ON THE CROSS-PRODUCT CONJECTURE FOR THE NUMBER OF LINEAR EXTENSIONS

SWEE HONG CHAN[⋆] , IGOR PAK[⋄] , AND GRETA PANOVA[♮]

Abstract. We prove a weak version of the *cross–product conjecture*: $\mathrm{F}(k+1,\ell)\,\mathrm{F}(k,\ell+1) \geq (\frac{1}{2}+\varepsilon)\,\mathrm{F}(k,\ell)\,\mathrm{F}(k+1,\ell+1)$, where $\mathrm{F}(k,\ell)$ is the number of linear extensions for which the values at fixed elements x,y,z are k and ℓ apart, respectively, and where $\varepsilon > 0$ depends on the poset. We also prove the converse inequality and disprove the *generalized cross–product conjecture*. The proofs use geometric inequalities for mixed volumes and combinatorics of words.

1. Introduction

This paper is centered around the *cross–product conjecture* (CPP) by Brightwell, Felsner and Trotter that gives the best known bound for the celebrated $\frac{1}{3}$–$\frac{2}{3}$ *Conjecture* [BFT95, Thm 1.3]. Here we prove several weak versions of the conjecture, and disprove a stronger version we conjectured earlier in [CPP22].

Let $P = (X, \prec)$ be a poset with $|X| = n$ elements. A *linear extension* of P is a bijection $L : X \to [n] = \{1, \ldots, n\}$, such that $L(x) < L(y)$ for all $x \prec y$. Denote by $\mathcal{E}(P)$ the set of linear extensions of P. Fix distinct elements $x, y, z \in X$. For $k, \ell \geq 1$, let

$$\mathcal{F}(k,\ell) \ := \ \big\{L \in \mathcal{E}(P) \,:\, L(y) - L(x) = k,\ L(z) - L(y) = \ell\big\},$$

and let $\mathrm{F}(k,\ell) := \big|\mathcal{F}(k,\ell)\big|$.

Conjecture 1.1 (*Cross–product conjecture* [BFT95, Conj. 3.1]). *We have:*

$$\text{(CPC)} \qquad \mathrm{F}(k+1,\ell)\,\mathrm{F}(k,\ell+1) \ \geq \ \mathrm{F}(k,\ell)\,\mathrm{F}(k+1,\ell+1).$$

The CPC was proved in [BFT95, Thm 3.2] for $k = \ell = 1$, and in [CPP22, Thm 1.4] for posets of width two. We also show in [CPP22, §3], that both the *Kahn–Saks* and the *Graham–Yao–Yao inequalities* follow from (CPC).

Theorem 1.2 (Main theorem). *Let $P = (X, \prec)$ be a poset on $|X| = n$ elements. Fix distinct elements $x, y, z \in X$. Suppose that $\mathrm{F}(k,\ell+2)\,\mathrm{F}(k+2,\ell) > 0$. Then:*

$$\mathrm{F}(k+1,\ell)\,\mathrm{F}(k,\ell+1) \ \geq \ \Big(\frac{1}{2} + \frac{1}{4n\sqrt{k\ell}}\Big)\,\mathrm{F}(k,\ell)\,\mathrm{F}(k+1,\ell+1). \tag{1.1}$$

Suppose that $\mathrm{F}(k,\ell+2) = 0$ and $\mathrm{F}(k+2,\ell) > 0$. Then:

$$\mathrm{F}(k+1,\ell)\,\mathrm{F}(k,\ell+1) \ \geq \ \big(\tfrac{1}{2} + \tfrac{1}{16nk\ell^2}\big)\,\mathrm{F}(k,\ell)\,\mathrm{F}(k+1,\ell+1). \tag{1.2}$$

Suppose that $\mathrm{F}(k+2,\ell) = 0$ and $\mathrm{F}(k,\ell+2) > 0$. Then:

$$\mathrm{F}(k+1,\ell)\,\mathrm{F}(k,\ell+1) \ \geq \ \big(\tfrac{1}{2} + \tfrac{1}{16nk^2\ell}\big)\,\mathrm{F}(k,\ell)\,\mathrm{F}(k+1,\ell+1). \tag{1.3}$$

Finally, suppose that $\mathrm{F}(k,\ell+2) = \mathrm{F}(k+2,\ell) = 0$ and $\mathrm{F}(k,\ell)\,\mathrm{F}(k+1,\ell+1) > 0$. Then:

$$\mathrm{F}(k+1,\ell)\,\mathrm{F}(k,\ell+1) \ = \ \mathrm{F}(k,\ell)\,\mathrm{F}(k+1,\ell+1)\,. \tag{1.4}$$

Date: January 1, 2024.

[⋆]Department of Mathematics, Rutgers University, Piscataway, NJ, 08854. Email: `sweehong.chan@rutgers.edu`.
[⋄]Department of Mathematics, UCLA, Los Angeles, CA, 90095. Email: `pak@math.ucla.edu`.
[♮]Department of Mathematics, USC, Los Angeles, CA 90089. Email: `gpanova@usc.edu`.

When $\mathrm{F}(k,\ell)\,\mathrm{F}(k+1,\ell+1)=0$, the inequality (CPC) holds trivially. Curiously, the equality (1.4) does not hold in that case since the LHS can be strictly positive (Example 4.5). Except for the natural symmetry between (1.3) and (1.2), the proof of remaining three cases are quite different and occupies much of the paper.

Note that computing the number $e(P)$ of linear extensions of P is #P-complete [BW91], even for posets of height two or dimension two [DP18]. Still, the vanishing assumptions which distinguish the cases in the Main Theorem 1.2, can be decided in polynomial time (see Theorem 4.2).

The proof of the Main Theorem 1.2 is a combination of geometric and combinatorial arguments. The former are fairly standard in the area, and used largely as a black box. The combinatorial part is where the paper becomes technical, as the translation of geometric ratios into the language of posets (following Stanley's pioneering approach in [Sta81]) leads to bounds on ratios of linear extensions that have not been investigated until now. Here we employ the *combinatorics of words* technology following our previous work [CPP22, CPP23a, CPP23b] (cf. §8.7)

Let us emphasize that getting an explicit constant above $\frac{1}{2}$ in the RHS is the main difficulty in the proof, as the $\frac{1}{2}$ constant is relatively straightforward to obtain from Favard's inequality. This was noticed independently by Yair Shenfeld who derived it from Theorem 2.4 in the same way we did in the proof of Theorem 3.1.[1] In another independent development, Julius Ross, Hendrik Süss and Thomas Wannerer gave a proof of the same $\frac{1}{2}$ lower bound using the technology of *Lorentzian polynomials* [BH20] combined with a technical result from [BLP23].[2]

Our combinatorial tools also allow us to inch closer to the CPC for two classes of posets. Fix a subset $A\subseteq X$. We say that a poset $P=(X,\prec)$ is *t-thin with respect to A*, if for every $u\in X\smallsetminus A$ there are at most t elements incomparable to u. For $A=\varnothing$, such posets are a subclass of posets of width t. This class is a generalization of t-thin posets (the case of $A=X$), studied in the context of the $\frac{1}{3}$–$\frac{2}{3}$ Conjecture [BW92, Pec08].

Similarly, we say that a poset $P=(X,\prec)$ is *t-flat with respect to A*, if for every $u\in A$ there are at most t elements comparable to u. For $A=X$, such posets are a subclass of posets of height t. Examples include *incidence posets* (see e.g. [Tro95, §10]), defined as follows. Let $G=(V,E)$ be a simple graph, let $X=V\cup E$, and let $v\prec e$ for all $e=(v,w)\in E$. For $A\subseteq E$, the corresponding poset P is 2-flat with respect to A. For $A\subseteq V$ and G is d-regular, the corresponding poset P is d-flat with respect to A.

Theorem 1.3. *Let $P=(X,\prec)$ be a finite poset. Fix distinct elements $x,y,z\in X$, and let $A:=\{x,y,z\}$. Suppose that P is either t-thin with respect to A, or t-flat with respect to A. Then:*

$$\mathrm{F}(k+1,\ell)\,\mathrm{F}(k,\ell+1)\;\geq\;\Big(\tfrac{1}{2}\,+\,\tfrac{1}{16\,t\,(t+1)^3}\Big)\,\mathrm{F}(k,\ell)\,\mathrm{F}(k+1,\ell+1). \tag{1.5}$$

Note that the constant in the RHS of (1.5) depends only on t, and thus holds for posets of arbitrary large size n, see also §8.3. We also have the following counterpart to the CPC.

Theorem 1.4 (*Converse cross–product inequality*). *Suppose that* $\mathrm{F}(k,\ell)\,\mathrm{F}(k+1,\ell+1)>0$. *Then:*

$$\mathrm{F}(k+1,\ell)\,\mathrm{F}(k,\ell+1)\;\leq\;2k\ell(\min\{k,\ell\}+1)n\,\cdot\,\mathrm{F}(k,\ell)\,\mathrm{F}(k+1,\ell+1).$$

Note that the inequality in the theorem is asymptotically tight, see Proposition 7.5. On the other hand, originally we believed in the following stronger version of the CPC:

Conjecture 1.5 (*Generalized cross–product conjecture* [CPP22, Conj. 3.2]). *We have:*

$$\mathrm{F}(k,\ell)\,\mathrm{F}(p,q)\;\;\leq\;\;\mathrm{F}(p,\ell)\,\mathrm{F}(k,q)\quad\textit{for all}\quad k\leq p,\;\ell\leq q. \tag{GCPC}$$

[1] Yair Shenfeld, personal communication (May 2, 2021).

[2] Julius Ross, personal communication (May 31, 2023).

For $p = k+1$ and $q = \ell+1$, where $k, \ell \geq 1$, this gives (CPC). In [CPP22, Thm. 3.3], the inequality (GCPC) was proved for posets of width two. However, here we show that it fails in full generality:

Theorem 1.6. *The inequality* (GCPC) *fails for an infinite family of posets of width three.*

Our final result further confirms that CPC is somehow special among similar families of inequalities. While these other inequalities are not always true, they are not simultaneously too far off in the following sense.

Theorem 1.7. *For every $P = (X, \prec)$, every distinct $x, y, z \in X$, and every $k, \ell \geq 1$, at least two of the inequalities* (CPC), (CPC1) *and* (CPC2) *are true, where*

$$\text{(CPC1)} \qquad \mathrm{F}(k+2, \ell)\, \mathrm{F}(k, \ell+1) \ \leq \ \mathrm{F}(k+1, \ell)\, \mathrm{F}(k+1, \ell+1),$$

$$\text{(CPC2)} \qquad \mathrm{F}(k, \ell+2)\, \mathrm{F}(k+1, \ell) \ \leq \ \mathrm{F}(k, \ell+1)\, \mathrm{F}(k+1, \ell+1).$$

We prove that inequalities (CPC1) and (CPC2) hold for posets of width two (Corollary 7.3). However, they are false on infinite families of counterexamples (Proposition 7.1). By Theorem 1.7, this means that the CPC holds in all these cases.

Paper structure. We start with a short background Section 2 on mixed volumes and variations on the Alexandrov–Fenchel inequalities. This section is self-contained in presentation, and uses several well-known results as a black box. In a lengthy Section 3 we show how cross product inequalities arise as mixed volume, and make some useful calculations. We also prove Theorem 1.7.

We begin our combinatorial study of linear extensions in Section 4, where we give explicit conditions for vanishing of $\mathrm{F}(k, \ell)$, and explore the consequences which include the equality (1.4). In Sections 5 and 6, we prove different cross product inequalities in the nonvanishing and vanishing case, respectively. We conclude with explicit examples (Section 7) and final remarks (Section 8).

2. Mixed volume inequalities

2.1. **Alexandrov–Fenchel inequalities.** Fix $n \geq 1$. For two sets $A, B \subset \mathbb{R}^n$ and constants $a, b > 0$, denote by

$$aA + bB \ := \ \big\{ a\,\mathbf{x} + b\,\mathbf{y} \ : \ \mathbf{x} \in A, \mathbf{y} \in B \big\}$$

the *Minkowski sum* of these sets. For a convex body $\mathrm{A} \subset \mathbb{R}^n$ with affine dimension d, denote by $\mathrm{Vol}_d(\mathrm{A})$ the volume of A. One of the basic result in convex geometry is *Minkowski's theorem* that the volume of convex bodies with affine dimension d behaves as a homogeneous polynomial of degree d with nonnegative coefficients:

Theorem 2.1 (Minkowski, see e.g. [BuZ88, §19.1])**.** *For all convex bodies $\mathrm{A}_1, \ldots, \mathrm{A}_r \subset \mathbb{R}^n$ and $\lambda_1, \ldots, \lambda_r > 0$, we have:*

$$\mathrm{Vol}_d(\lambda_1 \mathrm{A}_1 + \ldots + \lambda_r \mathrm{A}_r) \ = \sum_{1 \leq i_1, \ldots, i_d \leq r} \mathrm{V}\big(\mathrm{A}_{i_1}, \ldots, \mathrm{A}_{i_d}\big)\, \lambda_{i_1} \cdots \lambda_{i_d}\,, \tag{2.1}$$

where the functions $\mathrm{V}(\cdot)$ are nonnegative and symmetric, and where d is the affine dimension of $\lambda_1 \mathrm{A}_1 + \ldots + \lambda_r \mathrm{A}_r$ (which does not depend on the choice of $\lambda_1, \ldots, \lambda_r$).

The coefficients $\mathrm{V}(\mathrm{A}_{i_1}, \ldots, \mathrm{A}_{i_d})$ are called *mixed volumes* of $\mathrm{A}_{i_1}, \ldots, \mathrm{A}_{i_d}$. We use $d := d(\mathrm{A}_1, \ldots, \mathrm{A}_r)$ to denote the affine dimension of the Minkowski sum $\mathrm{A}_1 + \ldots + \mathrm{A}_r$.

There are many classical inequalities concerning mixed volumes, and here we list those that will be used in this paper. Let $\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{Q}_1, \ldots, \mathrm{Q}_{d-2}$ be convex bodies in $\mathbb{R}^n$. We denote $\mathbf{Q} = (\mathrm{Q}_1, \ldots, \mathrm{Q}_{d-2})$ and use $\mathrm{V}_{\mathbf{Q}}(\cdot, \cdot)$ as a shorthand for $\mathrm{V}(\cdot, \cdot, \mathrm{Q}_1, \ldots, \mathrm{Q}_{d-2})$.

Theorem 2.2 (*Alexandrov–Fenchel inequality*, see e.g. [BuZ88, §20])**.**

$$\mathrm{V}_{\mathbf{Q}}(\mathrm{A},\mathrm{B})^2 \ \geq \ \mathrm{V}_{\mathbf{Q}}(\mathrm{A},\mathrm{A})\, \mathrm{V}_{\mathbf{Q}}(\mathrm{B},\mathrm{B}). \tag{AF}$$

The following technical result generalizes Theorem 2.2 to inequalities involving differences in (AF); see e.g. [Sch14, §7.4].

