SKEW SHAPE ASYMPTOTICS, A CASE-BASED INTRODUCTION

IGOR PAK

ABSTRACT. We discuss various tools in the emerging area of Asymptotic Algebraic Combinatorics, as they apply to one running example of thick ribbons. Connections to other areas, exercises and open problems are also included.

1. INTRODUCTION

1.1. Foreword. This paper is a short tutorial with many exercises, on some ideas in Asymptotic Algebraic Combinatorics. We do not intend to be broad or thorough, but rather give a cross section of the area concentrated around a single example, which turned out to have rich connections to different results and problems in the area. This is not a substitute of a serious survey, but in the absence of such we envision it as a quick guide to the literature, and an easy entry point to the area.

1.2. Thick ribbons. Let \( \delta_k = (k-1, k-2, \ldots, 2, 1) \vdash \binom{k}{2} \) be a staircase shape of size \( k \). Let \( \tau_k = (\delta_k/\delta_k) \vdash n \) be a thick ribbon shape of size \( k \), see Figure 1. Here and below, we have \( n = \frac{k(3k-1)}{2} \).

Denote by \( a_k := |\text{SYT}(\tau_k)| \) the number of standard Young tableaux of shape \( \tau_k \). The sequence \( \{a_n\} \) is rapidly growing:

\[
1, 16, 101376, 689783212740930831360, \ldots
\]

For example, \( a_2 = |\text{SYT}(321/1)| = f^{321} = 16 \). For larger values, see [OEIS, A278289]. The main goal of the paper is to give lower and upper bounds on \( a_k \). Roughly, \( a_k \approx \sqrt{n!} \). More precisely, it is known and easy to see that

\[
\log a_k = \frac{3}{2} k^2 \log k + O(k^2) = \frac{1}{2} n \log n + O(n) \quad \text{as} \quad k \to \infty,
\]

see [MPP4]. The following result frames the answer in an asymptotic language.

**Theorem 1.1 (MPT).** There exists a universal constant \( \phi \), such that

\[
\log a_k = \frac{1}{2} n \log n + \phi n + o(n) \quad \text{as} \quad n = \frac{1}{2} k(3k-1) \to \infty.
\]

The rest of the paper is concerned with the following:

**Main Problem 1.2.** Find \( \phi \), i.e. give sharp rigorous estimates for \( \phi \).

The best bounds we present are:

\[
-0.2368 \leq \phi \leq -0.1648,
\]

given by Lower Bound 9.2 and Upper Bound 6.1. The result in [MPT] does not determine the exact value of \( \phi \), but rather presents it as a solution of a variational problem (cf. [Gor1]). It is unlikely that it can be computed exactly other than numerically.

The bounds (1.1) are remarkably close to each other, as opposed to the way these things usually go. In fact, the calculations by Pantone suggest that \( \phi \approx -0.18 \), see [OEIS, A278289], leaving very little room for improvement of the upper bound. Let us emphasize that getting best bounds is not really the point of this paper as we present a number of relatively weak bounds. The idea is to review the tools which in this particular case can give weaker bounds, but stronger in other cases perhaps, and often best used in combination.


*Department of Mathematics, UCLA, Los Angeles, CA 90095. Email: pak@math.ucla.edu.
1.3. Style and structure of the paper. In Section 2, we begin with an informal general discussion of how to obtain bounds for combinatorial numbers, notably how to approach them when some easy ideas fail. Then, one by one, we introduce tools of the area, starting with the classical and more established ideas and leading to the most recent work. We largely restrict ourselves to the running example of the skew shape $\tau_k$, leaving only a trail of crumbs for the interested reader to recover the full story from the references. We also heavily use figures, examples and exercises in place of formal general statements.

We do not present most definitions, standard results and notation, but instead assume that the reader is familiar with them or is able to quickly catch up using [Sag] and [S2, Ch. 7]. In every section, we supplement the results with exercises, which we believe will be helpful. In fact, each upper/lower bound is never carefully proved and can be viewed as exercises in their own right. Additional exercises are included in Section 11. We expect the reader to be committed to doing the exercises as this is probably the only way to get a grasp of the area.

In Section 12, we state several conjectures and open problems directly related to the subject. Although natural, they show both the power of tools in Algebraic Combinatorics, and their limitations in larger setting. We conclude with Section 13 where we give brief historical remarks, mostly aimed as a guide to the references.

Finally, let us relate the style of this paper to general goals of Asymptotic Algebraic Combinatorics. In the context of “two cultures in mathematics” [Gow], one often assumes that Combinatorics belong to the second, “problem solving” culture. This is far from the truth. Like other broad fields, Combinatorics spans both cultures, even if some areas in it are more at home in one than the other. Traditionally, the culture of ever improving estimates aiming towards the true value for a key benchmark problem (as in [Ber]), was foreign to the whole area of Algebraic Combinatorics. While the specific benchmark problem we chose is not of utmost importance, it serves a convenient battleground for several competing tools and techniques which can later be applied to other problems. With this style of exposition, we are aiming to lend further support to this important culture in the area.

2. Finding bounds

2.1. The basics. Suppose one is given a sequence $\{c_k\}$ to investigate. The first thing to do is to check whether $c_k$ has a nice product formula. This often works, e.g. for binomial coefficients, Catalan numbers, number of boxed plane partitions, alternating sign matrices, etc. When a product formula exists and can be proved, it is relatively straightforward to compute the exact asymptotics to everyone satisfaction. On the other hand, when there are some relatively large prime divisors appearing in $\{c_k\}$, one should look elsewhere. For example, in our sequence $\{a_k\}$, we have primes $251|a_4$ and $327317328039199|a_8$, making the prospects of a product formula rather unpromising.

The second approach is to look for a determinant formula, and this is where one finds an early success. Indeed, the following Feit determinant formula [Feit] is a standard result in the area, and applies to all skew shapes:

$$f^{\lambda/\mu} = n! \det \left( \frac{1}{(\lambda_i - \mu_j - i + j)!} \right)_{i, j = 1}^{\ell(\lambda)}.$$
see e.g. [Sag, S2]. The alternating sign nature of the formula allows only mediocre upper bounds in our case. To understand this, consider the leading (diagonal) term in the Laplace expansion of (2.1):

\[ \sum_{\lambda \vdash n} (f^\lambda)^2 = n! \]

We show in §4.2 that this product gives the right order of magnitude. Since we have only \( \ell(\lambda)! = \exp \Theta(k \log k) \) terms in the Laplace expansion, the product (2.2) rigorously implies an asymptotic upper bound (see Exercise 11.1). This also shows the limitations of the determinant approach in this case. Indeed, it is very hard to see how any nontrivial lower bound can be obtained in view of the sign cancellations, nor do other determinant tricks seem directly applicable [K1, K2].

2.2. Asymptotic thinking. While it is hard to give a broad description of what kind of arguments lead to good bounds, one natural approach is clear: clever use of identities and other summation formulas to bound one term in the summation (cf. [TV]). Let us give one important example to illustrate this approach.

