MONOTONE PARAMETERS ON CAYLEY GRAPHS OF FINITELY GENERATED GROUPS

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Abstract. We construct a new large family of finitely generated groups with uncountably many values of the following monotone parameters: spectral radius, critical probabilities, and asymptotic entropy. We also present several open problems on other monotone parameters.

1. Introduction

1.1. **Main results.** There are several probabilistic parameters of Cayley graphs of finitely generated groups that capture "global properties" of the graph. Typically, these parameters are *monotone*: as groups get "larger" the parameters increase/decrease, usually in a difficult to control way.

While there is an extensive literature on the bounds for these parameters and their relations to each other, computing them remains challenging. The exact values are known in only few examples and special families. Our main result shows that these parameters can be as diverse as the finitely generated group themselves.

Theorem 1.1 (Main theorem). *Let $G = \langle S \rangle$ be a finitely generated group, where $S = S^{-1}$ is a symmetric generating set, and let $f(G, S)$ denote one of the following parameters:*

- *spectral radius $\rho(G, S)$,*
- *asymptotic entropy $h(G, S)$,*
- *site percolation critical probability $p^{\mathrm{s}}_c(G, S)$,*
- *bond percolation critical probability $p^{\mathrm{b}}_c(G, S)$.*

Then, there is a family of 4-regular Cayley graphs $\{\mathrm{Cay}(G_\omega, S_\omega)\}$, such that the set of parameter values $\{f(G_\omega, S_\omega)\}$ has cardinality of the continuum.

Our proof further implies that $\{f(G_\omega, S_\omega)\}$ contains a copy of the Cantor set. However, as we explain in §7.4, this is equivalent to being uncountable since these sets are analytic. The construction of this family of Cayley graphs is based on properties of an uncountable family of *decorated Grigorchuk groups*, the setting we previously considered in [KP13] and related to the approach in [TZ25], see §7.1. See also §7.9 for an alternative approach to the theorem.

We also obtain the following result of independent interest.

Theorem 1.2. *In notation of Theorem 1.1, for $k \geq 3$ the sets*

$$X_{\rho,k} := \{\rho(G, S) : |S| = k\}$$

have no isolated points on $[\alpha_k, 1]$, where $\alpha_k := \frac{2\sqrt{k-1}}{k}$ is the spectral radius of the infinite k-regular tree, which can be viewed as the Cayley graph of a free product of k copies of $\mathbb{Z}_2$.

Our main argument (Lemma 5.1) only shows for any $\rho \in X_{\rho,k}$ such that $\rho > \alpha_k$ there exists an increasing sequence $\rho_i \in X_{\rho,k}$ which converges to ρ from below, in other words that $X_{\rho,k}$ has

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no isolated points in $(\alpha_k, 1]$. We use a computation from [Woe00] to shows that there exists a sequence in $X_{\rho,k}$ which converges to α_k from above. A version of Theorem 1.2 for the asymptotic entropy was given by Tamuz and Zheng in [TZ25], cf. §7.1.

1.2. **Monotone parameters.** Let $f : \{(G, S)\} \to \mathbb{R}$ be a function on *marked groups*, i.e., pairs of finitely generated groups G and finite symmetric generating sets $S = S^{-1}$ (see §2.2). We refer to f as a *parameter* (see the discussion in §7.8). We say that f is *decreasing* if the following conditions hold:

$$(*) \qquad f(G/N, S') \geq f(G, S) \quad \text{for all} \quad N \lhd G, \quad \text{and}$$

$$(**) \qquad (G_n, S_n) \underset{n\to\infty}{\to} (G, S) \implies \liminf_{n\to\infty} f(G_n, S_n) \geq f(G, S).$$

Here $N \lhd G$ denotes a normal subgroup N of G, and S' is the projection of S onto G/N. In $(**)$, the convergence on the left[1] is in the Chabauty topology for marked groups, see §2.5.

We say that f is *increasing* if the inequalities $(*)$ and $(**)$ are reversed. We say that f is *monotone* if it is either increasing or decreasing. We say that f is *strictly decreasing* (respectively, *strictly increasing*, *strictly monotone*) if the inequality $(*)$ is strict, provided that N is nonamenable.[2] Similarly, we say that f is *sharply decreasing* (respectively, *sharply increasing*, *sharply monotone*) if the inequality $(*)$ is strict if and only if N is nonamenable (see also §7.8).

1.3. **Spectral radius.** Let $\mathrm{Cay}(G, S)$ denote the Cayley graph of a finitely generated group $G = \langle S \rangle$. The *spectral radius* $\rho(G, S)$ is defined as:

$$\rho(G, S) := \limsup_{n\to\infty} \frac{1}{|S|} \sqrt[n]{c(n)}\,,$$

where the $\{c(n) = c(G, S; n)\}$ is *cogrowth sequence* of G, i.e., the number of words of length n in the alphabet S which are equal to the identity e in G. Equivalently, this is the number of loops in $\mathrm{Cay}(G, S)$ of length n, starting and ending at the identity.

Famously, it was shown by Kesten [Kes59], that $\rho(G, S) = 1$ if and only if G is nonamenable, and that

$$\alpha_k = \tfrac{2\sqrt{k-1}}{k} \leq \rho(G, S) \leq 1\,, \quad \text{where} \quad k = |S|.$$

For $k = 2m$, the lower bound is attained on a free group $\mathbb{F}_m = \langle x_1^{\pm 1}, \ldots, x_m^{\pm 1} \rangle$.

Unfortunately, relatively little is known about the spectral radius in general beyond these basic inequalities. Notably, it is open whether *every* $\alpha \in (0, 1]$ is a spectral radius of *some* finitely generated group, cf. [BLM23, Question 4.2]. On the other hand, until this paper it was not known if there is a single example of a group with a transcendental spectral radius, a problem discussed in [Pak18, §2.4], cf. §7.5.

For several families of nonamenable groups, the exact value of $\rho(G, S)$ is known, see e.g. [GH97, Woe00] (see also [BLM23, E+14, Kuk99]). In all these cases the spectral radii are algebraic. Let Γ_2 denote the surface group of genus 2, i.e.

$$\Gamma_2 = \langle a_1, a_2, b_1, b_2 \mid a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} = 1 \rangle.$$

After much effort, it was shown that $0.6627 \leq \rho(\Gamma_2, S) \leq 0.6629$, where S are standard generators as above [Gou15, Nag99] (see also [GH97, §7] for the background and further references). It is open whether this spectral radius is transcendental.

Proposition 1.3. *The spectral radius* $\rho : \{(G, S)\} \to (0, 1]$ *is a decreasing parameter.*

[1] The inequality $(**)$ states that f is *lower semi-continuous*, see Lemma 4.6; we include it in the definition to emphasize the direction of the inequality.

[2] Requiring that the inequality $(*)$ is strict, provided that N is nontrivial is a very strong condition which is satisfied by only a few commonly used parameters like critical probability, see Lemma 1.6.

This result is well known. The property (∗) is straightforward, while the property (∗∗) uses the fact that $\rho(G,S)$ is a limit of a supermultiplicative sequence: $c(n+m) \ge c(n)c(m)$. The following result by Kesten shows that the spectral radius is sharply decreasing:

Lemma 1.4 ([Kes59, Lemma 3.1])**.** *Let $G = \langle S\rangle$ and let N be a normal subgroup of G and S' be the projection of S onto G/N. Then $\rho(G,S) < \rho(G/N,S')$ if and only if N is nonamenable.*

Note that the corresponding inequality for the Cheeger constant remains open (Conjecture 6.2). There is an alternative notion of the *cogrowth sequence*, where only reduced words are considered, see e.g. [CD21, §14.6]. In the probability literature, these correspond to *non-backtracking random walks*, see e.g. [ABL07, OW07]. Much of what we present translates easily to this setting; we omit it to avoid the confusion.

1.4. **Critical probabilities.** In the *Bernoulli site percolation*, the vertices of the graph $\mathrm{Cay}(G,S)$ are open with probability p and closed with probability $(1-p)$, independently at random. The *Bernoulli bond percolation* is defined analogously, but now the edges are open/closed. These notions are different, yet closely related to each other.

Denote by $\theta^{\mathrm{s}}(G,S,p)$ and $\theta^{\mathrm{b}}(G,S,p)$ the probability that the identity element e is in an infinite connected component in the site and bond percolation, respectively. We omit the superscript when the notation or results hold for both site and bond percolations.

The *critical probability* is defined as follows:

$$p_c(G,S) := \sup\big\{p : \theta(G,S,p) = 0\big\}.$$

We refer the reader to [BS96] for an introduction to percolation on Cayley graphs, and to [BR06, Gri99, Wer09] for a thorough treatment of both classical and recent aspects, and to [Dum18] for a recent overview of the subject (see also §7.7).

It is easy to see that $p_c = 1$ for all groups of linear growth (which are all virtually $\mathbb{Z}$). It was conjectured in [BS96, Conj. 2], that $p_c < 1$ for all groups of superlinear growth. Special cases of this conjecture have been established in a long series of papers, until it was eventually proved in [D+20], see also [ET23]. The ultimate result, the remarkable "gap inequality" $p_c \le 1-\varepsilon$ for a universal constant $\varepsilon > 0$, was obtained in [PS23] for all groups of superlinear growth, see also [HT25]. This shows that Theorem 1.2 does not apply to critical probabilities.

