

# EQUALITY CASES OF THE STANLEY–YAN LOG-CONCAVE MATROID INEQUALITY

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ABSTRACT. The *Stanley–Yan (SY) inequality* gives the ultra-log-concavity for the numbers of bases of a matroid which have given sizes of intersections with  $k$  fixed disjoint sets. The inequality was proved by Stanley (1981) for regular matroids, and by Yan (2023) in full generality. In the original paper, Stanley asked for equality conditions of the SY inequality, and proved total equality conditions for regular matroids in the case  $k = 0$ .

In this paper, we completely resolve Stanley’s problem. First, we obtain an explicit description of the equality cases of the SY inequality for  $k = 0$ , extending Stanley’s results to general matroids and removing the “total equality” assumption. Second, for  $k \geq 1$ , we prove that the equality cases of the SY inequality cannot be described in a sense that they are not in the polynomial hierarchy unless the polynomial hierarchy collapses to a finite level.

## 1. INTRODUCTION

**1.1. Foreword.** Among combinatorial objects, *matroids* are fundamental and have been extensively studied in both combinatorics and their applications to other fields (see e.g. [Ox11, Sch03]). In recent years, a remarkable progress has been made towards understanding *log-concave* matroid inequalities for various matroid parameters (see e.g. [Huh18, Kal23]). Much less is known about their equality conditions as they remain inaccessible by algebraic techniques (see Section 2).

The *Stanley–Yan (SY) inequality* is a very general log-concave inequality for the numbers of bases of a matroid which have given sizes of intersections with  $k$  fixed sets. In this paper we completely resolve Stanley’s open problem [Sta81, p. 60], asking for equality conditions for the SY inequality, although probably not in the way Stanley had expected: we give a positive result for  $k = 0$  and a negative result for  $k \geq 1$ .

Since known proofs of the SY inequality are independent of  $k$ , it came as a surprise that the equality conditions have a completely different nature for different  $k$ . Curiously, our negative result is formalized and proved in the language of computational complexity. Even as a conjecture this was inconceivable until our recent work (cf. §15.1).

**1.2. Stanley’s problem.** Let  $\mathcal{M}$  be a matroid of rank  $r = \text{rk}(\mathcal{M})$ , with a ground set  $X$  of size  $|X| = n$ . Denote by  $\mathcal{B}(\mathcal{M})$  the set of bases of  $\mathcal{M}$ . This is a collection of  $r$ -subsets of  $X$ . Fix integers  $k \geq 0$  and  $0 \leq a, c_1, \dots, c_k \leq r$ . Additionally, fix disjoint subsets  $R, S_1, \dots, S_k \subset X$ . Define

$$\mathcal{B}_{\mathbf{Sc}}(\mathcal{M}, R, a) := \{A \in \mathcal{B}(\mathcal{M}) : |A \cap R| = a, |A \cap S_1| = c_1, \dots, |A \cap S_k| = c_k\},$$

where  $\mathbf{S} = (S_1, \dots, S_k)$  and  $\mathbf{c} = (c_1, \dots, c_k)$ . Denote  $\text{B}_{\mathbf{Sc}}(\mathcal{M}, R, a) := |\mathcal{B}_{\mathbf{Sc}}(\mathcal{M}, R, a)|$ , and let

$$\text{P}_{\mathbf{Sc}}(\mathcal{M}, R, a) := \text{B}_{\mathbf{Sc}}(\mathcal{M}, R, a) \binom{r}{a, c_1, \dots, c_k, v}^{-1},$$

where  $v = r - a - c_1 - \dots - c_k$ . See §3.2 for the matroid definition and notation.

**Theorem 1.1** (*Stanley–Yan inequality*, [Sta81, Thm 2.1] and [Yan23, Cor. 3.47]).

$$\text{(SY)} \quad \text{P}_{\mathbf{Sc}}(\mathcal{M}, R, a)^2 \geq \text{P}_{\mathbf{Sc}}(\mathcal{M}, R, a+1) \text{P}_{\mathbf{Sc}}(\mathcal{M}, R, a-1).$$

This inequality was discovered by Stanley who proved it for regular (unimodular) matroids using the *Alexandrov–Fenchel inequality*. The inequality was extended to general matroids by Yan [Yan23], using *Lorentzian polynomials*. Both proofs are independent of  $k$ .