Let \( D_n = \max\{f^\lambda, \lambda \vdash n\} \) denote the maximal dimension of the irreducible \( S_n \)-module, where \( f^\lambda := \chi^\lambda(1) = |\text{SYT}(\lambda)| \). For more on \( \{D_n\} \) see OEIS [A003040]. To get a bound on \( D_n \), recall the Burnside formula:

\[ \sum_{\lambda \vdash n} |f^\lambda|^2 = n! \]

This gives an upper bound \( D_n \leq \sqrt{n!} \), and a lower bound \( D_n \geq \sqrt{n!}/p(n) \), where \( p(n) = e^{O(\sqrt{n})} \) is the number of partitions. Putting these together and using Stirling’s formula gives:

\[ \log D_n = \frac{1}{2} n \log n - \frac{1}{2} n - O(\sqrt{n}) \]

Since \( \sqrt{n!} \) is the scale on which \( D_n \) is lying, it is the next term of the asymptotics which is most interesting. In other words, the ratio \( D_n/\sqrt{n!} \) is the right quantity to consider. Given (2.3), it is natural to conjecture that in fact we have

\[ \log D_n = \frac{1}{2} n \log n - \frac{1}{2} n - \frac{1}{2} n + o(\sqrt{n}) \]

This is the celebrated Vershik–Kerov–Pass (VKP) conjecture which remains open [KP, VK2].

To bring this discussion back to \( \{a_k\} \), one can make several conclusions. First, \( a_k \) lies on the same scale of \( \sqrt{n!} \), where \( n = |\tau_k| \), so the asymptotics of \( a_k/\sqrt{n!} \) is exactly the right quantity to consider. Second, the success of using the Burnside identity suggests one should consider numerous summation formulas involving \( f^{\lambda/\mu} \). In fact, this is the approach that works best for both the upper and the lower bounds. Third, the fact that \( \phi > -0.5 \) implies that \( a_k \) is exponentially larger than \( D_n \), making the summations very large and involving exponentially large terms. Finally, the mere fact that we know the existence of the limit \( \phi \) given by Theorem 1.1 suggests that this is an easier problem than the VKP conjecture. This turns out to be true as the problem is amenable to a variety of techniques and ideas.

2.3. Probabilistic thinking. In place of identities as above, one can ask a more delicate question: what is the shape of \( \lambda \) which attains the maximum \( f^\lambda = D_n \). This may seem vague, but it turns out that the limit shape of maximal \( \lambda \) is well defined after scaling the maximal shapes by \( 1/\sqrt{n} \). The answer was computed by Vershik–Kerov [VK1] and Logan–Shepp [LS], see also Rom. Note that when the row/column lengths of \( \lambda \) are constrained to \( \ell/\sqrt{n} \), the maximal dimension \( f^\lambda \) is given by a function \( D_n(t) \), which is exponentially smaller than \( D_n \), for all \( 1 \leq t < 2 \), see [LS]. The proof is based on the variational principle and uses [NHLE] in an essential way.

For our setting, one can ask for the limit shape or random \( A \in \text{SYT}(\lambda/\mu) \), where one considers scaled partitions \( \{i,j\} \in \lambda/\mu, A(i,j) \leq \alpha n \} \to \mathcal{L}_\alpha \). See Figure 2 for an example of limit curves in a \( k \times 2k \) rectangle. The existence of such limit curves is proved by Sun, see also Gor1, MPT. If one knows the exact shape of \( \{\mathcal{L}_\alpha, 0 < \alpha < 1\} \), one can estimate

\[ \log f^{\lambda/\mu} \approx \int_0^1 \log |\mathcal{L}\alpha|! \, d\alpha, \]

and an even better estimate can be obtained by taking adjacent ribbon hooks, see below.
Of course, getting closed formulas for the limit curves is a difficult problem, which often involves asymptotics of determinants of multivariate functions, see [BP2, Gor2]. In the absence of closed formulas for \( \{L_\alpha\} \), one can still use this approach in selecting which bounds fit the problem best. For example, for \( \lambda = (2k)^k \), we have \( |\text{SYT}(\lambda)| > |\text{SYT}(kk)|^2 \), but this is a rather poor lower bound as the square boundary cuts sharply across the limit shape curves. Indeed, one can check that the RHS is smaller than \( (k!)^{k+1} |\text{SYT}(\delta_k)|^2 \), which comes from a partition that is better aligned with the limit shape curves, see Figure 2.

3. Notation and basic asymptotics

As we mentioned earlier, we employ the standard notations in Algebraic Combinatorics and Representation Theory of \( S_n \), see e.g. [Sag] and [S2, Ch. 7]. We refer to [S2, Ch. 3] and [Tro] for the poset notation and standard results.

We write \( \mathbb{N} = \{0, 1, 2, \ldots\} \), \( [n] = \{1, \ldots, n\} \) and \( \mathbb{R}_+ = \{x \geq 0\} \). We use the standard asymptotics notations \( f \sim g \), \( f = o(g) \), \( f = O(g) \) and \( f = \Omega(g) \), see e.g. [FS, §A.2]. We use \( c \approx c' \) to approximate their numerical value with the usual rounding rules, e.g. \( \pi \approx 3.14 \).

14 Throughout the paper, we make heavy use of Stirling’s formula \( \log n! = n \log n - n + O(\log n) \). Here and everywhere below \( \log \) denotes the natural logarithm. We need four more products:

\[
(2n-1)!! := 1 \cdot 3 \cdot 5 \cdots (2n-1), \quad \Phi(n) := 1! \cdot 2! \ldots n!,
\]

\[
\Psi(n) := 1! \cdot 3! \cdot 5! \cdots (2n-1)!, \quad \Lambda(n) := 1!! \cdot 3!! \cdot 5!! \cdots (2n-1)!!
\]

These products have similar asymptotic formulas:

\[
\log(2n-1)!! = n \log n + (\log 2 - 1)n + O(1) \quad \text{OEIS A001147},
\]

\[
\log \Phi(n) = \frac{1}{2} n^2 \log n - \frac{3}{4} n^2 + O(n \log n) \quad \text{OEIS A000178},
\]

\[
\log \Psi(n) = n^2 \log n + \left( \log 2 - \frac{3}{2} \right) n^2 + O(n \log n) \quad \text{OEIS A168467},
\]

\[
\log \Lambda(n) = \frac{1}{2} n^2 \log n + \left( \log 2 - \frac{3}{2} \right) n^2 + O(n \log n) \quad \text{OEIS A057863}.
\]

4. Linear extensions of posets

4.1. Antichain partition. Observe that

\[
a_k \geq k! \cdot (k+1)! \cdots (2k-1)! = \frac{\Phi(2k-1)}{\Phi(k-1)}.
\]

This follows from counting standard Young tableaux with numbers in first antidiagonal smaller that numbers in second antidiagonal, etc., see Figure 3. These are in bijection with permutations in each antidiagonal, of sizes \( k, k+1, \ldots, 2k-1 \), respectively. This gives:

**Lower Bound 4.1 ([MPP4] §8.2).**

\[
\phi \geq \frac{11 \log 2}{6} - \frac{3}{2} \approx -0.7785.
\]

It is easy to see that this approach works for all posets, see e.g. [MPP4 §2]. It was shown in [BP2] that one can consider any antichain partition, or even slightly more involved Greene–Kleitman–Fomin (GKF) parameters.
Exercise 4.2. Prove that lower bound (4.1) is optimal over all antichain partitions.

