Journal of Combinatorial Theory, Series A **98**, 1–16 (2002)
doi:10.1006/jcta.2001.3208, available online at http://www.idealibrary.com on IDEAL®

Ribbon Tile Invariants from the Signed Area

Cristopher Moore

Department of Computer Science and Department of Physics and Astronomy, University of New Mexico, Albuquerque, New Mexico 87131
E-mail: moore@cs.unm.edu

and

Igor Pak

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
E-mail: pak@math.mit.edu

Received August 29, 2000

Ribbon tiles are polyominoes consisting of n squares laid out in a path, each step of which goes north or east. Tile invariants were first introduced by the second author (2000, *Trans. Amer. Math. Soc.* **352**, 5525–5561), where a full basis of invariants of ribbon tiles was conjectured. Here we present a complete proof of the conjecture, which works by associating ribbon tiles with certain polygons in the complex plane, and deriving invariants from the signed area of these polygons.

Key Words: polyomino tilings; tile invariants; Conway group; height representation.

1. INTRODUCTION

Polyomino tilings have been an object of attention of serious mathematicians as well as amateurs for many decades [G]. Recently, however, the interest in tiling problems has grown as some important ideas and techniques have been introduced. In [P1], the second author introduced a *tile counting group*, which appears to encode a large amount of information concerning the combinatorics of tilings. He made a conjecture on the group structure, and obtained several partial results. A special case of the conjecture was later resolved in [MP]. In this paper we continue this study and complete the proof of the conjecture.

0097-3165/02 $35.00

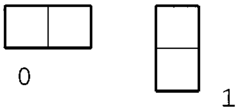

FIG. 1. Two dominoes.

Consider the set of *ribbon tiles* $\mathbf{T}_n$, defined as connected n-square tiles with no two squares in the same diagonal $x+y=c$ (as in Figs. 1–3). It is easy to see that $|\mathbf{T}_n|=2^{n-1}$, as each tile can be associated with a path of length $n-1$ in the square lattice, each step of which goes east or north. Recording these moves by **0** and **1** respectively, we obtain a sequence $\varepsilon=(\varepsilon_1, ..., \varepsilon_{n-1}) \in \{0, 1\}^{n-1}$, which uniquely encodes a ribbon tile. We will refer to this tile as τ_ε.

Now, let Γ be a finite simply connected region, and let v be a tiling of Γ by ribbon tiles in $\mathbf{T}_n$, $n \geqslant 2$. We denote by $a_\varepsilon(v)$ the number of times the ribbon tile τ_ε is used in v.

Conjecture 1.1 [P1]. Let Γ and v be as above. Then for every i, $1 \leqslant i < n/2$, we have

$$\sum_{\varepsilon:\, \varepsilon_i=0,\, \varepsilon_{n-i}=1} a_\varepsilon(v) - \sum_{\varepsilon:\, \varepsilon_i=1,\, \varepsilon_{n-i}=0} a_\varepsilon(v) = c_i(\Gamma),$$

where the $c_i(\Gamma)$ depend only on Γ and are independent of the tiling v of Γ. Furthermore, when n is even, we have

$$\sum_{\varepsilon:\, \varepsilon_{n/2}=1} a_\varepsilon(v) = c_*(\Gamma) \qquad \text{mod } 2,$$

where $c_*(\Gamma)$ is also independent of v.

The main result of the paper is a proof of this conjecture for all $n \geqslant 2$:

THEOREM 1.2. *Conjecture* 1.1 *holds for tilings by ribbon tiles* $\mathbf{T}_n$ *for all* $n \geqslant 2$*, and for all simply connected regions* Γ.

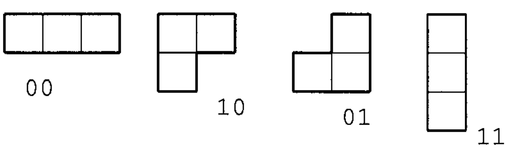

FIG. 2. Four ribbon trominoes.

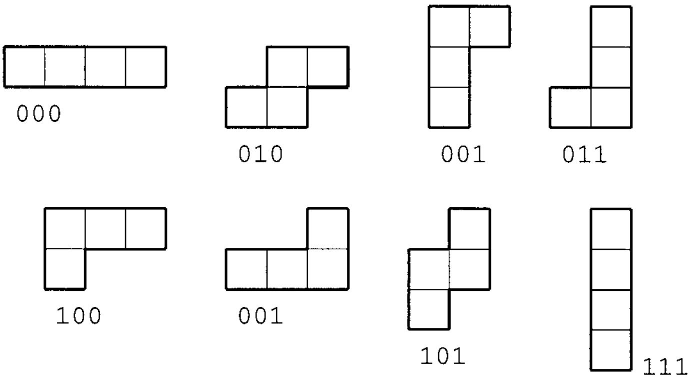

FIG. 3. Eight ribbon tetrominoes.

A few words about the history of this conjecture. For $n=2$, it implies that for every domino tiling of Γ, the parity of the number of vertical dominoes is always the same. This, in fact, holds for every region, not just the simply connected ones, and follows from a folklore coloring argument (see [G, P1] for details).

