Computation of Tutte polynomials of complete graphs

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1. Let $G=(V,E)$ be a graph, where V is the set of vertices of graph G , E is the set of edges of graph G. The <u>Tutte polynomial</u> of graph G is defined as follows :

$$T(G;x,y)= \sum (x-1)^{r(E)-r(S)}(y-1)^{\#S-r(s)} \tag{1}$$

where the summation is over the spanning subgraphs $G'=(V,S)$, $S \quad E$ and $r(S)$ is the number of connected components in graph G' (see e.g. [1,2]). Let us call $F_n(x,y)=T(K_{n+1};x,y)$ the Tutte polynomial of a complete graph K_{n+1} with $(n+1)$ vertices.

<u>Theorem</u>

$$F_n(x,y)= \sum_{k=1}^{n} \binom{n-1}{k-1} (x+y+y^2+...y^{k-1})\ F_{k-1}(1,y)\ F_{n-k}(x,y)$$

2. Proof of the Theorem.

Let $G=K_{n+1}=(V,E)$, $V=\{0,1,...,n\}$ and $E=2^V$. Let us define the lexicographical order "$\prec$" on the set E as follows: $(i,j)\prec(i',j')$ if $i<i'$ or $i=i'$, $j=j'$.

Let L_n be the set of spanning trees $t \in G$. We say that the edge $p\in t$ is <u>internally active</u> in t, $t\in L_n$ if $p\prec q$ for all $q\in E\backslash t$ such that $(t-p+q)\in L_n$. Similarly, we shall say that the edge $p\in t$ is <u>externally active</u> in t, $t\in L_n$ if $p\prec q$ for all $q\in t$ such that $(t-q+p)\in L_n$. The <u>internal</u> (<u>external</u>) <u>activity</u> of t, denoted $i(t)$ $(e(t))$, is number of elements internally (externally) active in t. Then, by the Tutte Theorem (see e.g. [1,2]):

$$F_n(x,y)=\sum_{t \in L_n} x^{i(t)}y^{e(t)} \tag{3}$$

Let us fix a tree $t \in L_n$. Let us consider an edge $(0,k) \in t$ dividing t into subtrees t' and t'' such that the vertex 0 lies in t''. Let $a=\#\{ j \mid j$ is a vertex in t'', $j<k \}$. Now we are going to prove that

$$i(t)=i(t')+\delta_{a,0}\ , \tag{4}$$

$$e(t)=e(t')+e(t'')+a, \tag{5}$$

where $\delta_{a,0}$ is Kronecker's delta.

It is clear that the edge (j_1,j_2) t'' is not internally active, because $(0,j_1) \prec (j_1,j_2)$, $(0,j_2) \prec (j_1,j_2)$ and $t-(j_1,j_2)+(0,j_1) \in L_n$ or $t-(j_1,j_2)+(0,j_2) \in L_n$. Also edge $(0,k) \in t$ is internally active in t iff $a=0$, because if there exists a vertex j of the subtree t'', $j<k$, then $(0,j) \prec (0,k)$ and $t-(0,k)+(0,j) \in L_n$. Thus, we have proved equality (4).

Let us consider the edge (j_1,j_2) with the vertex j_1 of the subtree t', the vertex j_2 of the subtree t'' and such that $j_1>0$. Then (j_1,j_2) is not externally active, because $(0,k) \prec (j_1,j_2)$ and $t-(j_1,j_2)+(0,k) \in L_n$. Similarly, the edge $(0,j)$, where j is a vertex of subtree t'', is externally active iff $j<k$. Therefore, we have proved equality (5).

Now the identity (2) is derived by substituting equalities (4), (5) in (3) and by summation over all pairs of subtrees t', t'' and all edges of type $(0,k)$. This completes the proof of the Theorem.

3. Remark. Similarly,

$F_n(1,y)=J_{n+1}(y)$ - is the inversion polynomial (see [3-5]),

$F_n(1,1+z)=C_{n+1}(z)$ -is the generating function for the labelled connected graphs with n+1 vertices by the number of edges (see [5-7]),

$F_n(x,0)=x(x+1)\ldots(x+n)$ - is the Poincare polynomial of the $\mathbb{C}^n \setminus \bigcup_{0<i<j<n+1} (z_i=z_j)$ (see [8-10]),

$F_n(1,0)=n!$ - is the number of increasing trees with n+1 vertices (see [3,11]),

$F_n(1,1)=(n+1)^{n-1}$ - is the number of spanning trees in complete graph K_{n+1} (see [2,3,11]),

$F_n(2,2)=2^{n(n+1)/2}$ - is the number of all spanning subgraphs of the graph K_{n+1},

$F_n(1,-1)=a_n$ - is the number of updown permutations $\sigma\in S_n$, such that $\sigma(1)<\sigma(2)>\sigma(3)<\ldots$ (see [3,11]).

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