To motivate the result, Stanley showed in [Sta81, Thm 2.9] (see also [Yan23, Thm 3.48]), that the Stanley–Yan (SY) inequality for  $k = 0$  implies the *Mason–Welsh conjecture* (1971, 1972), see (M1) in §2.2. This is a log-concave inequality for the number of independent sets of a matroid (see §2.2), which remained a conjecture until Adiprasito, Huh and Katz [AHK18] famously proved it in full generality using *combinatorial Hodge theory*.

In [Sta81, §2], Stanley asked for equality conditions for (SY) and proved partial results in this direction (see below). Despite major developments on matroid inequalities, no progress on this problem has been made until now. We give a mixture of both positive and negative results which completely resolve Stanley’s problem. We start with the latter.

**1.3. Negative results.** Let  $\mathcal{M}$  be a *binary matroid* given by its representation over  $\mathbb{F}_2$ , and let  $k \geq 0$ ,  $R \subseteq X$ ,  $\mathbf{S} \in X^k$ ,  $a \in \mathbb{N}$ ,  $\mathbf{c} \in \mathbb{N}^k$  be as above. Denote by  $\text{EQUALITYSY}_k$  the decision problem

$$\text{EQUALITYSY}_k := \{ \text{P}_{\mathbf{S}\mathbf{c}}(\mathcal{M}, R, a)^2 \stackrel{?}{=} \text{P}_{\mathbf{S}\mathbf{c}}(\mathcal{M}, R, a+1) \text{P}_{\mathbf{S}\mathbf{c}}(\mathcal{M}, R, a-1) \}.$$

Here the input to the problem consists of a representation of the matroid  $\mathcal{M}$ , subsets  $R, S_1, \dots, S_k$  of the ground set, and integers  $a, c_1, \dots, c_k$ . The integer  $k$  is the only fixed parameter.

**Theorem 1.2 ( $k \geq 1$  case).** *For all  $k \geq 1$ , we have:*

$$\text{EQUALITYSY}_k \in \text{PH} \implies \text{PH} = \Sigma_m^{\text{P}} \text{ for some } m,$$

*for binary matroids. Moreover, the result holds for  $a = 1$  and  $c_1 = r - 2$ .*

This gives a negative solution to Stanley’s problem for  $k \geq 1$ . Informally, the theorem states that equality cases of the Stanley–Yan inequality (SY) cannot be *described* using a finite number of alternating quantifiers  $\exists$  and  $\forall$ , unless a standard complexity assumptions fails (namely, that the *polynomial hierarchy* PH collapses to a finite level<sup>1</sup>). This is an unusual application of computational complexity to a problem in combinatorics (cf. §15.1). The proof of Theorem 1.2 is given in Section 8, and uses technical lemmas developed in Section 4–7.

The theorem does not say that no *geometric* description of (SY) can be obtained, or that some large family of equality cases cannot be described. In fact, the vanishing cases we present below (see Theorem 1.6), is an example of the latter.

The proof of Theorem 1.2 uses the *combinatorial coincidences* approach developed in [CP24d, CP23a]. We also use the analysis of the *spanning tree counting function* through continued fractions (see §15.6 below). Paper [CP23a] is especially notable, as it can be viewed both a philosophical and (to a lesser extent) a technical prequel to this paper. There, we prove that the equality cases of the Alexandrov–Fenchel inequality are not in PH for *order polytopes* (under the same assumptions). See §15.2 for possible variations of the theorem for other classes of matroids.

**1.4. Positive results.** For  $k = 0$ , we omit the subscripts:

$$\text{B}(\mathcal{M}, R, a) := |\{A \in \mathcal{B}(\mathcal{M}) : |A \cap R| = a\}| \quad \text{and} \quad \text{P}(\mathcal{M}, R, a) := \text{B}(\mathcal{M}, R, a) \binom{r}{a}^{-1}.$$

Denote by  $\text{NL}(\mathcal{M})$  the set of *non-loops* in  $\mathcal{M}$ , i.e. elements  $x \in X$  such that  $\{x\}$  is an independent set. For a non-loop  $x \in \text{NL}(\mathcal{M})$ , denote by  $\text{Par}_{\mathcal{M}}(x) \subseteq X$  the set of elements of  $\mathcal{M}$  that are *parallel* to  $x$ , i.e. elements  $y \in X$  such that  $\{x, y\}$  is not an independent set. The following two results (Theorem 1.3 and Proposition 1.4) give a positive solution to Stanley’s problem for  $k = 0$ .