Theorem 2.3 (see e.g. [Sch14, Lemma 7.4.1])**.** *We have*

$$\begin{aligned}\big(\mathrm{V}_{\mathbf{Q}}(\mathrm{A},\mathrm{C})^2 - \mathrm{V}_{\mathbf{Q}}(\mathrm{A},\mathrm{A})\,\mathrm{V}_{\mathbf{Q}}(\mathrm{C},\mathrm{C})\big)\,\big(\mathrm{V}_{\mathbf{Q}}(\mathrm{B},\mathrm{C})^2 - \mathrm{V}_{\mathbf{Q}}(\mathrm{B},\mathrm{B})\,\mathrm{V}_{\mathbf{Q}}(\mathrm{C},\mathrm{C})\big)&\\ \geq\ \big(\mathrm{V}_{\mathbf{Q}}(\mathrm{A},\mathrm{C})\,\mathrm{V}_{\mathbf{Q}}(\mathrm{B},\mathrm{C})\ -\ \mathrm{V}_{\mathbf{Q}}(\mathrm{A},\mathrm{B})\,\mathrm{V}_{\mathbf{Q}}(\mathrm{C},\mathrm{C})\big)^2&.\end{aligned} \tag{2.2}$$

2.2. **Favard's inequality for the cross-ratio.** Towards proving the Main Theorem 1.2, we are most interested in bounds on the *cross-ratio*

$$\Upsilon_{\mathbf{Q}}(\mathrm{A},\mathrm{B},\mathrm{C}) \ := \ \frac{\mathrm{V}_{\mathbf{Q}}(\mathrm{A},\mathrm{C})\,\mathrm{V}_{\mathbf{Q}}(\mathrm{B},\mathrm{C})}{\mathrm{V}_{\mathbf{Q}}(\mathrm{A},\mathrm{B})\,\mathrm{V}_{\mathbf{Q}}(\mathrm{C},\mathrm{C})}\,.$$

We start with the following well-known result which goes back to Favard (see §8.2).

Theorem 2.4 (*Favard's inequality*, see e.g. [BGL18, Lemma 5.1])**.** *Suppose we have*

$$\mathrm{V}_{\mathbf{Q}}(\mathrm{A},\mathrm{B})\,\mathrm{V}_{\mathbf{Q}}(\mathrm{C},\mathrm{C}) > 0\,.$$

Then:

$$\frac{\mathrm{V}_{\mathbf{Q}}(\mathrm{A},\mathrm{C})\,\mathrm{V}_{\mathbf{Q}}(\mathrm{B},\mathrm{C})}{\mathrm{V}_{\mathbf{Q}}(\mathrm{A},\mathrm{B})\,\mathrm{V}_{\mathbf{Q}}(\mathrm{C},\mathrm{C})} \ \geq \ \frac{1}{2}\,. \tag{2.3}$$

In the next section, we use order polytopes to write the cross product ratio in (CPC) into the cross-ratio Υ. Then Favard's inequality (2.3) $\Upsilon \geq \frac{1}{2}$ easily gives the constant $\frac{1}{2}$ in the inequalities in the Main Theorem 1.2 (see Theorem 3.1). To move beyond $\frac{1}{2}$ we need to strengthen (2.3), see below.

Remark 2.5. From geometric point of view, the constant $\frac{1}{2}$ in the inequality (2.3) is sharp. For example, take A and B non-collinear line segments, and $\mathrm{C} = \mathrm{A} + \mathrm{B}$, see e.g. [AFO14, Prop. 5.1] and [SZ16, Thm 6.1]. However, for various families of convex bodies, it is possible to improve the constant perhaps, although not to 1 as one would wish. For example, when C is a unit ball in $\mathbb{R}^2$ the constant can be improved to $\frac{2}{\pi}$ [AFO14, Prop. 5.3].

2.3. **Better cross-ratio inequalities.** The following two results follow from (2.2) by elementary arguments. They are variations on inequalities that are already known in the literature. We include simple proofs for completeness.

Proposition 2.6. *Suppose that* $\mathrm{V}_{\mathbf{Q}}(\mathrm{A},\mathrm{B})\,\mathrm{V}_{\mathbf{Q}}(\mathrm{C},\mathrm{C}) > 0$. *Then:*

$$\frac{\mathrm{V}_{\mathbf{Q}}(\mathrm{A},\mathrm{C})\,\mathrm{V}_{\mathbf{Q}}(\mathrm{B},\mathrm{C})}{\mathrm{V}_{\mathbf{Q}}(\mathrm{A},\mathrm{B})\mathrm{V}_{\mathbf{Q}}(\mathrm{C},\mathrm{C})} \ \geq \ \frac{1}{2}\left(1 + \frac{\sqrt{\mathrm{V}_{\mathbf{Q}}(\mathrm{A},\mathrm{A})\,\mathrm{V}_{\mathbf{Q}}(\mathrm{B},\mathrm{B})}}{\mathrm{V}_{\mathbf{Q}}(\mathrm{A},\mathrm{B})}\right). \tag{2.4}$$

Proof. Let $\alpha_1, \alpha_2, \beta_1, \beta_2$ be nonnegative real numbers given by

$$\alpha_1 \ := \ \frac{\mathrm{V}_{\mathbf{Q}}(\mathrm{A},\mathrm{C})}{\sqrt{\mathrm{V}_{\mathbf{Q}}(\mathrm{A},\mathrm{B})\,\mathrm{V}_{\mathbf{Q}}(\mathrm{C},\mathrm{C})}}\,, \qquad \alpha_2 \ := \ \frac{\mathrm{V}_{\mathbf{Q}}(\mathrm{B},\mathrm{C})}{\sqrt{\mathrm{V}_{\mathbf{Q}}(\mathrm{A},\mathrm{B})\,\mathrm{V}_{\mathbf{Q}}(\mathrm{C},\mathrm{C})}}\,,$$

$$\beta_1 \ := \ \frac{\mathrm{V}_{\mathbf{Q}}(\mathrm{A},\mathrm{A})}{\mathrm{V}_{\mathbf{Q}}(\mathrm{A},\mathrm{B})}\,, \qquad \beta_2 \ := \ \frac{\mathrm{V}_{\mathbf{Q}}(\mathrm{B},\mathrm{B})}{\mathrm{V}_{\mathbf{Q}}(\mathrm{A},\mathrm{B})}\,.$$

Note that $\beta_1\beta_2 \leq 1$ by (AF). By perturbing the convex bodies again if necessary, we can without loss of generality assume that $\beta_1\beta_2 < 1$.

In this notation, we can rewrite (2.2) as

$$(\alpha_1\alpha_2 - 1)^2 \ \leq \ (\alpha_1^2 - \beta_1)\,(\alpha_2^2 - \beta_2).$$

Rearranging the terms, this gives:

$$\alpha_1\alpha_2 \ \geq \ \frac{1}{2} + \frac{1}{2}\left(\alpha_1^2\beta_2 + \alpha_2^2\beta_1\right) - \frac{1}{2}\,\beta_1\beta_2\,. \tag{2.5}$$

By applying the AM–GM inequality to the terms $\left(\alpha_1^2\beta_2 + \alpha_2^2\beta_1\right)$, we get

$$\alpha_1\,\alpha_2 \ \geq \ \frac{1}{2} + \alpha_1\alpha_2\sqrt{\beta_1\,\beta_2} - \frac{1}{2}\,\beta_1\beta_2\,.$$

Rearranging the terms, this gives:

$$\left(1 - \sqrt{\beta_1\beta_2}\right)\alpha_1\alpha_2 \ \geq \ \frac{1}{2}\left(1 - \beta_1\beta_2\right).$$

Since $\beta_1\beta_2 < 1$, we can divide both side of the inequality above by $\left(1 - \sqrt{\beta_1\beta_2}\right)$ and get

$$\alpha_1\alpha_2 \ \geq \ \frac{1}{2}\left(1 + \sqrt{\beta_1\beta_2}\right).$$

This gives the desired (2.4). ◻

We now present a variant of Proposition 2.6 in a degenerate case.

Proposition 2.7. *Suppose that* $\mathrm{V}_{\mathbf{Q}}(\mathrm{A},\mathrm{B})\mathrm{V}_{\mathbf{Q}}(\mathrm{C},\mathrm{C}) > 0$ *and* $\mathrm{V}_{\mathbf{Q}}(\mathrm{B},\mathrm{B}) = 0$. *Then:*

$$\frac{\mathrm{V}_{\mathbf{Q}}(\mathrm{A},\mathrm{C})\,\mathrm{V}_{\mathbf{Q}}(\mathrm{B},\mathrm{C})}{\mathrm{V}_{\mathbf{Q}}(\mathrm{A},\mathrm{B})\,\mathrm{V}_{\mathbf{Q}}(\mathrm{C},\mathrm{C})} \ \geq \ \left(1 + \sqrt{1 - \frac{\mathrm{V}_{\mathbf{Q}}(\mathrm{A},\mathrm{A})\,\mathrm{V}_{\mathbf{Q}}(\mathrm{C},\mathrm{C})}{\mathrm{V}_{\mathbf{Q}}(\mathrm{A},\mathrm{C})^2}}\right)^{-1}. \tag{2.6}$$

Proof. First note that (2.2) gives:

$$\begin{aligned} &\left(\mathrm{V}_{\mathbf{Q}}(\mathrm{A},\mathrm{C})\,\mathrm{V}_{\mathbf{Q}}(\mathrm{B},\mathrm{C}) \ - \ \mathrm{V}_{\mathbf{Q}}(\mathrm{A},\mathrm{B})\,\mathrm{V}_{\mathbf{Q}}(\mathrm{C},\mathrm{C})\right)^2 \\ &\qquad \leq \ \left(\mathrm{V}_{\mathbf{Q}}(\mathrm{A},\mathrm{C})^2 - \mathrm{V}_{\mathbf{Q}}(\mathrm{A},\mathrm{A})\,\mathrm{V}_{\mathbf{Q}}(\mathrm{C},\mathrm{C})\right)\mathrm{V}_{\mathbf{Q}}(\mathrm{B},\mathrm{C})^2. \end{aligned} \tag{2.7}$$

We assume without loss of generality that

$$\mathrm{V}_{\mathbf{Q}}(\mathrm{A},\mathrm{C})\,\mathrm{V}_{\mathbf{Q}}(\mathrm{B},\mathrm{C}) \ < \ \mathrm{V}_{\mathbf{Q}}(\mathrm{A},\mathrm{B})\,\mathrm{V}_{\mathbf{Q}}(\mathrm{C},\mathrm{C})\,. \tag{2.8}$$

In fact, otherwise, since the right side of (2.6) is at most 1 we immediately have (2.6).

Now note that $\mathrm{V}_{\mathbf{Q}}(\mathrm{A},\mathrm{C})\,\mathrm{V}_{\mathbf{Q}}(\mathrm{B},\mathrm{C}) > 0$ by (2.3) and by the assumption of the theorem. Taking the square root of (2.7) using (2.8), and then dividing by $\mathrm{V}_{\mathbf{Q}}(\mathrm{A},\mathrm{C})\,\mathrm{V}_{\mathbf{Q}}(\mathrm{B},\mathrm{C})$, we get:

$$\frac{\mathrm{V}_{\mathbf{Q}}(\mathrm{A},\mathrm{B})\,\mathrm{V}_{\mathbf{Q}}(\mathrm{C},\mathrm{C})}{\mathrm{V}_{\mathbf{Q}}(\mathrm{A},\mathrm{C})\,\mathrm{V}_{\mathbf{Q}}(\mathrm{B},\mathrm{C})} - 1 \ \leq \ \sqrt{1 - \frac{\mathrm{V}_{\mathbf{Q}}(\mathrm{A},\mathrm{A})\,\mathrm{V}_{\mathbf{Q}}(\mathrm{C},\mathrm{C})}{\mathrm{V}_{\mathbf{Q}}(\mathrm{A},\mathrm{C})^2}}\,.$$

This is equivalent to (2.6). ◻

3. Poset inequalities via mixed volumes

3.1. **Definitions and notation.** We refer to [Tro95] for some standard posets notation. Let $P = (X, \prec)$ be a poset with $|X| = n$ elements. A *dual poset* is a poset $P^* = (X, \prec^*)$, where $x \prec^* y$ if and only if $y \prec x$.

We somewhat change the notation and fix distinct elements $z_1, z_2, z_3 \in X$ which we use throughout the paper. As in the introduction, for $k, \ell \geq 1$ let

$$\mathcal{F}(k,\ell) \ := \ \{L \in \mathcal{E}(P) \, : \, L(z_2) - L(z_1) = k, \, L(z_3) - L(z_2) = \ell\},$$

and let $\mathrm{F}(k,\ell) := \big|\mathcal{F}(k,\ell)\big|$. We will write $\mathrm{F}_{P,z_1,z_2,z_3}(k,\ell)$ in place of $\mathrm{F}(k,\ell)$ when there is a potential ambiguity in regards to the underlying poset P and the elements $z_1, z_2, z_3 \in X$.

3.2. **Half CPC.** We first prove that (CPC) holds up to a factor of 2. Formally, start with the following weak version of the Main Theorem 1.2:

Theorem 3.1. *For every $k, \ell \geq 1$, we have:*

$$\text{(half-CPC)} \qquad \mathrm{F}(k,\ell)\,\mathrm{F}(k+1,\ell+1) \ \leq \ 2\,\mathrm{F}(k+1,\ell)\,\mathrm{F}(k,\ell+1).$$

To prove Theorem 3.1, we will first interpret the quantity $\mathrm{F}(k,\ell)$ as in the language of mixed volumes. Here we follow Stanley's approach in [Sta81] (see also [KS84]).

Fix a poset $P = (X, \prec)$, and let $\mathbb{R}^X$ be the space of real vectors $\mathbf{v}$ that are indexed by elements $x \in X$. Throughout this section, the entries of the vector $\mathbf{v}$ that corresponds to $x \in X$ will be denoted by $\mathbf{v}(x)$, to maintain legibility when x are substituted with elements z_i. The *order polytope* $\mathrm{K} := \mathrm{K}(P) \subset \mathbb{R}^X$ is defined as follows:

$$\mathrm{K} \ := \ \big\{\, \mathbf{v} \in \mathbb{R}^X \ : \ \mathbf{v}(x) \leq \mathbf{v}(y) \text{ for all } x \prec y, \, x, y \in X\,, \quad \text{and} \quad 0 \leq \mathbf{v}(x) \leq 1 \text{ for all } x \in X \,\big\}.$$

Let $\mathrm{K}_1, \, \mathrm{K}_2, \, \mathrm{K}_3 \subseteq \mathrm{K}$ be the slices of the order polytope defined as follows:

$$\begin{aligned} \mathrm{K}_1 \ &:= \ \{\, \mathbf{v} \in \mathrm{K} \, : \, \mathbf{v}(z_2) - \mathbf{v}(z_1) \ = \ 1, \ \mathbf{v}(z_3) - \mathbf{v}(z_2) \ = \ 0 \,\}, \\ \mathrm{K}_2 \ &:= \ \{\, \mathbf{v} \in \mathrm{K} \, : \, \mathbf{v}(z_2) - \mathbf{v}(z_1) \ = \ 0, \ \mathbf{v}(z_3) - \mathbf{v}(z_2) \ = \ 1 \,\}, \\ \mathrm{K}_3 \ &:= \ \{\, \mathbf{v} \in \mathrm{K} \, : \, \mathbf{v}(z_2) - \mathbf{v}(z_1) \ = \ \mathbf{v}(z_3) - \mathbf{v}(z_2) \ = \ 0 \,\}. \end{aligned} \tag{3.1}$$

Note that all Minkowski sums of these three polytopes have affine dimension $d = n - 2$.