For $n=3$, the conjecture gives only one relation:

$$a_{01}(\nu)-a_{10}(\nu)=c_1(\Gamma).$$

This is the celebrated Conway–Lagarias relation for trominoes [CL]. Recently, the conjecture was established for $n=4$ [MP], using a combinatorial technique similar to [CL]. In this notation, it was shown in [MP] that

$$\begin{aligned} a_{001}+a_{011}-a_{101}-a_{111} &= c_1(\Gamma), \\ a_{010}+a_{011}+a_{110}+a_{111} &= c_*(\Gamma) \qquad \bmod 2. \end{aligned}$$

It was shown in [CL], in a certain rigorous sense, that even for $n=3$, the conjecture can't be proved by means of coloring arguments. This was extended by the second author to all $n \geqslant 4$ [P1]. It was observed in [P1], that for $n=3$ there exists a non-simply connected region for which the relations in the conjecture do not hold. Thus, there is little hope of generalizing the conjecture to all regions.

The conjecture originated in [P1], where the author considered only row (or column) convex regions Γ, and proved the linear relations in Conjecture 1.1 for all such Γ [P1, Theorem 1.4]. The technique used a connection with combinatorics of Young tableaux which could not be extended to all simply connected regions (see [P1] for details). The author in [P1] also showed that the linear relations in the conjecture are the only relations which can occur between the $a_\varepsilon(v)$, even for this smaller set of regions (see Section 2 below).

About the proof technique: We use notion of tile invariants, introduced in [P1], but here we define new real-valued invariants, which we call *adèle invariants*. As it turns out, these invariants imply all the integer-valued invariants that we need to establish. We then show the validity of the adèle invariants by presenting them as a signed area of a certain polygon corresponding to each tile. These two results together imply Theorem 1.2.

The rest of the paper is structured as follows. In Section 2 we introduce tile invariants and compute the tile counting group based on Theorem 1.2. Much of the material follows [P1], so we present only sketches of the proofs for completeness. In Section 3, we define and study the adèle invariants. Small examples are computed in Section 4. We exhibit the relationship between the adèle invariants and integer invariants in Section 5. This completes the proof of Theorem 1.2. We conclude with final remarks in Section 6.

2. TILE INVARIANTS

Let us start by defining tilings and tile invariants. Let Λ be a set of (closed) squares of a square grid $\mathbb{Z}^2$ on a plane. A *region* is a finite subset $\Gamma \subset \Lambda$. Region $\Gamma \subset \Lambda$ is called simply connected if its boundary $\partial\Gamma$ is connected. We say that two regions Γ and Γ' are *equivalent*, denoted $\Gamma \sim \Gamma'$, if Γ is a parallel translation of Γ' (rotations and reflections are not allowed). Let $\tilde{\Gamma} = \{\Gamma' \colon \Gamma' \sim \Gamma\}$ be the set of regions equivalent to Γ.

Let $\mathbf{T} = \{\tau_1, ..., \tau_r\}$ be a finite set of simply connected regions, which we call *tiles*. By $\tilde{\tau}_i$ we denote the set of their parallel translations, and let $\tilde{\mathbf{T}} = \bigcup_i \tilde{\tau}_i$. A *tiling* v of Γ, denoted $v \vdash \Gamma$, is a set of tiles $\tau \in \tilde{\mathbf{T}}$, such that their disjoint union is Γ:

$$\Gamma = \bigsqcup_{\tau \in v} \tau.$$

Here we ignore the intersection of the boundaries.

Let G be an abelian group, and let $\varphi\colon \mathbf{T} \to G$ be any map. We extend the definition of φ to all $\tau \in \tilde{\mathbf{T}}$, by setting $\varphi(\tau) = \varphi(\tau_i)$ for all $\tau \sim \tau_i$. We say that the map φ is a *tile invariant* of $\mathbf{T}$ if, for every simply connected region Γ and every tiling $v \vdash \Gamma$ by the set of tiles $\mathbf{T}$, we have

$$\sum_{\tau \in v} \varphi(\tau) = c(\Gamma),$$

where the constant on the r.h.s. depends only on the region Γ and is independent of ν. In this paper G is either $\mathbb{Z}$, or $\mathbb{Z}_n(=\mathbb{Z}/n\mathbb{Z})$, or $\mathbb{R}$ (with addition as the group operation).

Tile invariants are directly related to numerical relations between the respective numbers of times differently-shaped tiles occur in a tiling. Indeed, let $a_i(\nu) = |\nu \cap \tilde{\tau}_i|$ be the number of tiles $\tau \sim \tau_i$ in the tiling $\nu \vdash \Gamma$. We immediately have

$$\sum_{i=1}^{r} \varphi(\tau_i)\, a_i(\nu) = \sum_{\tau \in \nu} \varphi(\tau) = c(\Gamma).$$

In [P1], we introduced a *tile counting group* $\mathbb{G}(\mathbf{T})$, which is defined as a quotient,

$$\mathbb{G}(\mathbf{T}) = \mathbb{Z}^r / \langle (a_1(\nu) - a_1(\nu'), \ldots, a_r(\nu) - a_r(\nu')),\ \nu, \nu' \vdash \Gamma \rangle,$$

where ν, ν' are tilings of the same simply connected region Γ by the set of tiles $\mathbf{T}$. Computing the tile counting group $\mathbb{G}(\mathbf{T})$ is a difficult task, even in simple cases. The main result of this paper is a computation of $\mathbb{G}(\mathbf{T}_n)$ for the case of ribbon tiles:

THEOREM 2.1. *If $n = 2m+1$, then $\mathbb{G}(\mathbf{T}_n) \simeq \mathbb{Z}^{m+1}$. If $n = 2m$, then $\mathbb{G}(\mathbf{T}_n) \simeq \mathbb{Z}^m \times \mathbb{Z}_2$.*

Theorem 2.1 was stated as a conjecture in [P1]. It was shown in [P1] that it follows from Theorem 1.2. For completeness, we sketch the proof below.