<sup>1</sup>This is a standard assumption in theoretical computer science that is similar to  $\text{P} \neq \text{NP}$  (stronger, in fact), and is widely believed by the experts. If false it would bring revolutionary changes to the field, see e.g. [Aar16, Wig23].

**Theorem 1.3** ( *$k = 0$  case*). *Let  $\mathcal{M}$  be a matroid of rank  $r \geq 2$  with a ground set  $X$ . Let  $R \subset X$ , and let  $1 \leq a \leq r - 1$ . Suppose that  $P(\mathcal{M}, R, a) > 0$ . Then the equality*

$$(1.1) \quad P(\mathcal{M}, R, a)^2 = P(\mathcal{M}, R, a + 1) P(\mathcal{M}, R, a - 1)$$

*holds if and only if for every independent set  $A \subset X$  s.t.  $|A| = r - 2$  and  $|A \cap R| = a - 1$ , and every non-loop  $x \in \text{NL}(\mathcal{M}/A)$ , we have:*

$$(1.2) \quad |\text{Par}_{\mathcal{M}/A}(x) \cap R| = s |\text{Par}_{\mathcal{M}/A}(x) \cap (X - R)| \quad \text{for some } s > 0.$$

In particular, Theorem 1.3 resolves a conjecture of Yan (Conjecture 1.10 in §1.6 below). We prove the theorem in Section 10 using the *combinatorial atlas* technology, see §2.3. This is a technical linear algebraic approach we developed in [CP22a, CP24a] to prove both the inequalities and the equality cases of matroid and poset inequalities, as well as their generalizations. Notably, we obtain the equality cases of Mason’s ultra-log-concave inequality [CP24a, §1.6], and we use a closely related setup in this case.

**1.5. Vanishing conditions.** Note that when  $P(\mathcal{M}, R, a) = 0$ , we always have equality in (1.1). The following nonvanishing conditions give a description to such equality cases:

**Proposition 1.4** (*nonvanishing conditions for  $k = 0$* ). *Let  $\mathcal{M}$  be a matroid of rank  $r = \text{rk}(\mathcal{M})$  with a ground set  $X$ , and let  $R \subseteq X$ . Then, for every  $0 \leq a \leq r$ , we have:  $P(\mathcal{M}, R, a) > 0$  if and only if*

$$r - \text{rk}(X \setminus R) \leq a \leq \text{rk}(R).$$

The proposition is completely straightforward and is a special case of a more general Theorem 1.6, see below. Combined, Theorem 1.3 and Proposition 1.4 give a complete description of equality cases of the Stanley–Yan inequality (SY) for  $k = 0$ .

It is natural to compare our positive and negative results, in the complexity language. In particular, Theorem 1.2 shows that  $\text{EQUALITYSY}_k \notin \text{coNP}$ , for all  $k \geq 1$  (unless PH collapses). In other words, it is *very unlikely* that there is a *witness* for (SY) being strict that can be verified in polynomial time. This is in sharp contrast with the case  $k = 0$  :

**Corollary 1.5.** *Let  $\mathcal{M}$  be a matroid given by a succinct presentation. Then:*

$$\text{EQUALITYSY}_0 \in \text{coNP}.$$

Here by *succinct* we mean a presentation of a matroid with an oracle which computes the rank function (of a subset of the ground set) in polynomial time, see e.g. [KM22, §5.1]. Matroids with succinct presentation include graphical, transversal and bicircular matroids (see e.g. [Ox11, Wel76]), certain paving matroids based on Hamiltonian cycles [Jer06, §3], and matroids given by their representation over fields  $\mathbb{F}_q$  or  $\mathbb{Q}$ . By a mild abuse of notation, we use  $\text{EQUALITYSY}_k$  to denote the equality decision problem of the Stanley–Yan inequality (SY) for general matroids given by a succinct presentation.

Corollary 1.5 follows from the explicit description of the equality cases given in Theorem 1.3 and Proposition 1.4. See also Section 13 for several examples, and §15.4 for further discussion of computational hardness of  $\text{EQUALITYSY}_0$ .