Famously, it was shown in [GS98], that

$$p_c^{\mathrm{b}}(G,S) \le p_c^{\mathrm{s}}(G,S) \le 1-\big(1-p_c^{\mathrm{b}}\big)^{k-1} \quad \text{where} \quad k = |S|.$$

It is also known that the bond percolation can be simulated by the site percolation but not vice versa, see [GZ26] for the definitions and precise statements.[3] In general, these critical probabilities do not coincide. For example, the celebrated Kesten's theorem states that $p_c^{\mathrm{b}} = \frac{1}{2}$ for the square grid (Cayley graph of $\mathbb{Z}^2$ with standard generators), see e.g. [BR06, Gri99]. By comparison, $p_c^{\mathrm{s}} \approx 0.592746$ in this case [NZ01], although the exact value is not known.

In all known examples when the critical probabilities $p_c^{\mathrm{s}}(G,S)$ and $p_c^{\mathrm{b}}(G,S)$ are computed exactly, they are always algebraic, cf. [Koz08, SZ10]. For example, it is known that

$$p_c^{\mathrm{s}}(G,S) = p_c^{\mathrm{b}}(H,R) = 1 - 2\sin\big(\tfrac{\pi}{18}\big) \in \overline{\mathbb{Q}},$$

where

$$G = \langle x,y \mid x^3 = y^3 = (xy)^3 = 1\rangle, \quad S = \{x,x^{-1},y,y^{-1}\},$$
$$H = \langle a,b,c \mid a^2 = b^2 = c^2 = (abc)^2 = 1\rangle, \quad R = \{a,b,c\}.$$

[3]For example, we say that bond percolation on $G=(V,E)$ *can be simulated* by site percolation on $G'=(V',E')$, if there is an injection $\gamma : V \to V'$ such that connecting probabilities $v \to w$ coincide with connecting probabilities $\gamma(v) \to \gamma(w)$ for all $v,w \in V$.

In this case, $\mathrm{Cay}(G, S)$ is the *Kagomé lattice* and $\mathrm{Cay}(H, R)$ is the *hexagonal lattice*, see e.g. [BR06, §5.5]. It was asked in [PS08], whether there exist critical probabilities $p_c(G, S)$ that are transcendental. Theorem 1.1 gives a positive answer to this question. The theorem also resolves a closely related Häggström's question, see §7.3. As before, we start with the following observation.

Proposition 1.5. *The critical probabilities* $p_c^{\mathrm{s}}, p_c^{\mathrm{b}} : \{(G, S)\} \to (0, 1]$ *are decreasing parameters.*

The proof of this result is straightforward (cf. [ET23] for continuity). The following result by Martineau and Severo shows that critical probabilities are strictly decreasing:

Lemma 1.6 ([MS19])**.** *Let* $G = \langle S \rangle$ *and let* $N \neq \mathbf{1}$ *be a normal subgroup of* G*. Let* S' *be the projection of* S *onto* G/N*. Then:*

$$p_c^{\mathrm{s}}(G, S) < p_c^{\mathrm{s}}(G/N, S') \quad \textit{and} \quad p_c^{\mathrm{b}}(G, S) < p_c^{\mathrm{b}}(G/N, S').$$

Note that there is no assumption that N is nonamenable, and, in fact, the main result in [MS19] is stated in the greater generality of quasi-transitive group actions on graphs. This result resolved a well-known open problem by Benjamini and Schramm [BS96, Question 1]. Thus, for example, we have $1 > p_c(\mathbb{Z}^2, S') > p_c(\mathbb{Z}^3, S)$, so the critical probabilities are not sharply decreasing.

1.5. **Entropy.** Let G be a finitely generated group, and let $\mu : G \to \mathbb{R}_{\geq 0}$ be a probability distribution. *Shannon's entropy* is defined as

$$\mathrm{H}(\mu) := - \sum_{g \in \mathrm{supp}(\mu)} \mu(g) \log \mu(g).$$

As before, let $G = \langle S \rangle$, where $S = S^{-1}$. Denote by $\mu_n(g) := \mathbb{P}[x_n = g]$ the distribution of the simple random walk $\{x_n\}$ on $\mathrm{Cay}(G, S)$ starting at identity $x_0 = e$ (with respect to the uniform distribution on the finite set S). Finally, let

$$h(G, S) := \lim_{n \to \infty} \frac{\mathrm{H}(\mu_n)}{n}$$

denote the *asymptotic entropy*, see [KV83]. Recall that $h(G, S) > 0$ if and only if $\mathrm{Cay}(G, S)$ has the *non-Liouville property* (existence of non-constant bounded harmonic functions). Equivalently, $h(G, S) = 0$ if and only if the random walk $\{x_n\}$ has trivial Poisson boundary, ibid.

We note that there are solvable groups of exponential growth with positive asymptotic entropy; the *lamplighter group* $\mathbb{Z}_2 \wr \mathbb{Z}^d$ for $d \geq 3$ is the most famous example [KV83, §6] (see also [Pete24, §9.1] and an introduction in [Tab17]). Note also that the asymptotic entropy is known explicitly only in a few cases as it is so hard to compute. For example, it was computed in [Gri78, p. 22] and [Bis92, Prop. 2.11], that

$$h(\mathbb{F}_m, S) = \frac{m-1}{m} \log(2m - 1).$$

Here $\mathbb{F}_m = \langle z_1^{\pm 1}, \ldots, z_m^{\pm 1} \rangle$ denotes the free group with the standard generating set. This shows that the asymptotic entropy is transcendental in this case: $h(\mathbb{F}_m, S) \notin \overline{\mathbb{Q}}$.

Proposition 1.7. *The asymptotic entropy* $h : \{(G, S)\} \to \mathbb{R}_{\geq 0}$ *is an increasing parameter.*

The proof of this result is straightforward. The following result is a direct consequence of results by Kaimanovich–Vershik [KV83] and Kaimanovich [Kai02], which show that the asymptotic entropy is strictly increasing:

Lemma 1.8 (see [Kai02, Thm 2])**.** *Let* $G = \langle S \rangle$ *and let* N *be a normal subgroup of* G*. Suppose* N *is nonamenable, and let* S' *be the projection of* S *onto* G/N*. Then* $h(G, S) > h(G/N, S')$*.*

Since there are amenable groups with positive asymptotic entropy, we conclude that the asymptotic entropy is not sharply increasing. Also, note that the corresponding inequality for the speed of random walk $\{x_n\}$ remains open (Conjecture 6.5).

Let us mention other (closely related) entropy notions, such as the *Connes–Størmer entropy* (see e.g. [Bis92]), and the *tree entropy* [Lyo05]. These can also be viewed as parameters of infinite (vertex-transitive) graphs.

1.6. **Proof outline.** The proof of Theorem 1.1 is extremely general and only requires that the parameter is strictly monotone. The proof is based on a group theoretic construction and a set theoretic argument. Formally, Theorem 1.1 is an immediate consequence from following two complementary lemmas.

Lemma 1.9 (Combined Lemma)**.** *All parameters f in Theorem 1.1 are strictly monotone.*

The lemma is a combination of Propositions 1.3, 1.5, 1.7 and Lemmas 1.4, 1.6, 1.8. Let us reiterate that all three lemmas are well known in the literature.

Lemma 1.10 (Main Lemma)**.** *Let f be a strictly monotone parameter. Then, there is a family $\big\{(G_\omega, S_\omega)\big\}$ of marked groups, such that $\big\{f(G_\omega, S_\omega)\big\}$ has cardinality of the continuum.*

The proof of Lemma 1.10 is in turn an easy consequence of the following result of independent interest.

Theorem 1.11 (main construction)**.** *There exists a family $\big\{(G_J, S_J) : J \in 2^{\mathbb{N}}\big\}$ of 4-generated marked groups, satisfying the following:*

- (strict monotonicity) *For every subset $J \subsetneq J'$, there is a surjection of marked groups $G_{J'} \to G_J$, such that the kernel of the projection is nonamenable.*
- (continuity) *For every sequence $\{J_n \in 2^{\mathbb{N}} : n \in \mathbb{N}\}$ such that $J_n \to J$ in the Tychonoff topology of $2^{\mathbb{N}}$, we have $\big(G_{J_n}, S_{J_n}\big) \to (G_J, S_J)$ in the Chabauty topology.*

Equivalently, the continuity condition says that there exists a function $\lambda : \mathbb{N} \to \mathbb{N}$, such that the ball of radius n in the Cayley graph of (G_J, S_J), depend only on $J \cap \{1, \ldots, \lambda(n)\}$.

1.7. **Paper structure.** In Sections 2 and 3 we recall definitions of marked groups, Grigorchuk groups and their convergence. Most readers familiar with the area should be able to skip this sections. In Section 4 we give the proof of Theorem 1.11 and Lemma 1.10, which complete the proof of the main Theorem 1.1. In Section 5, we prove Theorem 1.2. We conclude with a discussion of other monotone parameters in Section 6 and final remarks in Section 7.

