a finite arithmetic progression, with  $a, m \geq 1$ , such that  $\gcd(a, m) = 1$  as above. Denote by  $\pi'_R$  the coefficients in

$$\sum_{n=0}^N \pi'_R(n) t^n = \prod_{r=0}^k (1 + t^{a+rm}),$$

where  $N = (k+1)a + \binom{k+1}{2}m$  is the largest degree with a nonzero coefficient. Note that the sequence  $\{\pi'_R(n)\}$  is symmetric:

$$\pi'_R(n) = \pi'_R(N-n).$$

The following special case of a general result by Odlyzko and Richmond [OR] is the key tool we use throughout the paper.

**Theorem 2.1** ([OR]). *For every  $a$  and  $m$  with  $\gcd(a, m) = 1$ , there exists a constant  $L = L(a, m)$ , such that for every  $R = R(a, m, k)$  as above,*

$$\pi'_R(n+1) > \pi'_R(n) > 0, \quad \text{for all } L \leq n < \lfloor N/2 \rfloor.$$

In other words,

$$\pi'_R(L) < \dots < \pi'_R\left(\frac{N}{2}-1\right) < \pi'_R\left(\frac{N}{2}\right) > \pi'_R\left(\frac{N}{2}+1\right) > \dots > \pi'_R(N-L)$$

for even  $N$ , and

$$\pi'_R(L) < \dots < \pi'_R\left(\frac{N-1}{2}\right) = \pi'_R\left(\frac{N+1}{2}\right) > \dots > \pi'_R(N-L)$$

for odd  $N$ . Note that  $k$  in the theorem has to be large enough to ensure that  $L < N/2$ ; otherwise, the theorem is trivially true (there is no such  $n$ ).

### 3. KRONECKER PRODUCTS AND CHARACTERS

**3.1. Young diagrams.** We assume the reader is familiar with the standard results in combinatorics and representation theory of the symmetric group (see e.g. [Mac, Sag, Sta2]). Let us review some notations, definitions and basic results.

Denote by  $\lambda'$  the conjugate partition of  $\lambda$ . Partition  $\lambda$  is called *self-conjugate* if  $\lambda = \lambda'$ . Denote by  $\ell(\lambda) = \lambda'_1$  the number of parts in  $\lambda$ . We use  $[\lambda]$  to denote the Young diagram corresponding to the partition  $\lambda$  and a *hook length* by  $h_{ij} = \lambda_i + \lambda'_j - i - j + 1$ , where  $(i, j) \in [\lambda]$ . We denote by  $d(\lambda)$  the *Durfee size* of  $\lambda$ , i.e. the size of the main diagonal in  $[\lambda]$ . Define a *principal hook partition*  $\widehat{\lambda} = (h_{1,1}, \dots, h_{s,s})$ , where  $s = d(\lambda)$ . Observe that  $\widehat{\lambda} \vdash n$ .

**3.2. Rim hook tableaux.** We use  $\chi^\lambda[\nu]$  to denote the value of an irreducible character  $\chi^\lambda$  on the conjugacy class of cycle type  $\nu$  of the symmetric group  $S_n$ . For a sequence  $\mathbf{a} = (a_1, \dots, a_\ell)$ ,  $\ell = \ell(\lambda)$ , denote by  $\text{RH}(\lambda, \mathbf{a})$  the set of *rim hook tableaux* of shape  $\lambda$  and weight  $\mathbf{a}$ , with rim hooks  $h_i$  of size  $|h_i| = a_i$ . The  $\text{sign}(A)$  of a tableaux  $A \in \text{RH}(\lambda, \mathbf{a})$  is the product of  $(-1)^{\ell(h_i)-1}$  over all rim hooks  $h_i \in A$ . The *Murnaghan–Nakayama rule* then says that for every permutation  $\mathbf{a}$  of  $\nu$ ,

$$\chi^\lambda[\nu] = \sum_{A \in \text{RH}(\lambda, \mathbf{a})} \text{sign}(A).$$

More generally, the result extend verbatim to skew shapes  $\lambda/\mu$  (see e.g. [Sta2, §7.17]).

**3.3. The Frobenius and Giambelli formulas.** Recall the *Frobenius formula* for the character  $\chi^\lambda$ ,  $\ell(\lambda) = 2$  :

$$\chi^{(n-\ell, \ell)} = \chi^{(n-\ell) \circ (\ell)} - \chi^{(n-\ell+1) \circ (\ell-1)}.$$

where  $\lambda \circ \mu$  is a skew partition as in the Figure 1.

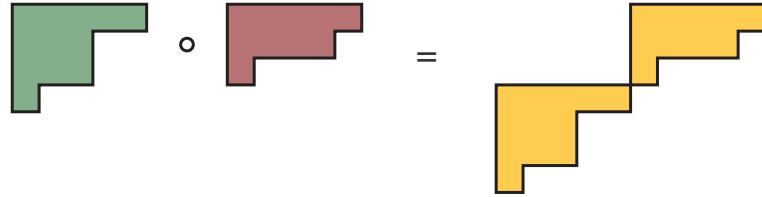


FIGURE 1. Partitions  $\lambda$ ,  $\mu$ , and  $\lambda \circ \mu$ , where  $\lambda = (5, 3, 3, 1)$ ,  $\mu = (5, 4, 1)$

Similarly, the *Giambelli formula* for the character  $\chi^\lambda$ , where  $d(\lambda) = 2$  :

$$\chi^{(a_1+1, a_2+2, 2^{b_2}, 1^{b_1-b_2-1})} = \chi^{(a_1+1, 1^{b_1}) \circ (a_2+1, 1^{b_2})} - \chi^{(a_1+1, 1^{b_2}) \circ (a_2+1, 1^{b_1})}.$$

where  $n = a_1 + a_2 + b_1 + b_2 + 2$ . The formula is illustrated in Figure 2 (here  $a_1 = 8$ ,  $a_2 = 2$ ,  $b_1 = 5$ , and  $b_2 = 3$ ).

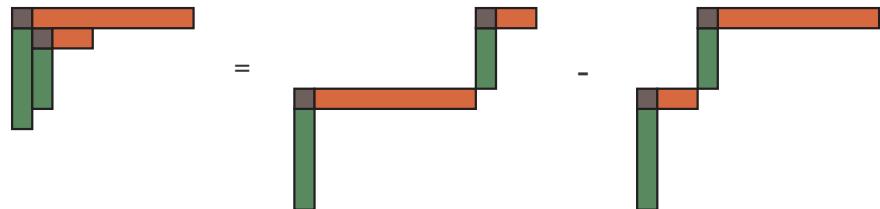


FIGURE 2. Partitions  $\lambda = (9, 4, 2^3, 1)$ ,  $(9, 1^5) \circ (3, 1^3)$ , and  $(3, 1^5) \circ (9, 1^3)$ .

**3.4. Kronecker products.** Let  $\lambda, \mu \vdash n$ . The *Kronecker product* of characters  $\chi^\lambda$  and  $\chi^\mu$  satisfies  $(\chi^\lambda \otimes \chi^\mu)[\nu] = \chi^\lambda[\nu] \cdot \chi^\mu[\nu]$ . *Kronecker coefficients* are defined as

$$g(\lambda, \mu, \nu) := \langle \chi^\lambda \otimes \chi^\mu, \chi^\nu \rangle = \langle 1, \chi^\lambda \otimes \chi^\mu \otimes \chi^\nu \rangle,$$

and thus they are symmetric for all  $\lambda, \mu, \nu \vdash n$ .

For a partition  $\lambda \vdash n$ , denote by  $\Phi(\lambda)$  the set of  $\mu \vdash n$  such that  $g(\mu, \lambda, \lambda) > 0$ . The tensor product conjecture says that  $\Phi(\lambda) = P_n$  for some  $\lambda$ . The Saxl conjecture says that  $\Phi(\rho_k) = P_n$ . We call  $\rho_k$  the *staircase shape* of order  $k$ .

**3.5. Chopped square.** Denote  $\eta_k = (k^{k-1}, k-1) \vdash k^2 - 1$ . We call  $\eta_k$  the *chopped square shape* of order  $k$ . Obviously,  $d(\eta_k) = k-1$ , and the scaled limit shape is a  $1/2 \times 1/2$  square.

**Conjecture 3.1.** *For all  $k \geq 2$ , we have  $\Phi(\eta_k) = P_n$ .*

This conjecture was checked by Andrew Soffer for  $k \leq 5$ .

**3.6. Caret shape.** Consider

$$\gamma_k = (3k-1, 3k-3, \dots, k+3, k+1, k, k-1, k-1, k-2, k-2, \dots, 2, 2, 1, 1),$$

which we call a *caret shape*. Note that  $\gamma'_k = \gamma_k$ ,  $n = |\gamma_k| = 3k^2$ ,  $d(\gamma_k) = k$ , and the principal hook partition  $\widehat{\gamma}_k = (6k-3, 6k-9, \dots, 3)$ . After  $1/\sqrt{n}$  scaling, the partition has a 4-gon limit shape as in Figure 3.



FIGURE 3. Partition  $\gamma_5$  and the limit shape of  $\gamma_k$ .

**Conjecture 3.2.** *For all  $k \geq 2$ , we have  $\Phi(\gamma_k) = P_n$ .*

**Remark 3.3.** Despite the less elegant shape of  $\gamma_k$  partitions, there seems to be nearly as much evidence in favor of Conjecture 3.2 as in favor of the Saxl Conjecture 1.2 (see corollaries 5.4 and 6.6). See Section 8 for more caret shapes and possibility of other self-conjugate shapes satisfying the Tensor Product Conjecture.