Lemma 3.2. *Let $k, \ell \geq 1, \ k + \ell \leq n$. We have:*

$$\mathrm{F}(k,\ell) \ = \ (n-2)! \ \mathrm{V}(\underbrace{\mathrm{K}_1, \ldots, \mathrm{K}_1}_{k-1}, \underbrace{\mathrm{K}_2, \ldots, \mathrm{K}_2}_{\ell-1}, \underbrace{\mathrm{K}_3, \ldots, \mathrm{K}_3}_{n-k-\ell}). \tag{3.2}$$

This lemma follows by a variation on the argument in the proof of [Sta81, Thm 3.2] and [KS84, Thm 2.5].

Proof. For $0 < s, t < 1, \ 0 < s + t < 1$, define

$$\mathrm{K}^{(s,t)} \ := \ \big\{ \ \mathbf{v} \in \mathrm{K} \, : \, \mathbf{v}(z_2) - \mathbf{v}(z_1) \ = \ s, \ \mathbf{v}(z_3) - \mathbf{v}(z_2) \ = \ t \,\big\}.$$

Note that $\mathrm{K}^{(s,t)} = s\mathrm{K}_1 + t\mathrm{K}_2 + (1-s-t)\mathrm{K}_3$. Let us now compute the volume of $\mathrm{K}^{(s,t)}$.

For every $L \in \mathcal{E}(P)$ we denote by $\Delta_L \subset \mathrm{K}^{(s,t)}$ the polytope

$$\Delta_L \ := \ \{\, \mathbf{v} \in \mathrm{K}^{(s,t)} \ | \ \mathbf{v}(x) \ \leq \ \mathbf{v}(y) \ \text{ whenever } \ L(x) \leq L(y) \,\}.$$

Note that $\mathrm{K}^{(s,t)}$ is the union of Δ_L's over all linear extensions L such that $L(z_1) < L(z_2) < L(z_3)$, and furthermore all Δ_L's have pairwise disjoint interiors. Hence it remains to compute the volume of Δ_L's.

Let $L \in \mathcal{F}(k,\ell)$ for some $k, \ell \geq 1$, let $h := L(z_1)$, and let x_i $(i \in \{1, \ldots, n\})$ be the i-th smallest element under the total order of L. Note that $z_1 = x_h, \ z_2 = x_{h+k}$, and $z_3 = x_{h+k+\ell}$. Then Δ_L

consists of $\mathbf{v} \in \mathbb{R}^X$ that satisfies these three inequalities: $0 \le \mathbf{v}(x_1) \le \mathbf{v}(x_2) \le \ldots \le \mathbf{v}(x_n) \le 1$, $\mathbf{v}(x_{h+k}) = \mathbf{v}(x_h)+s$, $\mathbf{v}(x_{h+k+\ell}) = \mathbf{v}(x_h)+s+t$. Denote by $\Phi : \mathbb{R}^X \to \mathbb{R}^X$ the (volume preserving) transformation defined as follows: $\Phi(\mathbf{v}) = \mathbf{w}$, where

$$\begin{aligned}
\mathbf{w}(x_i) &= \mathbf{v}(x_i) && \text{if} \quad i \le h, \\
\mathbf{w}(x_i) &= \mathbf{v}(x_i) - \mathbf{v}(x_h) && \text{if} \quad h < i \le h+k, \\
\mathbf{w}(x_i) &= \mathbf{v}(x_i) - \mathbf{v}(x_h) - s && \text{if} \quad h+k < i \le h+k+\ell, \\
\mathbf{w}(x_i) &= \mathbf{v}(x_i) - s - t && \text{if} \quad h+k+\ell < i \le n.
\end{aligned}$$

Then the image $\Phi(\Delta_L)$ is the set of $\mathbf{w} \in \mathbb{R}^X$ that satisfies

$$\begin{aligned}
& 0 \le \mathbf{w}(x_1) \le \ldots \le \mathbf{w}(x_h) \le \mathbf{w}(x_{h+k+\ell+1}) \le \ldots \le \mathbf{w}(x_n) \le 1-s-t, \\
& 0 \le \mathbf{w}(x_{h+1}) \le \ldots \le \mathbf{w}(x_{h+k}) = s, \quad \text{and} \\
& 0 \le \mathbf{w}(x_{h+k+1}) \le \ldots \le \mathbf{w}(x_{h+k+\ell}) = t.
\end{aligned}$$

This set is the direct product of three simplices and has volume

$$\rho(s,t) := \frac{s^{k-1}}{(k-1)!} \times \frac{t^{\ell-1}}{(\ell-1)!} \times \frac{(1-s-t)^{n-k-\ell}}{(n-k-\ell)!}.$$

It follows from here that

$$\begin{aligned}
\mathrm{Vol}_d\big(\mathrm{K}^{(s,t)}\big) &= \sum_{k,\ell\ge 1} \sum_{L\in\mathcal{F}(k,\ell)} \mathrm{Vol}_d(\Delta_L) = \sum_{k,\ell\ge 1} \sum_{L\in\mathcal{F}(k,\ell)} \rho(s,t) \\
&= \sum_{k,\ell\ge 1} \binom{n-2}{n-k-\ell,\, k-1,\, \ell-1} \frac{\mathrm{F}(k,\ell)}{(n-2)!}\, s^{k-1}\, t^{\ell-1}\, (1-s-t)^{n-k-\ell}.
\end{aligned}$$

Since the choice of s,t is arbitrary, equation (3.2) follows from the Minkowski Theorem 2.1. $\square$

Proof of Theorem 3.1. Let $d = n-2$, and let $\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{Q}_1, \ldots, \mathrm{Q}_{d-2} \subset \mathrm{K}$ be given by

$$\begin{aligned}
& \mathrm{A} \leftarrow \mathrm{K}_1, \quad \mathrm{B} \leftarrow \mathrm{K}_2, \quad \mathrm{C} \leftarrow \mathrm{K}_3, \quad \text{and} \\
& \mathrm{Q}_1, \ldots, \mathrm{Q}_{d-2} \leftarrow \underbrace{\mathrm{K}_1, \ldots, \mathrm{K}_1}_{k-1}, \underbrace{\mathrm{K}_2, \ldots, \mathrm{K}_2}_{\ell-1}, \underbrace{\mathrm{K}_3, \ldots, \mathrm{K}_3}_{n-k-\ell}.
\end{aligned} \tag{3.3}$$

The theorem now follows by applying Lemma 3.2 into Theorem 2.4. $\square$

3.3. **Applications to cross products.** We now quickly derive the key applications of mixed volume cross-ratio inequalities for the cross product inequalities.

Proposition 3.3. *Suppose that* $\mathrm{F}(k,\ell)\,\mathrm{F}(k+1,\ell+1) > 0$. *Then:*

$$\frac{\mathrm{F}(k+1,\ell)\,\mathrm{F}(k,\ell+1)}{\mathrm{F}(k,\ell)\,\mathrm{F}(k+1,\ell+1)} \ \ge\ \frac{1}{2} + \frac{\sqrt{\mathrm{F}(k,\ell+2)\,\mathrm{F}(k+2,\ell)}}{2\,\mathrm{F}(k+1,\ell+1)}.$$

Proof. Let $d = n-2$, and let $\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{Q}_1, \ldots, \mathrm{Q}_{d-2} \subset \mathrm{K}$ be given by (3.3). The conclusion of the proposition now follows from Lemma 3.2 and Proposition 2.6. $\square$

Proposition 3.4. *Suppose that* $\mathrm{F}(k,\ell)\,\mathrm{F}(k+1,\ell+1) > 0$ *and* $\mathrm{F}(k,\ell+2) = 0$. *Then:*

$$\frac{\mathrm{F}(k+1,\ell)\mathrm{F}(k,\ell+1)}{\mathrm{F}(k+1,\ell+1)\mathrm{F}(k,\ell)} \ \ge\ \left(1 + \sqrt{1 - \frac{\mathrm{F}(k,\ell)\mathrm{F}(k+2,\ell)}{\mathrm{F}(k+1,\ell)^2}}\right)^{-1}.$$

Proof. Let $d = n-2$, and let $\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{Q}_1, \ldots, \mathrm{Q}_{d-2} \subset \mathrm{K}$ be given by (3.3). The conclusion of the proposition now follows from Lemma 3.2 and Proposition 2.7. □

3.4. **More half-CPC inequalities.** We start with the following half-versions of (CPC1) and (CPC2). The proofs follow the proof of Theorem 3.1 given above.

Lemma 3.5. *For every $k, \ell \geq 1$, we have:*

$$\text{(half-CPC1)} \qquad \mathrm{F}(k+2,\ell)\,\mathrm{F}(k,\ell+1) \ \leq \ 2\,\mathrm{F}(k+1,\ell)\,\mathrm{F}(k+1,\ell+1),$$

$$\text{(half-CPC2)} \qquad \mathrm{F}(k,\ell+2)\,\mathrm{F}(k+1,\ell) \ \leq \ 2\,\mathrm{F}(k,\ell+1)\,\mathrm{F}(k+1,\ell+1).$$

Proof. We again let $d = n-2$ and let $\mathrm{Q}_1, \ldots, \mathrm{Q}_{d-2} \subset \mathrm{K}$ be given by (3.3). Then (half-CPC1) follows by applying Lemma 3.2 into Theorem 2.4, with the choice

$$\mathrm{A} \leftarrow \mathrm{K}_3\,, \quad \mathrm{B} \leftarrow \mathrm{K}_2 \quad \text{and} \quad \mathrm{C} \leftarrow \mathrm{K}_1\,.$$

Similarly, (half-CPC2) follows from the choice

$$\mathrm{A} \leftarrow \mathrm{K}_3\,, \quad \mathrm{B} \leftarrow \mathrm{K}_1 \quad \text{and} \quad \mathrm{C} \leftarrow \mathrm{K}_2\,.$$

This completes the proof. □

Note that (CPC1) is a dual inequality to (CPC2) in the following sense. Let $P^* := (X, \prec^*)$ be the *dual poset* of P, i.e. $x \prec^* y$ if and only if $x \succ y$. Let $z_1^* := z_3$, $z_2^* := z_2$, $z_3^* := z_1$. Then $\mathrm{F}_{P,z_1,z_2,z_3}(k,\ell) = \mathrm{F}_{P^*,z_1^*,z_2^*,z_3^*}(\ell,k)$ by the maps that send linear extensions of P to linear extensions of P^* by reversing the total order.

On the other hand, one can think of (CPC1) and (CPC2) as negative variants of (CPC), in the following sense. Let $z_1' := z_2$, $z_2' := z_1$, $z_3' := z_3$, and we write $\mathrm{F} = \mathrm{F}_{P,z_1,z_2,z_3}$ and $\mathrm{F}' = \mathrm{F}_{P,z_1',z_2',z_3'}$. Then, for every integer k, ℓ,

$$\begin{aligned} \mathrm{F}(k,\ell) \ &= \ \big|\{L \in \mathcal{E}(P) \,:\, L(z_2) - L(z_1) = k,\, L(z_3) - L(z_2) = \ell\}\big| \\ &= \ \big|\{L \in \mathcal{E}(P) \,:\, L(z_1) - L(z_2) = -k,\, L(z_3) - L(z_1) = \ell + k\}\big| \\ &= \ \mathrm{F}'(-k, \ell+k). \end{aligned}$$

Let $k' := -k-1$ and $\ell' := \ell + k$. Under this change of variable, (CPC) then becomes

$$\mathrm{F}'(k'+1,\ell')\,\mathrm{F}'(k',\ell'+2) \ \leq \ \mathrm{F}'(k',\ell'+1)\,\mathrm{F}'(k'+1,\ell'+1),$$

which coincides with (CPC2) in this case.

Note, however, that (CPC) does not imply (CPC1) and vice versa, since k' are necessarily negative under this transformation. In fact, as mentioned in the introduction, we will present counterexamples to (CPC1) in §7.2.

3.5. **Variations on the theme.** The following three inequalities are variations on (CPC).

Lemma 3.6. *For every $k, \ell \geq 1$ we have:*

$$\text{(LogC-1)} \qquad \mathrm{F}(k+1,\ell+1)^2 \ \geq \ \mathrm{F}(k+2,\ell)\,\mathrm{F}(k,\ell+2),$$

$$\text{(LogC-2)} \qquad \mathrm{F}(k,\ell+1)^2 \ \geq \ \mathrm{F}(k,\ell)\,\mathrm{F}(k,\ell+2),$$

$$\text{(LogC-3)} \qquad \mathrm{F}(k+1,\ell)^2 \ \geq \ \mathrm{F}(k,\ell)\,\mathrm{F}(k+2,\ell).$$

Proof. Let $d = n-2$, and let $\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{Q}_1, \ldots, \mathrm{Q}_{d-2} \subset \mathrm{K}$ be given by (3.3). It follows from the Alexandrov–Fenchel inequality (AF) that

$$\begin{aligned} \mathrm{V}_{\mathbf{Q}}(\mathrm{A},\mathrm{B})^2 \ &\geq \ \mathrm{V}(\mathrm{A},\mathrm{A})\, \mathrm{V}(\mathrm{B},\mathrm{B}), \\ \mathrm{V}_{\mathbf{Q}}(\mathrm{B},\mathrm{C})^2 \ &\geq \ \mathrm{V}(\mathrm{B},\mathrm{B})\, \mathrm{V}(\mathrm{C},\mathrm{C}), \\ \mathrm{V}_{\mathbf{Q}}(\mathrm{A},\mathrm{C})^2 \ &\geq \ \mathrm{V}(\mathrm{A},\mathrm{A})\, \mathrm{V}(\mathrm{C},\mathrm{C}). \end{aligned}$$

By applying Lemma 3.2, we get the desired inequalities. □

Remark 3.7. The inequalities (LogC-1), (LogC-2) and (LogC-3) can be viewed as extensions of Stanley's and Kahn–Saks inequalities, cf. [CPP22, CPP23b].