2. Marked groups and their limits

We recall standard definitions of marked groups. We stay close to [KP13] from which we heavily borrow the notation and some basic results.

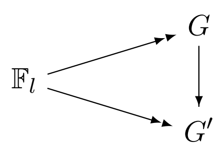
2.1. **Basic definitions and notation.** Denote $[n] = \{1, \ldots, n\}$ and $\mathbb{N} = \{1, 2, \ldots\}$. Let $\mathbb{R}_{\geq 0} = \{x \geq 0\}$, $\overline{\mathbb{Q}}$ the algebraic numbers, and $\mathbb{Q}_p$ the p-adic numbers. We use $\log x$ to denote the natural logarithm of a real number x.

We use both 1 and e to denote the identity in the group, and we use $\mathbf{1}$ to denote the trivial group. Let $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$ denote the group of integers modulo m, and let $\mathbb{F}_k$ denote the free group on k generators. Throughout the paper, all generating sets will be finite and *symmetric*: $S = S^{-1}$. We use $[x, y] = x^{-1}y^{-1}xy$ to denote the commutator of elements x and y.

2.2. **Definition of marked groups and their homomorphisms.** We consider only groups with ordered finite symmetric generating sets of the same size k. Whenever we mention a group G, we mean a pair (G, S) where $S = \{s_1, \ldots, s_k\}$ is an ordered symmetric generating (multi)set of G of size k. Although Cayley graph does not depends on the order of generators, the order is crucial for our results. We call these *marked groups*, and k will always denote the size of the symmetric generating set.

Throughout the paper, the homomorphisms between marked groups will send one generating set to the other. Formally, let (G, S) and (G', S') be marked groups, where $S = \{s_1, \ldots, s_k\}$ and $S' = \{s'_1, \ldots, s'_k\}$. Then $\phi : (G, S) \to (G', S')$ is a *marked group homomorphism* if $\phi(s_j) = s'_j$, and this map on generators extends to the (usual) homomorphism between groups: $\phi : G \to G'$.

Since we require that all generating sets S are symmetric, they naturally come with the involution $\theta : s \to s^{-1}$. Thus, the free marked group on a generating set S depends not only on the set S but also on the involution of $\theta : S \to S$. If $|S| = 2l$ and the involution θ does not have ant fixed points, then the free marked group on S will be the free group $\mathbb{F}_l$ on l generators. Therefore one can think of a marked group as an epimorphism $\mathbb{F}_l \twoheadrightarrow G$. In this picture, a map between marked groups G and G' correspond to a commutative diagram:



This means that for every two marked groups, there is at most one homomorphism $G \to G'$ which is necessarily surjective.

In general, one needs to replace the free group $\mathbb{F}_l$ with the group $F_{l,t}$ which is the free product of l copies of $\mathbb{Z}$ (corresponding to orbits of size 2 in S under the action of θ) and t copies of $\mathbb{Z}_2$ (corresponding to fixed points of θ), where $2l + t \geq 3$.

By a slight abuse of notation we do not include the involution θ in the notation. Additionally, we will often drop S and refer to a *marked group* G, when S is either clear from the context or not relevant.

Almost all groups in this paper are marked, however in a few occasions we use unmarked groups (mostly in the proof of Lemma 5.1 and Claim 5.5).

2.3. **Products of groups.** The *direct product* of groups G and H is denoted $G \oplus H$, rather than by the more standard $G \times H$. This notation allows us to write infinite product as $\bigoplus G_i$, where all but finitely many terms are trivial and we will typically omit the index of summation.

We denote by $\prod G_i$ the (usually uncountable) group of sequences of group elements, without any finiteness conditions. Of course, when the index set is infinite, the groups $\bigoplus G_i$ and $\prod G_i$ are not finitely generated (unless almost all groups G_i are trivial).

Finally, let $H \wr_X G = G \ltimes H^X$ denotes the permutational wreath product of the groups, where $\phi : G \to \Sigma(X)$ action of the group G on $|X|$ letters, in the case of permutation groups $G \subset \Sigma_\ell$ we will drop the subscript X.

2.4. **Products of marked groups.** Fix $I \subseteq \mathbb{N}$, and let $\{(G_i, S_i), i \in I\}$ be a sequence of marked groups with generating sets $S_i = \{s_{i1}, \ldots, s_{ik}\}$. Define $(\Gamma, S) = (\bigotimes G_i, S_i)$ to be the subgroup of $\prod G_i$ generated by diagonally embedding the generating sets of each G_i, i.e., $\bigotimes G_i = \langle s_1, \ldots, s_k \rangle$, where $s_j = \{s_{ij}\} \in \prod G_i$.

Note that Γ comes with canonical epimorphisms $\zeta_i : \Gamma \twoheadrightarrow G_i$. Often the generating sets will be clear from the context and will simply use $\Gamma = \bigotimes G_i$. When the index set contains only 2 elements we denote the product by $G_1 \otimes G_2$.

The product $\bigotimes G_i$ can be defined by universal properties and it is the "smallest" marked group which surjects onto each G_i. Thus, this is equivalent to the categorical product in the category of marked groups. We refer to §7.1 for some background and references.

2.5. **Limits of groups.** We say that the sequence of marked groups $\{(G_i, S_i) : i \in I\}$ *converges in the Chabauty topology*, to a group (G, S) if for any n there exists $m = m(n)$ such that for any $i > m$ the ball of radius n in G_i is the same as the ball of radius n in G. We write $\lim_{i\to\infty} G_i = G$.

Equivalently, this can be stated as follows: if $R_i = \ker(\mathbb{F}_k \twoheadrightarrow G_i)$ and $R = \ker(\mathbb{F}_k \twoheadrightarrow G)$ then

$$\lim_{i\to\infty} R_i \cap B_{\mathbb{F}_k}(n) \,=\, R \cap B_{\mathbb{F}_k}(n),$$

which means that for a fixed n and sufficiently large i the sets $R_i \cap B_{\mathbb{F}_k}(n)$ and $R \cap B_{\mathbb{F}_k}(n)$ must coincide.

Lemma 2.1 ([KP13, Lemma 4.6])**.** *Let $\{G_i\}$ be a sequence of marked groups which converge to a marked group G, and define $\Gamma := \bigotimes G_i$. Then there is an epimorphism $\pi : \Gamma \twoheadrightarrow G$. Moreover, the kernel of π is equal to the intersection $\Gamma \cap \bigoplus G_i$.*

Lemma 2.1 allows us to think of G as the *group at infinity* for the product Γ.

Example 2.2. Many group properties are not preserved under the limit. For example, it is easy to construct examples of amenable groups with a nonamenable limit. In fact, classic Margulis's (constant degree) expander construction produce Cayley graphs of $G_p = \mathrm{PSL}(2, \mathbb{Z}_p)$ with girth $\Omega(p)$ and virtually free group limit $\mathrm{PSL}(2, \mathbb{Z})$. See e.g. [HLW06, §11] and [Lub95, §7.3] for more on this and further references.

3. Grigorchuk groups

We now recall some definitions and results on Grigorchuk groups. Again, we stay close to notations and definitions in [KP13], and note that these results can be found throughout the literature.

3.1. **Free Grigorchuk group.** The *free Grigorchuk group* $\mathcal{G}$ with presentation

$$\mathcal{G} = \langle a, b, c, d \mid a^2 = b^2 = c^2 = d^2 = bcd = 1\rangle$$

will play a central role throughout the paper, as all our groups and also all Grigorchuk groups are homomorphic images of $\mathcal{G}$. This group is the free product of a group of order 2 and elementary abelian group of order 4, i.e. $\mathcal{G} \simeq \mathbb{Z}_2 * \mathbb{Z}_2^2$. It contains free subgroups and is nonamenable.

Notice that the generators of $\mathcal{G}$ are of order two, thus $\mathcal{G}$ can either be viewed as a marked group on the generating set $S = \{a, b, c, d\}$ of size for 4 where the involution θ fixes all generators, or as a marked group on a generating set $S' = \{a, \bar{a}, b, \bar{b}, c\bar{c}, d, \bar{d}\}$ of size 8 where the involution θ swaps x with $\bar{x}$. In this section we will only consider quotients of the group $\mathcal{G}$ thus the two viewpoints are equivalent.

3.2. **Family of Grigorchuk groups.** Below we present variations on standard results on the Grigorchuk groups $\mathbf{G}_\omega$. Rather than give standard definitions as a subgroup of the automorphism group of binary rooted tree, $\mathrm{Aut}(\mathrm{T}_2)$, we define $\mathbf{G}_\omega$ and $\mathbf{G} = \mathbf{G}_{(012)^\infty}$ via its properties. We refer to [Gri05, GP08, dlH00] for a more traditional introduction and most results in this subsection.