#### 4. KNOWN RESULTS AND SPECIAL CASES

**4.1. General results.** The following results are special cases of known results about Kronecker products, applied to our case.

**Lemma 4.1.** *Let  $\nu \vdash n$ . Then  $\chi^\mu \otimes \chi^{(n)} = \chi^\mu$  and  $\chi^\mu \otimes \chi^{(1^n)} = \chi^{\mu'}$ .*

*Proof.* Note that  $\chi^{(n)}$  is the trivial character and  $\chi^{(1^n)}$  is the sign character. The first identity is trivial; the second follows from the Murnaghan–Nakayama rule.  $\square$

**Corollary 4.2.** *Let  $\mu \vdash n$ . Then  $\mu = \mu'$  if and only if  $(1^n) \in \Phi(\mu)$ .*

*Proof.* By Lemma 4.1

$$\langle \chi^\mu \otimes \chi^\mu, \chi^{(1^n)} \rangle = \langle \chi^\mu, \chi^\mu \otimes \chi^{(1^n)} \rangle = \langle \chi^\mu, \chi^{\mu'} \rangle.$$

So, the claim follows.  $\square$

**Proposition 4.3.** *Let  $\mu \vdash n$ . If  $\mu = \mu'$ , then  $g(\lambda, \mu, \mu) = g(\lambda', \mu, \mu)$ .*

*Proof.* Simply observe that by Lemma 4.1 we have

$$\begin{aligned} g(\lambda, \mu, \mu) &= \langle \chi^\lambda, \chi^{\mu'} \otimes \chi^\mu \rangle = \langle \chi^\lambda, \chi^\mu \otimes \chi^{(1^n)} \otimes \chi^\mu \rangle \\ &= \langle \chi^\lambda, \chi^{(1^n)} \otimes \chi^\mu \otimes \chi^\mu \rangle = \langle \chi^\lambda \otimes \chi^{(1^n)}, \chi^{\mu'} \otimes \chi^\mu \rangle = g(\lambda', \mu, \mu), \end{aligned}$$

as desired.  $\square$

**Theorem 4.4** ([BR]). *If  $d(\lambda) > 2d(\mu)^2$ , then  $\lambda \notin \Phi(\mu)$ .*<sup>4</sup>

**Corollary 4.5.** *If  $d(\mu) < \frac{n^{1/4}}{\sqrt{2}}$ , then  $\Phi(\mu) \neq P_n$ .*

Therefore, if  $\mu \neq \mu'$  or  $2d(\mu)^2 < \sqrt{n}$ , then character  $\chi^\mu$  cannot be used in the tensor product conjecture.

**Theorem 4.6** ([BB]). *If  $\lambda = \lambda'$ , then  $\lambda \in \Phi(\lambda)$ , i.e.  $g(\lambda, \lambda, \lambda) > 0$ .*

**Remark 4.7.** Note that neither of the results in this section disproves the Saxl conjecture, nor conjectures 3.1 and 3.2. Indeed, all these partitions are self-conjugate and have Durfee size of the order  $\Theta(\sqrt{n})$ .

**4.2. Large  $\mu_1$ .** Let  $\nu$  be a composition and denote by  $r_\nu(\mu)$  the number of ways to remove a ribbon, which has  $\nu_i$  boxes on row  $i$ , from  $\mu$  such that  $\mu/\nu$  is a Young diagram. The following result is given in [Val3, §7.13] (some of these formulas were given earlier in [Saxl, Zis]).

**Theorem 4.8.** *For  $\lambda = (n - r, \tau)$  and  $\tau \vdash r$ , denote  $f(\tau, \mu) = g(\lambda, \mu, \mu)$ . Then:*

$$\begin{aligned} f(\emptyset, \mu) &= 1, & f(1, \mu) &= r_1(\mu) - 1, & f(1^2, \mu) &= (r_1(\mu) - 1)^2, \\ f(2, \mu) &= r_2(\mu) + r_{12}(\mu) + r_1(\mu)^2 - 2r_1(\mu), \\ f(3, \mu) &= r_3(\mu) + r_{13}(\mu) + r_{21}(\mu) + r_{12}(\mu) + (2r_1(\mu) - 3)(r_2(\mu) + r_{12}(\mu)) \\ &\quad + r_1(\mu)^3 - 4r_1(\mu)^2 + 3r_1(\mu), \\ f(21, \mu) &= r_{21}(\mu) + r_{12}(\mu) + (3r_1(\mu) - 4)(r_2(\mu) + r_{12}(\mu)) \\ &\quad + 2r_1(\mu)^3 - 8r_1(\mu)^2 + 7r_1(\mu), \\ f(1^3, \mu) &= r_{21}(\mu) + r_{12}(\mu) + (r_1(\mu) - 1)(r_2(\mu) + r_{12}(\mu)) \\ &\quad + r_1(\mu)^3 - 4r_1(\mu)^2 + 4r_1(\mu) - 1. \end{aligned}$$

Calculating these values explicitly, gives the following result:

**Corollary 4.9.** *Let  $\mu \vdash n$ . If  $\mu = \mu'$  and  $\mu$  is not a square, we have:*

$$(n), (n-1, 1), (n-2, 2), (n-2, 1^2), (n-3, 3), (n-3, 2, 1), (n-3, 1^3) \in \Phi(\mu).$$

However,  $(n-1, 1), (n-2, 1^2) \notin \Phi(k^k)$  for  $n = k^2$ .

In other words, the theorem rules out square partitions in the tensor product conjecture.

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<sup>4</sup>Thm. 3.26 in [BR] is more general, and applies to all diagonal lengths of constituents in all  $\chi^\lambda \otimes \chi^\nu$ .

*Proof.* For  $n = 3$  the statement follows from a direct calculation, so we assume  $n \geq 4$ . Since  $\mu$  is not a square,  $r_1(\mu) \geq 2$ . So, we have that  $(n)$ ,  $(n-1, 1)$  and  $(n-2, 1^2)$  are in  $\Phi(\mu)$ . Let us consider first the case  $r_1(\mu) = 2$ . If  $\mu$  is not a hook, since  $\mu = \mu'$ , one has either  $r_2(\mu) + r_{12}(\mu) \geq 4$  or  $r_2(\mu) + r_{12}(\mu) = 2$  and  $r_{21}(\mu) = 1$ ; so, the remaining partitions are in  $\Phi(\mu)$ . If  $\mu$  is a hook, since  $\mu = \mu'$ , one has  $r_2(\mu) + r_{12}(\mu) = 2$ , which implies that  $(n-2, 2)$ ,  $(n-3, 2, 1)$  and  $(n-3, 1^3)$  are in  $\Phi(\mu)$ . For  $(n-3, 3)$ , we must have  $n \geq 6$ , thus  $r_3(\mu) + r_{13}(\mu) = 2$ , and we obtain that  $(n-3, 3)$  is in  $\Phi(\mu)$ . Finally, if  $r_1(\mu) \geq 3$  all the polynomials in  $r_1$  at the end of the summations are nonnegative, and we have either  $r_2(\mu) + r_{12}(\mu) \geq 2$  or  $r_{21}(\mu) \geq 1$ . So, the claim follows.  $\square$

**Example 4.10.** A direct calculation gives

$$\chi^{(2^2)} \otimes \chi^{(2^2)} = \chi^{(4)} + \chi^{(2^2)} + \chi^{(1^4)}.$$

In other words,  $\Phi(2^2)$  is missing only  $(3, 1)$  and  $(2, 1^2)$ .

**4.3. Two rows.** There are many results on Kronecker products of characters with at least one partition with two row (see e.g. [BO1, BO2, BOR, RW, Ros]). Of these, only the result of Ballantine and Orellana [BO2] extends to general partitions. Here we briefly describe their statement.

An  $\alpha$ -lattice permutation is a sequence  $a = a_1 a_2 \dots$  of letters among  $1, 2, \dots$ , such that the sequence  $q^{\alpha_1} 2^{\alpha_2} \dots a_1 a_2 \dots$  is an ordinary lattice permutation (ballot sequence). A *Kronecker tableaux*, indexed by  $\lambda, \nu, \alpha$ , is a SSYT of shape  $\lambda/\alpha$  and type  $\nu/\alpha$ , whose reverse reading word (obtained by reading the entries in the tableaux from right to left along rows and top to bottom), is an  $\alpha$ -lattice permutation, and if  $\alpha_1 > \alpha_2$  we have that the number of 1s on the first row, or 2s on the second row is exactly  $\alpha_1 - \alpha_2$ . The result in [BO2] states that if  $\lambda_1 \geq 2p - 1$ , then  $g(\lambda, \mu, (n-p, p))$  is equal to the total number of Kronecker tableaux indexed by  $\lambda, \nu, \alpha$ , where  $\alpha$  runs over all partitions of  $p$ .