Sketch of Proof. Indeed, in [P1, Theorem 1.4] it was shown that $\mathbb{G}(\mathbf{T}) \subset \mathbb{Z}^{m+1}$ for $n = 2m+1$ and $\mathbb{G}(\mathbf{T}_n) \subset \mathbb{Z}^m \times \mathbb{Z}_2$ for $n = 2m$. Observe that one can view the relations in Conjecture 1.1 as elements of $\mathbb{G}(\mathbf{T}_n)$. Recall that these relations, together with the trivial *area invariant* f_0 (defined by $f_0(\tau) = 1$ for all $\tau \in \mathbf{T}_n$), are independent in $\mathbb{Z}^n$ (see the proof of Theorem 1.4 in [P1, Sect. 5]). Now Theorem 1.2 implies the result. ▮

Before we conclude this section, let us make a final observation on the relations in Conjecture 1.1 implied by previous work. Following [P1, Sect. 9], define the *shade invariant* as

$$f_{\blacktriangledown}(\tau_\varepsilon) = \sum_{k=1}^{n-1} k \cdot \varepsilon_k \qquad \text{mod } n,$$

where $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_{n-1})$. The fact that it is an invariant follows easily from an extended coloring argument [P1, Sect. 9]. Namely, consider a coloring of the squares $\zeta\colon \mathbb{Z}^2 \to \mathbb{Z}_n$ defined by $\zeta(x, y) = y \bmod n$. Note that the sum of the colors in each ribbon tile τ is equal to $f_{\blacktriangledown}(\tau)+C$, where $C = C(n) \in \mathbb{Z}_n$ is a constant which depends only on n. We omit the (easy) details.[1]

PROPOSITION 2.2. *When n is even, the relations in the first part of Conjecture 1.1 imply that in the second part.*

Proof. We will show that the mod 2 relation follows from the $m = n/2$ relations in the first part, and the shade invariant. In the language of invariants, consider the *k-convexity invariants* f_k, introduced in [P1]:

$$f_k(\tau_\varepsilon) = \varepsilon_k - \varepsilon_{n-k}, \qquad \text{where} \quad \varepsilon = (\varepsilon_1, \ldots, \varepsilon_{n-1}).$$

We need to show that the shade invariant and the k-convexity invariants generate the *parity invariant* f_*:

$$f_*(\tau_\varepsilon) = \varepsilon_m \qquad \text{mod } 2, \qquad \text{where} \quad n = 2m.$$

But this is immediate since

$$f_{\blacktriangledown} \text{ mod } 2 = (f_1 + 2f_2 + \cdots + (m-1)\, f_{m-1}) + f_* \qquad \text{mod } 2$$

(cf. [P1, Sect. 9]). This completes the proof. ∎

3. NEW RIBBON TILE INVARIANTS AND THE SIGNED AREA

Let $\mathbf{T}_n$ be the set of ribbon tiles, defined as above. From now on, we will also use a different encoding of $\mathbf{T}_n$, by sequences $\alpha = (\alpha_1, \ldots, \alpha_n) \in \{\pm 1\}^{n-1}$: $\mathbf{T}_n = \{\tau_\alpha\}$, where $\tau_\alpha = \tau_\varepsilon$, if $\alpha_i = 1 - 2\varepsilon_i$ for all $1 \leqslant i \leqslant n-1$ (i.e. $\mathbf{0} \to +1$ and $\mathbf{1} \to -1$).

For every $1 \leqslant \ell < n$ we define a function $\Phi_\ell \colon \mathbf{T}_n \to \mathbb{R}$ as

$$\Phi_\ell(\tau_\alpha) = \sum_{k=1}^{n-1} \alpha_k \sin \frac{2\pi k\, \ell}{n},$$

where $\alpha = (\alpha_1, \ldots, \alpha_{n-1})$, $\alpha_k \in \{\pm 1\}$ as above. The main result of this section is the following key observation:

THEOREM 3.1. *The function $\Phi_\ell \colon \mathbf{T}_n \to \mathbb{R}$ is a tile invariant for the set $\mathbf{T}_n$ of ribbon tiles, for all $1 \leqslant \ell < n$.*

[1] In contrast with other ribbon tile invariants we introduce, the shade invariant can be extended to *all* regions, not just the simply connected ones [P1, Theorem 9.1].

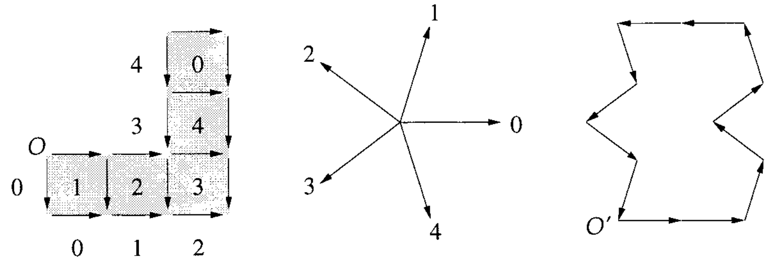

FIG. 4. The ribbon tile $\tau = \tau_{0011}$ with labels on the edges, the roots of unity $v_0, \ldots, v_4$, $v_k = e^{2\pi ik/5}$, and the closed loop $\eta_1(\tau)$.

We will call Φ_ℓ the *ℓ-th adèle invariant*. Note that when $n = 2m$, we have $\Phi_m(\tau_\alpha) = 0$ for all $\tau_\alpha \in \mathbf{T}_n$. The claim of the theorem is trivial in this case.