Corollary 3.8. *Suppose that* $\mathrm{F}(k,\ell)\,\mathrm{F}(k+1,\ell+1) > 0$. *Then we have:*

$$\frac{\mathrm{F}(k+1,\ell)\,\mathrm{F}(k,\ell+1)}{\mathrm{F}(k,\ell)\,\mathrm{F}(k+1,\ell+1)} \ \geq \ \frac{\mathrm{F}(k+2,\ell)\,\mathrm{F}(k,\ell+2)}{\mathrm{F}(k+1,\ell+1)^2}\,.$$

In particular, if (LogC-1) *is an equality, then the inequality* (CPC) *holds.*

Proof. Taking the product of (LogC-1), (LogC-2) and (LogC-3), we have:

$$\mathrm{F}(k+1,\ell)\,\mathrm{F}(k,\ell+1)\,\mathrm{F}(k+1,\ell+1) \ \geq \ \mathrm{F}(k,\ell)\,\mathrm{F}(k+2,\ell)\,\mathrm{F}(k,\ell+2).$$

By the assumptions, this implies the result.[3] □

Proof of Theorem 1.7. First, assume that both (CPC1) and (CPC2) are false:

$$\begin{aligned} \mathrm{F}(k+2,\ell)\,\mathrm{F}(k,\ell+1) \ &> \ \mathrm{F}(k+1,\ell)\,\mathrm{F}(k+1,\ell+1) \quad \text{and} \\ \mathrm{F}(k,\ell+2)\,\mathrm{F}(k+1,\ell) \ &> \ \mathrm{F}(k,\ell+1)\,\mathrm{F}(k+1,\ell+1). \end{aligned}$$

Taking the product of both inequalities, we then get

$$\mathrm{F}(k+2,\ell)\,\mathrm{F}(k,\ell+2) \ > \ \mathrm{F}(k+1,\ell+1)^2,$$

which contradicts (LogC-1). The proofs for the other cases are analogous. □

4. Vanishing of poset inequalities

4.1. **Poset parameters.** For an element $x \in X$, let $B(x) := \{y \in X \, : \, y \preccurlyeq x\}$ denote the *lower order ideal* generated by x, and let $b(x) := |B(x)|$. Similarly, let $B^*(x) := \{y \in X \, : \, y \succcurlyeq x\}$ denote the *upper order ideal* generated by x, and let $b^*(x) := |B^*(x)|$.

By analogy, let $B(x,y) = \{z \in X \, : \, x \preccurlyeq z \preccurlyeq y\}$ be the *interval* between x and y, and let $b(x,y) = |B(x,y)|$. Without loss of generality we can always assume that $z_1 \prec z_2 \prec z_3$, since otherwise these relations can be added to the poset. We then have $b(z_1,z_2)$, $b(z_2,z_3) \geq 2$.

Let $x,y \in X$ be two incomparable elements in P, write $y \| x$. Define

$$U(x,y) \ := \ \big\{ z \in X \, : \, z \| y\,, \ z \preccurlyeq x \big\} \quad \text{and} \quad u(x,y) \ := \ |U(x,y)|.$$

Similarly, define

$$U^*(x,y) \ := \ \big\{ z \in X \, : \, z \| y\,, \ z \succcurlyeq x \big\} \quad \text{and} \quad u^*(x,y) \ := \ |U^*(x,y)|.$$

Finally, let

$$t(x) \ := \ \max\big\{ u(x,y) \, : \, y \in X, \ y \| x \big\} \quad \text{and} \quad t^*(x) \ := \ \max\big\{ u^*(x,y) \, : \, y \in X, \ y \| x \big\},$$

[3] Alternatively, the corollary follows immediately from Proposition 3.3.

and we define $t(x) := 1$, $t^*(x) := 1$ if every element $y \in X$ is comparable to x. Clearly, $t(x) \leq b(x)$ and $t^*(x) \leq b^*(x)$, by definition.

In this notation, recall that a poset $P = (X, \prec)$ is *t-thin with respect to A*, if for every $u \in X \setminus A$ we have $n - b(u) - b^*(u) \leq t - 1$. Similarly, recall that a poset $P = (X, \prec)$ is *t-flat with respect to A*, if for every $u \in A$ we have $b(u) + b^*(u) \leq t + 1$. Note that $t(u), t^*(u) \leq t$ in either case.

4.2. **Vanishing conditions.** Recall the following conditions for existence of restricted linear extensions.

Theorem 4.1 ([CPP23a, Thm 1.12]). *Let $P = (X, \prec)$ be a poset with $|X| = n$ elements, and let $z_1, \ldots, z_r \in X$ be distinct elements such that $z_1 \prec z_2 \prec \cdots \prec z_r$. Fix integers $1 \leq a_1 < a_2 < \cdots < a_r \leq n$. Then there exists a linear extension $L \in \mathcal{E}(P)$ with $L(z_i) = a_i$ for all $1 \leq i \leq r$* <u>if and only if</u>

$$\begin{cases} b(z_i) \leq a_i\,, \quad b^*(z_i) \leq n - a_i + 1 \quad \textit{for all} \quad 1 \leq i \leq r\,, \quad \textit{and} \\ a_j - a_i \geq b(z_i, z_j) - 1 \quad \textit{for all} \quad 1 \leq i < j \leq r\,. \end{cases} \tag{4.1}$$

We apply this result to determine the vanishing conditions for $\mathrm{F}(k, \ell)$.

Theorem 4.2. *Let $P = (X, \prec)$ be a poset with $|X| = n$ elements, and let $z_1 \prec z_2 \prec z_3$ be distinct elements in X. Then $\mathrm{F}(k, \ell) > 0$* <u>if and only if</u>

$$\begin{aligned} b(z_1, z_2) - 1 \ &\leq \quad k \quad \leq \ n + 1 - b(z_1) - b^*(z_2), \\ b(z_2, z_3) - 1 \ &\leq \quad \ell \quad \leq \ n + 1 - b^*(z_3) - b(z_2), \\ b(z_1, z_3) - 1 \ &\leq \ k + \ell \ \leq \ n + 1 - b^*(z_3) - b(z_1). \end{aligned}$$

Note that conditions in the theorem can be viewed as 6 linear inequalities for $(k, \ell) \in \mathbb{N}^2$. These inequalities determine a convex polygon in $\mathbb{R}^2$ (see below).

Proof. We have that $\mathrm{F}(k, \ell) > 0$ if and only if there exists an integer a, such that the conditions of Theorem 4.1 are satisfied for the elements $z_1 \prec z_2 \prec z_3$ with $a_1 = a, a_2 = a + k, a_3 = a + k + \ell$. Rewriting the inequalities we obtain the following conditions

$$\begin{gathered} b(z_1, z_2) \leq k + 1\,, \quad b(z_2, z_3) \leq \ell + 1\,, \quad b(z_1, z_3) \leq k + \ell + 1 \qquad \text{and} \\ \max\{b(z_1), b(z_2) - k, b(z_3) - k - \ell\} \leq a \leq n + 1 - \max\{b^*(z_1), k + b^*(z_2), k + \ell + b^*(z_3)\} \end{gathered}$$

The integer a exists if and only if the last inequalities are consistent, which leads to

$$\begin{gathered} b(z_1, z_2) + 1 \leq k\,, \quad b(z_2, z_3) + 1 \leq \ell\,, \quad b(z_1, z_3) + 1 \leq k + \ell \qquad \text{and} \\ \max\{b(z_1), b(z_2) - k, b(z_3) - k - \ell\} + \max\{b^*(z_1), k + b^*(z_2), k + \ell + b^*(z_3)\} \leq n + 1 \end{gathered}$$

Noting that $b(z_i) + b^*(z_i) \leq n + 1$ for all i, the second inequality translates to 6 unconditional linear inequalities for k and ℓ, which can be written as

$$\begin{aligned} b(z_2) + b^*(z_1) - n - 1 \ &\leq \quad k \quad \leq \ n + 1 - b(z_1) - b^*(z_2), \\ b^*(z_2) + b(z_3) - n - 1 \ &\leq \quad \ell \quad \leq \ n + 1 - b^*(z_3) - b(z_2), \\ b^*(z_1) + b(z_3) - n - 1 \ &\leq \ k + \ell \ \leq \ n + 1 - b^*(z_3) - b(z_1). \end{aligned}$$

Finally, since $|X| = n$, we also have:

$$b(z_i) + b^*(z_j) - n \ \leq \ b(z_j, z_i) \quad \text{for all} \quad 1 \leq j < i \leq 3\,.$$

Combining with the previous inequalities, we obtain the desired conditions. $\square$

Corollary 4.3. *Suppose that $\mathrm{F}(k + 1, \ell)\, \mathrm{F}(k, \ell + 1) = 0$. Then $\mathrm{F}(k, \ell)\, \mathrm{F}(k + 1, \ell + 1) = 0$.*

Proof. Let $\mathcal{S} := \big\{(k,\ell) \in \mathbb{N}^2 \,:\, \mathrm{F}(k,\ell) > 0\big\}$ denote the support of $\mathrm{F}(\cdot,\cdot)$. By Theorem 4.2 we have $\mathcal{S}$ is a (possibly degenerate) hexagon with sides parallel to the axis and the line $k+\ell = 0$. Observe that if $(k,\ell), (k+1,\ell+1) \in \mathcal{S}$, then we also have $(k+1,\ell), (k,\ell+1) \in \mathcal{S}$. In other words, if $\mathrm{F}(k,\ell)\,\mathrm{F}(k+1,\ell+1) = 0$, then we also have $\mathrm{F}(k+1,\ell)\,\mathrm{F}(k,\ell+1) = 0$, as desired. □

4.3. **Cross product equality in the vanishing case.** We are now ready to prove (1.4) in the main theorem.

Lemma 4.4. *Let $P = (X, \prec)$ be a finite poset, and let $z_1 \prec z_2 \prec z_3$ be three distinct elements in X. Suppose that $\mathrm{F}(k,\ell+2) = \mathrm{F}(k+2,\ell) = 0$ and $\mathrm{F}(k,\ell)\,\mathrm{F}(k+1,\ell+1) > 0$. Then*

$$\mathrm{F}(k,\ell+1)\,\mathrm{F}(k+1,\ell) \;=\; \mathrm{F}(k+1,\ell+1)\,\mathrm{F}(k,\ell).$$

Proof. As in the proof of Corollary 4.3, let $\mathcal{S} := \big\{(k,\ell) \in \mathbb{N}^2 \,:\, \mathrm{F}(k,\ell) > 0\big\}$ denote the support of $\mathrm{F}(\cdot,\cdot)$. By the assumption, we have $(k,\ell+2), (k+2,\ell) \notin \mathcal{S}$ and $(k,\ell), (k+1,\ell+1) \in \mathcal{S}$. Theorem 4.2 then gives:

$$\begin{aligned} k+1 \;&\leq\; n+1-b(z_1)-b^*(z_2) \quad \text{and} \quad k+2 \;>\; n+1-b(z_1)-b^*(z_2),\\ \ell+1 \;&\leq\; n+1-b(z_2)-b^*(z_3) \quad \text{and} \quad \ell+2 \;>\; n+1-b(z_2)-b^*(z_3).\end{aligned}$$

Together these imply

$$(*) \qquad k \;=\; n-b(z_1)-b^*(z_2) \quad \text{and} \quad \ell \;=\; n-b(z_2)-b^*(z_3).$$

Theorem 4.2 also gives

$$k+\ell+2 \;\leq\; n+1-b^*(z_3)-b(z_1).$$

Substituting $(*)$ into this inequality, we get:

$$n-b(z_1)-b^*(z_2)+n-b(z_2)-b^*(z_3) \;\leq\; n-1-b^*(z_3)-b(z_1).$$

This simplifies to $n+1 \leq b(z_2)+b^*(z_2)$ and implies that all elements in X are comparable to z_2.

Let $S = B(z_2) - z_2$ and $T = B^*(z_2) - z_2$ be the lower set and upper sets of z_2, respectively. Denote $s := |S| = b(z_2) - 1$ and $t := |T| = b^*(z_2) - 1$. Note that $X = S \sqcup T \sqcup \{z_2\}$ by the argument above.

Let $1 \leq r \leq n$. Consider a subposet $(S, \prec)$ of $P = (X, \prec)$ and denote by N_r the number of linear extensions L of $(S, \prec)$ such that $L(z_1) = r$. Similarly, consider a subposet $(T, \prec)$ of $P = (X, \prec)$ and denote by N'_r the number of linear extensions L of $(S, \prec)$ such that $L(z_3) = r$.

Since $z_1 \prec z_2 \prec z_3$, we have $z_1 \in S$ and $z_3 \in T$. Therefore, for all $p, q \geq 1$ we have:

$$\mathrm{F}(p,q) \;=\; \mathrm{N}_{s-p+1}\,\mathrm{N}'_q\,.$$

This implies that

$$\begin{aligned}\mathrm{F}(k,\ell+1)\,\mathrm{F}(k+1,\ell) \;&=\; \mathrm{N}_{s-k+1}\,\mathrm{N}'_{\ell+1}\,\mathrm{N}_{s-k}\,\mathrm{N}'_{\ell}\\ &=\; \mathrm{N}_{s-k}\,\mathrm{N}'_{\ell+1}\,\mathrm{N}_{s-k+1}\,\mathrm{N}'_{\ell} \;=\; \mathrm{F}(k+1,\ell+1)\,\mathrm{F}(k,\ell),\end{aligned}$$

as desired. □

Example 4.5. For $k, \ell \geq 1$, let $X := \{x_1, \ldots, x_{k+\ell-1}, z_1, z_2, z_3\}$. Consider a poset $P = (X, \prec)$, where $A := \{x_1, \ldots, x_{k+\ell-1}, z_2\}$ is an antichain, and $z_1 \prec A \prec z_3$. Observe that

$$\begin{gathered}\mathrm{F}(k,\ell) = \mathrm{F}(k+1,\ell+1) = \mathrm{F}(k,\ell+2) = \mathrm{F}(k+2,\ell) = 0,\\ \mathrm{F}(k,\ell+1) \;=\; \tbinom{k+\ell-1}{k-1} \quad \text{and} \quad \mathrm{F}(k+1,\ell) \;=\; \tbinom{k+\ell-1}{k}.\end{gathered}$$

Then we have:

$$\mathrm{F}(k,\ell+1)\,\mathrm{F}(k+1,\ell) \;=\; \tbinom{k+\ell-1}{k-1}\tbinom{k+\ell-1}{k} \;>\; \mathrm{F}(k+1,\ell+1)\,\mathrm{F}(k,\ell) \;=\; 0.$$

This shows that the nonvanishing assumption $\mathrm{F}(k,\ell)\,\mathrm{F}(k+1,\ell+1) > 0$ in Lemma 4.4 cannot be dropped.