**Theorem 4.11.** *For every  $\mu \vdash n$ ,  $2 \leq p \leq \min\{\ell(\mu), \frac{1+\mu_1}{2}\}$ , we have  $(n-p, p) \in \Phi(\mu)$ .*

*Proof.* By the rule of [BO2, Thm, 3.2] just described, let  $\alpha = (1^p)$ . Consider the tableau  $T$  of shape  $(\mu/1^p)$  filled with numbers  $i$  in  $i$ -th row. It is then also of type  $\mu/\alpha$  and its reverse reading word is an  $\alpha$ -lattice permutation. Since  $p > 1$  we have that  $\alpha_1 = \alpha_2 = 1$ , so there are no further restrictions and  $T$  is a particular Kronecker tableaux indexed by  $\mu, \mu, \alpha$ . This immediately gives  $g(\mu, \mu, (n-p, p)) \geq 1$  as desired.  $\square$

For example, for  $\mu = \rho_k$ , we have  $\ell(\mu) = \mu_1 = k$ , and the result gives positive Kronecker coefficients  $k((n-p, p), \mu, \mu)$  for  $p \leq (k+1)/2$ . In Section 6, we improve the above bound to  $p \leq n/2$ , which covers all two row shapes.

**4.4. Hooks.** Of the extensive literature, Blasiak's combinatorial interpretation (see [Bla], Theorem 3.5), is perhaps the most convenient. We state it here briefly. We consider the sets of barred and unbarred numbers  $\bar{1}, \bar{2}, \dots, 1, 2, \dots$  with two possible orders:  $\bar{1} < 1 < \bar{2} < 2 < \dots$  and  $\bar{1} \prec \bar{2} \prec \dots \prec 1 \prec 2 \dots$ . A tableau with respect to any of these given orders is a filling of a Young diagram with barred and unbarred letters, such that the entries weakly increase along rows and down columns with respect to the given order, such that no two equal unbarred letters are in the same column, and no two equal barred letters are in the same row. A colored Yamanouchi tableau of shape  $\lambda$ , content  $\mu$  and total color  $m$  is a tableau of shape  $\lambda$  under the  $\prec$  order, filled with barred and unbarred letters, such that: the total number of  $i$  and  $\bar{i}$  entries is  $\mu_i$ ; there are exactly  $m$  barred entries; the reading word

obtained by reading first the unbarred numbers from right to left, top to bottom, and then the barred numbers from left to right, bottom to top, is a ballot sequence after removing all bars. The Theorem then states that  $g(\lambda, \mu, (n-m, 1^m))$  is equal to the number of colored Yamanouchi tableau of shape  $\lambda$ , content  $\mu$  and total color  $m$ , such that after performing a jeu-de-taquin operation on the barred letters and converting to a tableau under the order  $<$ , the southwest corner is an unbarred letter.

**Theorem 4.12.** *Let  $\mu = (\mu_1, \dots, \mu_\ell)$  be a partition of  $n$ , such that  $\mu_1 > \dots > \mu_r$  for some  $r \leq \ell$ . Then:*

$$(n-m, 1^m) \in \Phi(\mu) \quad \text{for all } m < r.$$

*Proof.* Consider the tableau  $T$  of shape  $\mu$ , filled with numbers  $i$  in  $i$ -th row. Now place a bar on the first  $m$  integers in the first column. Denote by  $w$  the word obtained by reading the unbarred skew shape from right to left, and then barred shape from left to right (in this case, just the first column from bottom to top). We have:

$$w = 1^{\mu_1-1} 2^{\mu_2-1} \dots m^{\mu_m-1} (m+1)^{\mu_{m+1}} \dots \ell^{\mu_\ell} m(m-1) \dots 21$$

The inequalities in the statement imply that word  $w$  is a *ballot sequence*, so  $T$  is a colored Yamanouchi tableau of shape  $\mu$ , type  $\mu$  and total color  $m$ .

Consider tableau  $C(T)$  obtained when  $T$  is converted into a tableau under the order  $<$  via jeu-de-taquin moves. Since  $T$  is also a tableau under the  $<$  order, no moves are necessary to convert it, so  $C(T) = T$ . By the construction and the restriction of  $m < \ell$ , we have the lower left corner of  $C(T)$  at  $(\ell, 1)$  unbarred. Therefore, tableau  $T$  gives the desired tableau in the combinatorial interpretation of  $g((n-m, 1^m), \mu, \mu)$ , which shows that  $g((n-m, 1^m), \mu, \mu) \geq 1$ .  $\square$

**4.5. Large Durfee size.** The following result is well known and easy to prove.

**Lemma 4.13.** *We have  $\chi^\mu[\hat{\mu}] = \pm 1$ , for all  $\mu \vdash n$ . Moreover, if  $\mu = \mu'$ , then*

$$\chi^\mu[\hat{\mu}] = (-1)^{(n-d(\mu))/2}.$$

*Proof.* The principal hook condition implies that there is a unique rim hook condition in the Murnaghan–Nakayama rule. The second part follows by taking the product of signs of all hooks.  $\square$

**Proposition 4.14.** *We have:  $|\Phi(\rho_k)| > 3^{\lceil k/2 \rceil - 1}$  and  $|\Phi(\gamma_k)| > 5^{k-1}$ .*

*Proof.* There are two sequences of principal hook partitions for  $\rho_k$ : For  $k = 2m + 1$  odd the sequence is  $(4m+1, 4m-3, \dots, 5, 1)$ . For  $k = 2m$  even the sequence is  $(4m-1, 4m-5, \dots, 7, 3)$ . In each case the proof is by induction on  $m$ . We show the even case. The odd one is similar. For each  $\lambda \in \{(3), (2, 1), (1^3)\}$  there is exactly one rim hook tableau of shape  $\lambda$  and weight  $(3)$ . So that, by the Murnaghan–Nakayama rule and the Main Lemma,  $|\Phi(\rho_2)| = 3 > 1$ . We assume, by induction hypothesis, that there are  $3^m$  partitions  $\lambda$  such that for each of them there is exactly one rim hook tableau of shape  $\lambda$  and weight  $(3, \dots, 4m-1)$ , and that the rim hook of size  $4m-1$  intersects the first row and the first column of  $[\lambda]$ . For each such  $\lambda$  we construct three partitions as follows: Let  $H$  be the rim hook in  $\lambda$  of size  $4m-1$  with end boxes  $(1, a)$  and  $(b, 1)$ . Define

$$\tilde{H} = \{(x+1, y+1) \mid (x, y) \in H\} \cup \{(1, a+1), (b+1, 1)\}.$$

Then  $|\tilde{H}| = 4m + 1$ . Define partitions of size  $(4m + 3) + (4m - 1) + \dots$  by

$$\begin{aligned} [\lambda(1)] &= [\lambda] \cup \tilde{H} \cup \{(1, a+2), (1, a+3)\}; \\ [\lambda(2)] &= [\lambda] \cup \tilde{H} \cup \{(1, a+2), (b+2, 1)\}; \\ [\lambda(3)] &= [\lambda] \cup \tilde{H} \cup \{(b+2, 1), (b+3, 1)\}. \end{aligned}$$

We claim that for each  $i = 1, 2, 3$ ,  $\lambda(i)$  has exactly one rim hook tableau of shape  $\lambda(i)$  and weight  $(3, 7, \dots, 4m + 3)$ . The southeast border of  $[\lambda(i)]$  is exactly a rim hook  $H(i)$  of size  $4m + 3$ . It intersects the first row and the first column of  $[\lambda(i)]$ . By construction  $[\lambda(i)] \setminus H(i) = [\lambda]$ . But, by induction hypothesis, there is only one rim hook tableau of shape  $\lambda$  and weight  $(3, 7, \dots, 4m - 1)$ . So, there is only one rim hook tableau of shape  $\lambda(i)$  and weight  $(3, 7, \dots, 4m + 3)$ . So, by the Main Lemma,  $\lambda(i) \in \Phi(\rho_{2m+2})$ .

It remains to show that the  $3^{m+1}$  partitions just constructed are all different. This follows also by induction and the fact that the construction of  $\lambda(i)$  from  $\lambda$  is reversible, since  $\lambda(i)$  has exactly  $3 - i$  parts of size 1. Thus,  $|\Phi(\rho_{2m+2})| \geq 3^{m+1} > 3^m$ .

For the caret shapes there is only one sequence of principal hook partitions  $\hat{\gamma}_k = (6k - 3, 6k - 9, \dots, 3)$ . Since  $d(\gamma_k) = k$ , a similar argument now proves the second claim.  $\square$

In other words, in both cases the number of irreducible constituents is weakly exponential  $\exp \Theta(\sqrt{n})$ . Indeed, the corollary gives the lower bound and the asymptotics for  $\pi(n)$  gives the upper bound. Note also that the lemma gives nothing for the chopped square shape  $\eta_k$ .

**4.6. Large principal hooks.** The following result is a trivial consequence of the classical Murnaghan–Nakayama rule.

**Lemma 4.15.** *Suppose  $\lambda, \mu \vdash n$  and  $\hat{\lambda}_1 < \hat{\mu}_1$ . Then  $\chi^\lambda[\hat{\mu}] = 0$ .*

From here we conclude the following counterpart of Proposition 4.14.

**Proposition 4.16.** *There are at least  $3^{\lceil k/2 \rceil - 3}$  partitions  $\lambda$  of  $n = k(k+1)/2$  such that  $\chi^\lambda[\hat{\rho}_k] = 0$ . Similarly, there are at least  $5^{k-3}$  partitions  $\lambda$  of  $n = 3k^2$  such that  $\chi^\lambda[\hat{\gamma}_k] = 0$ .*

*Proof.* Follow the construction as in the proof of Proposition 4.14, to construct  $3^{\lceil k/2 \rceil - 3}$  partitions  $\lambda$  with principal hooks of size

$$(4m - 1, 4m - 3, 4m - 7, 4m - 11, \dots, 5, 3) \quad \text{for } k = 2m + 1, \quad \text{and}$$

$$(4m - 3, 4m - 5, 4m - 9, 4m - 13, \dots, 7, 5) \quad \text{for } k = 2m.$$

Here  $3^2$  possibilities are lost when counting placements of the outer and the inner rim hooks. By the lemma above, all such characters  $\chi^\lambda[\hat{\rho}_k] = 0$ . The second part follows verbatim.  $\square$

The proposition implies that there is a weakly exponential number of partitions for which the Saxl conjecture and Conjecture 3.2 cannot be proved. Curiously, this approach does not apply to  $\eta_k$ . In Subsection 8.1 we prove a much stronger result about the number of partitions  $\lambda$  such that  $\chi^\lambda[\hat{\gamma}_k] = 0$ .