The proof of Theorem 3.1 is based on a new geometric construction. But first we need several definitions.

Let the squares of the grid have numbers written on them, from 0 to $n-1$, with the rule that $(x, y) \in \mathbb{Z}^2$ has the number $x + y \bmod n$. Let us orient edges of the grid eastward and southward as in Fig. 4. Set labels on the edges so that the edge between square k and $(k+1 \bmod n)$ has label k.

Let $\ell \neq n/2$ be fixed for the rest of this section. On a complex plane $V = \mathbb{C}$, fix n vectors $v_0, v_1, \ldots, v_{n-1}$, where $v_k = e^{2\pi ik\ell/n}$. We say that a loop in V is a *polygon* if it is a closed (perhaps self-intersecting) path with straight edges.

Now, let Γ be a simply connected region on a grid, and let $\partial\Gamma$ be the boundary of Γ. Fix any integer point $O \in \partial\Gamma$. Consider a sequence of edges on the grid obtained by moving along $\partial\Gamma$ counterclockwise, starting at O. Recall that these edges are oriented and labeled with integers modulo n.

We shall describe a map $\eta = \eta_\ell$, which maps simply connected regions Γ, tileable by $\mathbf{T}_n$, into polygons in V. First, fix any $O' \in V$. As one moves along the sequence of edges of $\partial\Gamma$, add a vector $\pm v_j \in V$, where j is a label of the edge in $\partial\Gamma$, and a sign $\pm$ is chosen depending on whether the edge in $\partial\Gamma$ is oriented counterclockwise or clockwise (see figures below). We denote the resulting path by $\eta(\gamma) = \eta_\ell(\Gamma)$. Note that it already has an induced orientation.

In Fig. 4 we present a **V**-pentomino (cf. [G]), which is encoded by $\alpha = (+1, +1, -1, -1)$ in our notation, along with 5 vectors $v_0, \ldots, v_4$, and the corresponding polygon. Note that a priori, it is unclear whether our map is well-defined, i.e., whether all tileable regions correspond to closed loops in V. By definition, $\eta(\Gamma)$ is only a path starting at O', with straight edges.

LEMMA 3.2. *The above map η_ℓ is well-defined, i.e. for any simply connected region Γ tileable by $\mathbf{T}_n$, the path $\eta_\ell(\Gamma)$ is a closed loop in V.*

Proof. We prove the result by induction on the area of Γ. Suppose τ is one of the ribbon tiles and let $(k+1 \bmod n)$ be the label of the square in the lower left corner. Let O be the point in the upper left corner of this square. Now observe that the sequence of edges in $\partial\tau$ has two labels k, then a sequence w of labels, then two labels k, and then the same sequence as w but in the opposite order. Observe also that the first two edges, with the label k, are directed counterclockwise, while the second two are clockwise. This implies that the pieces of $\eta(\tau)$, corresponding to these four edges, form two straight parallel intervals oriented in opposite directions.

Note also, that each edge in the first sequence w has an orientation which is *opposite* to that of a corresponding edge in the second (reversed) w. Therefore, the pieces corresponding to the two w are exactly parallel to each other, with a shift of $2v_k$. We conclude that $\eta(\tau)$ is a closed loop in V, so η is well-defined for ribbon tiles. This proves the base of our induction.

The induction step is straightforward. Let Γ be a region tileable by $\mathbf{T}_n$. Fix any tiling of Γ. Consider a tile τ in the tiling such that $\Gamma' = \Gamma \setminus \tau$ is simply connected. In [MP, Lemma 2.1] we prove that there always exists such a tile.[2] Now present $\partial\Gamma$ as a union of two regions, $\partial\Gamma'$ and $\partial\tau$ (intersections of these will cancel each other as they have opposite orientations). If both $\eta(\Gamma')$ and $\eta(\tau)$ are closed, then $\eta(\Gamma)$ is also closed. This completes the proof. ∎

Let us present now a standard inductive definition of a *signed area* $A(\gamma)$ of an oriented polygon γ in V (see, e.g., [GO]). If γ is not self-intersecting, define $A(\gamma)$ to be the usual area times ± 1 depending on whether γ is oriented counterclockwise or not. If γ is self-intersecting at point x, split γ into the disjoint union of two γ_1 and γ_2 (separated by the point x), and let $A(\gamma) = A(\gamma_1) + A(\gamma_2)$.

Now let Γ be a region tileable by $\mathbf{T}_n$. Let us show that for any ℓ, the signed area of $\gamma = \eta_\ell(\Gamma)$ is invariant under parallel translation of Γ (recall that the construction of η_ℓ involves a fixed labeling of the plane, so a priori it may differ for $\Gamma' \sim \Gamma$). Indeed, observe that for a parallel translation $\Gamma' \sim \Gamma$, we have a cyclic shift of the labels of the edges in $\partial\Gamma'$. Therefore $\eta_\ell(\Gamma')$ is simply a rotation of $\eta_\ell(\Gamma)$ by a multiple of $\frac{2\pi\ell}{n}$. Thus these two loops have the same signed area $A(\eta_\ell(\Gamma')) = A(\eta_\ell(\Gamma))$. Similarly, the choice of the starting point $O \in \partial\Gamma$ (and $O' \in V$) doesn't change the signed area of γ. We shall prove now that there exists a closed formula for $A(\gamma)$ when Γ is a ribbon tile.

[2] Versions of this result were also used in [CL, Pr].