5. Cross product inequalities in the nonvanishing case

5.1. **Algebraic setup.** We employ the algebraic framework from [CPP23b, §6]. With every linear extension $L \in \mathcal{E}(P)$ we associate a word $\boldsymbol{x}_L = x_1 \ldots x_n \in X^*$, such that $L(x_i) = i$ for all $1 \leq i \leq n$. In the notation of the previous section, this says that $X = \{x_1, \ldots, x_n\}$ is a *natural labeling* corresponding to L.

We can now define the following action of the group G_n on $\mathcal{E}(P)$ as the right action on the words $\boldsymbol{x}_L$, $L \in \mathcal{E}(P)$. For $\boldsymbol{x}_L = x_1 \ldots x_n$ as above, let

$$(x_1 \ldots x_n)\, \tau_i \ := \ \begin{cases} x_1 \ldots x_n, & \text{if} \quad x_i \prec x_{i+1}\,, \\ x_1 \ldots x_{i+1} x_i \ldots x_n\,, & \text{if} \quad x_i \parallel x_{i+1}\,. \end{cases} \tag{5.1}$$

5.2. **Single element ratio bounds.** Let $P = (X, \prec)$ be a poset with $|X| = n$ elements, and fix an element $a \in X$ of the poset. Let $\mathcal{N}_k$ be the set of linear extensions $L \in \mathcal{E}(P)$ such that $L(a) = k$, and let $\mathrm{N}_k := |\mathcal{N}_k|$.

Lemma 5.1. *We have:*

$$\frac{\mathrm{N}_k}{\mathrm{N}_{k-1}} \ \leq \ t(a) \quad \textit{if} \quad \mathrm{N}_{k-1} > 0, \quad \textit{and}$$
$$\frac{\mathrm{N}_k}{\mathrm{N}_{k+1}} \ \leq \ t^*(a) \quad \textit{if} \quad \mathrm{N}_{k+1} > 0.$$

The idea and basic setup of the proof will be used throughout.

Proof. Consider the first inequality. The main idea is to construct an explicit injection $\phi : \mathcal{N}_k \to \mathcal{N}_{k-1} \times I$, where $I := \{1, \ldots, t(a)\}$. This will show that $\mathrm{N}_k = |\mathcal{N}_k| \leq |\mathcal{N}_{k-1} \times I| = \mathrm{N}_{k-1} t(a)$.

We identify a linear extension L where $L(a) = k$ with a word $\boldsymbol{x} \in \mathcal{N}_k$ where $x_k = a$. Let x_i be the last element in $\boldsymbol{x}$ appearing before a which is incomparable to a, that is set $i := \max\{i : i < k,\, x_i \not\prec x_k\}$. Such element exists because $\mathrm{N}_{k-1} > 0$ implies that $b(a) \leq k-1$ and so among $x_1, \ldots, x_{k-1}$ there is at least one $x_i \not\prec a$. Moreover, since i is maximal, we must have $x_j \prec x_k$ for $j \in [i+1, k]$. Also, for $j \in [i+1, k]$ we must have $x_j \| x_i$, as otherwise we would have $x_i \prec x_j \prec x_k = a$. Thus, we have $x_j \in U(a, x_i)$ for $i < j < k$ and so $1 \leq k - i \leq t(a)$.

We now define $\phi(\boldsymbol{x}) := (\boldsymbol{x}\tau_i \cdots \tau_{k-1}, k-i)$. Since $x_i \| x_j$ for $j \in [i+1, \ldots, k]$ we have that x_i is transposed consecutively with $x_{i+1}, \ldots, x_k$, so $\boldsymbol{x}\tau_i \cdots \tau_{k-1} = x_1 \ldots x_{i-1} x_{i+1} \ldots x_k x_i x_{k+1} \ldots \in \mathcal{N}_{k-1}$. We record the original position of x_i via $k - i$.

To see this is an injection we construct ϕ^{-1}, if it exists. Namely, $\phi^{-1}(\boldsymbol{x}', r)$ moves the element x'_k after $x'_{k-1} = a$ forward by $r = (k-i)$ positions as long as $x'_k \| x_j$ for $j \in [k-r, k-1]$. This completes the proof of the first inequality. The second inequality follows by applying the same argument to the dual poset P^*. $\square$

Corollary 5.2. *We have:*

$$\frac{\mathrm{N}_k}{\mathrm{N}_{k-1}} \ \leq \ k-1 \quad \textit{if} \quad \mathrm{N}_{k-1} > 0, \quad \textit{and}$$
$$\frac{\mathrm{N}_k}{\mathrm{N}_{k+1}} \ \leq \ n-k \quad \textit{if} \quad \mathrm{N}_{k+1} > 0.$$

Note that the inequalities in the corollary are tight, see Proposition 7.4.

Proof. Observe that $t(a) \leq k-1$ since there are at most $(k-1)$ elements less than or equal to a by the assumption that $\mathrm{N}_{k-1} > 0$. Similarly, observe that $t^*(a) \leq n-k$ since there are at most $(n-k)$ elements greater than or equal to a by the assumption that $\mathrm{N}_{k+1} > 0$. These imply the result. $\square$

5.3. **Double element ratio bounds.** We now give bounds for nonzero ratios of $\mathrm{F}(k,\ell)$. For the degenerate case, see Section 4.

Lemma 5.3. *Suppose that* $\mathrm{F}(k,\ell+2)>0$. *Then we have:*

$$\frac{\mathrm{F}(k+1,\ell+1)}{\mathrm{F}(k,\ell+2)} \le \min\{t(z_2),k\} + \min\big\{b(z_1,z_2)-2,t^*(z_1)\big\}\cdot\big(t^*(z_3)+t(z_2)\big).$$

Similarly, suppose that $\mathrm{F}(k+2,\ell)>0$. *Then we have:*

$$\frac{\mathrm{F}(k+1,\ell+1)}{\mathrm{F}(k+2,\ell)} \le \min\big\{t^*(z_2),\ell\big\} + \min\big\{b(z_2,z_3)-2,t(z_3)\big\}\cdot\big(t(z_1)+t^*(z_2)\big).$$

Proof. For the first inequality, we construct an injection $\psi:\mathcal{F}(k+1,\ell+1)\to I\times\mathcal{F}(k+2,\ell)$, where $I=I_1\sqcup I_2\sqcup I_3$ and I_i are intervals of lengths given by the RHS (see below). We use notation $[p,q]=\{i\in\mathbb{N}: p\le i\le q\}$ to denote the integer interval.

Let $\boldsymbol{x}\in\mathcal{F}(k+1,\ell+1)$ be a word, such that $x_i=z_1$, $x_{i+k+1}=z_2$ and $x_{i+k+\ell+2}=z_3$. We consider several cases.

Case 1: Suppose there exists an element $x_j\not\prec z_2$ for some $j\in[i+1,i+k]$. Let j be the maximal such index. Then for every $r\in[j+1,i+k]$ we have that $x_r\in U(z_2,x_j)$. Set $\psi(\boldsymbol{x})=(\boldsymbol{x}\tau_j\cdots\tau_{i+k},i+k+1-j)$, i.e. ψ moves x_j to the position after z_2, so that z_2 is now in position $i+k$. Observe that the inverse of ψ exists for all $\boldsymbol{y}\in\mathcal{F}(k,\ell+2)$, since $y_{i+k}=z_2\,\|\,y_{i+k+1}$. Note that $i+k+1-j\le\min\{u(z_2,x_j),k\}$. Thus, we can record the value $(i+k+1-j)$ in the first interval $I_1=[1,\min\{t(z_2),k\}]$.

Case 2: Suppose that we have $x_j\prec z_2$ for all $j\in[i,i+k]$. Then there exists an element $x_j\not\succ z_1$. Indeed, otherwise $x_j\in B(z_1,z_2)$ for all $j\in[i,i+k+1]$, which gives $k+2\le|B(z_1,z_2)|$ and implies $\mathrm{F}(k,\ell+2)=0$ contradicting the assumption. As above, let j be the smallest possible index such that $x_j\,\|\,z_1$, so we can move x_j in front of z_1. Note that $j-i\le\min\{b(z_1,z_2)-2,u^*(z_1,x_j)\}$. We now have a word $\boldsymbol{x}'\in\mathcal{F}(k,\ell+1)$. We split this case into two subcases.

Subcase 2.1: Suppose there exists $x_r\not\succ z_3$ for $r>i+k+\ell+2$. Let r be the minimal such index, and move x_r in front of z_3, creating a word $\boldsymbol{x}''\in\mathcal{F}(k,\ell+2)$. Note that $r-(k+\ell+2+i)\le u^*(z_3,x_r)$. Thus, we can record the value $(j-i,r-(k+\ell+2+i))$ in the second interval $I_2=[1,\min\{b(z_1,z_2)-2,t^*(z_1)\}t^*(z_3)]$.

Subcase 2.2: Suppose $x_s\succ z_3$ for all $s>i+k+\ell+2$. Then, since $\mathrm{F}(k,\ell+2)\ne 0$, there must be some $x_s\not\prec z_2$, for $s<i+k+1$. Since we are in Case 2, we have $s<i$. Let s be the largest such index. Thus $x_{s+1},\dots,x_{i+k}\prec x_{i+k+1}=z_2$. We can then move x_s past all these entries to right past z_2 and obtain a word in $\mathcal{F}(k,\ell+2)$. Note that $i-s\le u(z_2,x_s)$. Thus, we can record the value $(j-i,i-s)$ in the third interval $I_3=[\min\{b(z_1,z_2)-2,t^*(z_1)\}t(z_2)]$.

Gathering these cases, and noting that $t(x)\ge u(x,y)$ and $t^*(x)\ge u^*(x,y)$ for all $x,y\in X$, we obtain the desired first inequality. For the second inequality, we apply the analogous argument to the dual poset P^*. □

5.4. **Bounds on cross product ratios.** We can now bound the cross product ratios in the nonvanishing case.

Corollary 5.4. *Let* $P=(X,\prec)$ *be either a t-thin or t-flat poset with respect to* $\{z_1,z_2,z_3\}$. *Suppose that* $\mathrm{F}(k,\ell+2)>0$. *Then we have:*

$$\mathrm{F}(k+1,\ell+1) \le \mathrm{F}(k,\ell+2)\cdot\min\big\{k(2t+1),2t^2+t\big\}.$$

Similarly, suppose that $\mathrm{F}(k+2,\ell)>0$. *Then we have:*

$$\mathrm{F}(k+1,\ell+1) \le \mathrm{F}(k+2,\ell)\cdot\min\big\{\ell(2t+1),2t^2+t\big\}.$$

Proof. These inequalities come from different choices in the minima on the RHS of inequalities in Lemma 5.3. □

Theorem 5.5. *Let $P = (X, \prec)$ be either a t-thin or t-flat poset with respect to $\{z_1, z_2, z_3\}$. Suppose also that* $\mathrm{F}(k, \ell+2)\, \mathrm{F}(k+2, \ell) > 0$. *Then:*

$$\frac{\mathrm{F}(k+1,\ell)\mathrm{F}(k,\ell+1)}{\mathrm{F}(k,\ell)\mathrm{F}(k+1,\ell+1)} \ \geq \ \max\left\{\frac{1}{2} + \frac{1}{2\sqrt{k\ell}\,(2t+1)}\ , \frac{1}{2} + \frac{1}{2\,(2t^2+t)}\right\}.$$

Proof. These inequalities follow from Proposition 3.3 and the inequalities in Corollary 5.4. □

Theorem 5.6. *Suppose that* $\mathrm{F}(k, \ell+2)\, \mathrm{F}(k+2, \ell) > 0$. *Then:*

$$\mathrm{F}(k+1,\ell)\,\mathrm{F}(k,\ell+1) \ \geq \ \mathrm{F}(k,\ell)\,\mathrm{F}(k+1,\ell+1)\left(\frac{1}{2} + \frac{1}{2\sqrt{(2nk-2n-k+2)(2n\ell-2n-\ell+2)}}\right).$$

Proof. It follows from the definition that $t(x), t^*(x) \leq n-1$ for every $x \in X$. The nonvanishing condition in the assumption, combined with Theorem 4.2 implies that $b(z_1, z_2) \leq k+1$ and $b(z_2, z_3) \leq \ell+1$. It then follows from Lemma 5.3 that

$$\frac{\mathrm{F}(k+1,\ell+1)}{\mathrm{F}(k,\ell+2)} \ \leq \ k + (k-1)(2t) \ \leq \ k + (k-1)(2n-2) \ = \ 2nk - 2n - k + 2.$$

Similarly, we have:

$$\frac{\mathrm{F}(k+1,\ell+1)}{\mathrm{F}(k+2,\ell)} \ \leq \ 2n\ell - 2n + 1.$$

The theorem now follows from Proposition 3.3. □

6. Cross product inequalities in the vanishing case

6.1. **Double element ratio bounds.** As before, let $P = (X, \prec)$ be a poset with $|X| = n$ elements, and let $z_1 \prec z_2 \prec z_3$ be distinct elements in X. The following are the counterparts of the cross product inequalities in §5.3.

Lemma 6.1. *Suppose that* $\mathrm{F}(k, \ell) > 0$. *Then:*

$$\begin{aligned}\frac{\mathrm{F}(k+1,\ell)}{\mathrm{F}(k,\ell)} \ \leq \ & \min\{k, t^*(z_1)\} \ + \ \min\{k, t(z_3) - 1\} \ + \\ & + \ \min\{b(z_1,z_2) - 1, t(z_2)\}\,\big(\min\{\ell - 1, t(z_3)\} + \min\{\ell - 1, t^*(z_1) - 1\}\big).\end{aligned}$$

Note that the nonvanishing condition implies that $b(z_1, z_2) \leq k+1$ and $b(z_2, z_3) \leq \ell+1$.

Proof. We proceed as in the proof of Lemma 5.3, constructing an injection $\psi : \mathcal{F}(k+1,\ell) \to I \times \mathcal{F}(k,\ell)$, where $I = I_1 \sqcup I_2 \sqcup I_3 \sqcup I_4$ are intervals of lengths specified by the RHS, each of them given in the corresponding case below.

Let $\boldsymbol{x} \in \mathcal{F}(k+1,\ell)$ be a word (corresponding to a linear extension) with $x_i = z_1$, $x_{i+k+1} = z_2$ and $x_{i+k+\ell+1} = z_3$. We consider several independent cases, which correspond to different parts of the interval I:

Case 1: Suppose that there exists $x_j \,\|\, z_1$ with $j \in [i+1, i+k]$ and let j be the minimal such index. Then $\{x_i, \ldots, x_{j-1}\} \subseteq U^*(z_1, x_j)$ and $j - i \leq \min\{k, t^*(z_1)\}$. Take $\boldsymbol{x}\tau_{j-1}\cdots\tau_i$, which moves x_j to position i and z_1 to position $i+1$. Then the resulting word is in $\mathcal{F}(k,\ell)$, and we record the value $(j-i)$ in the first interval $I_1 = [1, \min\{k, t^*(z_1)\}]$.