**Example 4.17.** Of course, just because  $\chi^\lambda[\hat{\mu}] = 0$  it does not mean that  $\lambda \notin \Phi(\mu)$ . For example  $\chi^{(5,1)} \in \Phi(\rho_3)$ , even though

$$\chi^{(5,1)}[\hat{\rho}_3] = \chi^{(5,1)}[5, 1] = 0.$$

## 5. HOOKS IN TENSOR SQUARES

**5.1. Chopped square shape.** Let  $n = k^2 - 1$ , so that  $\eta_k = (k^{k-1}, k-1) \vdash n$ . Recall that  $\widehat{\eta}_k = (2k-1, 2k-3, \dots, 7, 5, 3)$ .

**Lemma 5.1.** *There exists a constant  $L$ , s.t.  $(n-\ell, 1^\ell) \in \Phi(\eta_k)$ , for all  $L \leq \ell < n/2$ .*

*Proof.* By the Main Lemma (Lemma 1.3), it suffices to show that

$$\chi^{(n-\ell, 1^\ell)}[\widehat{\eta}_k] > 0 \quad \text{for } \ell \text{ large enough.}$$

We claim that the above character is equal to

$$(*) \quad \pi'_R(\ell) - \pi'_R(\ell-1) + \pi'_R(\ell-2)$$

where  $R = \{5, 7, \dots, 2k-1\}$  (see §2.4 for notations). By Theorem 2.1,

$$\pi'_R(\ell) > \pi'_R(\ell-1) > \pi'_R(\ell-2) > 0$$

for  $\ell \leq n/2$  large enough, this would prove the theorem.

For  $(*)$ , by the Murnaghan–Nakayama rule, the character is equal to the sum over all rim hook tableaux of shape  $(n-\ell, 1^\ell)$  and weight  $\widehat{\eta}_k$  of the sign of the tableaux. For convenience, order the parts of  $\widehat{\eta}_k$  in increasing order. Note that the sign of every rim hook which fits inside in either the leg or the arm of the hook is positive. There are 3 ways to place a 3-hook, with the foot of size  $\ell$ ,  $\ell-1$  and  $\ell-2$ , respectively. Therefore, the number of rim hook tableaux is equal to the number of partitions into distinct parts in  $R$ , as in  $(*)$ .  $\square$

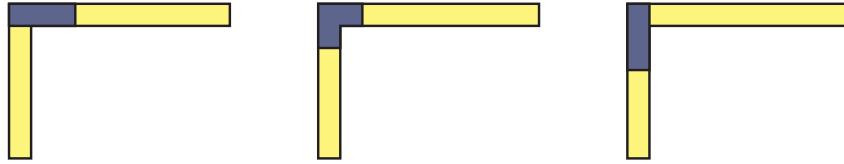


FIGURE 4. Three ways to place a 3-hook into a hook diagram.

**Example 5.2.** Although we made no attempt to find constant  $L$  in the lemma, we know that it is rather large even for  $k \rightarrow \infty$ . For example,

$$\pi'_{5,2}(21) - \pi'_{5,2}(20) + \pi'_{5,2}(19) = 0,$$

which implies  $\chi^{(n-21, 1^{21})}[\widehat{\eta}_k] = 0$  for all  $n \geq 21$ ,  $k \geq 11$ . Note also that function  $\pi'_{5,2}(n)$  continues to be non-monotone for larger  $n$ , i.e.  $\pi'_{5,2}(41) = 15$  and  $\pi'_{5,2}(42) = 14$ .

**5.2. Caret shape.** Let  $n = 3k^2$ , so that  $\gamma_k \vdash n$ . Recall that  $\widehat{\gamma}_k = (6k-3, 6k-9, \dots, 9, 3)$ . The following is the analogue of Lemma 5.1 for the caret shape.

**Lemma 5.3.** *There exists a constant  $L$ , such that  $(n-\ell, 1^\ell) \in \Phi(\gamma_k)$ , for all  $L \leq \ell \leq n/2$ .*

Combined with Theorem 4.12 as above, we obtain:

**Corollary 5.4.** *For  $k$  large enough, we have  $(n-\ell, 1^\ell) \in \Phi(\gamma_k)$ , for all  $1 \leq \ell \leq n-1$ .*

*Proof.* Let  $k > L$ . Theorem 4.12 proves the case  $\ell \leq k-1$  and the lemma gives  $L \leq \ell \leq n/2$ . In total, these cover all  $0 \leq \ell \leq n/2$ . Now Proposition 4.3 prove the remaining cases  $n/2 < \ell \leq n-1$ .  $\square$

*Proof of Lemma 5.3.* The proof follows the argument in the proof of Lemma 5.1. The difference is that the removed 3-rim hook can be removed only one way, as the other two values are not zero mod 3. This simplifies the character evaluation and gives

$$|\chi^{(n-\ell, 1^\ell)}[\widehat{\gamma}_k]| = \pi'_R(\lfloor \ell/3 \rfloor),$$

where  $R = \{3, 5, \dots, 2k-1\}$  is obtained from  $\widehat{\gamma}_k$  by removing the smallest part and then dividing by 3. Thus, the above character is nonzero, and the Main Lemma implies the result.  $\square$

**5.3. Staircase shape.** Let  $n = \binom{k+1}{2}$ , so that  $\rho_k \vdash n$ . The following result is the analogue of Lemma 5.1 for the staircase shape. Note that  $\widehat{\rho}_k = (2k-1, 2k-5, 2k-9, \dots)$ . There are two different cases: odd  $k$  and even  $k$ , which correspond to the smallest principal hooks  $(\dots, 9, 5, 1)$  and  $(\dots, 11, 7, 3)$ , respectively.

**Lemma 5.5.** *There exists a constant  $L$ , such that  $(n-\ell, 1^\ell) \in \Phi(\rho_k)$ , for all  $L \leq \ell < n/2$ .*

Combined with Theorem 4.12 and Proposition 4.3, we obtain:

**Corollary 5.6.** *For  $k$  large enough, we have  $(n-\ell, 1^\ell) \in \Phi(\rho_k)$ , for all  $1 \leq \ell \leq n-1$ .*

The proof of the corollary follows verbatim the proof of Corollary 5.4.

*Proof of Lemma 5.5.* Let  $\lambda = (n-\ell, 1^\ell)$  as above. Treat the even  $k$  case in the same way as that of Lemma 5.1 above. Then  $\widehat{\rho}_k = \{2k-1, \dots, 7, 3\}$ . Again, there are three ways to fit a 3-rim hook in the hook  $\lambda$ , so

$$\chi^\lambda[\widehat{\rho}_k] = \pi'_R(\ell) - \pi'_R(\ell-1) + \pi'_R(\ell-2),$$

where  $R = \{2k-1, \dots, 7, 3\}$ . By Theorem 2.1 we have  $\pi'_R(\ell) - \pi'_R(\ell-1) > 0$ , so the character is nonzero.

The odd  $k$  case is even easier: hook 1 can be placed in  $[\lambda]$  in a unique way, after which we get  $\chi^\lambda[\widehat{\rho}_k] = \pi'_R(\ell)$ , which is the number of partitions into distinct parts  $R = \{5, 9, 13, \dots, 2k-1\}$ . This is again nonzero by Theorem 2.1.

In both cases  $\chi^\lambda[\widehat{\rho}_k] > 0$  and the Main Lemma now implies the result.  $\square$

## 6. TWO ROW SHAPES IN TENSOR SQUARES

**6.1. Chopped square shape.** Let  $\eta_k = (k^{k-1}, k-1) \vdash n$ ,  $n = k^2 - 1$ , be as above.

**Lemma 6.1.** *There exists a constant  $L$ , such that  $(n-\ell, \ell) \in \Phi(\eta_k)$ , for all  $L \leq \ell \leq n/2$ .*

This immediately gives:

**Corollary 6.2.** *For  $k$  large enough, we have  $(n-\ell, \ell) \in \Phi(\eta_k)$ , for all  $0 \leq \ell \leq n/2$ .*

*Proof.* By Lemma 4.1 and the symmetry of Kronecker coefficients, we have  $\ell = 0$  case. By Theorem 4.11, we have the result for  $\ell \leq k/2$ . Finally, the lemma gives  $L \leq \ell \leq n/2$  case. Taking  $k \geq 2L$ , completes the proof.  $\square$

*Proof of Lemma 6.1.* Recall the Frobenius formula

$$\chi^{(n-\ell,\ell)} = \chi^{(n-\ell)\circ(\ell)} - \chi^{(n-\ell+1)\circ(\ell-1)}.$$

By the Murnaghan–Nakayama rule for skew shapes, we have:

$$\chi^{(n-m)\circ(m)}[\widehat{\eta}_k] = \pi'_R(m),$$

where  $R = \{3, 5, \dots, 2k-1\}$ . Therefore, for  $L \leq \ell \leq n/2$ , by Theorem 2.1 we have

$$\chi^{n-\ell,\ell}[\widehat{\eta}_k] = \pi'_R(\ell) - \pi'_R(\ell-1) > 0.$$

Now the Main Lemma implies the result.  $\square$

**6.2. Staircase shape.** Let  $\rho_k = (k, k-1, \dots, 1) \vdash n$ ,  $n = k(k+1)/2$ , be as above.

**Lemma 6.3.** *There exists a constant  $L$ , such that for all  $L < \ell \leq n/2$  we have  $(n-\ell,\ell) \in \Phi(\rho_k)$ .*

*Proof.* There are two cases to consider: even  $k$  and odd  $k$ . Each case follows verbatim the proof of Lemma 6.1. We omit the details.  $\square$

Combined with Theorem 4.11, this immediately gives:

**Corollary 6.4.** *For  $k$  large enough, we have  $(n-\ell,\ell) \in \Phi(\rho_k)$ , for all  $0 \leq \ell \leq n/2$ .*

**6.3. Caret shape.** Let  $\gamma_k = (3k-1, 3k-3, \dots, 2^2, 1^2) \vdash n$ ,  $n = 3k^2$ , be the caret shape defined above.