**Theorem 1.6** (*nonvanishing conditions for all  $k \geq 0$* ). *Let  $\mathcal{M} = (X, \mathcal{I})$  be a matroid with a ground set  $X$  and independent sets  $\mathcal{I} \subset 2^X$ . Let  $r = \text{rk}(\mathcal{M})$  be the rank of  $\mathcal{M}$ . Let  $\mathbf{S} = (S_1, \dots, S_\ell)$  be a set partition of  $X$ , i.e. we have  $X = \cup_i S_i$  and  $S_i \cap S_j = \emptyset$  for all  $1 \leq i < j \leq \ell$ . Finally, let  $\mathbf{c} = (c_1, \dots, c_\ell) \in \mathbb{N}^\ell$ . Then, there exists an independent set  $A \in \mathcal{I}$  such that*

$$|A \cap S_i| = c_i \quad \text{for all } i \in [\ell]$$

*if and only if*

$$\text{rk}\left(\bigcup_{i \in L} S_i\right) \geq \sum_{i \in L} c_i \quad \text{for all } L \subseteq [\ell], L \neq \emptyset.$$

One can think of this result as a positive counterpart to the (negative) Theorem 1.2. In the language of Shenfeld and van Handel [SvH22, SvH23], the *vanishing conditions* are “trivial” equality cases of the SY inequality, in a sense of having a simple geometric meaning rather than ease of the proof. We prove the Theorem 1.6 in Section 11 using the *discrete polymatroid theory*.

Note that Proposition 1.4 follows from Theorem 1.6, by taking  $S_1 \leftarrow R$ ,  $S_2 \leftarrow X \setminus R$ ,  $c_1 \leftarrow a$ , and  $c_2 \leftarrow (r-a)$ . More generally, the complexity of the vanishing for all  $k \geq 0$  follows immediately from Theorem 1.6, and is worth emphasizing:

**Corollary 1.7.** *Let  $\mathcal{M}$  be a matroid given by a succinct presentation. Then, for all fixed  $k \geq 0$ , the problem  $\{\text{Bsc}(\mathcal{M}, R, a) >^? 0\}$  is in P.*

**1.6. Total equality cases.** Throughout this section, we let  $k = 0$ . We start with a simple observation whose proof is well-known and applies to all positive log-concave sequences.

**Corollary 1.8.** *Let  $\mathcal{M}$  be a matroid of rank  $r \geq 2$  with a ground set  $X$ , and let  $R \subseteq X$ . Suppose  $P(\mathcal{M}, R, 0) > 0$  and  $P(\mathcal{M}, R, r) > 0$ . Then:*

$$(1.3) \quad P(\mathcal{M}, R, 1)^r \geq P(\mathcal{M}, R, 0)^{r-1} P(\mathcal{M}, R, r).$$

Moreover, the equality in (1.3) holds if and only if (SY) is an equality for all  $1 \leq a \leq r-1$ .

For completeness, we include a short proof in §12.1. This motivates the following result that is more surprising than it may seem at first:

**Theorem 1.9** (*total equality conditions*, [Sta81] and [Yan23]). *Let  $\mathcal{M}$  be a loopless regular matroid of rank  $r \geq 2$  with a ground set  $X$ , and let  $R \subseteq X$ . Suppose that  $P(\mathcal{M}, R, 0) > 0$  and  $P(\mathcal{M}, R, r) > 0$ . Then the following are equivalent :*

- (i)  $P(\mathcal{M}, R, 1)^r = P(\mathcal{M}, R, 0)^{r-1} P(\mathcal{M}, R, r)$ ,
- (ii)  $P(\mathcal{M}, R, a)^2 = P(\mathcal{M}, R, a+1) P(\mathcal{M}, R, a-1)$  for all  $a \in \{1, \dots, r-1\}$ ,
- (iii)  $P(\mathcal{M}, R, a)^2 = P(\mathcal{M}, R, a+1) P(\mathcal{M}, R, a-1)$  for some  $a \in \{1, \dots, r-1\}$ ,
- (iv)  $|\text{Par}_{\mathcal{M}}(x) \cap R| = s |\text{Par}_{\mathcal{M}}(x) \cap (X - R)|$  for all  $x \in X$  and some  $s > 0$ .