Case 2: Suppose that $x_j \succ z_1$ for all $j \in [i+1, i+k]$. Furthermore, suppose that $x_r \succ z_1$ and $x_r \prec z_3$ for all $r \in [i+k+2, i+k+\ell]$. These assumptions imply that there exists $j \in [i+1, i+k]$ such that $x_j \| z_3$, as otherwise we have $\{x_i, \ldots, x_{i+k+\ell+1}\} \in B(z_1, z_3)$, contradicting the assumption that $\mathrm{F}(k, \ell) > 0$. Assume that j is the maximal such index j. It then follows that $\{x_{j+1}, \ldots, x_{i+k+\ell+1}\} \subseteq U(z_3, x_j)$. This implies that $i+k+\ell+1-j \leq t(z_3)$, which in turn implies that $i+k+1-j \leq t(z_3) - \ell \leq t(z_3) - 1$. Then we take $\boldsymbol{x}' = \boldsymbol{x}\tau_j \cdots \tau_{i+k+\ell+1} \in \mathcal{F}(k, \ell)$ and record the value $(i+k+1-j)$ in the second interval $I_2 = [1, \min\{k, t(z_3)-1\}]$.

Case 3: Suppose again that $x_j \succ z_1$ for all $j \in [i+1, i+k]$, but now that there exists $r \in [i+k+2, i+k+\ell]$ such that either $x_r \| z_1$ or $x_r \| z_3$. The first condition implies that there exists $x_j \| z_2$ with $j \in [i+1, i+k]$, as otherwise we would have $\mathrm{F}(k, \ell) = 0$. Let j be the maximal such index. Then $\{x_{j+1}, \ldots, x_{i+k+1}\} \subseteq B(z_1, z_2) - z_1$, and thus $i+k+1-j \leq b(z_1, z_2) - 1$. Also note that $\{x_{j+1}, \ldots, x_{i+k+1}\} \subseteq U(z_2, x_j)$, and thus $i+k+1-j \leq t(z_2)$. Move x_j right past z_2 via $\boldsymbol{x}\tau_j \cdots \tau_{i+k+1}$ and record that move with $s := i+k+1-j \leq \min\{b(z_1, z_2) - 1, t(z_2)\}$. We now consider the new word $\boldsymbol{x}' \in \mathcal{F}(k, \ell+1)$. We split this case into two subcases.

Subcase 3.1: Suppose that there exists an element $x'_r = x_r \| z_3$ for some $r \in [i+k+2, i+k+\ell]$. Let r be the maximal such index. Then $\{x'_{r+1}, \ldots, x'_{i+k+\ell+1}\} \subseteq U(z_3, x'_r)$ and $i+k+\ell+1-r \leq t(z_3)$. We then create the word $\boldsymbol{x}'\tau_r \cdots \tau_{i+k+\ell+1} \in \mathcal{F}(k, \ell)$ where x'_r is moved past z_3. We record the pair $(s, i+k+\ell+1-r)$ in the product of intervals $I_3 = [1, \min\{b(z_1, z_2) - 1, t(z_2)\}] \times [1, \min\{\ell - 1, t(z_3)\}]$.

Subcase 3.2: Suppose that there exists $x'_r = x_r \| z_1$ for $r \in [i+k+2, i+k+\ell]$. We take the minimal such r. Then $\{x'_i, \ldots, x'_{r-1}\} \subseteq U^*(z_1, x'_r)$ and thus $r - i \leq t^*(z_1)$. This in turn implies that $r - i - k - 1 \leq t^*(z_1) - k - 1 \leq t^*(z_1) - 1$. Take a word $\boldsymbol{x}'' \in \mathcal{F}(k, \ell)$ by moving x'_r to the position before z_1 and record the pair $(s, r-i-k-1)$ in the product of intervals $I_4 = [1, \min\{b(z_1, z_2) - 1, t(z_2)\}] \times [1, \min\{\ell - 1, t^*(z_1) - 1\}]$.

Gathering these cases we obtain the desired inequality in the lemma. □

Lemma 6.2. *Suppose that* $\mathrm{F}(k+2, \ell) > 0$. *Then:*

$$\frac{\mathrm{F}(k+1, \ell)}{\mathrm{F}(k+2, \ell)} \ \leq \ t(z_1) \ + \ \big(t^*(z_2) - 1\big) \ + \ \min\big\{\ell - 1, t^*(z_2)\big\}\, t^*(z_3).$$

Proof. We proceed as in the proof of Lemma 5.3, constructing an injection $\psi : \mathcal{F}(k+1, \ell) \to I \times \mathcal{F}(k+2, \ell)$, where $I = I_1 \sqcup I_2 \sqcup I_3$ are intervals of lengths specified by the RHS corresponding to each case below.

Let $\boldsymbol{x} \in \mathcal{F}(k+1, \ell)$ be a word (corresponding to a linear extension) with $x_i = z_1$, $x_{i+k+1} = z_2$ and $x_{i+k+\ell+1} = z_3$. We consider three independent cases, which correspond to different intervals I_i (see below).

Case 1: Suppose that there exists $x_j \| z_1$ with $j \in [1, i-1]$, and let j be the maximal such index. Then $\{x_{j+1}, \ldots, x_i\} \subset U(z_1, x_j)$ and so $i - j \leq t(z_1)$. We take $\boldsymbol{x}\tau_j \cdots \tau_{i-1}$, which moves x_j to position i and z_1 to position $i-1$. Then the resulting word is in $\mathcal{F}(k+2, \ell)$, and we record the value $(i-j)$ in the first interval $I_1 = [1, t(z_1)]$.

Case 2: Suppose that $x_j \prec z_1$ for all $j \in [1, i-1]$. Since $\mathrm{F}(k+2, \ell) > 0$, there exists $x_j \| z_2$ with $j \in [i+k+2, n]$. Let j be the minimal such index. Then $\{x_{i+k+1}, \ldots, x_{j-1}\} \subset U^*(z_2, x_j)$, and thus $j - i - k - 1 \leq t^*(z_2)$. Move x_j to the front of z_2 via $\boldsymbol{x}\tau_{j-1} \cdots \tau_{i+k+1}$ to get a new word $\boldsymbol{x}'$. We split this case into two subcases:

Subcase 2.1: Suppose that $j \in [i+k+\ell+2, n]$. Then $\boldsymbol{x}' \in \mathcal{F}(k+2, \ell)$. Also note that $j - i - k - \ell - 1 \leq t^*(z_2) - \ell \leq t^*(z_2) - 1$. We then record the value $(j - i - k - \ell - 1)$ in the second interval $I_2 = [1, t^*(z_2) - 1]$.

Subcase 2.2: Suppose that $j \in [i+k+2, i+k+\ell]$. Then $\boldsymbol{x}' \in \mathcal{F}(k+2, \ell-1)$. By the assumption of Case 2 and the fact that $\mathrm{F}(k+2,\ell) > 0$, there exists $r \in [i+k+\ell+2, n]$ such that $x'_r \,\|\, z_3$. Assume that r is the minimal such index. It then follows that $\{x'_{i+k+\ell+1}, \ldots, x'_{r-1}\} \subseteq U^*(z_3, x'_r)$. This implies that $(r-i-k-\ell-1) \leq t^*(z_3)$. Move x'_r to the front of z_3 to obtain a new word $\boldsymbol{x}'' \in \mathcal{F}(k+2,\ell)$, and we record the value $(j-i-k-1, r-i-k-\ell-1)$ to the product of intervals $I_3 = [1, \min\{\ell-1, t^*(z_2)\}] \times [1, t^*(z_3)]$.

Gathering these cases we obtain the desired inequality in the lemma. □

6.2. **Bounds on cross product ratios.** We are now ready to obtain bounds on the cross product ratios in the vanishing case.

Proposition 6.3. *Suppose that* $\mathrm{F}(k,\ell)\,\mathrm{F}(k+2,\ell) > 0$. *Then*

$$\frac{\mathrm{F}(k,\ell)\,\mathrm{F}(k+2,\ell)}{\mathrm{F}(k+1,\ell)^2} \;\geq\; \frac{1}{2n\ell^2 k}\,.$$

Proof. First, observe that $b(z_1,z_2) \leq k+1$ and $t(z_1)+t^*(z_2) \leq b(z_1)+b^*(z_2) \leq n$. We then have:

$$\tag{6.1} \begin{aligned} &\min\{b(z_1,z_2)-1, t(z_2)\}\big(\min\{\ell-1, t(z_3)\} \;+\; \min\{\ell-1, t^*(z_1)-1\}\big) \\ &\quad + \min\{k, t^*(z_1)\} \;+\; \min\{k, t(z_3)-1\} \;\leq\; k(2\ell-2)+2k \;=\; 2k\ell \end{aligned}$$

and

$$\tag{6.2} t(z_1) \;+\; (t^*(z_2)-1) \;+\; \min\big\{\ell-1, t^*(z_2)\big\}\, t^*(z_3) \;\leq\; n-1 \;+\; (\ell-1)(n-1) \;<\; n\ell.$$

Lemmas 6.1 and 6.2 now give:

$$\frac{\mathrm{F}(k,\ell)\,\mathrm{F}(k+2,\ell)}{\mathrm{F}(k+1,\ell)^2} \;\geq\; \left(\frac{1}{n\ell}\right)\cdot\left(\frac{1}{2k\ell}\right),$$

as desired. □

We also need the following variation on this proposition.

Proposition 6.4. *Let* $P = (X, \prec)$ *be either a* t*-thin or* t*-flat poset with respect to* $\{z_1, z_2, z_3\}$. *Suppose also that* $\mathrm{F}(k,\ell)\,\mathrm{F}(k+2,\ell) > 0$. *Then we have:*

$$\frac{\mathrm{F}(k,\ell)\mathrm{F}(k+2,\ell)}{\mathrm{F}(k+1,\ell)^2} \;\geq\; \max\left\{\frac{1}{2k\ell(\ell+1)t}\,,\,\frac{1}{2t(t+1)^3}\right\}.$$

Proof. We follow the proof of the proposition above with the following adjustments. For the first inequality in the maximum, we replace the bound (6.2) with the following:

$$\tag{6.3} t(z_1) \;+\; (t^*(z_2)-1) \;+\; \min\big\{\ell-1, t^*(z_2)\big\}\, t^*(z_3) \;\leq\; 2t-1 \;+\; (\ell-1)(t-1) \;<\; (\ell+1)t.$$

Now the first inequality follows from Lemmas 6.1 and 6.2, with the parameters bounded by (6.1) and (6.3).

For the second inequality in the maximum, we replace the bound (6.1) and (6.2) with the following:

$$\tag{6.4} \begin{aligned} &\min\{b(z_1,z_2)-1, t(z_2)\}\big(\min\{\ell-1, t(z_3)\} \;+\; \min\{\ell-1, t^*(z_1)-1\}\big) \\ &\quad + \min\{k, t^*(z_1)\} \;+\; \min\{k, t(z_3)-1\} \;\leq\; t(2t-1)+t+(t-1) \;<\; 2t(t+1) \end{aligned}$$

and

$$\tag{6.5} t(z_1) \;+\; (t^*(z_2)-1) \;+\; \min\big\{\ell-1, t^*(z_2)\big\}\, t^*(z_3) \;\leq\; 2t-1+t^2 \;<\; (t+1)^2.$$

Now the second inequality follows from Lemmas 6.1 and 6.2, with the parameters bounded by (6.4) and (6.5). □

Theorem 6.5. *Suppose that* $\mathrm{F}(k+2,\ell)>0$ *and* $\mathrm{F}(k,\ell+2)=0$. *Then we have:*

$$\mathrm{F}(k+1,\ell)\,\mathrm{F}(k,\ell+1) \;\geq\; \mathrm{F}(k,\ell)\,\mathrm{F}(k+1,\ell+1)\left(\frac{1}{2}+\frac{1}{16nk\ell^2}\right).$$

Proof. We can assume that $\mathrm{F}(k,\ell)\,\mathrm{F}(k+1,\ell+1)>0$ as otherwise the result is trivial. Propositions 3.4 and 6.3 then give:

$$\frac{\mathrm{F}(k+1,\ell)\mathrm{F}(k,\ell+1)}{\mathrm{F}(k+1,\ell+1)\mathrm{F}(k,\ell)} \;\geq\; \left(1+\sqrt{1-\frac{1}{2nk\ell^2}}\right)^{-1} \;\geq\; \frac{1}{2}+\frac{1}{16nk\ell^2}\,,$$

where the last inequality follows from $\frac{1}{1+\sqrt{1-\alpha}}\geq\frac{1}{2}+\frac{\alpha}{8}$ for $0\leq\alpha<1$. □

Theorem 6.6. *Let* $P=(X,\prec)$ *be either a t-thin or t-flat poset with respect to* $\{z_1,z_2,z_3\}$. *Suppose also that* $\mathrm{F}(k,\ell+2)=0$ *and* $\mathrm{F}(k+2,\ell)>0$. *Then we have:*

$$\mathrm{F}(k+1,\ell)\,\mathrm{F}(k,\ell+1) \;\geq\; \mathrm{F}(k,\ell)\,\mathrm{F}(k+1,\ell+1)\,\max\left\{\frac{1}{2}+\frac{1}{16k\ell(\ell+1)t}\,,\,\frac{1}{2}+\frac{1}{16t(t+1)^3}\right\}.$$

Proof. The proof follows the same argument as in Theorem 6.5, where Proposition 6.4 is used in place of Proposition 6.3. □

6.3. **Putting everything together.** We can now combine the results to finish the proofs.