**Lemma 6.5.** *There exists a constant  $L$ , such that  $(n-\ell,\ell) \in \Phi(\gamma_k)$ , for all  $\ell = 0, 1 \pmod{3}$ ,  $L \leq \ell \leq n/2$ .*

Again, combined with Theorem 4.11, this immediately gives:

**Corollary 6.6.** *For  $k$  large enough, we have  $(n-\ell,\ell) \in \Phi(\gamma_k)$ , for all  $0 \leq \ell \leq n/2$ ,  $\ell = 0, 1 \pmod{3}$ .*

*Proof of Lemma 6.5.* Recall the Frobenius formula

$$\chi^{(n-\ell,\ell)} = \chi^{(n-\ell)\circ(\ell)} - \chi^{(n-\ell+1)\circ(\ell-1)}.$$

By the Murnaghan–Nakayama rule for skew shapes, each of the characters on the right is equal to the number of partitions  $\pi'_R(\ell) - \pi'_R(\ell-1)$  into distinct parts in  $R = \{3, 9, \dots, 6k-3\}$ . Since, under the given constraints, exactly one of  $\{\ell, \ell-1\}$  is divisible by 3, we conclude that  $\chi^{n-\ell,\ell}[\gamma_k]$  is equal to either  $\pi'_R(\ell)$  or  $-\pi'_R(\ell-1)$ . Therefore,

$$\chi^{n-\ell,\ell}[\gamma_k] = \pm \pi'_S \left( \left\lfloor \frac{\ell-1}{3} \right\rfloor \right), \quad \text{where } S = \{1, 3, \dots, 2k-1\}.$$

Now Theorem 2.1 and the Main Lemma imply the result.  $\square$

## 7. VARIATIONS ON THE THEME

**7.1. Near-hooks.** Let  $\lambda = (n-\ell-m, m, 1^\ell)$ , which we call the *near-hook*, for small  $m \geq 2$ . We summarize the results in the following theorem.

**Theorem 7.1** (Near-hooks in staircase shapes). *There is a constant  $L > 0$ , such that for all  $\ell, k \geq L$ ,  $\ell < n/2 - 5$ ,  $n = k(k+1)/2$ , we have:*

$$(n-\ell-2, 2, 1^\ell), \dots, (n-\ell-10, 10, 1^\ell) \in \Phi(\rho_k)$$

*Sketch of proof.* For  $\lambda = (n-\ell-2, 2, 1^\ell)$ , use Giambelli's formula to obtain

$$\chi^{(n-\ell-2, 2, 1^\ell)} = \chi^{(n-\ell-2, 1^{\ell+1}) \circ (1)} - \chi^{(n-\ell-2) \circ (1^{\ell+2})}.$$

In the odd case, when the first summand of the right side of the equation is evaluated at  $[\hat{\rho}_k]$  using the Murnaghan–Nakayama rule, part (1) is placed uniquely, and there are 5 choices for the 5-rim hook. Thus, the first character evaluated at  $\hat{\rho}_k$ , is equal to

$$\pi'_R(\ell+1) - \pi'_R(\ell) + \pi'_R(\ell-1) - \pi'_R(\ell-2) + \pi'_R(\ell-3),$$

where  $R = \{9, 13, \dots, 2k-1\}$ . Similarly, the second character evaluated at  $\hat{\rho}_k$ , is equal to

$$\pi'_R(\ell+2) + \pi'_R(\ell+1) + \pi'_R(\ell-3) + \pi'_R(\ell-4),$$

depending on the placement of parts (1) and (5). For the difference, we have

$$\begin{aligned} & -\pi'_R(\ell+2) - \pi'_R(\ell) + \pi'_R(\ell-1) - \pi'_R(\ell-2) - \pi'_R(\ell-4) \\ &= -[\pi'_R(\ell) - \pi'_R(\ell-1)] - [\pi'_R(\ell+2) + \pi'_R(\ell-2) + \pi'_R(\ell-4)] < 0, \end{aligned}$$

for  $L \leq \ell \leq n/2 - 1$ , by Theorem 2.1. Now the Main Lemma implies the odd  $k$  case. In the even  $k$  case, there is no part (1) and the first character is zero; by Theorem 2.1 the second is positive for  $\ell$  as above, and the proof follows.

For  $\lambda = (n-\ell-3, 3, 1^\ell)$ , use Giambelli's formula to obtain

$$\chi^{(n-\ell-3, 3, 1^\ell)} = \chi^{(n-\ell-3, 1^{\ell+1}) \circ (2)} - \chi^{(n-\ell-3) \circ (2, 1^{\ell+1})}.$$

Since there no part of size (2), the first character evaluated at  $\hat{\rho}_k$ , is equal to zero. For the second character, if  $k$  is odd, the part of size 1 cannot be placed in  $(2, 1^{\ell+1})$ , and should be in the left corner of  $(n-\ell-3)$ . The other parts can be placed in either the hook  $(2, 1^{\ell+1})$  or the rest of  $(n-\ell-3)$ , as long as they partition  $2 + (\ell+1)$ , so

$$\chi^{(n-\ell-3) \circ (2, 1^{\ell+1})} = -\pi'_R(\ell+3), \quad \text{where } R = \{5, 9, \dots\}.$$

In the even  $k$  case, any part of size  $\geq 3$  can be placed in either the hook  $(2, 1^{\ell+1})$  or the strip  $(n-\ell-3)$ , so as before

$$\chi^{(n-\ell-3) \circ (2, 1^{\ell+1})} = -\pi'_R(\ell+3), \quad \text{where } R = \{3, 7, 11, \dots\}.$$

The rest of the proof follows verbatim.

For  $\lambda = (n-\ell-4, 4, 1^\ell)$ , there is no part (3) in the odd  $k$ , and the role of odd/even  $k$  are interchanged. The details are straightforward. Other results are similar as well.

The character values are summarized in the table below. The fact that they are nonzero follows from Theorem 2.1. The character value at  $m = 7, k - \text{odd}$  can be expressed as an alternating sum of  $\pi'$  functions and again the total character is nonzero.  $\square$

$m$	$\chi^\lambda[\widehat{\rho}_k], k - \text{odd}$	$\chi^\lambda[\widehat{\rho}_k], k - \text{even}$
4	$-\pi'_R(\ell+4), R = \{5, 9, \dots\}$	$-\pi'_R(\ell-6) + \pi'_R(\ell-5) - \pi'_R(\ell-4) - \pi'_R(\ell-2) + \pi'_R(\ell-1) - \pi'_R(\ell) - \pi'_R(\ell+4), R = \{11, 15, \dots\}$
5	$-\pi'_R(\ell+5), R = \{5, 9, \dots\}$	$-\pi'_R(\ell+5), R = \{7, 11, \dots\}$
6	$\pi'_R(\ell+1) - \pi'_{R \cup 5}(\ell+6), R = \{9, 13, \dots\}$	$-\pi'_R(\ell+6), R = \{7, 11, \dots\}$
7	$\chi^{(n-\ell-7, 1^{\ell+1})}[R] - \pi'_R(\ell+1) - \pi'_R(\ell+7), R = \{9, 13, \dots\}$	$-\pi'_R(\ell+7), R = \{7, 11, \dots\}$
8	$-\pi'_R(\ell+8), R = \{9, 13, \dots\}$	$\pi'_R(\ell-1) - \pi'_R(\ell) - \pi'_R(\ell+8), R = \{11, 15, \dots\}$
9	$-\pi'_R(\ell+9), R = \{9, 13, \dots\}$	$-\pi'_R(\ell+9), R = \{11, 15, \dots\}$
10	$\pi'_R(\ell-4) - \pi'_R(\ell+10), R = \{13, 17, \dots\}$	$\pi'_R(\ell-1) - \pi'_R(\ell+10), R = \{11, 15, \dots\}$

**Remark 7.2.** This sequence of results can be continued for a while, with computations of multiplicities of  $\lambda = (n - \ell - m, m, 1^\ell)$  becoming more complicated as  $m$  grows. Beyond some point, the naive estimates as above no longer apply and we need stronger estimates on the numbers of partitions. Finally, the characters in cases of chopped square and the caret shapes can be analyzed in a similar way. We omit them for brevity.