Figure 3. Skew shape $\tau_5 = \delta_{10}/\delta_5$, antichain partition, chain partition, and a lower order ideal of $(3,5) \in \tau_5$ with $br(3,5) = 6$.

4.2. Chain partition. Observe that

$$a_k \leq \binom{n}{1,2,\ldots,k-1,k,k,\ldots,k} = \frac{n!}{(k!)^{k-1} \Phi(k)}.$$  

To prove this, simply observe that SYT($\tau_k$) is a subset of column-strict tableaux, and that the columns have lengths $1,2,\ldots,k-1,k,k,\ldots,k$ ($k$ columns of length $k$).

Upper Bound 4.3 ([MPP4 §8.2]).

$$\phi \leq \frac{1}{6} - \frac{\log 2}{2} + \frac{\log 3}{2} \approx 0.3694.$$  

It is easy to see that this approach works for all chain partition of posets, see e.g. [MPP4 §2]. It was shown in [BP1] that one can also consider somewhat more involved GKF parameters.

Exercise 4.4. Prove that lower bound (4.2) is optimal over all chain partitions.

4.3. Lower order ideals. Observe that

$$a_k \geq \frac{n!}{1^{2k-1} \cdot 3^{2k-3} \cdot 6^{2k-5} \ldots \left(\frac{k}{2}\right)^{k}}.$$

For general posets, this was first observed by Stanley [S2 Exc. 3.57] and proved by Hammett and Pittel [HP, eq. (1.1)]. Heuristically, to see this, observe that $br(i,j) := \left(\frac{2k-i+j+2}{2}\right)$ is the number of squares $(p,q) \in \tau_k$ such that $i \leq p$, $j \leq q$. Thus, the probability that $(i,j)$ is the smallest of these squares in a random permutation. While these events are not independent, it is known that they do have positive correlations, giving (4.3).

Exercise 4.5. Give a formal proof of (4.3). Check that it does not give a nontrivial lower bound on $\phi$.

5. Thin ribbons and slim diagrams

5.1. Alternating permutations. Denote by $E_m$ the $m$-th Euler number, defined as

$$E_m := |\{\sigma \in S_m, \sigma(1) < \sigma(2) > \sigma(3) < \sigma(4) > \ldots\}|$$

the number of alternating permutations in $S_m$, see [S3] and [Oeis A000111]). Now observe that for $k = 0 \mod 2$, we have:

$$a_k \geq E_{4k-3} \cdot E_{4k-11} \cdot E_{4k-19} \cdot \ldots \cdot E_{2k+1}.$$  

To see this, break the ribbon shape $\tau_k$ into thin ribbons as in Figure 4 and consider standard Young tableaux with numbers in smaller ribbons smaller than numbers in larger ribbons. Recall that

$$E_m \sim \frac{4}{\pi} \left(\frac{2}{\pi}\right)^m m!$$

(see e.g. [FS S3]). This and (5.1) gives:
Lower Bound 5.1.
\[ \phi \geq \alpha_2 := -\frac{3}{2} - \frac{23 \log 2}{6} - \frac{\log 3}{2} - \log \pi \approx -0.5370. \]

**Exercise 5.2.** Compute the lower bound corresponding to the ribbon partition in Figure 4. Prove that of all patterns \( \sigma(1) \circ \sigma(2) \circ \sigma(3) \cdots \), where \( \circ \in \{<,>\} \), the alternating permutations form the largest class. Use Exercise 4.2 to show that Lower Bound 5.1 cannot be improved by a better partition of \( \tau_k \) into ribbons.

![Figure 4. Partitions of \( \tau_6 = \delta_{12}/\delta_6 \) into zigzag ribbons, 3-ribbons and 2-row diagrams.](image)

5.2. \( r \)-ribbons. Denote by \( F^{(r)}_m = \text{SYT}(\delta_m/\delta_{m-r}) \) the number of standard Young tableaux of the \( r \)-ribbon shape. For example, \( \delta_m/\delta_{m-2} \) are the usual zigzag ribbons, so \( F^{(2)}_m = E_{2m-1} \). It follows from [BR] that:

\[
F^{(3)}_m = \frac{(3m)! \ E_{2m} m^{\Theta(1)}}{(2m)! 2^{2m}} , \quad F^{(5)}_m = \frac{(5m)! (E_{2m})^2 m^{\Theta(1)}}{(2m)! 2^{2m}}.
\]

Now observe that for \( k = 0 \mod 3 \), we have:

\[
a_k \geq F^{(3)}_{6k-6} \cdot F^{(3)}_{6k-18} \cdot F^{(3)}_{6k-30} \cdots F^{(3)}_{3k+3}.
\]

To see this, consider partition of \( \tau_k \), \( k = 0 \mod 3 \), into 3-ribbons as Figure 4. Now formulas (5.2) and (5.3) give:

**Lower Bound 5.3.**
\[ \phi \geq \alpha_3 := -\frac{3}{2} - \frac{\log 3}{2} + \frac{11 \log 2}{6} - \frac{2 \log \pi}{3} \approx -0.4431. \]

**Exercise 5.4.** Find the analogue of (5.3) for \( F^{(5)}_m \). Use it to derive the following lower bound.

**Lower Bound 5.5.**
\[ \phi \geq \alpha_5 := -\frac{3}{2} + \frac{43 \log 2}{30} - \frac{\log 3}{2} + \log 5 - \frac{4 \log \pi}{5} \approx -0.3621. \]

**Exercise 5.6.** Denote by \( \alpha_r \) the bound obtained from taking \( r \)-ribbons. Prove that \( \{\alpha_r\} \) is strictly increasing and thus has a limit \( \alpha := \lim_{r \to \infty} \alpha_r \), s.t. \( \alpha \leq \phi \). Prove or disprove that \( \alpha = \phi \).

5.3. **Slim diagrams.** Observe that partition of \( \tau_k \) into 2-row diagrams in Figure 4 gives

\[
a_k \leq \binom{n}{1,5,9,\ldots,2k,2k,\ldots,2k} \cdot [C_1 \cdot C_3 \cdot C_5 \cdots] \cdot (k+1)^{\frac{k}{2}},
\]

where \( C_m = \frac{1}{m+1} \binom{2m}{m} \) is the **Catalan number**. To see this, follow the argument in §4.2

**Exercise 5.7.** Check that (5.4) improves the upper bound in (4.2), but does not improve the bound on \( \phi \) over that in the Upper Bound 4.3. Prove that the same holds for partitions of \( \tau_k \) into \( r \)-row slim diagrams, for every fixed \( r \). Explain formally why these slim diagrams are less effective for upper bounds compared to ribbons for lower bounds.
6. Hook-length formula

6.1. The setup. Let \( \lambda \vdash n \). Denote by \( f^\lambda = |\text{SYT}(\lambda)| \) the number of standard Young tableaux of shape \( \lambda \). The hook-length formula states:

\[
(\text{HLF}) \quad f^\lambda = n! \prod_{(i,j) \in \lambda} \frac{1}{h_{\lambda}(i,j)},
\]

where \( h_{\lambda}(i,j) = \lambda_i - i + \lambda'_j - j + 1 \) is the hook-length of the square \((i,j)\). See [NPS, PT] for some of our favorite proofs.