Proof of Main Theorem 1.2. The first inequality (1.1) follows immediately from Theorem 5.6. The second inequality (1.2) follows immediately from Theorem 6.5. The third inequality (1.3) follows by the symmetry $P\leftrightarrow P^*$, $z_1\leftrightarrow z_3$ and $k\leftrightarrow\ell$. Finally, the equality (1.4) is the equality in Lemma 4.4. □

Proof of Theorem 1.3. The proof of (1.5) follows the previous proof. The result is trivial in the case $\mathrm{F}(k,\ell)\,\mathrm{F}(k+1,\ell+1)=0$. In the vanishing case $\mathrm{F}(k,\ell+2)=\mathrm{F}(k+2,\ell)=0$ and $\mathrm{F}(k,\ell)\,\mathrm{F}(k+1,\ell+1)>0$ the result follows from the equality in Lemma 4.4. In the case when only one of the terms is vanishing: $\mathrm{F}(k,\ell+2)=0$ and $\mathrm{F}(k+2,\ell)>0$, the result is given by Theorem 6.6. The case $\mathrm{F}(k+2,\ell)=0$ and $\mathrm{F}(k,\ell+2)>0$ follows via poset duality as in the proof above. Finally, the nonvanishing case $\mathrm{F}(k,\ell+2)=\mathrm{F}(k+2,\ell)>0$ is given by the second inequality in Theorem 5.5. □

Proof of Theorem 1.4. Lemma 6.1 combined with (6.1), gives

$$\frac{\mathrm{F}(k+1,\ell)}{\mathrm{F}(k,\ell)} \;\leq\; 2k\ell.$$

Similarly, Lemma 6.2 for $k'=k-1$ and $\ell'=\ell+1$, combined with (6.2), gives:

$$\frac{\mathrm{F}(k,\ell+1)}{\mathrm{F}(k+1,\ell+1)} \;=\; \frac{\mathrm{F}(k'+1,\ell')}{\mathrm{F}(k'+2,\ell')} \;\leq\; n\ell' \;=\; n(\ell+1).$$

Multiplying these inequalities, we obtain the first term in the minimum of the desired upper bound. Via poset duality, see the proof of Theorem 1.2 above, we can exchange the k and ℓ terms and obtain the other inequality. □

7. Examples and counterexamples

7.1. **Inequalities (CPC1) and (CPC2).** Recall that by Theorem 1.7 at least one of these two inequalities must hold. We now show that for some posets (CPC2) does not hold. By the poset duality, the inequality (CPC1) also does not hold.

Proposition 7.1. *The inequality (CPC2) fails for an infinite family of posets of width three.*

Proof. Fix $k \geq 1$ and $\ell \geq 2$, and let $P := (X, \prec)$ be the poset given by

$$\begin{aligned} &X \ := \ \{x_1, \ldots, x_{k-1}\} \ \sqcup \ \{y_1, \ldots, y_{\ell-2}\} \ \sqcup \ \{z_1, z_2, z_3\} \ \ \sqcup \ \ \{u, v, w\}\,, \\ &z_1 \ \prec \ x_1 \ \prec \ x_2 \ \prec \ \cdots \ \prec \ x_{k-1} \ \prec \ z_2 \ \prec \ y_1 \ \prec \ y_2 \ \prec \ \cdots \ \prec \ y_{\ell-2} \ \prec \ z_3\,, \\ &x_{k-1} \ \prec u \ \prec \ y_1\,, \quad v \ \succ \ z_2\,, \quad w \ \succ \ z_2\,. \end{aligned}$$

Note that this is a poset of width three. Let us now compute all four terms in (CPC2):

First, observe that $L \in \mathcal{F}(k, \ell+2)$ if and only if $L(z_2) < L(u) < L(y_1)$ and $L(v), L(w) < L(z_3)$. Thus, there is a bijection between these linear extensions and the pairs (i,j) satisfying $1 \leq i \neq j \leq \ell+1$, through the map $L \mapsto \big(L(v) - L(z_2), L(w) - L(z_2)\big)$. Therefore, we have $\mathrm{F}(k, \ell+2) = (\ell+1)\ell$.

Second, observe that $L \in \mathcal{F}(k+1, \ell)$ if and only if $L(x_{k-1}) < L(u) < L(z_2)$, and either $L(v) < L(z_3) < L(w)$ or $L(w) < L(z_3) < L(v)$. Note that there is a bijection between those linear extensions satisfying $L(v) < L(z_3) < L(w)$ and the integers in $[1, \ell-1]$, through the map $L \mapsto L(v) - L(z_2)$. Therefore, we have $\mathrm{F}(k+1, \ell) = 2(\ell-1)$.

Third, observe that $L \in \mathcal{F}(k, \ell+1)$ if and only if $L(z_2) < L(u) < L(y_1)$, and either $L(v) < L(z_3) < L(w)$ or $L(w) < L(z_3) < L(v)$. Note that there is a bijection between those linear extensions satisfying $L(v) < L(z_3) < L(w)$ and the integers in $[1, \ell]$, through the map $L \mapsto L(v) - L(z_2)$. Therefore, we have $\mathrm{F}(k, \ell+1) = 2\ell$.

Fourth, observe that $L \in \mathcal{F}(k+1, \ell+1)$ if and only if $L(x_{k-1}) < L(u) < L(z_2)$ and $L(v), L(w) < L(z_3)$. Note that there is a bijection between these linear extensions and pairs (i,j) satisfying $1 \leq i \neq j \leq \ell$. Therefore, we have $\mathrm{F}(k+1, \ell+1) = \ell(\ell-1)$.

Combining these observations, we obtain:

$$\frac{\mathrm{F}(k, \ell+1)\,\mathrm{F}(k+1, \ell+1)}{\mathrm{F}(k, \ell+2)\,\mathrm{F}(k+1, \ell)} \ = \ \frac{\ell}{\ell+1} \ < \ 1\,.$$

This contradicts (CPC2), as desired. □

7.2. **Counterexamples to the generalized CPC.** We now show that the examples in proof of Proposition 7.2 are also counterexamples to Conjecture 1.5, thus proving Theorem 1.6.

Proposition 7.2. *Inequality (GCPC) implies (CPC2).*

Proof. Suppose (CPC2) fails for a poset $P = (X, \prec)$, elements $z_1, z_2, z_3 \in X$, and integers $k, \ell \geq 1$.

Let $z_1' := z_2$, $z_2' := z_1$, and $z_3' := z_3$. To avoid the clash of notation, let $\mathrm{F}'(k,\ell)$ be defined by

$$\mathrm{F}'(k, \ell) \ := \ \big|\{L \in \mathcal{E}(P) \ : \ L(z_2') - L(z_1') = k, \ L(z_3') - L(z_2') = \ell\}\big|.$$

By definition, we have

$$\mathrm{F}'(a,b) \ = \ \mathrm{F}(-a, a+b).$$

Now let $a := -k-1$ and $b := \ell+k+1$. Note aside that $a < 0$ for all $k > 0$. It then follows that

$$\begin{aligned} \mathrm{F}'(a,b) \ &= \ \mathrm{F}(-a, a+b) \ = \ \mathrm{F}(k+1, \ell), \\ \mathrm{F}'(a+1, b+1) \ &= \ \mathrm{F}(-a-1, a+b+2) \ = \ \mathrm{F}(k, \ell+2), \\ \mathrm{F}'(a, b+1) \ &= \ \mathrm{F}(-a, a+b+1) \ = \ \mathrm{F}(k+1, \ell+1), \\ \mathrm{F}'(a+1, b) \ &= \ \mathrm{F}(-a-1, a+b+1) \ = \ \mathrm{F}(k, \ell+1). \end{aligned}$$

In the new notation, the inequality (CPC2) is equivalent to

$$\mathrm{F}'(a,b)\,\mathrm{F}'(a+1,b+1) \ \leq \ \mathrm{F}'(a,b+1)\,\mathrm{F}'(a+1,b),$$

and note that $a<0,\ b>0$ whenever $k,\ell>0$. This shows that a counterexample for (CPC2) is also a counterexample to (GCPC). $\square$

Corollary 7.3. *Inequalities* (CPC1) *and* (CPC2) *hold for posets of width two.*

This follows from Proposition 7.2 and Theorem 3.3 in [CPP22] which proves (GCPC) for posets of width two.

7.3. **Stanley ratio.** It follows from Corollary 5.2, the following bound on the *Stanley ratio*:

$$\frac{\mathrm{N}_k^2}{\mathrm{N}_{k-1}\,\mathrm{N}_{k+1}} \ \leq \ (k-1)(n-k), \tag{7.1}$$

whenever the LHS is well defined. The following example shows that both the inequality (7.1) and Corollary 5.2 are tight.

In the notation of §5.2, fix $1\leq k\leq n$. Let $P_k:=(X,\prec)$ be the width two poset given by

$$\begin{aligned}
&X \ := \ \{x_1,\ldots,x_{k-2}\} \ \sqcup \ \{y_1,\ldots,y_{n-k-1}\} \ \sqcup \ \{a,v,w\},\\
&x_1 \ \prec \ x_2 \ \prec \ \cdots \ \prec \ x_{k-2} \ \prec \ a \ \prec \ y_1 \ \prec \ y_2 \ \prec \ \cdots \ \prec \ y_{n-k-1},\\
&v \ \prec \ y_1\,, \quad w \ \succ \ x_1\,, \quad v \ \prec \ w.
\end{aligned}$$

Proposition 7.4. *For posets P_k defined above the inequality* (7.1) *is an equality.*

Proof. Note that for all linear extensions $L\in\mathcal{N}_{k-1}$, we have $L(a)<L(v)=k<L(w)$, where $k+1\leq L(w)\leq n$. Similarly, for all linear extensions $L\in\mathcal{N}_k$, we have $L(v)<L(a)<L(w)$, where $1\leq L(v)\leq k-1$ and $k+1\leq L(w)\leq n$. Finally, for all linear extensions $L\in\mathcal{N}_{k+1}$, we have $L(v)<L(w)=k<L(a)$, where $1\leq L(v)\leq k-1$. These three observation imply thati

$$\mathrm{N}_{k-1} \ = \ n-k, \qquad \mathrm{N}_k \ = \ (k-1)(n-k), \qquad \mathrm{N}_{k+1} \ = \ k-1.$$

Thus, for posets P_k the inequality (7.1) is an equality. $\square$

7.4. **Converse cross product ratio.** The following example shows that Theorem 1.4 is essentially tight, up to a multiplicative factor of 2ℓ. Fix $k\geq 2,\ \ell\geq 1$, and denote $m:=n-k-\ell-3$. Let $P_{k,\ell}:=(X,\prec)$ be the poset given by

$$\begin{aligned}
&X \ := \ \{a_1,\ldots,a_{k-2}\} \ \sqcup \ \{b_1,\ldots,b_{\ell-1}\} \ \sqcup \ \{c_1,\ldots,c_m\} \ \sqcup \ \{z_1,z_2,z_3\} \ \sqcup \ \{u,v,w\},\\
&z_1 \ \prec \ a_1 \ \prec \ \cdots \ \prec \ a_{k-2} \ \prec \ z_2 \ \prec \ b_1 \ \prec \ \cdots \ \prec \ b_{\ell-1} \ \prec \ z_3 \ \prec \ c_1 \ \prec \ \cdots \ \prec \ c_m,\\
&u \ \prec \ z_2\,, \quad a_{k-2} \ \prec \ v \ \prec \ z_3\,, \quad w \ \succ \ b_{\ell-1}\,, \quad u \ \prec v \ \prec \ w.
\end{aligned}$$

Proposition 7.5. *Fix $k\geq 2,\ \ell\geq 1$. For posets $P_{k,\ell}$ defined above, we have:*

$$\frac{F(k,\ell+1)\,F(k+1,\ell)}{F(k,\ell)\,F(k+1,\ell+1)} \ = \ k\ell n\bigl(1+o(1)\bigr) \quad \text{as} \quad n\to\infty. \tag{7.2}$$

Proof. Note that for every linear extension $L\in\mathcal{E}(P_{k,\ell})$, we have:

$$L(z_2)-L(z_1) \ \geq \ |B(z_1,z_2)-z_1| \ = \ |\{a_1,\ldots,a_{k-2},z_2\}| \ = \ k-1, \tag{7.3}$$

$$L(z_3)-L(z_2) \ \geq \ |B(z_2,z_3)-z_2| \ = \ |\{b_1,\ldots,b_{\ell-1},z_3\}| \ = \ \ell. \tag{7.4}$$

Note also that

$$\text{either} \quad L(u) = 1 \quad \text{or} \quad L(z_1) < L(u) < L(z_2), \tag{7.5}$$

$$\text{either} \quad L(v) = L(z_2) - 1 \quad \text{or} \quad L(z_2) < L(v) < L(z_3), \tag{7.6}$$

$$\text{either} \quad L(w) = L(z_3) - 1 \quad \text{or} \quad L(w) > L(z_3). \tag{7.7}$$

We now compute the cross–product ratio of $P_{k,\ell}$ consider the following four cases.

Case 1. Let $L \in \mathcal{F}(k, \ell)$. Since $L(z_3) - L(z_2) = \ell$, it then follows from (7.4) that both $L(v)$ and $L(w)$ are not contained in the interval $[L(z_2), L(z_3)]$. It then follows from (7.6) and (7.7) that $L(v) = L(z_2) - 1$ and $L(w) > L(z_3)$, respectively. Now, since $L(z_2) - L(z_1) = k$ and $L(v) \in \big[L(z_1), L(z_2)\big]$, it then follows from (7.3) that $L(u)$ is not contained in the interval $[L(z_1), L(z_2)]$. It then follows from (7.5) that $L(u) = 1$ which in turn implies that $L(z_1) = 2$. We conclude that $L \in \mathcal{F}(k, \ell)$ satisfy:

$$\begin{aligned} &L(z_1) = 2, \quad L(z_2) = k+2, \quad L(z_3) = k+\ell+2, \\ &L(u) = 1, \quad L(v) = k+1, \quad L(w) \in [k+\ell+3, n]. \end{aligned}$$

This implies that $\mathrm{F}(k, \ell) = n - k - \ell - 2$, as desired.

Case 2. Let $L \in \mathcal{F}(k+1, \ell)$. Since $L(z_3) - L(z_2) = \ell$, it then follows from (7.4) that both $L(v)$ and $L(w)$ are not contained in the interval $[L(z_2), L(z_3)]$. It then follows from (7.6) and (7.7) that $L(v) = L(z_2) - 1$ and $L(w) > L(z_3)$, respectively. Now, since $L(z_2) - L(z_1) = k+1$ and $L(v) \in \big[L(z_1), L(z_2)\big]$, it then follows from (7.3) that $L(u)$ is contained in the interval $[L(z_1), L(z_2)]$. It then follows that $L(z_1) = 1$. We conclude that $L \in \mathcal{F}(k+1, \ell)$ satisfy:

$$\begin{aligned} &L(z_1) = 1, \quad L(z_2) = k+2, \quad L(z_3) = k+\ell+2, \\ &L(u) \in [2, k], \quad L(v) = k+1, \quad L(w) \in [k+\ell+3, n]. \end{aligned}$$

This implies that $\mathrm{F}(k+1, \ell) = (k-1)(n-k-\ell-2)$.

Case 3. We have $\mathrm{F}(k, \ell+1) = 1 + (k-1)\,\ell(n-k-\ell-2)$ by the following argument. Let $L \in \mathcal{F}(k, \ell+1)$. By (7.5) either $L(u) = 1$ or $L(u) \in [L(z_1), L(z_2)]$.