**Conjecture 1.10** ([Yan23, Conj. 3.40]). *The conclusion of Theorem 1.9 holds for all loopless matroids.*

The equivalence (i)  $\Leftrightarrow$  (ii) is the second part of Corollary 1.8 and holds for all matroids. The implication (ii)  $\Rightarrow$  (iii) is trivial. The equivalence (i)  $\Leftrightarrow$  (iv) was proved by Stanley for regular matroids [Sta81, Thm 2.8] (see also [Yan23, Thm 3.34]). Similarly, the implication (iii)  $\Rightarrow$  (ii) was proved in [Yan23, Lem 3.39] for regular matroids. The implication (iv)  $\Rightarrow$  (ii) was proved in [Yan23, Thm 3.41] for all matroids. The following result completely resolves the remaining implications of Yan’s Conjecture 1.10.

**Theorem 1.11.** *In the notation of Theorem 1.9, we have:*

- (1) (i)  $\Leftrightarrow$  (ii)  $\Leftrightarrow$  (iv) for all loopless matroids, and
- (2) there exists a loopless binary matroid  $\mathcal{M}$  s.t. (iii) holds but not (ii).

The theorem is another example of the phenomenon that regular matroids satisfy certain matroid inequalities that general binary matroids do not (see e.g. [HSW22] and §15.4). The proof of Theorem 1.11 is given in Section 12, and is based on Theorem 1.3. The example in Theorem 1.11 part (2) can be found in §13.3.

**1.7. Counting spanning trees.** At a crucial step in the proof of Theorem 1.2, we give bounds for the relative version of the tree counting function, see below. This surprising obstacle occupies a substantial part of the proof (Sections 5 and 6). It is also of independent interest and closely related to the following combinatorial problem.

Let  $G = (V, E)$  be a connected simple graph. Denote by  $\tau(G)$  the number of spanning trees in  $G$ . Sedláček [Sed70] considered the smallest number of vertices  $\alpha(N)$  of a planar graph  $G$  with exactly  $N$  spanning trees:  $\tau(G) = N$ .<sup>2</sup>

**Theorem 1.12** (Stong [Sto22, Cor. 7.3.1]). *For all  $N \geq 3$ , there is a simple planar graph  $G$  with  $O((\log N)^{3/2}/(\log \log N))$  vertices and exactly  $\tau(G) = N$  spanning trees.*

Until this breakthrough, even  $\alpha(N) = o(N)$  remained out of reach, see [AŠ13]. As a warmup, Stong first proves this bound in [Sto22, Cor. 5.2.2], and this proof already involves a delicate number theoretic argument. Naturally, Stong’s Theorem 1.12 is much stronger. The following result is a variation of Stong’s theorem, and has the advantage of having an elementary proof.

**Theorem 1.13.** *For all  $N \geq 3$ , there is a planar graph  $G$  with  $O(\log N \log \log N)$  edges and exactly  $\tau(G) = N$  spanning trees.*

Compared to Theorem 1.12, note that graphs are not required to be simple, but the upper bound in terms of edges is much sharper (in fact, it is nearly optimal, see §15.6). Indeed, the planarity implies the same asymptotic bound for the number of vertices and edges in  $G$ .<sup>3</sup> Theorem 1.13 is a byproduct of the proof of the following lemma that is an intermediate result in the proof of Theorem 1.2.

For an edge  $e \in E$ , denote by  $G - e$  and  $G/e$  the deletion of  $e$  and the contraction along  $e$ . Define the *spanning tree ratio* as follows:

$$\rho(G, e) := \frac{\tau(G - e)}{\tau(G/e)}.$$

**Lemma 1.14** (spanning tree ratios lemma). *Let  $A, B \in \mathbb{N}$  such that  $1 \leq B \leq A \leq 2B \leq N$ . Then there is a planar graph  $G$  with  $O((\log N)(\log \log N)^2)$  edges and  $\rho(G, e) = A/B$  spanning tree ratio.*

Note that the spanning tree ratios are not attainable by the tools in [Sto22]. This is why we need a new approach to the analysis of the spanning tree counting function, giving the proof of both Theorem 1.13 and Lemma 1.14 in Section 6.

Our approach follows general outlines in [CP23a, CP24c], although technical details are largely different. Here we use a variation on the celebrated *Hajós construction* [Haj61] (see also [Urq97]), introduced in the context of graph colorings. Also, in place of the Yao–Knuth [YK75] “average case” asymptotics for continued fractions used in [CP23a], we use more delicate “best case” bounds by Larcher [Lar86].

Finally, note that the lemma gives the spanning tree ratio  $\rho(G, e)$  in the interval  $[1, 2]$ . In the proof of Theorem 1.2, we consider more general ratios. We are able to avoid extending Lemma 1.14 by combining combinatorial recurrences and complexity ideas.