6.2. Staircase shape. Denote by \( b_m := f^\delta_m = |\text{SYT}(\delta_m)| \) the number of standard Young tableaux of the staircase shape, see [OEIS, A005118]. It follows from the HLF that

\[
(6.1) \quad b_m = \frac{(m/2)!}{1^{m-1} \cdot 3^{m-2} \cdot 5^{m-3} \cdots (2m-1)} = \frac{(m)!}{\Lambda(m)}.
\]

Observe that

\[
(6.2) \quad a_k \cdot b_k \leq b_{2k},
\]

since together two tableaux of shape \( \delta_k \) and \( \delta_{2k}/\delta_k \) can form a single tableau of shape \( \delta_{2k} \). This gives:

Upper Bound 6.1.

\[
\phi \leq \frac{1}{2} - \frac{\log 2}{6} - \frac{\log 3}{2} \approx -0.1648.
\]

6.3. Three staircases. Observe that

\[
(6.3) \quad a_k \geq b_k^2 \cdot b_{k+1} \cdot \binom{2k}{k}.
\]

This follows immediately from the partition of \( \tau_k \) into shapes \( \delta_k, \delta_k \) and \( \delta_{k+1} \) as in Figure 5. Combined with (6.1), this gives:

Lower Bound 6.2.

\[
\phi \geq \frac{1}{2} - \frac{5 \log 2}{6} - \frac{\log 3}{2} \approx -0.6269.
\]

![Figure 5. Partition of \( \tau_6 = \delta_{12}/\delta_6 \) into three staircases \( \delta_6, \delta_6 \) and \( \delta_7 \). Partition of \( \tau_6 \) into \( \delta_6 \) and \( \zeta_6 \). DeWitt shape \( \varsigma_3 = \delta_{12}/\rho_3 \) and its partition into \( \delta_3, \delta_3 \) and \( \tau_6 \).](image)

Exercise 6.3. Let \( \xi_k = (2k-1, \ldots, k) \). Partition \( \delta_{2k} \) into two staircases \( \delta_k \) and one skew shape \( \zeta_k := \xi_k/\delta_k \), see Figure 5. Denote \( z_k := |\text{SYT}(\zeta_k)| \). Use the HLF applied to shape \( (2k-1, \ldots, k) \) to get an upper bound on \( z_k \). Note that

\[
a_k \leq b_k \cdot z_k \left( \binom{k^2}{k} \binom{k}{2} \right).
\]

Use this and (6.1) to obtain an upper bound on \( a_k \).
6.4. DeWitt shape. Denote by $\rho_m = (k^k)$ the $k \times k$ square diagram, and define the DeWitt shape $\varsigma_m := \delta_{4m}/\rho_m$, see Figure 5. It was shown in [DeW] (see also [KS, MPP3]), that

$$d_k := |\text{SYT}(\varsigma_m)| = n! \frac{\Phi(m)^3 \Phi(3m) \Psi(m) \Psi(3m)}{\Phi(2m)^3 \Psi(2m)^2 \Psi(4m)} ,$$

where $n = 14m^2 = |\varsigma_m|$. Now observe that

$$(6.4) \quad a_{2k} \cdot b_k^2 \leq d_k .$$

Using this and (6.1), we get:

Upper Bound 6.4.

$$\phi \leq -\frac{17 \log 2}{3} + \log 3 + \frac{7 \log 7}{6} + \frac{1}{2} \approx -0.0590 .$$

7. Littlewood–Richardson coefficients

7.1. Basic formulas. Let $\lambda \vdash n$, $\mu \vdash m$, $\nu \vdash n - m$. Recall two properties of the LR–coefficients:

$$(7.1) \quad f^\mu f^{\nu} \left(\begin{array}{c} n \\ m \end{array}\right) = \sum_{\lambda \vdash n} c_{\mu \nu}^\lambda f^\lambda \quad \text{and} \quad f^{\lambda/\mu} = \sum_{\nu \vdash n - m} c_{\mu \nu}^\lambda f^{\nu} .$$

It was observed in [MPP4, Prop. 2.4], that

$$(7.2) \quad f^{\lambda/\mu} \leq \frac{n! f^\mu}{m! f^\lambda} .$$

Indeed, putting together equations in (7.1) gives:

$$f^{\lambda/\mu} = \sum_{\nu \vdash n - m} c_{\mu \nu}^\lambda f^{\nu} \leq \sum_{\nu \vdash n - m} \left(\begin{array}{c} n \\ m \end{array}\right) f^\mu f^{\nu} f^{\nu} = f^\mu f^{\lambda} \left(\begin{array}{c} n \\ m \end{array}\right) \sum_{\nu \vdash n - m} (f^{\nu})^2 = \frac{n! f^\mu}{m! f^\lambda} .$$

Now, applying (7.2) to $\tau_k = \delta_{2k}/\delta_k$, we get:

Upper Bound 7.1.

$$\phi \leq \frac{3}{2} - \frac{\log 2}{2} \approx 1.1534 .$$

Exercise 7.2. Explain why this upper bound is so poor.

7.2. Evaluations. Let $\lambda = \delta_{2k}$, $\mu = \delta_k$, $\nu = \xi_k := (2k - 1, \ldots, k)$ and $x_k := |\text{SYT}(\xi_k)|$. Note that $\lambda/\nu = \mu$, which implies $c_{\mu \nu}^\lambda = 1$. Then (7.1) gives $a_k \geq x_k$. Applying (HLF), we have:

Upper Bound 7.3.

$$\phi \geq \frac{1}{2} - \frac{\log 2}{6} - \log 3 \approx -0.7141 .$$

Exercise 7.4. Recall from [Sag], [S2], [LL], a combinatorial interpretation of LR–coefficients $c_{\mu \nu}^\lambda$ as the number $x_k := |\text{LR}(\lambda/\mu, \nu)|$ of lattice tableaux of shape $\lambda/\mu$ and weight $\nu$. Observe that there is a unique lattice tableau in $\text{LR}(\tau_k, \nu)$, see Figure 6. Deduce that $c_{\mu \nu}^\lambda = 1$, for $\lambda/\mu = \tau_k$ as above.

Figure 6. The unique lattice tableau in $\text{LR}(\tau_5, \xi_5)$. 