Definition 3.1. Let $\varphi : \mathcal{G} \twoheadrightarrow \mathcal{G}$ denote the automorphism of order 3 of the group $\mathcal{G}$ which cyclicly permutes the generators b, c and d, i.e.,

$$\varphi(a) = a, \quad \varphi(b) = c, \quad \varphi(c) = d, \quad \varphi(d) = b.$$

Let $\pi : \mathcal{G} \twoheadrightarrow H$ be an epimorphism, i.e., suppose group H comes with generating set consisting of 4 involutions $\{a, b, c, d\}$ which satisfy $bcd = 1$. By $F(H, \pi)$ we define the subgroup of $H \wr \mathbb{Z}_2 = \mathbb{Z}_2 \ltimes (H \oplus H)$ generated by the elements A, B, C, D defined as

$$A = (\xi; 1, 1), \quad B = (1; a, b), \quad C = (1; a, c) \quad \text{and} \quad D = (1; 1, d),$$

where $\xi^2 = 1$ is the generator of $\mathbb{Z}_2$. It is easy to verify that A, B, C, D are involutions which satisfy $BCD = 1$, which allows us to define an epimorphism $\widetilde{F}(\pi) : \mathcal{G} \to F(H, \pi)$.

The construction can be twisted by the powers of automorphism φ for $x = 0, 1, 2$

$$\widetilde{F}_x(\pi) := \widetilde{F}(\pi \circ \varphi^{-x}) \circ \varphi^x,$$

where $\widetilde{F}_x(\pi) : \mathcal{G} \to F_x(H, \pi)$.

In order to simplify the notation we use use $F(H)$ and $F_x(H)$ to denote $F(H, \pi)$ and $F_x(H, \pi)$.

An equivalent way of defining the group $F_x(H)$ is as the subgroups generated by

$$\begin{aligned}
A_0 &= (\xi; 1, 1), & B_0 &= (1; a, b), & C_0 &= (1; a, c), & D_0 &= (1; 1, d),\\
A_1 &= (\xi; 1, 1), & B_1 &= (1; a, b), & C_1 &= (1; 1, c), & D_1 &= (1; a, d),\\
A_2 &= (\xi; 1, 1), & B_2 &= (1; 1, b), & C_2 &= (1; a, c), & D_2 &= (1; a, d).
\end{aligned}$$

Here all groups H are marked, i.e., come with an epimorphism $\mathcal{G} \twoheadrightarrow H$. This allows us to slightly simplify the notation as above.

Proposition 3.2. *Each $\widetilde{F}_x$ is a functor $\mathcal{C} \to \mathcal{C}$, where $\mathcal{C}$ denotes the under category with objects morphisms from $\mathcal{G}$ to any marked group. By a slight abuse of language we will view each F_x as a functor form the category of homomorphic images of $\mathcal{G}$ to itself, i.e., any group homomorphism $H_1 \to H_2$ which sends generators to generators, induces a group homomorphism $F_x(H_1) \to F_x(H_2)$.*

Proposition 3.3. *The functors F_x commute with the products of marked groups, i.e.,*

$$F_x\left(\bigotimes H_j\right) = \bigotimes F_x(H_j).$$

Proof. This is immediate consequence of the functoriality of F_i and the universal property of the products of marked groups. Equivalently one can check directly from the definitions. □

Definition 3.4. One can define the functor F_ω for any finite word $\omega \in \{0, 1, 2\}^*$ as follows

$$F_{x_1 x_2 \dots x_i}(H) := F_{x_1}(F_{x_2}(\dots F_{x_i}(H) \dots))$$

If ω is an infinite word on the letters $\{0, 1, 2\}$ by F^i_ω we will denote the functor F_{ω_i} where ω_i is the prefix of ω of length i.

In [Gri85], Grigorchuk defined a group $\mathbf{G}_\omega$ for any infinite word ω. One way to characterize these groups is by $\mathbf{G}_{x\omega} = F_x(\mathbf{G}_\omega)$, where x is any letter in $\{0, 1, 2\}$ – this together with the contracting property of the functors F_x give that $\mathbf{G}_{x\omega} = \lim_i F_{\omega_i}(H_i)$ where the limit is taken in the Chabauty topology and H_i are arbitrary quotients of $\mathcal{G}$, satisfying mild restrictions, see Corollary 3.7. The *first Grigorchuk group* is denoted $\mathbf{G} = \mathbf{G}_{(012)^\infty}$, which corresponds to a periodic infinite word, see e.g. [Gri85, Gri05].

3.3. **Contraction in Grigorchuk groups.** Let $\mathbf{G}_{\omega,i} = F^i_\omega(\mathbf{1})$, where $\mathbf{1}$ denotes the trivial group with one element (with the trivial map $\mathcal{G} \twoheadrightarrow \mathbf{1}$).

Proposition 3.5 ([KP13, Prop. 5.9])**.** *There is a canonical epimorphism $\mathbf{G}_\omega \twoheadrightarrow \mathbf{G}_{\omega,i}$. For every i, the groups $\mathbf{G}_{\omega,i}$ are finite and naturally act on finite binary rooted tree of depth i and this action is transitive on the leaves. These actions come from the standard action of the Grigorchuk group $\mathbf{G}_\omega$ on the infinite rooted binary tree T_2.*

Here the group $F^i_\omega(H)$ is a subgroup of the permutational wreath product $H \wr_{X_i} \mathbf{G}_{\omega,i}$, where X_i is the set of leaves of the binary tree of depth i.

Lemma 3.6 ([KP13, Lemma 5.11])**.** *Let $\pi : \mathcal{G} \twoheadrightarrow H$ be an epimorphism, i.e., group H is generated by* 4 nontrivial *involutions which satisfy $bcd = 1$. If the word $\omega \in \{0, 1, 2\}^*$ does not stabilize, then for any $m > 1$ the balls of radius $\le 2^m - 1$ in the groups $F^m_\omega(H)$ and $\mathbf{G}_\omega$ coincide.*

We conclude with an immediate corollary of the Proposition 3.5 and Lemma 3.6, which can also be found in [Gri11].

Corollary 3.7. *Let* $\{\mathcal{G} \twoheadrightarrow H_i\}$ *be any sequence of groups generated by* $k = 4$ *nontrivial involutions. Then the sequence of marked groups converges:* $\lim_{i\to\infty} F^i_\omega(H_i) = \mathbf{G}_\omega$*, provided that the infinite word* $\omega \in \{0,1,2\}^*$ *does not stabilize.*

4. Main construction

4.1. **A nonamenable group.** Our main construction uses that the free Grigorchuk group $\mathcal{G}$ is close to a free group and thus has many nonamenable quotients which are very different from the groups $\mathbf{G}_\omega$. One such quotient is generated by following matrices in $\mathrm{PSL}\big(2,\mathbb{Z}[\boldsymbol{i},1/2]\big)$, where $\boldsymbol{i}^2=-1$,

$$a = \begin{pmatrix} \boldsymbol{i} & \boldsymbol{i}/4 \\ 0 & -\boldsymbol{i} \end{pmatrix}, \quad b = \begin{pmatrix} 0 & \boldsymbol{i} \\ \boldsymbol{i} & 0 \end{pmatrix}, \quad c = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad d = \begin{pmatrix} \boldsymbol{i} & 0 \\ 0 & -\boldsymbol{i} \end{pmatrix}.$$

A direct computation shows that $a^2 = b^2 = c^2 = d^2 = bcd = 1$, i.e., there is a (non-surjective) homomorphism $\iota : \mathcal{G} \to \mathrm{PSL}\big(2,\mathbb{Z}[\boldsymbol{i},1/2]\big)$, see [KP13, §6].

Let $\mathcal{H} := \langle a,b,c,d\rangle$ denote the marked group generated by the above matrices; as always we consider it as marked group. The group $\mathcal{G}$ contains a normal subgroup of index 2 generated by $\{c, ad\}$,[4] which yields a subgroup $N := \langle c, ad\rangle$ of index 2 of $\mathcal{H}$. The generators of this subgroup are

$$c = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad ad = \begin{pmatrix} 1 & -1/4 \\ 0 & 1 \end{pmatrix}.$$

It can be verified that $\langle c, ad\rangle = \mathrm{PSL}\big(2,\mathbb{Z}[1/2]\big)$, and thus we have $N = \mathrm{PSL}\big(2,\mathbb{Z}[1/2]\big)$, cf. [KP13, §6]. Another computation shows that

$$(ad)^4 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad h := [c,[d,[b,(ad)^4]]] = \begin{pmatrix} -1 & 2 \\ 2 & -5 \end{pmatrix},$$

see [KP13, §6].

Lemma 4.1. *The element* h *normally generates the group* N*.*

Proof. Let K be the normal subgroup of N generated by h. Since N can be viewed as an irreducible lattice in $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{Q}_2)$, by Margulis's normal subgroup theorem, we have that subgroup K is either central or of finite index, see e.g. [Mar91, Ch. IV]. Moreover, it is known that N has congruence subgroup property, see e.g. [PRR23], and it is easy to see that h is not contained in any proper congruence subgroup. □

4.2. **Technical lemma.** Let r denote the element $[c,[d,[b,(ad)^4]]] \in \mathcal{G}$, so we have $\iota(r) = h$. Let $r_x = \varphi^x(r)$ denote its twists by powers of the automorphism φ described in Definition 3.1. The following lemma is a variation on our [KP13, Lemma 5.16], adjusted to this setting.