**7.2. Near two rows.** Let  $\lambda = (n - \ell - m, \ell, m)$ , which we call the *near two rows*, for small  $m \geq 1$ . We summarize the results in the following theorem.

**Theorem 7.3** (Near two rows in staircase shapes). *There is a constant  $L > 0$ , such that for all  $\ell \geq L$ ,  $\ell + m \leq n/2$ ,  $n = k(k+1)/2$ , we have  $(n - \ell - m, \ell, m) \in \Phi(\rho_k)$*

- 1) for  $m = 1, 3, 5, 7, 8, 9$ ,
- 2) for  $m = 2, 4$ , and  $k$  even.

*Sketch of proof.* For  $\lambda = (n - \ell - 1, \ell, 1)$ , use Giambelli's formula to obtain

$$\chi^{(n-\ell-1, \ell, 1)} = \chi^{(n-\ell-1, 1^2) \circ (\ell-1)} - \chi^{(\ell-1, 1^2) \circ (n-\ell-1)}.$$

Again, use skew Murnaghan–Nakayama rule to evaluate the characters at  $\widehat{\rho}_k$ , where the parts are ordered in increasing order. Note that each skew partition is a composition of a hook and a row. For odd  $k$ , part 1 is placed uniquely, into a single row, and regardless how the rim hook fits the foot of the hook, it will have positive sign.

We conclude that the difference of character values is equal to

$$\begin{aligned} \pi'_R(\ell-2) - \pi'_R(\ell+1), &\quad \text{where } R = \{5, 9, \dots, 2k-1\}, k - \text{odd}, \\ \pi'_R(\ell-1) - \pi'_R(\ell+1), &\quad \text{where } R = \{3, 7, \dots, 2k-1\}, k - \text{even}. \end{aligned}$$

Thus, by Theorem 2.1, in both cases we have the difference  $< 0$  for  $\ell$  as above. The character evaluated at  $\widehat{\rho}_k$  is then nonzero and the Main Lemma implies the result in this case.

The  $m = 2$  case is similar, with a global change of sign as one of the hooks has even height. We omit the easy details.

For  $\lambda = (n - \ell - 3, \ell, 3)$ , use Frobenius formula<sup>5</sup> for 3 rows [Sag, Sta2]:

$$\begin{aligned} \chi^{(n-\ell-3, \ell, 3)} &= \chi^{(n-\ell-3) \circ (\ell) \circ (3)} - \chi^{(n-\ell-2) \circ (\ell-1) \circ (3)} \\ &\quad - \chi^{(n-\ell-3) \circ (\ell+1) \circ (2)} + \chi^{(n-\ell-1) \circ (\ell-1) \circ (2)} \\ &\quad + \chi^{(n-\ell-2) \circ (\ell+1) \circ (1)} - \chi^{(n-\ell-1) \circ (\ell) \circ (1)}. \end{aligned}$$

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<sup>5</sup>In the context of Schur functions, Frobenius formula is often called the *Jacobi-Trudi identity* [Mac].

Let  $k$  be odd. Evaluated at  $\rho_k$ , only the last difference is nonzero, giving  $\pi'_R(\ell+1) - \pi'_R(\ell)$ ,  $R = \{5, 9, \dots\}$ , which is positive for large enough  $\ell$ . Similarly, for even  $k$ , only the first difference is nonzero, giving  $\pi'_R(\ell) - \pi'_R(\ell-1)$  which is positive for large enough  $\ell$ . Other cases  $m \leq 9$  are similar. We omit the details.  $\square$

**Remark 7.4.** In the theorem we omit three cases ( $m = 2, 4$  for odd  $k$ ), since there are no skew rim hook tableaux in that case. This implies that the corresponding character is zero, and the Main Lemma tells us nothing. Of course, one can increase  $m$  and/or the number of rows. Both the Frobenius and the Giambelli formulas become more involved and it becomes more difficult to compute the characters in terms of partitions into distinct parts in arithmetic progressions.

## 8. RANDOM CHARACTERS

**8.1. Caret shape.** We start with the following curious result.

**Proposition 8.1.** *Let  $n = 3k^2$  and  $\lambda \in P_n$  be a random partition. Then  $\chi^\lambda[\hat{\gamma}_k] = 0$  w.h.p., as  $n \rightarrow \infty$ .*

*Proof.* Since all parts of  $\hat{\gamma}_k$  are divisible by 3, we have  $\chi^\lambda[\hat{\gamma}_k] = 0$  unless  $\lambda$  has an empty 3-core. By the asymptotics given in §2.2, the probability of that is

$$\frac{\pi_3(n)}{\pi(3n)} = O\left(\frac{1}{\sqrt{n}}\right),$$

as desired.  $\square$

**Remark 8.2.** The proposition states that almost all character values are zero in this case. This suggests that either Conjecture 3.2 is false for large  $k$ , or the Main Lemma is too weak when it comes to this conjecture even for a constant fraction of the partitions.

**8.2. Staircase shape.** Keeping in mind Proposition 4.14 and Proposition 4.16, we conjecture that almost all characters are not equal to zero:

**Conjecture 8.3.** *Let  $n = k(k+1)/2$  and  $\lambda \in P_n$  be a random partition. Then  $\chi^\lambda[\hat{\rho}_k] \neq 0$  with high probability (w.h.p.), as  $n \rightarrow \infty$ .*

**8.3. Random shapes.** Note that the random partitions  $\lambda \vdash n$  are (approximately) self-conjugate and by the limit shape results (see §2.3), have Durfee size

$$d(\lambda) \sim \frac{(\ln 2)\sqrt{6n}}{\pi} \approx 0.54\sqrt{n}.$$

This raises the question that perhaps random self-conjugate partitions satisfy the tensor product conjecture.

**Open Problem 8.4.** *Let  $\mu \vdash n$  be a random self-conjugate partition of  $n$ . Prove or disprove:  $\Phi(\mu) = P_n$  w.h.p., as  $n \rightarrow \infty$ .*

The following result is a partial evidence in support of the positive solution of the problem.

**Theorem 8.5.** *Let  $\mu \vdash n$  be a random self-conjugate partition of  $n$ . Then  $\Phi(\mu)$  contains all hooks  $(n - \ell, 1^\ell)$ ,  $0 \leq \ell < n$ , w.h.p., as  $n \rightarrow \infty$ .*

*Sketch of proof.* We follow the proof of Lemma 5.1. First, recall that self-conjugate partitions are in natural bijection with partitions into distinct odd parts:  $\mu \leftrightarrow \widehat{\mu}$  (see e.g. [Pak]). This implies that  $\ell(\widehat{\mu}) = \Omega(\sqrt{n})$  and the smallest part  $s = \widehat{\mu}_d = O(1)$  w.h.p., where  $d = d(\mu)$ . Now observe that the  $\gcd(\widehat{\mu}_1, \dots, \widehat{\mu}_d) = 1$  w.h.p. Then, by [OR], there exists an integer  $L$  such that  $\pi'_R(r)$  are positive and monotone for  $L < r < n - L$ , where  $R = \{\mu_1, \dots, \mu_s\}$ . Now, the Murnaghan–Nakayama rule implies that for  $L < \ell n - L$ , there are  $s = s(\mu)$  ways to remove the smallest part, giving

$$\chi^{(n-\ell, 1^\ell)}[\widehat{\mu}] = \pi'_R(\ell) - \pi'_R(\ell-1) + \pi'_R(\ell-2) - \dots + \pi'_R(\ell-s) > 0.$$

Now the Main Lemma gives the result for  $\ell$  as above, and for small  $\ell = O(\sqrt{n})$ , the result follows from Theorem 4.12.  $\square$

In case Open Problem 8.4 is too strong, here is a weaker, asymptotic version of this claim.

**Open Problem 8.6.** *Let  $\mu \vdash n$  be a uniformly random self-conjugate partition of  $n$ . Prove or disprove:*

$$\frac{|\Phi(\mu)|}{|P_n|} \rightarrow 1 \quad \text{as } n \rightarrow \infty.$$

**8.4. Random characters.** We believe the following claim closely related to open problems above.

**Conjecture 8.7.** *Let  $\lambda \vdash n$  be a random partition of  $n$ , and let  $\mu$  be a random self-conjugate partition of  $n$ . Then  $\chi^\lambda[\widehat{\mu}] = 0$  w.h.p., as  $n \rightarrow \infty$ .*

Heuristically, the first principal hook  $\widehat{\mu}_1$  is greater than  $\widehat{\lambda}_1$  with probability  $1/2$ , and by Lemma 4.15 the above character is zero. If  $\widehat{\mu}_1 \leq \widehat{\lambda}_1$ , with good probability the  $\widehat{\mu}_1$ -rim hook can be removed in a unique way, after which the process is repeated for smaller principal hooks, in a manner similar to Proposition 4.14. This gives that the probability there exists no rim hook tableaux  $\rightarrow 0$  as  $n \rightarrow \infty$ .

Now, the above argument is heuristic and may be hard to formalize, since the “good probability” is rather hard to estimate and in principle it might be close to 0 and lead to many rim hook tableaux. However, sharp bound on the distribution of the largest part of the random partitions (cf. §2.3) combined with the first step of the argument can be formalized to prove the following result.<sup>6</sup>

**Proposition 8.8.** *Let  $\lambda, \mu \vdash n$  be as in the Conjecture above. Then there exists  $\varepsilon > 0$  such that  $\chi^\lambda[\widehat{\mu}] = 0$  with probability  $< 1 - \varepsilon$ , as  $n \rightarrow \infty$ .*

This implies our Main Lemma is too weak to establish even the weaker Open Problem 8.6, since a constant fraction of characters are zero.