1 1 1 1 1
1 2 2 2 2
1 2 3 3 3
1 2 3 4 4
1 2 3 4 5
1 3 4 5 6
7.3. **Upper bound on LR–coefficients.** Recall from [PPY] Thm 1.5 that
\[
\frac{c_{\mu\nu}^\lambda}{\sqrt{\binom{n}{m}}}, \quad \text{for all } \lambda \vdash n, \mu \vdash m, \nu \vdash n - m.
\]
To see this, use (7.1) to obtain
\[
\sum_{\lambda \vdash n} (c_{\mu\nu}^\lambda)^2 \leq \sum_{\lambda \vdash n} c_{\mu\nu}^\lambda \frac{f^\lambda_{\mu\nu}}{f^\mu f^\nu} = \frac{1}{f^\mu f^\nu} \cdot \frac{n}{k} \binom{n}{k} = \binom{n}{k}.
\]
This implies:
\[
f_{\frac{\lambda}{\mu}} = \sum_{\nu \vdash n-m} c_{\mu\nu}^\lambda f^\nu \leq p(n-m) \sqrt{\binom{n}{m}} \cdot \sqrt{(n-m)!} < p(n) \sqrt{n! m!},
\]
where \(p(n)\) denotes the number of partitions of \(n\). Applying this to \(\lambda/\mu = \tau_k\) and using \(p(n) = e^{O(\sqrt{n})}\), we get:

**Upper Bound 7.5.**
\[
\phi \leq 4 \log 2 - \log 3 - \frac{1}{2} \approx -0.1251.
\]

**Exercise 7.6 ([PPY] §4.1).** In the opposite direction, prove that for every \(0 \leq k \leq n\), we have:
\[
\sum_{\lambda \vdash n} \sum_{\mu \vdash k, \nu \vdash n-k} (c_{\mu\nu}^\lambda)^2 \geq \binom{n}{k}.
\]
What does this formula say about how sharp is (7.3)?

8. **Naruse hook-length formula**

8.1. **The setup.** Let \(\lambda/\mu\) be a skew shape and \(D\) be a subset of the Young diagram of \(\lambda\). A square \((i, j) \in D\) is called **active** if \((i+1, j), (i, j+1)\) and \((i+1, j+1)\) are all in \(\lambda \setminus D\). An **excited move** is a replacement of an active \((i, j) \in D\) with \((i+1, j+1)\). An **excited diagram** of \(\lambda/\mu\) is a subset of squares in \(\lambda\) obtained from the Young diagram \(\mu\) after a sequence of excited moves on active cells. Let \(\mathcal{E}(\lambda/\mu)\) be the set of excited diagrams of \(\lambda/\mu\), see Figure 7. The **Naruse hook-length formula** (NHLF) states that for every skew shape \(\lambda/\mu\) we have:

\[
(\text{NHLF}) \quad |\text{SYT}(\lambda/\mu)| = n! \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j) \in \lambda \setminus D} \frac{1}{h^\lambda_{\lambda}(i,j)},
\]
where \(n = |\lambda/\mu|\).

![Figure 7. Set \(\mathcal{E}(\lambda/\mu)\), for \(\lambda = (5, 4, 4, 1)\) and \(\mu = (2, 1)\). The arrows indicate excited moves.](image)

**Exercise 8.1 ([MPP] §12.1, see also [PPS]).** Denote by \(\lambda^*\) the skew shape obtained by rotating the diagram \(\lambda\) at 180 degrees. Apply the NHLF to \(\lambda^*\). Conclude the following inequality:
\[
\prod_{(i,j) \in \lambda} h^\lambda_{\lambda}(i,j) \leq \prod_{(i,j) \in \lambda} h^\lambda_{\lambda^*}(i,j),
\]
where \( h^*_λ(i, j) = (i + j - 1) \). Compare this with

\[
\sum_{(i,j)\in λ} h_λ(i,j) = \sum_{(i,j)\in λ} h^*_λ(i,j),
\]

and explain the discrepancy.

8.2. **Basic bounds.** Observe that since \( μ \in \mathcal{E}(λ/μ) \), we have:

\[
|\text{SYT}(λ/μ)| \geq F(λ/μ) := n! \prod_{(i,j)\in λ/μ} \frac{1}{h(i,j)}.
\]

Taking \( λ = δ_{2k} \) and \( μ = δ_k \) gives

**Lower Bound 8.2.**

\[
φ \geq \frac{1}{6} - \frac{3 \log 2}{2} + \frac{\log 3}{2} \approx -0.3237.
\]

Now, for the upper bound observe that the hooks decrease under excited moves. This gives

\[
|\text{SYT}(λ/μ)| \leq F(λ/μ) \cdot |\mathcal{E}(λ/μ)|.
\]

For \( λ = δ_{2k}, \ μ = δ_k, \) and \( k \) even, observe that the excited diagrams are in the bijection with non-intersecting paths as shown in Figure 8. Thus,

\[
|\mathcal{E}(τ_k)| \leq \left(\frac{2k-2}{k-1}\right)^{k/2}.
\]

Combined with (8.2), this gives:

**Upper Bound 8.3.**

\[
φ \leq \frac{1}{6} - \frac{5 \log 2}{6} + \frac{\log 3}{2} \approx 0.1384.
\]

**Exercise 8.4.** Before moving on, show that both the lower and the upper bounds are not exact, i.e. they can be improved by some \( ε > 0 \), using better counting over excited diagrams.

![Figure 8. Non-intersecting paths in the bijection with excited diagrams, in the beginning and after three moves.](image)

8.3. **Non-intersecting paths.** Recall that non-intersecting paths with fixed start and end points are counted with a determinant. For thick ribbons, this determinant was computed by Proctor:

\[
|\mathcal{E}(τ_k)| = \prod_{1 \leq i < j \leq k} \frac{k+i+j-1}{i+j-1} = \left[\frac{Φ(3k-1) Φ(k-1)^3 (2k-1)! (k-1)!}{Φ(2k-1)^3 (3k-1)!}\right]^{1/2},
\]

see [MPP4 Lemma 8.1] and [OEIS A181119]. Combined with (8.2), this gives:

**Upper Bound 8.5.**

\[
φ \leq \frac{1}{6} - \frac{7 \log 2}{2} + 2 \log 3 \approx -0.0621.
\]
Exercise 8.6 (see [MPP3 §7]). Prove that the excited diagrams in this case are in bijection with lozenge tilings of half of a \( (k-1) \times (k-1) \times k \) hexagon. Recall the MacMahon box formula for the number \( M(a,b,c) \) of lozenge tilings of an \( a \times b \times c \) hexagon. Use this to get an upper bound \( |E(\tau_k)|^2 \leq M(k-1,k-1,k) \). Show that this inequality gives the same bound on \( \phi \) as in Upper Bound 8.5. Explain why.

9. Flipped HLF

9.1. The setup. A skew shape \( \lambda/\mu \) is called slim if \( \lambda_\ell \geq \mu_1 + \ell - 1 \), where \( \ell = \lambda_1' \) is the number of parts in \( \lambda \). A subset \( D \) of \( \lambda \) is called a flipped excited diagram if after vertical flipping it is the (usual) exited diagram, see Figure 9. Let \( E^\circ(\lambda/\mu) \) be the set of flipped excited diagrams of \( \lambda/\mu \). Clearly, \( |E^\circ(\lambda/\mu)| = |E(\lambda/\mu)| \). In [MPP3, §3.4], we show that

\[
\text{(Flipped-HLF)} \quad |\text{SYT}(\lambda/\mu)| = n! \sum_{D \in E^\circ(\lambda/\mu)} \prod_{(i,j) \in \lambda \setminus D} \frac{1}{h_\lambda(i,j)},
\]

where \( n = |\lambda/\mu| \).