Lemma 4.2. *Let* $\iota : \mathcal{G} \twoheadrightarrow \mathcal{H}$*, and let* $N \lhd \mathcal{H}$ *be a normal subgroup of index* 2 *described in the previous subsection. Then the kernel of the map* $F^k_\omega(\mathcal{H}) \twoheadrightarrow \mathbf{G}_{\omega,k}$ *induced by* F^k_ω *from the trivial homomorphism* $\mathcal{H} \twoheadrightarrow \mathbf{1}$*, contains* $N^{\oplus 2^k}$*. Moreover, there exists a word* $\eta_{\omega,k} \in \mathcal{G}$ *of length less than* $C \cdot 2^k$*, for some universal constant* C*, such that the image of* $\eta_{\omega,k}$ *in* $F^k_\omega(\mathcal{H})$ *normally generates* $N^{\oplus 2^k}$ *and* $\eta_{\omega,k}$ *is trivial in* $F^{k+1}_\omega(H)$*, for every* $\mathcal{G} \twoheadrightarrow H$*.*

[4] The group $\mathcal{G}$ is 4 generated and the standard Schreier algorithm gives that any subgroup of index 2 is generated by 7 elements - in the case of the kernel of $\mathcal{G} \to \mathbb{Z}_2$ which sends a, b, d to notrivial element, several of these generators are trivial, and others are redundant.

Proof. Consider the substitutions σ, τ (endomorphisms $\mathcal{G} \to \mathcal{G}$), defined as follows:

- $\sigma(a) = aca$ and $\sigma(s) = s$, for $s \in \{b, c, d\}$,
- $\tau(a) = c$, $\tau(b) = \tau(c) = a$ and $\tau(d) = 1$.

It is easy to see that for any $\eta \in \mathcal{G}$, the evaluation of $\sigma(\eta)$ in $F(\mathcal{G})$ is equal to

$$\big(1; \tau(\eta), \eta\big) \;\in\; \{1\} \times \mathcal{G} \times \mathcal{G} \;\subset\; \mathcal{G} \wr \mathbb{Z}_2 .$$

Define $w_i \in \mathcal{G}$ for $i = 0, \ldots, k$ as follows: $w_0 = r_{x_{k+1}}$ and $w_{i+1} = \sigma_{x_{k-i}}(w_i)$ where $\sigma_{x_i} = \varphi^{x_i} \sigma \varphi^{-x_i}$ the twist of the substitution σ. Notice that all these words have the form $[c, [d, [b, *]]]$ because σ_{x_i} fixes b, c and d. Therefore $\tau_x(w_i) = 1$.

By construction the word $\eta_{\omega,k} = w_k$ evaluates in $F^k_\omega(\mathcal{G})$ to $r_{x_{k+1}}$ in one of the copies of $\mathcal{G}$, viewed as a subgroup of $F^k_\omega(\mathcal{G}) \subset \mathcal{G} \wr_{X_k} \mathbf{G}_{\omega,k}$. Therefore the evaluation of w_k in $F^k_\omega(\mathcal{H})$ is in the kernel of $F^k_\omega(\mathcal{H}) \to \mathbf{G}_{\omega,k}$ is the element h in one of the copies of $\mathcal{H}$. This together with the transitivity of the action of $\mathbf{G}_{\omega,k}$ on X_k shows that the kernel contains $N^{\oplus 2^k}$. Finally, the word η is trivial in $F^k_\omega(\mathcal{G})$ since $(ad)^4 = 1$ in $F_x(\mathcal{G})$ for any x. □

4.3. **Main construction.** Let $\mathcal{H}$ be the marked group described in §4.1. Denote G_i the marked group $F^i_{(012)^\infty}(\mathcal{H})$, which surjects onto $F^i_{(012)^\infty}(\mathbf{1}) = \mathbf{G}_{(012)^\infty,i}$. Using Corollary 3.7 we can see that G_i converge in the Chabauty topology to $\mathbf{G}_{(012)^\infty}$.

Definition 4.3. Let $J \subseteq 2^{\mathbb{N}}$ be a fixed subset of $\mathbb{N}$. Denote

$$\widetilde{G}_J \,:=\, \bigotimes_{i \in J} G_i \quad \text{and} \quad G_J \,:=\, \widetilde{G}_J \otimes \mathbf{G}_{(012)^\infty} .$$

By construction these are 4-marked groups that are quotients of $\mathcal{G}$, and such that $G_\varnothing = \mathbf{G}_\omega$.

Using the definition on G_J one can see that the group G_J can be defined as $\bigotimes_{i \in \mathbb{N}} \Gamma_{i,J}$ where

$$\Gamma_{i,J} = \begin{cases} G_i = F^i_{(012)^\infty}(\mathcal{H}) & \text{if } i \in J, \\ \mathbf{G}_{(012)^\infty,i} = F^i_{(012)^\infty}(\mathbf{1}) & \text{if } i \notin J. \end{cases}$$

Lemma 4.4. *The map $I \to G_I$ defines a continuous map from $2^{\mathbb{N}}$ to the space of marked groups.*

Proof. This follows from the convergence $G_i \to \mathbf{G}_{(012)^\infty}$ in Chabauty topology and the observation that the ball of radius n in $\bigotimes \Gamma_i$ depend only on the balls of radius n in Γ_i, for every i. More precisely, for all n, there exists N such that the ball of radius n in marked groups G_k, $\mathbf{G}_{(012)^\infty,k}$ and $\mathbf{G}_{(012)^\infty}$ coincide for all $\kappa \geq N$. This implies that the ball of radius n in G_J coincides with the one in $\bigotimes_{i \in \mathbb{N}} \widetilde{\Gamma}_{i,J}$, where

$$\widetilde{\Gamma}_{i,J} = \begin{cases} G_i = F^i_{(012)^\infty}(\mathcal{H}) & \text{if } i \in J, i \leq N \\ \mathbf{G}_{(012)^\infty,i} = F^i_{(012)^\infty}(\mathbf{1}) & \text{if } i \notin J, i \leq N \\ \mathbf{G}_{(012)^\infty} & \text{if } i > N. \end{cases}$$

Therefore, the ball of radius n in G_J depends only on $J \cap \{1, \ldots, N\}$, which implies that the function $J \to G_J$ is continuous. □

For any set $J \subset J'$ there is a surjection $G_{J'} \twoheadrightarrow G_J$. The main step in proving Theorem 1.11 is to show that if J is a proper subset of J' then this map has a large kernel.

Lemma 4.5. *The kernel of the map $G_J \to \mathbf{G}_{(012)^\infty}$ contains*

$$\bigoplus_{i \in J} N_i^{2^i} \subset \bigoplus_{i \in J} H_i^{2^i} \subset \bigoplus_{i \in I} G_J .$$

Proof. We will use induction on k to show that

$$\sum_{j\in J, j\leq k} N^{2^j} \subset G_J \cap \bigoplus_{i\leq k} \Gamma_{i,J}$$

The base case of the induction $k=0$ is trivial. Using Lemma 4.2 we can see that element $\eta_{\omega,k}$ is trivial in G_i for $i>k$, therefore it corresponds to an element in $G_J \cap \bigoplus_{i\leq k}\Gamma_{i,J}$. For $k\in J$, the projection of this element in $G_k=\Gamma_{k,J}$ is a normal generators of N^{2^k}, and for $i\leq k$ and $i\in J$ the projection in $G_i=\Gamma_{i,J}$ is some element in N^{2^i}. This observation imples that the normal subgroup generated by $\eta_{\omega,k}$ containes $\sum_{j\in J,j\leq k} N^{2^j}$. □

Proof of Theorem 1.11. The continuity is equivalent to Lemma 4.4. The existence of the surjection $G_{J'} \twoheadrightarrow G_J$ for $J\subset J'$ follows from the construction of the groups G_J. The strict monotonicity follows from Lemma 4.5 which gives that the kernel of $G_{J'}\twoheadrightarrow G_J$ contains the non amenable group N^{2^i} for any $i\in J'\setminus J$. □

4.4. **Proof of Lemma 1.10.** The first step is to show that for any decreasing parameter f, the mapping $J\to f(G_J)$ is right continuous in the restriction to the totally ordered subsets in $2^{\mathbb{N}}$.

Lemma 4.6. *Suppose that I_n is decreasing sequence of subsets of $\mathbb{N}$ which converges to $I=\bigcap I_n$ in the Tychonoff topology. In other words, suppose for every $n\in\mathbb{N}$, there exists $k=k(n)$ such that $I_m\cap[n]=I\cap[n]$ for all $m>k(n)$. Then for any decreasing parameter f we have*

$$f(G_I) = \lim_{n\to\infty} f\big(G_{I_n}\big).$$

To state it differently, the assumption of the lemma is equivalent to saying that the sequence of subsets $\{I_n\}$ satisfies *pointwise convergence*. This implies that a decreasing parameter f also converges.

Proof. The sequence $f\big(G_{I_n}\big)$ is monotone since the parameter f is monotone and G_{I_n} form a "decreasing" sequence of groups by Lemma 1.10, therefore the limit on the right side exists.

The mapping $I\to G_I$ defines a function from $2^{\mathbb{N}}$ to the space of marked groups, which is continuous with respect to the Tychnoff topology on $2^{\mathbb{N}}$ and the Chabauty topology on the space of marked groups. Therefore, for any l the balls in the Cayley graphs of the groups G_{I_n} of radius l converge to the ball of radius l in G_I, which implies that $f(G_I)\leq \lim_{n\to\infty} f(G_{I_n})$ by property ($**$).