**8.5. Character table of the symmetric group.** Now, Conjecture 8.7 raises a more simple and natural question about random entries of the character table of  $S_n$ .

**Conjecture 8.9.** *Let  $\lambda, \mu \vdash n$  be random partitions of  $n$ . Then  $\chi^\lambda[\mu] \neq 0$  w.h.p. as  $n \rightarrow \infty$ .*

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<sup>6</sup>The proof will appear elsewhere.

Andrew Soffer's calculations show gradual decrease of the probability  $p(n)$  of zeroes in the character table of  $S_n$ , for  $n > 20$ , from  $p(20) \approx 0.394$  to  $p(39) \approx 0.359$ . Furthermore, for large partitions the probability that the character values are small seems to be rapidly decreasing. For example,  $q(20) \approx 0.06275$  and  $q(37) \approx 0.020375$ , where  $q(n)$  is the probability that the character is equal to 1.

**Open Problem 8.10.** *Let  $\lambda, \mu \vdash n$  be random partitions of  $n$ . Find the asymptotic behavior of  $p_n := P(\chi^\lambda[\mu] = 1)$ .*

The data we have suggests that the probability in the open problem decreases mildly exponentially:  $p_n < \exp[-n^\alpha]$ , for some  $\alpha > 0$ .

## 9. PROOF OF THE MAIN LEMMA

Let us first restate the lemma using our notation.

**Lemma 1.3 (Main Lemma)** *Let  $\lambda, \mu \vdash n$ , such that  $\mu = \mu'$  and  $\chi^\lambda[\widehat{\mu}] \neq 0$ . Then  $\lambda \in \Phi(\mu)$ .*

*Proof.* Recall Lemma 4.13, and let

$$\varepsilon_\mu = \chi^\mu[\widehat{\mu}] = (-1)^{(n - d(\mu))/2}.$$

Recall also that the  $S_n$  conjugacy class of cycle type  $\zeta$ , when  $\zeta$  is a partition into distinct odd parts, splits into two conjugacy classes in the alternating group  $A_n$ , which we denote by  $\zeta^1$  and  $\zeta^2$ . There are two kinds of irreducible characters of  $A_n$ . For each partition  $\nu$  of  $n$  such that  $\nu = \nu'$  there are two irreducible characters associated to  $\nu$ , which we denote by  $\alpha^{\nu+}$  and  $\alpha^{\nu-}$ ; and for each partition  $\nu$  of  $n$  such that  $\nu \neq \nu'$  there is an irreducible character associated to the pair  $\nu, \nu'$ , which we denote by  $\alpha^\nu$ . These characters are related to irreducible characters of  $S_n$  as indicated below. We will need the following standard results (see e.g. [JK, Section 2.5]).

**1.** If  $\nu \neq \nu'$ , then

$$\text{Res}_{A_n}^{S_n}(\chi^\nu) = \text{Res}_{A_n}^{S_n}(\chi^{\nu'}) = \alpha^\nu,$$

is an irreducible character of  $A_n$ .

**2.** If  $\nu = \nu'$ , then

$$\text{Res}_{A_n}^{S_n}(\chi^\nu) = \alpha^{\nu+} + \alpha^{\nu-},$$

is the sum of two different irreducible characters of  $A_n$ . Moreover, both characters are conjugate, that is, for any  $\sigma \in A_n$  we have

$$\alpha^{\nu+}[(12)\sigma(12)] = \alpha^{\nu-}[\sigma].$$

**3.** The characters  $\alpha^\nu$ ,  $\nu \neq \nu'$  and  $\alpha^{\nu+}$ ,  $\alpha^{\nu-}$ , where  $\nu = \nu'$  are all different and form a complete set of irreducible characters of  $A_n$ .

**4.** If  $\nu = \nu'$ , and  $\gamma$  is a conjugacy class of  $A_n$  different from  $\widehat{\nu}^1$  or  $\widehat{\nu}^2$ , then

$$\alpha^{\nu+}[\gamma] = \alpha^{\nu-}[\gamma] = \frac{1}{2} \chi^\nu[\gamma].$$

We also have

$$\begin{aligned}\alpha^{\nu+}[\widehat{\nu}^1] &= \alpha^{\nu-}[\widehat{\nu}^2] = \frac{1}{2} \left( \varepsilon_\nu + \sqrt{\varepsilon_\nu \prod_i \widehat{\nu}_i} \right), \\ \alpha^{\nu+}[\widehat{\nu}^2] &= \alpha^{\nu-}[\widehat{\nu}^1] = \frac{1}{2} \left( \varepsilon_\nu - \sqrt{\varepsilon_\nu \prod_i \widehat{\nu}_i} \right).\end{aligned}$$

In other words, for any self-conjugate partition  $\nu$ , the only irreducible characters of  $A_n$  that differ on the classes  $\widehat{\nu}^1$  and  $\widehat{\nu}^2$  are precisely  $\alpha^{\nu\pm}$ .

There are two cases to consider with respect to whether  $\lambda$  is self-conjugate or not.

First, assume that  $\lambda \neq \lambda'$ . Then  $\alpha^\lambda$  is an irreducible character of  $A_n$ . Since

$$\alpha^\lambda[\widehat{\mu}] = \alpha^\lambda[\widehat{\mu}^2] = \chi^\lambda(\widehat{\mu}),$$

we obtain:

$$\begin{aligned}(\alpha^{\mu+} \otimes \alpha^\lambda)[\widehat{\mu}^1] - (\alpha^{\mu+} \otimes \alpha^\lambda)[\widehat{\mu}^2] &= (\alpha^{\mu+}[\widehat{\mu}^1] - \alpha^{\mu+}[\widehat{\mu}^2]) \cdot \chi^\lambda(\widehat{\mu}) \\ &= \left( \sqrt{\varepsilon_\mu \prod_i \widehat{\mu}_i} \right) \cdot \chi^\lambda(\widehat{\mu}) \neq 0.\end{aligned}$$

Therefore, either  $\alpha^{\mu+}$  or  $\alpha^{\mu-}$  is a component of  $\alpha^{\mu+} \otimes \alpha^\lambda$ . In other words,

$$\text{either } \langle \alpha^{\mu+} \otimes \alpha^\lambda, \alpha^{\mu+} \rangle \neq 0 \quad \text{or} \quad \langle \alpha^{\mu+} \otimes \alpha^\lambda, \alpha^{\mu-} \rangle \neq 0.$$

We claim that the terms in these product can be interchanged. Formally, we claim that:

$$(*) \quad \text{either } \langle \alpha^{\mu+} \otimes \alpha^{\mu+}, \alpha^\lambda \rangle \neq 0 \quad \text{or} \quad \langle \alpha^{\mu+} \otimes \alpha^{\mu-}, \alpha^\lambda \rangle \neq 0.$$

There are two cases. If  $\varepsilon_\mu = 1$ , then both  $\alpha^{\mu+}$  and  $\alpha^{\mu-}$  take real values. Thus

$$\langle \alpha^{\mu+} \otimes \alpha^{\mu\pm}, \alpha^\lambda \rangle = \langle \alpha^{\mu+} \otimes \alpha^\lambda, \alpha^{\mu\pm} \rangle \neq 0,$$

which implies  $(*)$  in this case.

If  $\varepsilon_\mu = -1$ , then

$$\text{Im}(\alpha^{\mu+}[\widehat{\mu}^1]) = -\text{Im}(\alpha^{\mu-}[\widehat{\mu}^1]) \quad \text{and} \quad \text{Im}(\alpha^{\mu+}[\widehat{\mu}^2]) = -\text{Im}(\alpha^{\mu-}[\widehat{\mu}^2]).$$

Therefore,  $\overline{\alpha^{\mu+}} = \alpha^{\mu-}$ , since all other character values are real. Thus,

$$\langle \alpha^{\mu+} \otimes \alpha^{\mu\pm}, \alpha^\lambda \rangle = \langle \alpha^{\mu+} \otimes \alpha^\lambda, \alpha^{\mu\mp} \rangle \neq 0,$$

which implies  $(*)$  in this case.

In summary, we have both cases in  $(*)$  imply that  $\alpha^\lambda$  is a component of  $\text{Res}_{A_n}^{S_n}(\chi^\mu \otimes \chi^\mu)$ . Therefore, either  $\chi^\lambda$  or  $\chi^{\lambda'}$  is a component of  $\chi^\mu \otimes \chi^\mu$ . Since  $\mu = \mu'$ , we have, by Proposition 4.3, that  $\chi^\lambda$  and  $\chi^{\lambda'}$  are components of  $\chi^\mu \otimes \chi^\mu$ , as desired. This completes the proof of the  $\lambda \neq \lambda'$  case.