**1.8. Paper structure.** In Section 2, we give an extensive historical background of many strains leading to the two main results (Theorems 1.2 and 1.3). The material is much too rich to give a proper review in one section, so we tried to highlight the results that are most relevant to our work, leaving unmentioned many major developments.

In Section 3, we give basic definitions and notation, covering both matroid theory and computational complexity. In a short Section 4, we give a key reduction to the SY equality problem from

<sup>2</sup>The original problem considered general rather than planar graphs, see §2.5.

<sup>3</sup>Although Stong does not explicitly mention planarity in [Sto22], his construction involves only planar graphs.



the matroid basis coincidence problem. In Sections 5 and 6 we relate spanning trees in planar graphs to continuous fractions, and prove Theorem 1.13 and Lemma 1.14 in §6.1 along the way.

Sections 7 and 8 contain the proof of Theorem 1.2. Here we start with the proof of the Verification lemma (Lemma 7.1), which uses our spanning tree results and number theoretic estimates, and prove the theorem using a complexity theoretic argument.

In Sections 9 and 10, we give a proof of Theorem 1.3. We start with an overview of our combinatorial atlas technology (Sections 9). We give a construction of the atlas for this problem in §10.1 and proceed to prove the theorem. These two sections are completely independent from the rest of the paper.

In Section 11, we discuss vanishing conditions and prove Theorem 1.6. We give examples and counterexamples to equality conditions of (SY) in Section 13. In a short Section 14, we present the *generalized Mason inequality*, a natural variation of the Stanley–Yan inequality for the independent sets. We conclude with final remarks and several open problems in Section 15, all of them in connection with matroid inequalities and computational complexity.

## 2. BACKGROUND

**2.1. Log-concave inequalities.** Log-concavity is a classical analytic property going back to Maclaurin (1929) and Newton (1732). Log-concavity is closely related to negative correlations, which also has a long history going back to Rayleigh and Kirchhoff, see e.g. [BBL09]. Log-concave inequalities for matroids and their generalizations (morphisms of matroids, antimatroids, greedoids) is an emerging area in its own right, see [CP24a, Yan23] for detailed overview.

Stanley was a pioneer in the area of unimodal and log-concave inequalities in combinatorics, as he introduced both algebraic and geometric techniques [Sta89], see also [Brä15, Bre89]. In [Sta81], he gave two applications of the *Alexandrov–Fenchel* (AF) *inequality* for mixed volumes of convex bodies, to log-concavity of combinatorial sequences. One is the (SY) for regular matroids. The other is *Stanley’s poset inequality* for the number of linear extensions [Sta81, Thm 3.1] that is extremely well studied in recent years, see a survey in [CP23b]. Among many variations, we note the *Kahn–Saks inequality* which was used to prove the first major breakthrough towards the  $\frac{1}{3} - \frac{2}{3}$  *conjecture* [KS84].

Formally, let  $P = (X, <)$  be a poset with  $|X| = n$  elements. A *linear extension* of  $P$  is an order-preserving bijection  $f : X \rightarrow [n]$ . Denote by  $\mathcal{E}(P)$  the set of linear extensions of  $P$ . Fix  $x, z_1, \dots, z_k \in X$  and  $a, c_1, \dots, c_k \in [n]$ . Let  $\mathcal{E}_{\mathbf{zc}}(P, x, a)$  be the set of linear extensions  $f \in \mathcal{E}(P)$ , s.t.  $f(x) = a$  and  $f(z_i) = c_i$  for all  $1 \leq i \leq k$ . *Stanley’s poset inequality* is the log-concavity of numbers  $N_{\mathbf{zc}}(P, x, a) := |\mathcal{E}_{\mathbf{zc}}(P, x, a)|$  :

$$(Sta) \quad N_{\mathbf{zc}}(P, x, a)^2 \geq N_{\mathbf{zc}}(P, x, a+1) \cdot N_{\mathbf{zc}}(P, x, a-1).$$

These Stanley’s inequalities (SY) and (Sta) have superficial similarities as they were obtained in the same manner, via construction of combinatorial polytopes whose volumes and mixed volumes have a combinatorial interpretation. For regular matroids, Stanley used *zonotopes* spanned by the vectors of a unimodular representation, while for posets he used *order polytopes* [Sta86]. Partly motivated by Stanley’s paper, Schneider [Sch88] gives equality conditions for the AF inequality in case of zonotopes. In principle, one should be able to derive Theorem 1.3 in the case of regular matroids from Schneider’s result as well.