PROPOSITION 3.3. *Let* $\gamma = \eta_\ell(\tau_\alpha)$, *where* $\alpha = (\alpha_1, \ldots, \alpha_{n-1}) \in \{\pm 1\}^{n-1}$. *Then*

$$A(\gamma) = 2 \sum_{k=1}^{n-1} \alpha_k \sin \frac{2\pi k \ell}{n}.$$

Proof. This follows immediately from the analysis used in the induction step in the proof of Lemma 3.2. Indeed, let us translate the tile τ so that the lower left square has label 1. Also, choose point $O \in \partial\tau$ as in the proof above. Recall that the signed area remains unchanged. Observe that the signed area is exactly the area of the parallelogram whose vertices are the endpoints of two horizontal intervals of length 2. Therefore $A(\gamma) = 2 \cdot \mathit{height}$, where *height* is the height of the image of a sequence of labels w, defined as in the proof above. Now, the height of the image of w is the sum of the heights of each of the vectors v_k, taken with a sign α_k, for $k = 1, \ldots, n-1$. This implies the formula in the proposition. ∎

PROPOSITION 3.4. *Let* $\nu \vdash \Gamma$ *be a tiling of* Γ *by ribbon tiles in* $\mathbf{T}_n$. *Then*

$$\sum_{\tau \in \nu} A(\eta_\ell(\tau)) = A(\eta_\ell(\Gamma)), \qquad \textit{for all} \quad 1 \leqslant \ell \leqslant n-1.$$

Proof. This is an immediate corollary of the induction step in the proof of Lemma 3.2. Indeed, let us prove the claim by induction on the area of Γ. The claim is trivial when $\Gamma = \tau \in \mathbf{T}_n$.

Now, by construction, γ is a union of γ_1 and γ_2, where $\gamma = \eta_\ell(\Gamma)$, $\gamma_1 = \eta_\ell(\Gamma')$, and $\gamma_2 = \eta_\ell(\tau)$. By definition, this implies that $A(\gamma) = A(\gamma_1) + A(\gamma_2)$. This completes the inductive step and finishes the proof. ∎

Proof of Theorem 3.1. This is a corollary of Propositions 3.3 and 3.4. Indeed, Proposition 3.4 implies that

$$\Phi_\ell(\tau) = \tfrac{1}{2} A(\eta_\ell(\tau))$$

for every ribbon tile $\tau \in \mathbf{T}_n$, and every $1 \leqslant \ell \leqslant n-1$. Now Proposition 3.4 implies that Φ_ℓ satisfies the definition of a tile invariant. ∎

4. EXAMPLES

Let $n = 3$. In Fig. 5, we show all four ribbon trominoes τ_α, along with the corresponding polygons $\eta_1(\tau_\alpha) \in V$. Let us calculate the values of the adèle invariant Φ_1. Consider the straight trominoes first. Observe that the signed area of the corresponding polygons is zero. Indeed, the two equilateral

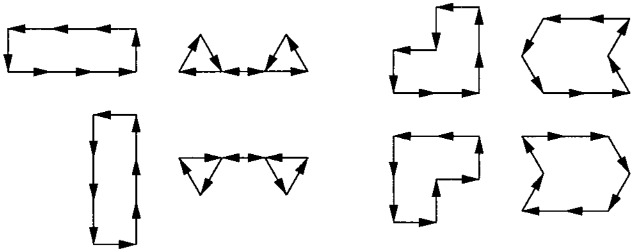

FIG. 5. Four ribbon trominoes τ_α and the corresponding closed loops $\eta_1(\tau_\alpha)$.

triangles cancel each other, since we circle one equilateral triangle clockwise and the other counterclockwise. On the other hand, for the two right trominoes the adèle invariant $\Phi_1 = \pm\sqrt{3}$. Indeed, in both cases these polygons circle eight equilateral triangles, in the first case counterclockwise and in the other clockwise. Thus the signed area is $A = \pm 8\,\frac{\sqrt{3}}{4} = \pm 2\sqrt{3}$, which implies the claim.

Now observe that $\frac{1}{\sqrt{3}}\cdot\Phi_1$ coincides with the Conway–Lagarias invariant (see Section 1). This gives a new interpretation of this remarkable invariant in terms of an "area," rather than the "winding number" as defined in [CL].

Let us note here that for $n = 3, 4$ the group of translations of $V = \mathbb{C}$ by integer linear combinations of vectors v_i is a lattice in V. Thus the corresponding polygons $\eta(\tau)$ have a natural combinatorial group structure and can be described by the technique of [CL]. However, for other values of n these vectors do not form a lattice, and instead form a dense set in the plane. This explains the reason why [MP] were able to completely resolve the case $n = 4$, and why the case $n = 5$ has remained mysterious until now. (We note that signed area on the square grid is used to study other tetrominoes in [Pr].)

Consider the case $n = 5$. Let us calculate the adèle invariant of several ribbon pentominoes. First, let τ be the **V**-pentomino, which corresponds to $\alpha = (+1, +1, -1, -1)$. We have

$$\begin{aligned}\Phi_1(\tau) &= \frac{1}{2}\,A(\eta_1(\tau_\alpha)) = \sin\frac{2\pi}{5} + \sin\frac{4\pi}{5} - \sin\frac{6\pi}{5} - \sin\frac{8\pi}{5}\\ &= 2\sin\frac{2\pi}{5} + 2\sin\frac{4\pi}{5} = \sqrt{\frac{5+\sqrt{5}}{2}} + \sqrt{\frac{5-\sqrt{5}}{2}}.\end{aligned}$$

The same calculation can be done for all remaining ribbon pentominoes (see Fig. 6). For example, for **I**- and **Z**-pentominoes, which correspond to