Figure 9. Slim skew shape \( \lambda/\mu \), where \( \lambda = (9,8,6) \), \( \mu = (2,1) \), and the set \( E^\circ(\lambda/\mu) \) of flipped excited diagrams.

Exercise 9.1. In the context of the lower bound (8.1), prove that

\[
n! \prod_{(i,j) \in \lambda/\mu} \frac{1}{h_\lambda(i,j)} \geq n! \prod_{(i,j) \in \mu^\circ \setminus \lambda} \frac{1}{h_\lambda(i,j)},
\]

where \( \mu^\circ \subset \lambda \) denotes the subset of squares of \( \lambda \) obtained by the vertical flip of \( \mu \). Conclude that [Flipped-HLF] gives a better upper bound than (8.2).

9.2. The staircase phenomenon. In this section, we follow [MPP5]. Clearly, diagram \( \tau_k \) is not slim. However, we can create two smaller slim diagrams by partitioning \( \delta_k \) into a square \( \rho_k/2 \) and two smaller staircases \( \delta_k/2 \), where \( k \) is even. In the RHS of (NHLF), consider only excited diagrams which do not move \( \rho_k/2 \), which are thus restricted to slim shapes. Since the sum over excited diagrams in each slim shape is equal to the sum over flipped excited diagrams, this gives a lower bound on \( a_k \).

There are two additional properties of thick ribbons \( \tau_k = \delta_{2k}/\delta_k \) which play a role here. First, because the hooks in \( \delta_{2k} \) are invariant under flipped excited moves, we conclude that the terms we are summing in the RHS of (Flipped-HLF) are all equal. Second, by the Exercise 9.3 below, the number of such terms is equal to \( 2^{(k-1)/2} \), for each of the two slim diagrams. Putting these together, we obtain:

\[
a_{2k} \geq \frac{n! \cdot 2^{k(k-1)}}{\Psi(2k+1) \left[(2k+1)! (2k+5)! \cdots (6k-3)! \right]^2},
\]

where \( n = |\nu_{2k}| = k(6k+1) \). This gives:

**Lower Bound 9.2 ([MPP5]).**

\[
\phi \geq \frac{1}{2} - \frac{2 \log 2}{3} - \frac{\log 3}{4} \approx -0.2368.
\]

Exercise 9.3. Prove that for every slim shape \( \lambda/\mu \), s.t. \( \mu = \delta_\ell \), we have \( |E(\lambda/\mu)| = 2^{(\ell)} \), as in Figure 9. Hint: Use non-intersecting paths as above to give a bijection with domino tilings of the Aztec diamond, see OEIS A006125.
10. SLANTED HLF

10.1. The setup. For a skew shape $\lambda/\mu$, define a slanted shape $\lambda^\nabla/\mu^\nabla$ as in Figure 11. The slanted excited moves are now vertical moves as in Figure 11 while the squares of $\lambda/\mu$ move diagonally as before. Slanted excited diagrams are defined similarly, see an example is Figure 12 where slated diagrams are in dark blue.

In [MZ] and earlier in a different language in [KT, OO], the authors show that

\[
\left| \text{SYT}(\lambda/\mu) \right| \geq G(\lambda/\mu) := \frac{n! f^\lambda}{m!} \prod_{(i,j) \in \mu^\nabla} a_{\lambda}(i,j),
\]

where $m = |\lambda|$, $n = |\lambda/\mu|$, $f^\lambda = |\text{SYT}(\lambda)|$ as above, and $a_{\lambda}(i,j) := \lambda_i - j + 1$ is the arm length of the square $(i,j) \in \lambda$.

Exercise 10.1. Suppose the smallest part $\lambda_\ell \geq \mu_1$. Show that in this case $E^\nabla(\lambda/\mu)$ are in bijection with $E^\nabla(\lambda/\mu)$. Then compare (Slanted-HLF) vs. (Flipped-HLF).

10.2. Lower and upper bounds. We follow [MZ, §9.6] in this section. For a slanted excited diagram $D = \mu^\nabla$, suppose square $(i,j) \in D$ is an image of $(p,q) \in \mu$. It follows from (Slanted-HLF) that

\[
\left| \text{SYT}(\lambda/\mu) \right| \geq G(\lambda/\mu) := \frac{n! f^\lambda}{m!} \prod_{(i,j) \in \mu^\nabla} a_{\lambda}(i,j).
\]

Taking $\lambda = \delta_{2k}$ and $\mu = \delta_k$ gives:

Lower Bound 10.2 ([MZ Thm 1.4]),

\[
\phi \geq \frac{1}{2} - \frac{9 \log 2}{2} + 2 \log 3 \approx -0.4219.
\]
Observe that arm length \( a_\lambda(i,j) \) are non-increasing under slanted excited moves. This gives
\[
|\text{SYT}(\lambda/\mu)| \leq G(\lambda/\mu) \cdot |\mathcal{E}^\varphi(\lambda/\mu)|.
\]
By employing ad hoc estimates on \( |\mathcal{E}^\varphi(\tau_k)| \), one can get:
\[
\text{Upper Bound } 10.3 \text{ ([MZ Thm 1.4])}.
\]
\[
\phi \leq \frac{1}{2} - \frac{13 \log 2}{2} + \frac{7 \log 3}{2} \approx -0.1603.
\]

**Exercise 10.4.** In notation above, observe that \( a_\lambda(i,j) \leq h_\lambda(p,q) \). Conclude that \( G(\lambda/\mu) \leq F(\lambda/\mu) \) for every \( \lambda/\mu \), i.e. the lower bound in (10.1) is never better than the lower bound in (8.1).

**Exercise 10.5.** Let \( \lambda = \rho_{2k} \) and \( \mu = \rho_k \) be the difference of two squares. Compare the bounds given by (Slanted-HLF) vs. (NHLF) for the thick hook \( \varrho_k = \lambda/\mu \).

### 11. Additional exercises

#### 11.1. General bounds.

**Exercise 11.1.** Follow up on the discussion in §2.1 and prove an upper bound for general \( f^{\lambda/\mu} \) from Feit’s determinant formula (2.1).

**Exercise 11.2 (McK).** Recall the identity
\[
\sum_{\lambda \vdash n} f^\lambda = v_n,
\]
where \( v_n = |\{ \sigma \in S_n, \sigma^2 = 1 \} | \) is the number of involutions, see e.g. [OEIS A000085]. Use this identity to conclude (2.3). Which approach gives a better upper bound for the constant \( c \) in (2.4)? Explain why.

**Exercise 11.3 (VK2).** Prove that
\[
D_n \leq \sqrt{n!} e^{-c\sqrt{n}} \quad \text{for some } c > 0,
\]
cf. (2.4). Conclude that “most” dimensions \( f^\lambda \) are much smaller than \( D_n \).