Now, suppose  $\lambda = \lambda'$ . The case  $\lambda = \mu$  is given by Theorem 4.6. If  $\lambda \neq \mu$ , then

$$\alpha^{\lambda\pm}[\widehat{\mu}^1] = \alpha^{\lambda\pm}[\widehat{\mu}^2] = \frac{1}{2} \chi^\lambda(\widehat{\mu}) \neq 0.$$

By a similar argument as above applied to  $\lambda+$  and  $\lambda-$  in place of  $\lambda$ , we have the following analogue of  $(*)$ :

$$\begin{aligned}\text{either } \langle \alpha^{\mu+} \otimes \alpha^{\mu+}, \alpha^{\lambda+} \rangle &= \langle \alpha^{\mu+} \otimes \alpha^{\mu+}, \alpha^{\lambda-} \rangle \neq 0, \\ \text{or } \langle \alpha^{\mu+} \otimes \alpha^{\mu-}, \alpha^{\lambda+} \rangle &= \langle \alpha^{\mu+} \otimes \alpha^{\mu-}, \alpha^{\lambda-} \rangle \neq 0.\end{aligned}$$

This implies that  $\alpha^{\lambda+}$  and  $\alpha^{\lambda-}$  are components of  $\text{Res}_{A_n}^{S_n}(\chi^\mu \otimes \chi^\mu)$ . Therefore,  $\chi^\lambda$  is a component of  $\chi^\mu \otimes \chi^\mu$ , as desired. This completes the proof of the  $\lambda = \lambda'$  case, and finishes the proof of the lemma.  $\square$

## 10. CONCLUSIONS AND FINAL REMARKS

**10.1.** For the staircase shapes  $\rho_k$ , the number  $|\Phi(\rho_k)|$  of irreducible constituents is exponential by Proposition 4.14. From this point of view, theorems 1.4, 7.1 and 7.3 barely make a dent: they add  $O(k^2)$  additional constituents. On the other hand, for the chopped square shape  $\eta_k$  there is no obvious weakly exponential lower bound. Although we believe that finding such bound should not be difficult by an ad hoc construction,  $\Omega(k^2)$  is the best bound we currently have.

**10.2.** There is a curious characterization of positivity Littlewood–Richardson coefficients of the staircase shape. Namely, Berenstein and Zelevinsky proved in [BZ2] the former *Kostant Conjecture*, which states that the Littlewood–Richardson coefficient,  $LR(\rho_k, \rho_k, \lambda)$ , is positive if and only if the Kostka coefficient,  $K(2\rho_k, \lambda)$ , is positive. For the proof, they defined *BZ-triangles*, which proved crucial in [KT].

**10.3.** A natural question would be to ask whether the dimensions of  $\rho_k$ ,  $\gamma_k$  and  $\eta_k$  are large enough to contain all irreducible representations of  $S_n$ . Same question for the limit shape defined in §2.3 discussed also in Open Problem 8.4. The answer is yes in all cases, as can be seen by a direct application of the hook-length and Stirling formulas. Given that all these shapes are far from the Kerov–Vershik shape which has the maximal (and most likely) dimension [VK], this might seem puzzling. The explanation is that the dimensions are asymptotically greater than the number of partitions  $\pi(n) = \exp \Theta(\sqrt{n})$ .

More precisely, it is well known and easy to see that the sum of all dimensions is equal to  $a_n = \#\{\sigma \in S_n \mid \sigma^2 = 1\}$ . The sequence  $\{a_n\}$  is A000085 in [OEIS], and is equal to  $\exp[\frac{1}{2}n \log n]$ , up to an  $\exp O(n)$  factor (see [Rob]). Thus, the squares of dimensions of random irreducible representations of  $S_n$  are *much larger* than  $a_n$ . This also suggests a positive answer to Open Problem 8.4. More relevant to the Tensor Square Conjecture, the dimensions of  $\rho_k$ ,  $\gamma_k$  and  $\eta_k$  are also equal to  $\exp[\frac{1}{2}n \log n]$ , up to an  $\exp O(n)$  factor, which supports the Saxl Conjecture and conjectures 3.1, 3.2.

**10.4.** Although there are very strong results on the monotonicity of the partition function (see e.g. [BE]), for partitions into distinct parts much less is known (see [OR, RS]). Curiously, some results follow from Dynkin’s result in Lie Theory. For example, the unimodality of  $\pi'_R(n)$  for  $R = \{1, 2, \dots, k\}$ , i.e. the unimodality of the coefficients in

$$\prod_{i=1}^k (1 + t^i),$$

is called *Hughes theorem* and corresponds to root system  $C_k$ . We refer to [Bre, Sta1] for more on this approach and general surveys on unimodality.

**10.5.** Constants  $L$  in theorems 7.1 and 7.3 should be possible to estimate explicitly, in the same manner as was done in [OR], to reprove the Hughes theorem (see above). However, because of Example 5.2, there is no unimodality for small values of  $\ell$ , so without advances in the study of Kronecker coefficients it is unlikely that the gap can be bridged by an explicit computation.

We should mention also that after the results of this paper were obtained, we found obtain an advance extension of Theorem 4.12, based again on a combinatorial interpretation given in [Bla]. Among other things, we prove that  $\Phi(\rho_k)$  contain *all* hooks, effectively removing constant  $L$  in this case. Independently, Blasiak also obtained the same result by a different method.<sup>7</sup>

**10.6.** The reason conjectures 8.7 and 8.9 are not in direct contradiction has to do with a difference between limit shapes of random partitions  $\mu$  and random partitions into distinct odd part  $\widehat{\mu}$  (see [Ver]). Using the asymptotics for the number of the latter, Conjecture 8.7 implies that at least  $\exp \Theta(n^{1/4})$  many columns of the character table  $M_n$ , most entries are zero. In a different direction, the Main Lemma can be reversed to show that at least  $\exp \Theta(n^{1/4})$  many rows have most entries zero, by taking partitions with Durfee squares  $O(n^{1/4})$ . We omit the details.

**10.7.** Curiously, and quite coincidentally, the plane can be tiled with copies of parallel translations and rotations of  $\gamma_k$  shapes, and the same is true also for  $\rho_k$  and  $\eta_k$  (see Figure 5).

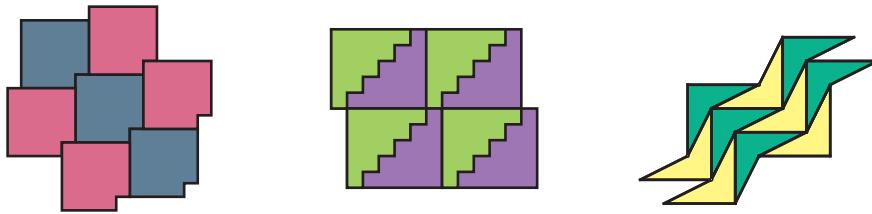


FIGURE 5. Tiling of the plane with chopped square, staircase and caret shapes.

**10.8.** John Thompson's conjecture states that every finite simple group  $G$  has a conjugacy class whose square is the whole  $G$ . For  $A_n$ , this is a well known result [Ber]. We refer to [Sha] for a survey of recent progress towards Thompson's conjecture, and related results.

Passman's problem is concerned with the conjugation action on conjugacy classes of general simple groups. For  $A_n$ , it was positively resolved by Heide and Zalesski in [HZ].

**Theorem 10.1** ([HZ]). *For every  $n \geq 5$  there is a conjugacy class  $C_\lambda$  of  $A_n$ , such that the action of  $A_n$  on  $C_\lambda$  by conjugation as a permutation module, contains every irreducible character as a constituent.*

**10.9.** It is known that the problem KRON of computing  $g(\lambda, \mu, \nu)$  is  $\#P$ -hard (see [BI]). However, the same also holds for the Kostka numbers  $K_{\lambda, \mu}$  and the Littlewood–Richardson coefficients  $LR(\lambda, \mu, \nu)$ , so this is not the main obstacle [Nar]. The difference is, for the latter there exists several combinatorial interpretations in the form of counting certain Young tableaux, which are (relatively) easy to work with (see e.g. [KT, KTW, PV, Sta2]). Of course, it is not known whether KRON is in  $\#P$ , as this would imply a combinatorial interpretation for the Kronecker coefficients, but it was shown in [BI] that KRON  $\in$  GAPP.

<sup>7</sup>Personal communication.

Recall that by the Knutson–Tao theorem [KT] (formerly the *saturation conjecture* [Zel]), the problem whether  $LR(\lambda, \mu, \nu) = 0$  is equivalent to the problem whether the corresponding LR-polytope is nonempty, and thus can be solved in polynomial time by linear programming [MNS]. Similarly, by the Knutson–Tao–Woodward theorem [KTW] (formerly *Fulton’s conjecture*), the problem whether  $LR(\lambda, \mu, \nu) = 1$  is equivalent to the problem whether the corresponding LR-polytope consists of exactly one point, and thus also can be solved in polynomial time (cf. [BZ1]). Finally, Narayanan showed that the corresponding problems for Kostka numbers reduce to LR-coefficients [Nar]. Together these results imply that all four decision problems can be solved in polynomial time.

On the other hand, it is not known whether the corresponding decision problems for Kronecker coefficients are in P [Mul]. In the words of Peter Bürgisser [Bür], “deciding positivity of Kronecker coefficients [...] is a major obstacle for proceeding with geometric complexity theory” of Mülmuley and Sohoni [MS]. Special cases of the problem are resolved in [PP]. We refer to [Mul] for the detailed overview of the role Kronecker coefficients play in this approach (see also [BOR, Ike]).

Let us mention that when  $n$  is in unary, computing individual character values  $\chi^\lambda[\mu]$  can be done in Probabilistic Polynomial time [Hep]. This gives an  $\exp O(\sqrt{n})$  time algorithm for computing Kronecker coefficients, via the scalar product of characters. On the other hand, the Main Lemma now can be viewed as a “polynomial witness” for positivity of Kronecker coefficients, which, if Conjecture 8.3 holds, works for most partitions.

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<sup>8</sup>Specifically, the references in the answers to this MathOverflow question proved very useful: <http://mathoverflow.net/questions/111507/>

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