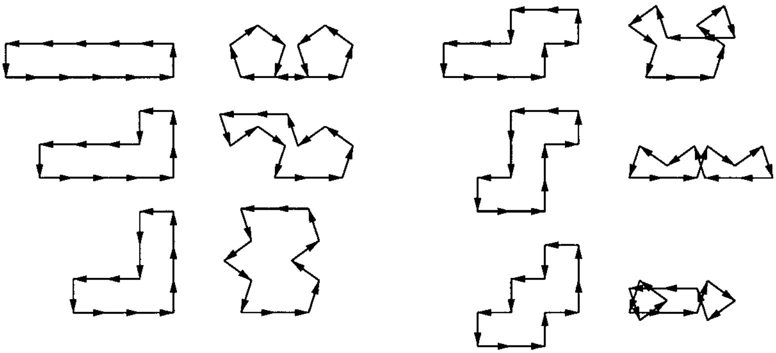

FIG. 6. Several ribbon pentominoes τ_α and the corresponding closed loops $\eta_1(\tau_\alpha)$. The remaining ribbon pentominoes, as well as the corresponding closed loops, can be obtained from these by rotation, reflection, etc.

$(+1, +1, +1, +1)$ and $(-1, +1, +1, -1)$, all adèle invariants are zero. In general, we have:

PROPOSITION 4.1. *Let τ be a ribbon tile with a $180°$ rotational symmetry. Then $\Phi_\ell(\tau) = 0$ for all $1 \leqslant \ell \leqslant n-1$.*

Proof. Having $180°$ symmetry implies that $\alpha_k = \alpha_{n-k}$ for all $k < \frac{n}{2}$. On the other hand, we have $\sin \frac{2\pi k\,\ell}{n} = -\sin \frac{2\pi(n-k)\,\ell}{n}$, i.e., all the sign terms in the expression for $\Phi_\ell(\cdot)$ cancel each other. This implies the result. ∎

Before we conclude, let us state two possible ways of deriving the linear relations in Conjecture 1.1 from adèle invariants.

We consider only the case $n = 5$. Recall that $\sin \frac{\pi}{5}$ and $\sin \frac{2\pi}{5}$ are rationally independent. Observe that for all regions Γ tileable by $\mathbf{T}_5$, we have

$$(\Diamond) \qquad \Phi_1(\Gamma) = -2c_1 \sin \frac{2\pi}{5} - 2c_2 \sin \frac{4\pi}{5},$$

where $c_1 = c_1(\Gamma)$ and $c_2(\Gamma)$ are as in Conjecture 1.1. Indeed, this holds for all ribbon tiles $\tau \in \mathbf{T}_5$, and thus by additivity for all tileable simply connected regions Γ. Since c_1 and c_2 are integers, by rational independence, the adèle invariant then induces two integer-valued invariants.

Another approach is based on using both Φ_1 and Φ_2. We have

$$(\Diamond\Diamond) \qquad \Phi_2(\tau) = -2c_1 \sin \frac{4\pi}{5} + 2\,c_2 \sin \frac{2\pi}{5}.$$

We can write both ($\Diamond$) and ($\Diamond\Diamond$) as

$$(\Phi_1, \Phi_2) = -2\,(c_1, c_2) \begin{pmatrix} \sin\dfrac{2\pi}{5} & \sin\dfrac{4\pi}{5} \\ \sin\dfrac{4\pi}{5} & -\sin\dfrac{2\pi}{5} \end{pmatrix}.$$

Since the matrix on the r.h.s. is invertible, we can obtain c_1 and c_2 as a linear combination of Φ_1, Φ_2 (the same for every tile $\tau_\alpha \in \mathbf{T}_5$).

We will show in the next section that we can generalize this argument for any n, and prove Theorem 1.2.

5. PROOF OF THEOREM 1.2

Let $n = 2m+1$ be an odd integer, $n \geqslant 3$. We claim that in this case the functions $\Phi_\ell(\tau)$, $1 \leqslant \ell \leqslant m$, are linearly independent (as real functions on $\mathbf{T}_n$). Similarly, when $n = 2m$ is an even integer, the functions $\Phi_\ell(\tau)$, $1 \leqslant \ell < m$, are linearly independent (note that $\Phi_m \equiv 0$ in this case). Let us state this as follows:

LEMMA 5.1. *For all n, we have $\dim\langle \Phi_1, ..., \Phi_m \rangle = m$, where $m = \lfloor \frac{(n-1)}{2} \rfloor$.*

Proof of Theorem 1.2. By Proposition 2.3, it suffices to prove only the first part of Conjecture 1.1. We claim that this part follows from Lemma 5.1. Indeed, let $W = \langle f_1, ..., f_m \rangle$, where f_k is a k-convexity invariant defined in the proof of Proposition 2.3.