**Exercise 11.4.** In the context of §2.3 check that
\[
|\text{SYT}(k^{2k})| > (k!)^{k+1} |\text{SYT}(\delta_k)|^2 > |\text{SYT}(k^k)|^2.
\]

**Exercise 11.5.** Prove that for every series parallel poset \( P = (P, \prec) \), we have:
\[
e(P) = \prod_{x \in X} \frac{1}{\text{br}(x)},
\]
where \( e(P) \) is the number of linear extensions of \( P \). Conclude the HLF for trees.

**Exercise 11.6 (SK).** Let \( B_n \) denote the Boolean lattice, a poset of all subsets of \([n]\) ordered by inclusion. Use induction to prove that \( B_n \) has a partition into \( \binom{n}{\lfloor n/2 \rfloor} \) saturated chains. Use this to partition to obtain an upper bound for \( e(B_n) \). Compare this bound with the lower bound obtained via the natural partition of \( B_n \) into \((n+1)\) antichains.

**Exercise 11.7.** In §7.3 we used a simple \( f^\nu \leq \sqrt{(n-m)!} \) inequality to obtain an upper bound. Using the fact that we must have \( \nu_1, \ell(\nu) \leq 2k \) and the result in [LS], give a better upper bound for \( f^\nu \), and then for \( \phi \).

**Exercise 11.8.** Let \( \nu \) be a collection of squares in \( \mathbb{N}^2 \), such that \( \nu \subset [k] \times [k] \) and \( n := |\nu| > \epsilon k^2 \) for some fixed \( \epsilon > 0 \). Use chains and antichains bounds to prove that
\[
\log |\text{SYT}(\nu)| = \frac{1}{2} n \log n + O(n).
\]
Exercise 11.9. Fix $d \geq 3$, and let $\nu$ be a collection of squares in $[k]^d$, and $n := |\nu| > \epsilon k^d$ for some fixed $\epsilon > 0$. Generalize standard Young tableaux to $d$-dimensional space, and prove that $\log |\text{SYT}(\nu)| = \frac{d-1}{d} n \log n + O(n)$ bound.

Exercise 11.10. Let $H_k \subset \mathbb{N}^3$ be the $k \times k \times k$ cube. Use partition of $H_k$ into $k$ layers of $k \times k$ squares $\rho_k$ to give an upper bound for the constant implied by the $O(n)$ notation in the previous exercise. Compare it with the chains upper bound.

Exercise 11.11. What is the cost of computing $a_k$ using Feit’s determinant formula? Do this carefully.


Exercise 11.12. Note that $c_k := |\text{SYT}(\rho_{2k-1}/\delta_k)|$ can be computed by the HLF. Show that $a_k \cdot \delta_{2k-1} \leq c_k$. Use this inequality to derive an explicit upper bound on $\phi$.

Exercise 11.13. Recall (or compute directly) the asymptotics for $|\text{SYT}(\rho_k)|$, see e.g. [OEIS, A039622]. Consider two more partitions of a square as in Figure 13 and find the upper bounds on $\phi$. Compare them with the bound in the previous exercise. Is it clear a priori which bound is the best?

Exercise 11.14. Use the approach in (6.2) to get an upper bound for $|\text{SYT}(\rho_k)|$ for the thick hook $\varrho_k := \rho_{2k}/\rho_k$. Compare with the true value given by the HLF.

Exercise 11.15. Use the approach in (6.2) to get an upper bound for $|\text{SYT}(\varsigma_k)|$ for the DeWitt shape $\varsigma_k = \delta_{4k}/\rho_k$. Compare with the actual asymptotics.

![Figure 13. Three partitions of a square and a thick hook $\varrho_k := \rho_{2k}/\rho_k$.](image)

Exercise 11.16 (see [MPP], §9.2). Use (8.1) for the thick hook $\varrho_k := \rho_{2k}/\rho_k$. How good is this? Compare with upper bound first using non-intersecting paths, and then with the exact asymptotics given by MacMahon box formula for $M(k,k,k)$, see e.g. [OEIS, A008793].

Exercise 11.17. Explain the similarity of asymptotics in Exercise 8.6 and 11.16 on the level of lozenge tilings of the hexagon, and then on the level of standard Young tableaux. Can you make the connection formal, at least in one direction?

11.3. Other shapes in the plane.

Exercise 11.19. Recall (Shifted-HLF), the analogue of the HLF for shifted shapes, see e.g. [S2, §7.21]. Use it to compute the asymptotics of the shifted staircase as in Figure 14. Check your calculations here: [OEIS A003121]

Exercise 11.20. Define a $\theta$-truncated square $\eta_{k,\theta}$ as in Figure 14, obtained by removing shifted triangle of size $\theta k$ from a $k$-square. Use the chain and the antichain partitions to prove that

$$\log |\text{SYT}(\eta_{k,\theta})| = \frac{1}{2} n \log n + O(n),$$

where $n = k^2 - \theta k/2$ is the size of $\eta_{k,\theta}$, and the constants implied by the $O(\cdot)$ notation can depend on $\theta$.

Now use formulas in [Pan] (see also [AR, §8.2]), to prove that $O(n)$ can be replaced with $c(\theta) + o(n)$, and compute the exact formula for $c(\theta)$. Explain why $c(\theta)$ is monotone on $[0,1]$.

Exercise 11.21. Partition the square into a shifted staircase and a $\theta$-truncated square, see Figure 14. Use this to get an upper bound for $c(\theta)$. Explain why this upper bound is sharper than the corresponding chain partition bound.

Exercise 11.22. Consider a $k$-octagon $\omega_k$, defined as the difference of a square $\rho_{3k}$ and four rotated staircases $\delta_k$. Use two partitions in Figure 14 to obtain lower bounds on the number of standard Young tableaux of the $k$-octagon via exact formulas for shifted staircases and truncated squares, see above. Try to guess which bound is sharper before the calculation, and check if your guess is confirmed by the calculation.

Exercise 11.23. Use 5-ribbon partition to get yet another lower bound for $k$-octagons $\omega_k$. Compare with the bound in the previous exercise.

Exercise 11.24. Use an outpartition in Figure 14 to get an upper bound for $k$-octagons $\omega_k$. Compare with the chain upper bound.

Exercise 11.25. Consider plane region $B_k$ of squares which fit the circle of radius $k$. Read about the Gauss circle problem. Prove that

$$\log |\text{SYT}(B_k)| = \frac{1}{2} n \log n + O(n),$$

where $n = |B_k| \sim \pi k^2$. Do you think that $O(n)$ can be replaced with $cn + o(n)$?

Exercise 11.26. Find two subsets of squares $D, D' \subset \mathbb{N}^2$ which are equal up to a permutation of rows and columns, and $|\text{SYT}(D)| \neq |\text{SYT}(D')|$. Generalize the Young symmetrizer to obtain the corresponding $S_n$-characters $\chi^D = \chi^{D'}$. Conclude that we do not always have $\chi^D(1) = |\text{SYT}(D)|$, and the Algebraic Combinatorics technology no longer applies.