On other hand, we have that G_I is a quotient of G_{I_n} for each n. Therefore, $f(G_I)\geq f(G_{I_n})$ by property ($*$). Passing to the limit gives $f(G_I)\geq \lim_{n\to\infty} f(G_{I_n})$. This completes the proof. □

One way to prove that the set of all possible values of f is large is to show that $f(G_I)$ is a strictly decreasing function with respect to the lex order of $2^{\mathbb{N}}$. Unfortunately this is not the case in general, however this becomes true if we restrict to sufficiently sparse subsets of $\mathbb{N}$:

Lemma 4.7. *Let f be a strictly decreasing parameter, then there exists a function $\mu_f:\mathbb{N}\to\mathbb{N}$ such that the following holds: for every infinite set $M=\{m_1,m_2,\dots,m_n,\dots\}$ with $m_{n+1}>\mu_f(m_n)$, the function $J\to f(G_J)$ is a strictly decreasing function from $2^M\to\mathbb{R}$ with respect to the lex total order on 2^M.*

Proof. Let n be an integer and let K be any subset of $[n-1]:=\{1,\dots,n-1\}$. Denote $K'=K\cup\{n\}$. Since $K\subsetneq K'$, by Theorem 1.11 the kernel of $G_{K'}\twoheadrightarrow G_K$ is nonamenable and the strict monotonicity of f implies that $f(G_K)>f(G_{K'})$. Let K_m denote the set $K_m=K\cup\{m,m+1,\dots,\}$ for $m>n$. Clearly we have that $\bigcap_m K_m=K$ and the sets K_m converge to K from above. The semi-continuity of f (Lemma 4.6) implies that

$$f(G_K) = \lim_{m\to\infty} f(G_{K_m}).$$

Therefore, there exist $\Lambda_f(n,K) \in \mathbb{N}$ such that $f(G_{K'}) < f(G_{K_m})$ for all $m > \Lambda_f(n,K)$. Denote $\mu_f(n) := \max_K\{\Lambda_f(n,K)\} \in \mathbb{N}$, where the maximum is taken over all subsets of $[n-1]$.

Let $M = \{m_1, m_2, \ldots, m_n, \ldots\} \subset \mathbb{N}$ be an infinite set of integers, such that $m_{n+1} > \mu_f(m_n)$ for all n. Let J', J'' be subsets of J such that $J' < J''$ in the lex order on 2^M. By the definition of the lex order there exists k such that $J' \cap [m_k - 1] = J'' \cap [m_k - 1]$ and $m_k \in J''$ but $m_k \notin J'$. Using the notation from the previous paragraph with $n = m_k$ and $K = J' \cap [m_k - 1]$, we have $K' \subset J''$ and $J' \subset K_{j_{n+1}}$. By the choice of the function μ and the sequence m_k, we have that

$$f(G_{J'}) \geq f(G_{K_{\Lambda_f(n,K)}}) > f(G_{K'}) \geq f(G_{J''}),$$

where the first and the last inequality follow from the inclusions $J' \subset K_{\Lambda_f(n,K)}$ and $K' \subset J''$, and the the second inequality follows from the definition of the constant $\Lambda_f(n,K)$. This implies that f is a strictly decreasing function on 2^M, as desired. □

Proof of Lemma 1.10. Let M be a sparse set satisfying the conditions in the previous lemma, so the set 2^M has cardinality of a continuum. By Lemma 4.7, the function $J \to f(G_J)$ is strictly decreasing function with respect to the total lex order on 2^M. Therefore the set of all possible values of f on the groups G_J has cardinality of the continuum. □

5. Isolated points

We now return to the set of spectral radii discussed in Theorem 1.2. The same approach applies for the asymptotic entropy.

Lemma 5.1. *Let S be a set with an involution $\theta : S \to S$, such that $|S| \geq 3$, and let (Γ, S) be a k-generated marked group which is not free as a marked group. Then there exists a sequence H_i of nonamenable k-generated marked groups such that H_i converge to an amenable marked group H of subexponential growth, and for any i the projection*

$$\Gamma \otimes H_i \twoheadrightarrow \Gamma$$

has nonamenable kernel.

Remark 5.2. The condition that Γ is not free is necessary since if $\Gamma = \mathbb{F}_l$, then for every l-generated group H we have that the map $\mathbb{F}_l \otimes H \twoheadrightarrow \mathbb{F}_l$ is an isomorphism.

Remark 5.3. For $|S| = 4$ and θ fixes all elements in S, if we drop the last condition in the lemma, one can simply take the group $G_i = F^i_{(012)^\infty}(\mathcal{H})$ constructed in Section 4. However, since all these groups are quotients of $\mathcal{G}$, we note that the last condition fails for $\Gamma = \mathcal{G}$.

As an immediate corollary of this lemma we obtain the following result.

Theorem 5.4. *Let f be a sharply decreasing parameter. Then, for any marked group (Γ, S) which is not free, there exists a sequence of marked groups $\{(\Gamma_i, S_i)\}$, such that $f(\Gamma_i, S_i) < f(\Gamma, S)$ and $\lim_{i\to\infty} f(\Gamma_i, S_i) = f(\Gamma, S)$. Therefore, the set $X_{f,S}$ of values of f on all marked groups with a generating set S has no isolated points, except possibly at $f(\mathbb{F}_S)$, where $\mathbb{F}_S$ denotes the free group on the generating set S.*

Proof. This is an immediate consequence of Lemma 5.1, with $\Gamma_i = \Gamma \otimes H_i$ – the inequality $f(\Gamma_i, S_i) < f(\Gamma, S)$ follows from the strict property of f and the fact that the kernel of $\Gamma \otimes H_i \twoheadrightarrow \Gamma$ is nonamenable. The convergence $H_i \to H$ in the Chabauty topology implies that $\Gamma \otimes H_i$ converge to $\to \Gamma \otimes H$ in the same topology. Therefore, $\limsup_{i\to\infty} f(\Gamma \otimes H_i) \geq f(\Gamma \otimes H)$. However the sharpness of f implies that $f(\Gamma \otimes H) = f(\Gamma)$ since the kernel of $\Gamma \otimes H \twoheadrightarrow H$ is a subgroup of H, and therefore it is amenable. □

Proof of Lemma 5.1. Let $\mathbb{F}_S$ denote the free group on the marked generating set S, as mentioned in Section 2 this group is a free product of several copies of $\mathbb{Z}$ and $\mathbb{Z}_2$. The condition $|S| \geq 3$ implies the group $\mathbb{F}_S$ is nonamenable.

Claim 5.5. *For any nontrivial normal subgroup $\Delta \lhd \mathbb{F}_S$ of the free marked group $\mathbb{F}_S$, where $|S| > 1$, there exists a finite marked group (K, S) such that the kernel $R = \ker(\mathbb{F}_S \twoheadrightarrow K)$ admits a surjective homomorphism $\psi : R \twoheadrightarrow \mathcal{G}$ for which $\psi(\Delta \cap R)$ contains the generator a of $\mathcal{G}$.*[5]

Proof of Claim 5.5. Since Δ is nontrivial there exists a nontrivial $r \in \Delta \cap [\mathbb{F}_S, \mathbb{F}_S]$. Consider the 2-Frattini series for $\mathbb{F}_S$ defined by $T_0 = \mathbb{F}_S$ and $T_i = T_i^2[T_i, T_i]$. Each T_i is a finite index normal subgroup in $\mathbb{F}_S$, moreover for $i \geq 1$ the group T_i is free on more than 3 generators, since $|S| \geq 3$.[6] The free marked group $\mathbb{F}_S$ is a residually 2-group, which implies that $\bigcap T_i = 1$. Therefore, there exists an index i such that $r \in T_i \setminus T_{i+1}$ and by the choice of r we have that $i \geq 1$. We can take $K = \mathbb{F}_S/T_i$, the kernel $R = \ker(\mathbb{F}_S \twoheadrightarrow K)$ is T_i therefore R is a free group of rank at least 3. Since r is outside the 2-Frattini subgroup of R, there exists a homomorphism $R \twoheadrightarrow \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ which maps r to one of the generators. Finally ψ is the composition of that homomorphism and the projection $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 \twoheadrightarrow \mathcal{G}$. □

Let ψ be the homomorphism from Claim 5.5 applied to $\Delta = \ker(\mathbb{F}_S \twoheadrightarrow \Gamma)$. Denote by ψ_i the composition of ψ and the projection $\mathcal{G} \twoheadrightarrow F^i_{(012)^\infty}(\mathcal{H})$ and ψ_∞ the composition of ψ and the projection $\mathcal{G} \twoheadrightarrow \mathbf{G}_{(012)^\infty}$, where $\mathcal{H}$ in the group constructed in subsection 4.1

Let L_i be in intersection of all conjugates of $\ker \psi_i$ in $\mathbb{F}_l$ (there are exactly $|K|$ of them since $\ker \psi_i$ is a normal subgroup of R), similarly define L_∞. By construction we have that $H_i = \mathbb{F}_l / L_i$ fits into exact sequence

$$1 \to M_i \to H_i \to K \to 1,$$

where M_i is naturally a subgroup of $F^i_{(012)^\infty}(\mathcal{H})^{\oplus |K|}$, and a similar statement holds for H_∞.