Using $\sin 2\pi k\ell/n = -\sin 2\pi(n-k)\,\ell/n$, we can rewrite the ℓth adèle invariant as

$$\Phi_\ell(\tau_\alpha) = \sum_{k=1}^{m} (\alpha_k - \alpha_{n-k}) \sin\frac{2\pi k\,\ell}{n} = -2 \sum_{k=1}^{m} (\varepsilon_k - \varepsilon_{n-k}) \sin\frac{2\pi k\,\ell}{n}$$
$$= -2 \sum_{k=1}^{m} f_k \sin\frac{2\pi k\,\ell}{n},$$

where $\alpha = (\alpha_1, ..., \alpha_{n-1}) \in \{\pm 1\}^{n-1}$, $\varepsilon = (\varepsilon_1, ..., \varepsilon_{n-1}) \in \{0, 1\}^{n-1}$, $\alpha_k = 1 - 2\varepsilon_k$, for all $1 \leqslant k \leqslant n-1$ (so that $\tau_\alpha = \tau_\varepsilon$). This implies that $\Phi_\ell \in W$. From Lemma 5.1 we obtain

$$m = \dim\langle \Phi_1, ..., \Phi_m \rangle \leqslant \dim\langle f_1, ..., f_m \rangle = \dim W \leqslant m,$$

and therefore $\langle \Phi_1, ..., \Phi_m \rangle = W$. We conclude $f_k \in \langle \Phi_1, ..., \Phi_m \rangle$ for all $1 \leqslant k \leqslant m$. The linearity of tile invariants implies that f_k is a tile invariant of the set $\mathbf{T}_n$ of ribbon tiles (cf. proof of Proposition 2.3). This completes the proof of Theorem 1.2. ∎

Proof of Lemma 5.1. Suppose $n=2m+1$ is odd. Consider two $n\times n$ matrices $X=(x_{k,\ell})$, $Y=(y_{k,\ell})$, $0\leqslant k,\ell\leqslant n-1$, defined as

$$x_{k,\ell}=\cos\frac{2\pi k\,\ell}{n},\qquad y_{k,\ell}=\sin\frac{2\pi k\,\ell}{n}.$$

Since $Z=X+i\cdot Y$ is a Vandermonde matrix $Z=(z_{k,\ell})$, $z_{k,\ell}=\exp(2\pi ik\,\ell/n)$, we immediately have

$$\det(Z)=\prod_{0\leqslant k<\ell\leqslant n-1}(e^{2\pi ik/n}-e^{2\pi i\ell/n})\neq 0.$$

Thus $\operatorname{rk}(Z)=n$.

From $y_{k,\ell}=-y_{n-k,\ell}$, $y_{0,\ell}=0$, and $x_{k,\ell}=x_{n-k,\ell}$, we obtain $\operatorname{rk}(Y)\leqslant m$, and $\operatorname{rk}(X)\leqslant m+1$. Since $2m+1=\operatorname{rk}(Z)=\operatorname{rk}(X+iY)\leqslant\operatorname{rk}(X)+\operatorname{rk}(Y)$, we immediately have $\operatorname{rk}(Y)=m$. From $y_{k,\ell}=-y_{k,n-\ell}$, $1\leqslant\ell\leqslant m$, we conclude that an $m\times(n-1)$ submatrix $Y'=(x_{k,\ell})$, where $1\leqslant k\leqslant n-1$, $1\leqslant\ell\leqslant m$, has rank $\operatorname{rk}(Y')=m$. One can think of $\alpha\in\{\pm 1\}^{n-1}$ as vectors $\mathbb{R}^{n-1}$. Since

$$(\Phi_1(\tau_\alpha),\ldots,\Phi_m(\tau_\alpha))=(\alpha)\cdot Y'$$

and $\dim\langle\alpha\rangle=n-1$, we get $\dim\langle\Phi_1,\ldots,\Phi_m\rangle=m$.

When $n=2m$, the proof follows verbatim, except that in this case $y_{m,\ell}=\pm y_{0,\ell}=\pm 1$ (depending on the parity of ℓ). Then $\operatorname{rk}(X)=\operatorname{rk}(Y)=m$, and the result follows. ∎

6. FINAL REMARKS

The main result in this paper can be viewed as an existence of a large number of invariants for tilings by ribbon tiles. Still, the source of these invariants remains something of a mystery, yet to be discovered. It seems that such a rich structure of invariants is an exception rather than the rule, and these sets of tiles enjoy some special properties others do not. In this section we shall speculate on the possible explanations for these questions.

Let us start by saying, that although we do not pursue here the 'rational independence' approach (see Section 4), it can in fact be used. In fact, it is quite straightforward for prime n, while for composite n one has to employ Φ_d, for each $d\mid n$ and Möbius inversion. In the original version of the paper the authors favored this idea, while at the end we chose to employ an elementary linear algebra approach. Let us mention here that the arguments in Section 5, while elementary, were influenced by the ideas in [BF]. As the referee pointed out, one can think of the proof as an application of the discrete Fourier transform.

We shall note here, that miraculously, for any n, the real-valued tile invariant Φ_1 already induces a large number $\phi(n) = \Omega(n/\log\log n)$ of linearly independent integer-valued ribbon tile invariants. It would be interesting to find other examples of this phenomenon.

Let us now state the following conjecture, which seems more plausible now in view of Theorem 1.2.

Conjecture 6.1 [P1]. Define 2-flips to be transformations of tilings by $\mathbf{T}_n$ which involve exactly two tiles. Then for any simply connected region Γ, and any two tilings v, v' of Γ, there is a sequence of 2-flips which moves v into v'.

The are several reasons behind this conjecture. For $n = 2$ the truth of the assertion is well known (see, e.g., [T1]). For $n \geqslant 3$ it has been established when Γ has the shape of a Young diagram [P1] or skew Young diagram [P2]. For $n = 3$ it was also proved by an ad hoc argument for a very special set of regions [W]. There is also a topological reason in favor of the conjecture [T2]. Perhaps the most compelling reason,[3] however, is given by the following result:

PROPOSITION 6.2 [P1]. *Conjecture* 6.1 *implies Theorem* 1.2.

Indeed, assume the conjecture. Then to prove Theorem 1.2 one needs only to check that the invariants are preserved along the 2-flips. As the structure of the flips is known, this is straightforward. We refer to [P1] for details.