Case 3.1 Assume that $L(u) = 1$. This implies that $L(z_1) = 2$. Since $L(u) \notin [L(z_1), L(z_2)]$, it then follows from $L(z_2) - L(z_1) = k$ and (7.3) that $L(v)$ is contained in the interval $[L(z_1), L(z_2)]$. It then follows from (7.6) that $L(v) = L(z_2) - 1$. Since $L(z_3) - L(z_2) = \ell + 1$ and $L(v) \notin [L(z_2), L(z_3)]$, it then follows from (7.4) that $L(w)$ is contained in the interval $[L(z_2), L(z_3)]$. By (7.7), this implies that $L(w) = L(z_3) - 1$. We conclude:

$$\begin{aligned} &L(z_1) = 2, \quad L(z_2) = k+2, \quad L(z_3) = k+\ell+3, \\ &L(u) = 1, \quad L(v) = k+1, \quad L(w) = k+\ell+2. \end{aligned}$$

Thus, there is exactly one such linear extension.

Case 3.2 Assume that $L(u) \in [L(z_1), L(z_2)]$. This implies that $L(z_1) = 1$. Since $L(z_2) - L(z_1) = k$ and $L(u) \in [L(z_1), L(z_2)]$, it then follows from (7.3) that $L(v)$ is not contained in the interval $[L(z_1), L(z_2)]$. By (7.6), this implies that $L(v)$ is contained in the interval $[L(z_2), L(z_3)]$. Since $L(z_3) - L(z_2) = \ell + 1$, it then follows from (7.4) that $L(w)$ is not contained in the interval $[L(z_2), L(z_3)]$. By (7.7), this implies that $L(w) > L(z_3)$. We conclude:

$$\begin{aligned} &L(z_1) = 1, \quad L(z_2) = k+1, \quad L(z_3) = k+\ell+2, \\ &L(u) \in [2, k], \quad L(v) \in [k+2, k+\ell+1], \quad L(w) \in [k+\ell+3, n]. \end{aligned}$$

Thus, there are exactly $(k-1)\ell(n-k-\ell-2)$ such linear extensions.

Case 4. Let $L \in \mathcal{F}(k+1, \ell+1)$. Since $L(z_2) - L(z_1) = k+1$, it follows from (7.3) that both $L(u)$ and $L(v)$ are contained in the interval $[L(z_1), L(z_2)]$. This implies that $L(z_1) = 1$. Since $L(v) \in [L(z_1), L(z_2)]$, it then follows from (7.6) that $L(v) = L(z_2) - 1$. Now, since $L(z_3) - L(z_2) = \ell + 1$ and $L(v) \notin [L(z_2), L(z_3)]$, it then follows from (7.4) that $L(w)$ is contained in the interval

$[L(z_2), L(z_3)]$. By (7.7) this implies that $L(w) = L(z_3) - 1$. We conclude that $L \in \mathcal{F}(k+1, \ell+1)$ satisfy:

$$\begin{aligned} &L(z_1) = 1, && L(z_2) = k+2, && L(z_3) = k+\ell+3, \\ &L(u) \in [2, k], && L(v) = k+1, && L(w) = k+\ell+2. \end{aligned}$$

This implies that $\mathrm{F}(k+1, \ell+1) = k-1$.

In summary, for the poset $P_{k,\ell}$, we have:

$$\frac{\mathrm{F}(k, \ell+1)\, \mathrm{F}(k+1, \ell)}{\mathrm{F}(k, \ell)\, \mathrm{F}(k+1, \ell+1)} \ = \ 1 + (k-1)\ell(n-k-\ell-2) \ = \ k\ell n\big(1+o(1)\big) \ \text{ as } \ n \to \infty,$$

as desired. □

8. Final remarks

8.1. The *cross-product conjecture* (Conjecture 1.1) has been a major open problem in the area for the past three decades, albeit with relatively little progress to show for it, see [CP23b] for the background. The following quote about a closely related problem seems applicable:

> *"As sometimes happens, we cannot point to written evidence that the problem has received much attention; we can only say that a number of conversations over the last 10 years suggest that the absence of progress on the problem was not due to absence of effort."* [KY98, p. 87]

8.2. Theorem 2.4 is well-known in the area and can be traced back to the works of Jean Favard in the early 1930s.[4] Of course, this is not the only *Favard's inequality* known in the literature. In fact, Theorem 2.3 which goes back to Matsumura (1932) and Fenchel (1936), seem to also have been inspired by Favard's work.[5] In a closely related context of *Lorentzian polynomials*, Favard's inequality appears in [BH20, Prop. 2.17]. For more on Theorem 2.3, see [BF87, §51] and references therein.

8.3. As we mentioned in the introduction, the $\Upsilon \geq \frac{1}{2}$ lower bound derived from Favard's inequality (Theorem 2.4) easily implies the $\frac{1}{2}$ lower bound on the cross product. Given the straightforward nature of this implication, one can think of this paper as the first attempt to finding the best $\varepsilon \geq 0$, such that

$$\frac{\mathrm{F}(k+1, \ell)\, \mathrm{F}(k, \ell+1)}{\mathrm{F}(k, \ell)\, \mathrm{F}(k+1, \ell+1)} - \frac{1}{2} \ \geq \ \varepsilon.$$

In this notation, the CPC states that $\varepsilon = \frac{1}{2}$. Our Main Theorem 1.2 and especially the "t-thin or t-flat" Theorem 1.3 are the first effective bounds for $\varepsilon > 0$. More precisely, here we prove $\varepsilon = \Omega\big(\frac{1}{n}\big)$ for all posets, and a constant lower bound on ε for posets with bounded parameter t. Improving these bounds seems an interesting challenging problem even if the CPC ultimately fails.

8.4. The constant $\frac{1}{2}$ in Favard's inequality has the same nature as the constant 2 in [RSW23, Cor. 1.5] which also follows from Favard's inequality written in terms of the Lorentzian polynomials technology. The relationships to the constant 2 in [CP22b, Thm 1.1] and [HSW22, Thm 5] are more distant, but fundamentally of the same nature. While in the former case it is tight, in the latter is likely much smaller, see [Huh18, §2.3].

[4] Ramon van Handel, personal communication (May 3, 2021).

[5] Ramon van Handel, personal communication (June 12, 2023).

8.5. The reader might find surprising the discrepancy between the vanishing and the nonvanishing cases in the Main Theorem 1.2. Note that the vanishing case actually implies a *worse bound* (1.2) compared to the bound (1.1) in the nonvanishing case, instead of making things simpler. This is an artifact of the mixed volume inequalities and combinatorial ratios. Proposition 6.3 gives a better bound than Proposition 6.4 simply because the ratio of $\mathrm{F}(\cdot,\cdot)$'s in the former is under a square root which decreases the order. However, these combinatorial bounds can only be applied when the corresponding terms are nonzero.

Clearly, there is no way to justify this discrepancy, as otherwise we would know how to disprove the CPC. Still, one can ask if there is another approach to the vanishing case which would improve the bound? We caution the reader that sometimes nonvanishing does indeed make a difference (see e.g. Example 4.5).

8.6. Theorem 4.1 gives the vanishing conditions for the *generalized Stanley inequalities*. It was first stated without a proof in [DD85, Thm 8.2], and it seems the authors were aware of a combinatorial proof by analogy with their proof of the corresponding results for the order polynomial. The theorem was rediscovered in [CPP23a, Thm 1.12], where it was proved via combinatorics of words. Independently, it was also proved in [MS24, Thm 5.3] by a geometric argument.

8.7. There is a large literature on the negative dependence in a combinatorial context, see e.g. [BBL09, Huh18, Pem00], and in the context of linear extensions [KY98, She82]. When it comes to correlation inequalities for linear extensions of posets, this paper can be viewed as the third in a series after [CP22b] and [CP23a] by the first two authors. These papers differ by the tools involved. In [CP22b], we use the *combinatorial atlas* technology (see [CP21, CP22a]), while in [CP23a] we use the *FKG-type inequalities*.

The idea of this paper was to use geometric inequalities for mixed volumes, to obtain new cross product type inequalities. As we mentioned in the introduction, it transfers the difficulty to the *combinatorics of words*. This is the approach introduced in [Hai92, MR94] (see also [Sta09]), and advanced in [CPP22, CPP23a, CPP23b] in a closely related context.

8.8. Despite the apparent symmetry between the t-thin and t-flat notions, there is a fundamental difference between them. For posets $P = (X, \prec)$ which are t-thin with respect to a set A of bounded size, the number $e(P)$ of linear extension can be computed in polynomial time, since $P' := (X \smallsetminus A, \prec)$ has width at most t. On the other hand, for posets which have bounded height, computing $e(P)$ is #P-complete [BW91, DP18], and the same holds for posets which are t-flat with respect to a set of bounded size.

Acknowledgements. We are grateful to Jeff Kahn, Yair Shenfeld and Ramon van Handel for many helpful discussions and remarks on the subject, and to Julius Ross for telling us about [RSW23]. We are thankful to Per Alexandersson for providing `Mathematica` packages to test the conjectures. The first author was partially supported by the Simons Foundation. The second and third authors were partially supported by the NSF.

References

[AFO14] Shiri Artstein-Avidan, Dan Florentin and Yaron Ostrover, Remarks about mixed discriminants and volumes, *Commun. Contemp. Math.* **16** (2014), no. 2, 1350031, 14 pp.

[BF87] Tommy Bonnesen and Werner Fenchel, *Theory of convex bodies* (originally published in 1934 in German), BCS Associates, Moscow, ID, 1987, 172 pp.

[BBL09] Julius Borcea, Petter Brändén and Thomas M. Liggett, Negative dependence and the geometry of polynomials, *J. Amer. Math. Soc.* **22** (2009), 521–567.

[BH20] Petter Brändén and June Huh, Lorentzian polynomials, *Annals of Math.* **192** (2020), 821–891.

[BLP23] Petter Brändén, Jonathan Leake and Igor Pak, Lower bounds for contingency tables via Lorentzian polynomials, *Israel J. Math.* **253** (2023), 43–90.

[BGL18] Silouanos Brazitikos, Apostolos Giannopoulos and Dimitris-Marios Liakopoulos, Uniform cover inequalities for the volume of coordinate sections and projections of convex bodies, *Adv. Geom.* **18** (2018), 345–354.

[BFT95] Graham Brightwell, Stefan Felsner and William T. Trotter, Balancing pairs and the cross product conjecture, *Order* **12** (1995), 327–349.

[BW91] Graham Brightwell and Peter Winkler, Counting linear extensions, *Order* **8** (1991), 225–247.

[BW92] Graham Brightwell and Colin Wright, The 1/3–2/3 conjecture for 5-thin posets, *SIAM J. Discrete Math.* **5** (1992), 467–474.

[BuZ88] Yuri D. Burago and Victor A. Zalgaller, *Geometric inequalities*, Springer, Berlin, 1988, 331 pp.

[CP21] Swee Hong Chan and Igor Pak, Log-concave poset inequalities, preprint (2021), 71 pp; `arXiv:2110.10740`.

[CP22a] Swee Hong Chan and Igor Pak, Introduction to the combinatorial atlas, *Expo. Math.* **40** (2022), 1014–1048.

[CP22b] Swee Hong Chan and Igor Pak, Correlation inequalities for linear extensions, preprint (2022), 23 pp; `arXiv:2211.16637`.

[CP23a] Swee Hong Chan and Igor Pak, Multivariate correlation inequalities for P-partitions, *Pacific J. Math.* **323** (2023), 223–252.

[CP23b] Swee Hong Chan and Igor Pak, Linear extensions of finite posets, preprint (2023), 55 pp; `arXiv:2311.02743`.

[CPP22] Swee Hong Chan, Igor Pak and Greta Panova, The cross–product conjecture for width two posets, *Trans. AMS* **375** (2022), 5923–5961.

[CPP23a] Swee Hong Chan, Igor Pak, and Greta Panova, Effective poset inequalities, *SIAM J. Discrete Math.* **37** (2023), 1842–1880.

[CPP23b] Swee Hong Chan, Igor Pak and Greta Panova, Extensions of the Kahn–Saks inequality for posets of width two, *Combinatorial Theory* **3** (2023), no. 1, Paper No. 8, 35 pp.

[DD85] David E. Daykin and Jacqueline W. Daykin, Order preserving maps and linear extensions of a finite poset, *SIAM J. Algebraic Discrete Methods* **6** (1985), 738–748.

[DP18] Samuel Dittmer and Igor Pak, Counting linear extensions of restricted posets, preprint (2018), 33 pp.; `arXiv:1802.06312`.

[Hai92] Mark D. Haiman, Dual equivalence with applications, including a conjecture of Proctor, *Discrete Math.* **99** (1992), 79–113.

[Huh18] June Huh, Combinatorial applications of the Hodge–Riemann relations, in *Proc. ICM Rio de Janeiro*, vol. IV, World Sci., Hackensack, NJ, 2018, 3093–3111.

[HSW22] June Huh, Benjamin Schröter and Botong Wang, Correlation bounds for fields and matroids, *Jour. Eur. Math. Soc.* **24** (2022), 1335–1351.

[KS84] Jeff Kahn and Michael Saks, Balancing poset extensions, *Order* **1** (1984), 113–126.

[KY98] Jeff Kahn and Yang Yu, Log-concave functions and poset probabilities, *Combinatorica* **18** (1998), 85–99.

[MS24] Zhao Yu Ma and Yair Shenfeld, The extremals of Stanley's inequalities for partially ordered sets, *Adv. Math.* **436** (2024), Paper No. 109404., 72 pp.

[MR94] Claudia Malvenuto and Christophe Reutenauer, Evacuation of labelled graphs, *Discrete Math.* **132** (1994), 137–143.

[Pec08] Marcin Peczarski, The gold partition conjecture for 6-thin posets, *Order* **25** (2008), 91–103.

[Pem00] Robin Pemantle, Towards a theory of negative dependence, *J. Math. Phys.* **41** (2000), 1371–1390.

[RSW23] Julius Ross, Hendrick Suss and Thomas Wannerer, A note on Lorentzian polyomials and the Kahn–Saks inequality, preprint (2023), 3 pp.

[Sch14] Rolf Schneider, *Convex bodies: the Brunn–Minkowski theory* (second edition), Cambridge Univ. Press, Cambridge, UK, 2014, 736 pp.

[SvH23] Yair Shenfeld and Ramon van Handel, The extremals of the Alexandrov–Fenchel inequality for convex polytopes, *Acta Math.* **231** (2023), 89–204.

[She82] Lawrence A. Shepp, The XYZ conjecture and the FKG inequality, *Ann. Probab.* **10** (1982), 824–827.

[SZ16] Ivan Soprunov and Artem Zvavitch, Bezout inequality for mixed volumes, *Int. Math. Res. Not.* **2016**, no. 23, 7230–7252.

[Sta81] Richard P. Stanley, Two combinatorial applications of the Aleksandrov–Fenchel inequalities, *J. Combin. Theory, Ser. A* **31** (1981), 56–65.

[Sta09] Richard P. Stanley, Promotion and evacuation, *Electron. J. Combin.* **16** (2009), no. 2, RP 9, 24 pp.

[Tro95] Wiliam T. Trotter, Partially ordered sets, in *Handbook of combinatorics*, vol. 1, Elsevier, Amsterdam, 1995, 433–480.