By construction there is a relation in Γ which maps to a generator of $\mathcal{G}$, therefore there exists a relation in Γ which maps to the word $\eta_{(012)^\infty, i}$ constructed in the proof of Lemma 4.2. This implies that the kernel of the map

$$\Gamma \otimes H_i \twoheadrightarrow \Gamma$$

contains N^{2^i} and is nonamenable.

The convergence $\mathcal{G} \twoheadrightarrow F^i_{(012)^\infty}(\mathcal{H}) \to \mathbf{G}_{(012)^\infty}$ in the Chabauty topology implies that H_i converge to H_∞ which is an amenable group of subexponential growth. This completes the proof of Lemma 5.1. □

Proof of Theorem 1.2. For k even and $\rho > \alpha_k$, the theorem follows immediately from Theorem 5.4 since the spectral radius is sharply decreasing by Lemma 1.4. The case for k odd follows from a variant of Lemma 5.1 for groups where one of the generators has order 2.

For $k = 2l \geq 4$, it is known that α_k is not an isolated point because

$$\alpha_k = \lim_{n\to\infty} \rho\big(G_{l,n}, S\big),$$

where $G_{l,n}$ is the free product of l copies of $\mathbb{Z}_n$ with a natural generating set S, see e.g. [Woe00, Ex. 9.25(2)]. The argument in the reference above can be extended to show that for all $k = 2l + 1 \geq 3$, we have:

$$\alpha_k = \lim_{n\to\infty} \rho\big(Y_{l,n}, S\big), \quad \text{where} \quad Y_{l,n} := \underbrace{\mathbb{Z}_2 * \cdots * \mathbb{Z}_2}_{2l-1} * \mathbb{Z}_n.$$

This shows that α_k are not isolated points, for all $k \geq 3$. This completes the proof. □

[5] Here R is *not* considered a marked group, and ψ is *not* a homomorphism between marked groups.

[6] If $\mathbb{F}_S$ is the free group $\mathbb{F}_l$, it is well know that each T_i is a free group on $(l-1)[\mathbb{F}_l : T_i] + 1$ generators. Same result holds in general with l replaced by $|S|/2$.

6. Other monotone parameters

6.1. **Cheeger constant.** Let $|S| = k$ and define the *Cheeger constant* (also called the *isoperimetric constant*):

$$\phi(G,S) \;:=\; \inf_{X\subset G} \frac{|E(X, G \smallsetminus X)|}{k|X|}\,,$$

where the infimum is over all nonempty finite X, and $E(X,Y)$ is the set of edges (x,y) in $\mathrm{Cay}(G,S)$ such that $x\in X$, $y \in Y$. Note that $0 \le \phi(G,S) \le 1$.

The following celebrated inequality relates the spectral radius and the Cheeger constant:

$$\tfrac{1}{2}\,\phi(G,S)^2 \;\le\; 1 \,-\, \rho(G,S) \;\le\; \phi(G,S).$$

This inequality was discovered independently by a number of authors in different contexts, including Kesten (1959), Cheeger (1970), Dodziuk (1984), Alon and Milman (1985), and Mohar (1988); see e.g. [Pete24, §7.2] for the exact references and discussion. In particular, we have $\phi(G,S)>0$ if and only if G is nonamenable.

Similar to other probabilistic parameters, computing the Cheeger constant is very difficult and the exact values are known only in a few special cases. For example, $\phi(\mathbb{F}_k, S) = \frac{k-1}{k}$ for a free group with standard generators (see e.g. [LP16, §6.1]). Additionally, the Cheeger constant is computed for nonamenable hyperbolic tessellations where it is always algebraic [HJL02] (see also [LP16, §5.3]).

Proposition 6.1. *The Cheeger constant* $\phi : \{(G,S)\} \to \mathbb{R}_{\ge 0}$ *is an increasing parameter.*

The proof of this result is straightforward. By analogy with Lemma 1.4, one can ask if ϕ is strictly increasing? Perhaps even sharply increasing? Unfortunately, this remains open:

Conjecture 6.2. *Let* $G = \langle S\rangle$*, let* N *be a finitely generated normal subgroup of* G*, and let* S' *be the projection of* S *onto* G/N*. Then* $\phi(G,S) < \phi(G/N, S')$ *if and only if* N *is nonamenable.*

By analogy with the proof of our Theorem 1.1, the *if* part of the conjecture would imply:

Proposition 6.3. *Conjecture 6.2 implies that there is a family of Cayley graphs* $\big\{(G_\omega, S_\omega)\big\}$*, such that the set of values* $\big\{\phi(G_\omega,S_\omega)\big\}$ *has cardinality of the continuum.*

The conclusion of the proposition remains an open problem in the area. The analogue of Theorem 1.2 for $X_{\phi,k}$ would follow from the conjecture. We omit the details.

6.2. **Speed of simple random walks.** As in the introduction, let $\{x_n\}$ denote simple random walk on $\Gamma = \mathrm{Cay}(G,S)$ starting at $x_0 = e$. For all $z\in G$, denote by $|z|$ the distance from e to z in Γ. The *speed* (also called *drift* and *rate of escape*) of $\{x_n\}$ is defined as follows:

$$\sigma(G,S) \;:=\; \lim_{n\to\infty} \frac{\mathbb{E}[|x_n|]}{n}\,,$$

see e.g. [Pete24, §9.1] and [Woe00, §II.8]. Note that the limit is well defined and the same for almost all sample paths. Famously, it follows from [Var85], that $\sigma(G,S)>0$ if and only if there are no nonconstant bounded harmonic function on Γ. Combined with the same characterization of the asymptotic entropy [KV83], this implies that $\sigma(G,S)>0$ if and only if $h(G,S)>0$ (see also [KL07]). We also have:

Proposition 6.4. *Speed* $\sigma : \{(G,S)\} \to \mathbb{R}_{\ge 0}$ *is an increasing parameter.*

The proof of this result is straightforward. Unfortunately, the following natural analogue of Lemma 1.8 remains open:

Conjecture 6.5. *Let* $G = \langle S\rangle$ *and let* N *be a finitely generated normal subgroup of* G*. Suppose* N *is nonamenable, and let* S' *be the projection of* S *onto* G/N*. Then* $\sigma(G,S) < \sigma(G/N,S')$*.*

By analogy with the proof of our Theorem 1.1 and Proposition 6.3 above, we have:

Proposition 6.6. *Conjecture 6.5 implies that there is a family of Cayley graphs* $\{(G_\omega, S_\omega)\}$, *such that the set of values* $\{\sigma(G_\omega, S_\omega)\}$ *has cardinality of the continuum.*

6.3. **Rate of exponential growth.** Let $b(n) := |\{z \in G : |z| \leq n\}|$ denote the number of elements at distance at most n, and define the *rate of exponential growth* as follows:

$$\gamma(G,S) \,:=\, \frac{1}{|S|-1} \lim_{n\to\infty} \frac{\log b(n)}{n}\,.$$

Clearly, $\gamma(G,S) \in [0,1)$, and $\gamma(G,S) > 0$ is independent of the generating set S. See e.g. [Gri05, GP08, dlH00] for accessible introductions.

Proposition 6.7. *Rate of exponential growth* $\gamma : \{(G,S)\} \to \mathbb{R}_{\geq 0}$ *is an increasing parameter.*

The proof of this result is straightforward. It is easy to see that the rate of exponential growth is not strictly increasing, see e.g. [GH97, §2] with $G = F_k \times F_k$. In fact, the notion of strictly increasing parameters for the rate of exponential growth is closely related to the notion of *growth tightness* introduced in [GH97, §2] and established in [AL02] for word hyperbolic groups. In [Ers04], Erschler was able to modify the notion of "strict monotonicity" to obtain the following natural analogue of Theorem 1.1 for the rate of exponential growth:

Theorem 6.8 ([Ers04])**.** *The set of rates of exponential growth* $\{\gamma(G,S)\}$ *has cardinality of the continuum.*

It is worth comparing this question with Grigorchuk's celebrated result in [Gri85], that there is an uncountable family of marked groups with incomparable growth functions. The proof of Theorem 6.8 is also somewhat related to Bowditch's elementary construction in [Bow98], of an uncountable family of marked groups with pairwise non-quasi-isometric Cayley graphs. See also the most recent result by Louvaris, Wise and Yehuda [LWY24], which proves that the set of (unscaled) growth rates of subgroups of the free group $\mathbb{F}_k$ is dense on $[1, 2k-1]$.

6.4. **Connective constant.** Let $c(n)$ denote the number of *self-avoiding walks* of length n in the Cayley graph $\mathrm{Cay}(G,S)$, and define the *connective constant* as follows:

$$\mu(G,S) \,:=\, \lim_{n\to\infty} \sqrt[n]{\upsilon(n)}\,.$$

Here the limit exists by submultiplicativity: $\upsilon(m+n) \leq \upsilon(m)\upsilon(n)$, see e.g. [MS93, §1.2]. The exact value is known for the hexagonal lattice [DS12] and only few other special cases, see e.g. [GL19, §1.2].