Exercise 11.27. Use the #P-completeness of $|\text{SYT}(D)|$ proved in [DP], to argue why the determinant formula is unlikely to extend for general $D$. 

\[
\text{Figure 14. Shifted staircase, truncated square, partition of a square, 4-octagon, two partitions and one outpartition of an octagon.}
\]
12. **Conjectures and open problems**

12.1. **Exciting diagrams.** We start with some unfinished business:

**Open Problem 12.1** (see §10.2). *Improve the upper bound on $|E^\varnothing(\tau_k)|$.*

Even a relatively small improvement in the Upper Bound 10.3 can perhaps give a bound better than Upper Bound 6.1.

Now, consider the number $|E(\lambda/\mu)|$ of excited diagrams of general skew shapes obtained from piecewise linear regions scaled by $\sqrt{n}$. In this case they are in bijection with lozenge tilings [MPP3], and the existence of $\lim_{n \to \infty} \frac{1}{n} \log |E(\lambda/\mu)|$ follows from the existing technology [Gor2]. This suggests the following:

**Open Problem 12.2.** *Prove the existence, and compute the limit*

$$\lim_{n \to \infty} \frac{1}{n} \log |E^\varnothing(\tau_k)|.$$

12.2. **Limits for general shapes.** The main result in [MPT] of which Theorem 1.1 is a corollary, proves existence of the limit

$$(12.1) \quad \lim_{n \to \infty} \frac{1}{n} \left[ \log |\text{SYT}(\lambda/\mu)| - \frac{1}{2} n \log n \right],$$

where the skew shape $\lambda/\mu$ is obtained by a piecewise smooth regions in the plane, scaled by $\sqrt{n}$. For (usual) Young diagram shapes it follows easily from the (HLF), see [MPP3].

**Conjecture 12.3.** *The limit (12.3) holds for all regions, not just skew shapes.*

This conjecture extends Exercise 11.8. It is open even in the special case of k-octagons:

**Conjecture 12.4.** *Prove that for k-octagons $\omega_k$ with $n = 9k^2 - 4{k \choose 2}$ we have:

$$\log |\text{SYT}(\omega_k)| = \frac{1}{2} n \log n + cn + o(n) \quad \text{as} \quad k \to \infty,$$

for some $c \in \mathbb{R}$.

In $\mathbb{R}^d$, the analogous conjecture is likely to hold as well but is open even for the 3-dimensional Young diagram shapes:

**Conjecture 12.5.** Let $S \subset \mathbb{R}_+^3$ be a 3-dimensional piecewise linear shape, such that $(a, b, c) \in S \implies (x, y, z) \in S$, for all $x \leq a$, $y \leq b$, $z \leq c$. Then there is a limit:

$$(12.2) \quad \lim_{n \to \infty} \frac{1}{n} \left[ \log |\text{SYT}(\Lambda_n)| - \frac{2}{3} n \log n \right],$$

where the shape $\Lambda_n$ is obtained by a scaling of $S$ by $\sqrt[n]{n}$.

Even in the special case of a 3-dimensional cubes this conjecture is open, and is especially attractive:

**Conjecture 12.6.** Let $H_k \subset \mathbb{N}_+^3$ be a $k \times k \times k$ cube. Then there is a limit:

$$(12.3) \quad \lim_{n \to \infty} \frac{1}{k^3} \left[ \log |\text{SYT}(H_k)| - 2k^3 \log k \right].$$

In view of (2.3), we also conjecture the existence of the limit surfaces for the shapes of integers $\leq \alpha n$ in random $A \in \text{SYT}(\Lambda_n)$. It would be exciting to prove this even for the cube, but this goes outside the scope of this paper.
13. Brief historical remarks

The study of Young tableaux goes back to the works of Alfred Young (c. 1900), and is so extensive that a quick overview would not do it justice. We refer to the extensive Ch. 7 in [S2] which gives a thorough overview of the subject, textbook [Sag] for the friendly introduction, and to [AR] for an extensive survey of enumerative results. See also [DP] for complexity aspects and the recent \#P-completeness of $|\text{SYT}(D)|$ for general $D \subset \mathbb{N}^2$ (cf. Exercise 11.27).

The hook-length formula goes back to Thrall (1952) and Frame–Robinson–Thrall (1954). It has been reproved and generalized in numerous ways. We refer to [CKP, §6.2] for a survey. Similarly, the Littlewood–Richardson coefficients are classical and go back to 1930s. They are the subject of intense investigation in its own right with many generalizations and variations. We refer to [vL] for a somewhat dated but helpful entry point, and to [PPY] for a recent asymptotic analysis based in part on some earlier observations by Stanley.

The study of limit shapes and the Vershik–Kerov–Logan–Shepp theory is in itself a subject of intense investigation. We refer to [Rom] for a thorough treatment and connections to longest increasing subsequences in random permutations. A related followup story of random lozenge tilings, the Arctic circle phenomenon, etc., is well presented in [Gor2]. It relates to the subject of this paper in connection with excited diagrams, and emerges a great motivation for many of the limit questions and conjectures above.

The equation (NHLF) is in an unpublished work by Naruse, and is discussed at length in [MPP1, MPP2]. We refer to [Kon] [MPP2] for elementary proofs, to [MPP1, §9] for a survey of other formulas for $f^{\lambda/\mu} = |\text{SYT}(\lambda/\mu)|$, and to [MPP3, §9] for a variety of other exact formulas (some conjectured and proved in a later work by other authors). The asymptotic applications of (NHLF) were first introduced in [MPP3], and advanced in [MPP3, MZ].

Despite superficial similarities, our two variations on (NHLF) have very different nature. Equation (Flipped-HLF) is given in [MPP3, §3] and further explored in [MPP5]. A simple proof via reduction to (NHLF) is given in [PP]. Equation (Slanted-HLF) is a Morales–Zhu reformulation [MZ] of the Okounkov–Olshanski formula [OO] (see also [S1]). In [MZ], the authors also give a simple proof, and establish a previously conjectured equivalence to the rule in [KT].

In conclusion, let us mention that while the style of this paper is somewhat unusual, it borrows ideas from several sources. First, we are heavily influenced by the brief exercise-based presentation by Lovász [Lov], which is itself a continuation of a long tradition, see e.g. the celebrated problem books by Pólya–Szegő (1925) and Yaglom–Yaglom (1954). Second, we learned the value of worked out publishable examples clarifying the theory from the Pemantle–Wilson papers, see [PW]. Third, while the idea of a “running example” is standard, we personally were influenced by Krattenthaler’s presentation [K4].

Acknowledgements. We are grateful to Alejandro Morales, Greta Panova and Martin Tassy for numerous interesting conversations. Much of this paper grew from our previous work in the area. We are also thankful to Sam Dittmer, Anna Gordenko, Vadim Gorin, Victor Kleptsyn, Fёdor Petrov, Leo Petrov, Dan Romik, Richard Stanley, Anatoly Vershik and Damir Yeliussizov for helpful comments on the subject. The author was partially supported by the NSF.
References


[Gor1] A. Gordenko, Limit shapes of large skew Young tableaux and a modification of the TASEP process, in preparation (2020–).


R. Pemantle and M. C. Wilson, Twenty combinatorial examples of asymptotics derived from multivariate generating functions, SIAM Rev. 50 (2008), 199–272.