*A word of caution:* Although it may seem that inequalities (Sta) and (SY) are both consequences of the AF inequality, and that this paper and [CP23a] cover the same or similar ground, in fact the opposite is true. While (Sta) is a direct consequence of the AF inequality, only the *computationally easy* part of the (SY) follows from the AF inequality. It took Lorentzian polynomials to prove the *computationally hard* part of (SY). See Proposition 15.5 for the formal statement.

In [Sta81], Stanley asked for equality conditions for both matroid and poset inequalities that he studied. He noted that the AF inequality has equality conditions known only in a few special cases. He used one such known special case (dating back to Alexandrov), to describe equality

cases of his matroid log-concave inequality for regular matroids (Theorem 1.9). The equality conditions for (Sta) are now largely understood, see below.

Stanley’s inequality (SY) led to many subsequent developments. Notably, Godsil [God84] resolved Stanley’s question to show that the generating polynomial

$$\sum_a P_{\mathbf{Sc}}(\mathcal{M}, R, a) t^a$$

has only real nonpositive roots (this easily implies log-concavity). Choe and Wagner [CW06] proved that  $\{P_{\mathbf{Sc}}(\mathcal{M}, R, a) : 0 < a < r\}$  is log-concave for a larger family of matroids with the *half-plane property* (HPP), see also [Brä15, §9.1].

In a remarkable series of papers, Huh and coauthors developed a highly technical algebraic approach to log-concave inequalities for various classes of matroids, see an overview in [Huh18, Kal23]. Most famously, Adiprasito, Huh and Katz [AHK18], proved a number of log-concave inequalities for *general matroids*, some of which were conjectured many decades earlier. These results established log-concavity for the number of independent sets of a matroid according to the size (*Mason–Welsh conjecture* implied by the (SY) inequality, see below), and of the coefficients of the characteristic polynomial (*Heron–Rota–Welsh conjecture*). After that much progress followed, eventually leading to the proof of a host of other matroid inequalities.

**2.2. Lorentzian polynomials.** *Lorentzian polynomials* were introduced by Brändén and Huh [BH20], and independently by Anari, Oveis Gharan and Vintant [AOV18]. This approach led to a substantial extension of earlier algebraic and analytic notions, as well as a major simplification of the earlier proofs. Specifically, they showed that the homogeneous *multivariate Tutte polynomial* of a matroid is a Lorentzian polynomial (see [ALOV24, Thm 4.1] and [BH20, Thm 4.10]). This implied the ultra-log-concave inequality conjectured by Mason, i.e. the strongest of the Mason’s conjectures.

Formally, let  $I(\mathcal{M}, a)$  denotes the number of independent sets in matroid  $\mathcal{M}$  of size  $a$ . Mason’s weakest conjecture (the Mason–Welsh conjecture mentioned above) is the log-concave inequality

$$(M1) \quad I(\mathcal{M}, a)^2 \geq I(\mathcal{M}, a+1) I(\mathcal{M}, a-1) \quad \text{for all } 1 \leq a \leq (r-1).$$

Similarly, Mason’s strongest conjecture (we skip the intermediate one), is the ultra-log-concave inequality

$$(M2) \quad I(\mathcal{M}, a)^2 \geq \left(1 + \frac{1}{a}\right) \left(1 + \frac{1}{n-a}\right) I(\mathcal{M}, a+1) I(\mathcal{M}, a-1) \quad \text{for all } 1 \leq a \leq (r-1),$$

where  $n = |X|$  is the size of the ground set (see Section 14).

Most recently, Yan [Yan23] used Lorentzian polynomials to extend Stanley’s result from regular to general matroids (Theorem 1.1). The resulting Stanley–Yan inequality (SY) is one of the most general matroid results proved by a direct application of Lorentzian polynomials, and it easily implies (M1) (we include this argument in §14 for the reader’s convenience).