Theorem 6.9 ([GL14, Cor. 4.1])**.** *Connective constant* $\mu : \{(G,S)\} \to \mathbb{R}_{\geq 1}$ *is a strictly increasing parameter. Moreover, the inequality* $(*)$ *is strict for all* $N \neq \mathbf{1}$.

Now Lemma 1.10 gives a new proof of the following known result:

Theorem 6.10 ([Mar17])**.** *There is a family of* 4*-regular Cayley graphs* $\{\mathrm{Cay}(G_\omega, S_\omega)\}$, *such that the set of connective constants* $\{\mu(G_\omega, S_\omega)\}$ *has cardinality of the continuum.*

The original proof by Martineau [Mar17] is similar in nature to our approach but uses weaker tools, namely [dlH00, Lemma III.40] which suffices for the case of connective constants (cf. §7.8).

7. Final remarks and open problems

7.1. **Historical notes.** The results of this paper (for the spectral radius) were announced in 2016.[7] Later on, we extended this approach to other monotone parameters, and created an alternative to the small cancellation theory approach (see below). In the meantime, our spectral radius construction influenced [TZ25] mentioned in the introduction who independently derived the asymptotic entropy part of Theorem 1.1. While this creates a messy timeline, we would like to acknowledge that this part of Theorem 1.1 can be attributed to Tamuz and Zheng at least as much as to this work.

Let us also mention that the idea of taking products of marked groups goes back to B.H. Neumann [Neu37], and was repeatedly used in the last two decades. Notably, Pyber in [Pyb03, Pyb04] used a variation of this idea to construct an uncountable family of groups related to the *Grothendieck problem*, Erschler [Ers06] has a closely related construction based on decorated Grigorchuk groups, and most recently Brieussel and Zheng used them in their breakthrough work [BZ21].

7.2. **Graph theoretic aspects.** In [LM06], Leader and Markström constructed a simple uncountable family of pairwise nonisomorphic 4-regular Cayley graphs. They were clearly unaware of the earlier works by Grigorchuk [Gri85], Bowditch [Bow98] and Erschler [Ers04] which prove stronger results. However, the elementary nature of their construction is of independent interest.

7.3. **Computability aspects.** In [Häg08], Häggström showed that the critical probability $p_c^{\mathrm{s}}(\mathbb{Z}^2)$ is *computable* in the sense of the *Church–Turing thesis*: there exists a Turing machine which computes the digits in the binary expansion. This resolved Toom's question. Häggström then asked if $p_c(G,S)$ is always computable (ibid., p. 323). The negative answer follows immediately from our Theorem 1.1 and the observation that there are countably many Turing machines.

7.4. **Set theoretic aspects.** As in the introduction, let $X_{\rho,k} \subset (0,1]$ denote the set of spectral radii of marked groups with k generators, and let $X_\rho := \bigcup X_{\rho,k}$.

Open Problem 7.1. We have: $X_\rho = (0,1]$.

Between ourselves, we disagree whether one should believe or disbelieve this claim. While our results seem to suggest a positive answer, they give no intuition whether X_ρ is closed or dense.

Now, Main Theorem 1.1 shows that $X_{\rho,8}$ has cardinality of the continuum. As we mention in the introduction, our proof implies a stronger result, that $X_{\rho,8}$ has an embedding of the *Cantor set* (see Theorem 1.11). On the surface, this appears a stronger claim since there is a natural construction of the *Bernstein set* which has cardinality of the continuum and no embedding of the Cantor set, see e.g. [Kec95, Ex. 8.24].

In fact, there are two ways to see that $X_{\rho,k}$ having cardinality of the continuum is equivalent to having an embedding of the *Cantor set.* First, for $k \geq 3$, we prove in Theorem 1.2 that $X_{\rho,k}$ has no isolated points in the interval $[\alpha_k, 1]$. If $X_{\rho,k}$ is closed, then it is a perfect Polish space that always contains the Cantor set, see e.g. [Kec95, Thm 6.2].

Second, it is easy to see that $X_{\rho,k}$ is a projection of a Borel set, and thus *analytic*, see e.g. [Kec95, §14.A]. It is known that every analytic set that is a subset of a Polish space either is countable, or contains a Cantor set, see e.g. [Kec95, Ex. 14.13].

In summary, we have two different set theoretic arguments which imply that set $X_{\rho,k}$ is either countable or contains a Cantor set, and thus has cardinality of the continuum. The second argument applies to other monotone parameters in Theorem 1.1 as well, explaining the embedding of the Cantor set conclusion in [Bow98, Mar17, TZ25].

[7] Martin Kassabov, *A nice trick involving amenable groups*, MSRI talk (Dec. 9, 2016), video and slides available at www.slmath.org/workshops/770/schedules/21638

7.5. **Explicit constructions.** The proof of Theorem 1.1 is fundamentally set theoretic and does not allow an *explicit construction* of the Cayley graph with a transcendental spectral radius: $\rho(G, S) \notin \overline{\mathbb{Q}}$. Here we are intentionally vague about the notion of "explicit construction", as opposed to the setting in graph theory where it is well defined, see e.g. [HLW06, §2.1] and [Wig19, §9.2]. This leads to a host of interrelated open problems corresponding to different possible meanings in our context.

Question 7.2. *Is there a finitely presented group with a transcendental spectral radius? What about recursively presented groups? Similarly, what about graph automata groups?*

Despite our efforts, we are unable to resolve either of these questions using our tools. Note that there is a closely related but weaker notion of *D-transcendental cogrowth series*, see e.g. [GH97, GP17]. We refer to [Pak18] for some context about problems of counting walks in graphs and further references.

Remark 7.3. Soon after this paper was written, a positive answer to the first two questions was given by Bodart [Bod24].

7.6. **Asymptotic versions.** In the case of amenable groups, one can ask about the asymptotics of the return probability and the (closely related) *isoperimetric profile.* Similarly, when the Cayley graph has no non-constant bounded harmonic functions, one can ask about the asymptotics of the speed and entropy functions. In these setting, there is a large literature on both the exact and oscillating growth of these functions, too large to be reviewed here. We refer to [BZ21] for the recent breakthrough and many references therein.

For the critical probability, one can ask about asymptotics of the number of cuts, see e.g. [Tim07]. We note that this asymptotic version is always exponential for vertex-transitive graphs, and is largely of interest for families of graphs with nearly linear growth.

7.7. **Critical probability on general graphs.** The problem of describing critical probability constants on general graphs was introduced by van den Berg [vdB82], who showed that every $p \in [0, 1]$ is a critical probability using a probabilistic method. Famously, Grimmett [Gri83] showed that natural subgraphs of $\mathbb{Z}^2$ can be used to obtain every $p \in [\frac{1}{2}, 1]$ as a critical probability of the bond percolation. In a different direction, Ord, Whittington and Wilker [OWW84] construct a countable family of graphs using decorations of $\mathbb{Z}^2$, which has p_c^{b} dense on $(0, 1)$. While neither of these constructions is vertex-transitive, they suggest the following:

Question 7.4. *Let* $Y := \bigcup_k X_{p_c^{\mathrm{b}},k}$ *denote the set of critical probabilities of all Cayley graphs of bounded degrees. Does there exist a constant* $\alpha > 0$ *such that* $Y \cap (0, \alpha)$ *is dense? Does there exist a constant* $\beta > 0$ *such that* $(0, \beta) \subset Y$*?*

7.8. **Monotone properties.** The notions of *monotone properties* are modeled after standard notions of monotone and hereditary properties in probabilistic combinatorics, which describe set systems closed under taking subsets. Typical examples include properties of graphs that are invariant under deletion of edges or vertices, see e.g. [AS16, §6.3, §17.4]. Similarly, *parameters* are standard in graph theory, and describe any of the numerous quantitative graph functions, see e.g. [Bon95, Lov12]. We use *probabilistic parameters* in the introduction to indicate that our parameters have probabilistic nature.

Finally, note that both critical probabilities and connective constants are *completely monotone*, i.e., the inequality ($*$) is strict for *all* nontrivial N (see Lemma 1.6 and Theorem 6.9). For the completely monotone parameters, the proof of Theorem 1.10 substantially simplifies (cf. [Mar17]), while Theorem 5.4 is no longer valid. As we mentioned in the introduction, it is false for critical probabilities.

7.9. **Small cancellation groups.** There is an alternative approach to monotone parameters coming from the small cancellation theory. Notable highlights include Bowditch's work [Bow98] mentioned in §6.3, and Erschler's Theorem 6.8.

After the results of this paper were obtained, Erschler showed us how to prove both Theorems 1.1 and 1.11 using a construction from [Ers04] combined with strictly monotone properties.[8] Furthermore, Osin suggested that this approach could also be used to have groups satisfy additional properties, such as being acylindrically hyperbolic, lacunary hyperbolic, and having Kazhdan's Property (T).[9] It would be interesting to see how much further this construction can be developed. We note, however, that our approach is nearly self-contained (modulo several lemmas proved in [KP13]). Note also that Theorem 1.2 seems not attainable via small cancellation groups.

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