To conclude, let us speculate on how Conjecture 6.1 can be proved. The most promising and relevant method seem the "height representation" approach, pioneered in this context by Thurston [T1].[4] In view of importance of the subject, let us elaborate on this.

A *height representation* is a way of assigning a height to each site in the lattice so that a given tiling corresponds to a surface, i.e., a function from the lattice to the space in which the heights take their values. While the best-known height representations are integer-valued, in general they can be two- or more-dimensional vectors, or elements of a non-Abelian group (see [K, KK, MP, Pr, T1].

[3] As the referee validly points out, this is rather a reason for *wishing* that Conjecture 6.1 were true. While we agree, we leave the final judgement to the reader.

[4] Interestingly, Thurston's paper [T1] was inspired by [CL].

Height representations have many uses. If one desires to sample randomly from the set of tilings of a given simply connected region, these representations can be used to prove that this set is connected under some set of local moves [K, R], to devise exact sampling Monte Carlo algorithms based on these moves [PW], and to place upper limits on the mixing time of these algorithms [LRS]. They can also be used to develop an efficient algorithm to tell whether a given region can be tiled at all [K, R], which is interesting since this problem is NP-complete in general, even for some simple sets of tiles (see, e.g., [MR]).

For tilings, the standard approach is to define how the height changes, by small increments, as we move along the boundary between one tile and another. In order for the height to be a single-valued function, it must return to its original value whenever we travel around a loop. Therefore, each type of tile induces a relation in the height group [CL, T1], or, in the Abelian case, a linear constraint on the amount by which the height increases or decreases as we traverse different kinds of edges.

For instance, domino tilings of the square lattice have a height representation which can be thought of as follows. We color the lattice as a checkerboard, with white and black squares alternating. Whenever we move along an edge of the lattice, we change the height by $+1$ if the square on our left is black, and -1 if it is white. The reader can easily check that a set of moves encircling a horizontal or vertical domino will have a total height change of $+1+1+1-1-1-1=0$. In fact, this is our mapping η in the case $n=2$. We refer the reader to [KK, R, T1] for other examples and details.

Now consider what happens in our case. We define a complex-valued height function which is defined by local rules. It seem likely that our height function is a projection onto two dimension of the height function with values in an n-dimensional lattice [T2], but we were unable to make this observation precise. If only we could show a "nice" behavior under 2-flips, we would be able to prove Conjecture 6.1 and perhaps even give a linear time algorithm for checking tileability by ribbon tiles. So far, this remains a fantasy, so we leave the reader here until further developments.

ACKNOWLEDGMENTS

We thank Jim Propp and an anonymous referee for helpful remarks and pointing out errors in the original version of the paper. Discussions with Ezra Miller and Ravi Vakil helped to simplify the proof. The first author thanks the Sandia University Research Program. The second author thanks the Santa Fe Institute for the hospitality during a visit in which this work was done. The first author thanks Molly Rose and Spootie the Cat. The second author was supported by an NSF Mathematical Research Fellowship.

Note added in proof. Conjecture 6.1 was recently resolved by Scott Sheffield in "Ribbon tilings and multidimensional height functions," to appear in *Trans. Amer. Math. Soc.*

REFERENCES

[BF] L. Babai and P. Frankl, Linear algebra methods in combinatorics, with applications to geometry and computer science (preliminary version 2), preprint, University of Chicago, 1992.

[CL] J. H. Conway and J. C. Lagarias, Tilings with polyominoes and combinatorial group theory, *J. Comb. in Theory Ser. A* **53** (1990), 183–208.

[G] S. Golomb, "Polyominoes," Scribners, New York, 1965.

[GO] J. E. Goodman and J. O'Rourke (Eds.), "Handbook of Discrete and Computational Geometry," CRC Press, Boca Raton, FL, 1997.

[KK] C. Kenyon and R. Kenyon, Tiling a polygon with rectangles, *in* "Proc. 33rd Symp. Foundations of Computer Science, 1992," pp. 610–619.

[K] R. Kenyon, A note on tiling with integer-sided rectangles, *J. Combin. Theory Ser. A* **74** (1996), 321–332

[LRS] M. Luby, D. Randall, and A. Sinclair, Markov chain algorithms for planar lattice structures, *in* "Proc. 36th Symp. Foundations of Computer Science, 1995," pp. 150–159.

[MR] C. Moore and J. M. Robson, Hard tiling problems with simple tiles, preprint, Santa Fe Institute, 2000.

[MP] R. Muchnik and I. Pak, On tilings by ribbon tetrominoes, *J. Combin. Theory Ser. A* **88** (1999), 199–193.

[P1] I. Pak, Ribbon tile invariants, *Trans. Amer. Math. Soc.* **352** (2000), 5525–5561.

[P2] I. Pak, unpublished.

[Pr] J. Propp, A pedestrian approach to a method of Conway, or, A tale of two cities, *Math. Mag.* **70** (1997), 327–340.

[PW] J. Propp and D. Wilson, Exact sampling with coupled Markov chains and applications to statistical mechanics, *Random Structures Algorithms* **9** (1996), 223–252.

[R] E. Rémila, Tiling groups: New applications in the triangular lattice, *Discrete Combin. Geom.* **20** (1998), 189–204.

[T1] W. Thurston, Conway's tiling groups, *Amer. Math. Monthly* **97** (1990), 757–773.

[T2] W. Thurston, personal communication, 2000.

[W] D. C. West, An elementary proof of two triangle-tiling theorems of Conway and Lagarias, unpublished manuscript, 1990.