**2.3. Later developments.** Recently, the authors introduced a linear algebra based *combinatorial atlas* technology in [CP24a], which includes Lorentzian polynomials as a special case [CP22a, §5]. The authors proved equality conditions and various extensions of both Mason’s ultra-log-concave inequality (for the number of independent sets of matroids), and for Stanley’s poset inequality (Sta). Most recently, the authors used combinatorial atlases to establish correlation inequalities for the numbers of linear extensions [CP24b]. These results parallel earlier correlation inequalities by Huh, Schröter and Wang [HSW22].

In a separate development, Brändén and Leake introduced *Lorentzian polynomials on cones* [BL23]. They were able to give an elementary proof of the Heron–Rota–Welsh conjecture. Note that both combinatorial atlas and this new technology give new proofs of the Alexandrov–Fenchel inequality, see [CP22a, §6] and [BL23, §6]. This is a central and most general inequality in convex geometry, with many proofs none of which are truly simple, see e.g. [BZ88, §20].

Shenfeld and van Handel [SvH23] undertook a major study of equality cases of the AF inequality. They obtained a (very technical) geometric characterization in the case of convex polytopes, making a progress on a long-standing open problem in convex geometry, see [Sch85]. They gave a complete description of equality cases for (Sta) in the  $k = 0$  case and showed that the equality decision problem is in P.

The  $k = 0$  equality cases of (Sta) were rederived in [CP24a] using combinatorial atlas, where the result was further extended to *weighted linear extensions*. Shenfeld and van Handel's approach was further extended in [vHYZ23] to the Kahn–Saks inequality (a diagonal slice of the  $k = 1$  case), and in [CP23a] to the full  $k = 1$  case of (Sta). For general  $k \geq 2$ , Ma and Shenfeld [MS24] gave a technical combinatorial description of the equality cases of (Sta). Notably, this description involves  $\#P$  oracles, and therefore is not naturally in PH.

**2.4. Negative results.** In a surprising development, the authors in [CP23a] showed that for  $k \geq 2$ , the equality conditions of Stanley's poset inequality are not in PH unless PH collapses to a finite level. In particular, this implied that the equality cases of the AF inequality for  $H$ -polytopes with a concise description, are also not in PH, unless PH collapses to a finite level.

Prior to [CP23a], there were very few results on computational complexity of (equality cases) of combinatorial inequalities. The approach was introduced by the second author in [Pak19], as a way to show that certain combinatorial numbers do not have a *combinatorial interpretation*. This was formalized as counting functions not being in  $\#P$ , see survey [Pak22]. Various examples of functions not in  $\#P$  were given in [IP22], based on an assortment of complexity theoretic assumptions.

It was shown by Ikenmeyer, Panova and the second author in [IPP24], that vanishing of  $S_n$  characters problem  $[\chi^\lambda(\mu) =? 0]$  is  $C=P$ -complete. This implies that this problem is not in PH unless PH collapses to the second level (ibid.). Finally, a key technical lemma in [CP23a] is based on the analysis of the *combinatorial coincidence problem*. This is a family of decision problems introduced and studied in [CP24d]. They are also characterised by a collapse of PH.

**2.5. Spanning trees.** Sedláček [Sed70] and Azarija–Škrekovski [AŠ13] considered two closely related functions  $\alpha'(N)$  and  $\beta'(N)$ , defined to be the minimal number of vertices and edges, respectively, over *all* (i.e., not necessarily planar) graphs  $G$  with  $\tau(G) = N$  spanning trees. For connected planar graphs, the number of edges is linear in the number of vertices, so this distinction disappears.

In recent years, there were several applications of continued fractions to problems in combinatorics. Notably, Kravitz and Sah [KS21] used continuous fractions to study a similar problem for the number  $|\mathcal{E}(P)|$  of linear extensions of a poset, see also [CP24c]. An earlier construction by Schiffler [Sch19] which appeared in connection with cluster algebras, related continued fractions and perfect matchings. We also mention a large literature on enumeration of lattice paths via continued fractions, see e.g. [GJ83, Ch. 5].

**2.6. Counting complexity.** The problem of counting the number of bases and more generally, the number of independent sets of given size, been heavily studied for various classes of matroids. Even more generally, both problems are evaluations of the Tutte polynomial, and other evaluations have also been considered. We refer to [Wel93] for both an introduction to the subject and a detailed, though somewhat dated, survey of known results.

Among more recent work, let us mention  $\#P$ -completeness for the number of trees (of all sizes) in a graph [Jer94], the number of bases in bicircular matroids [GN06], in balanced paving matroids [Jer06], rational matroids [Sno12], and most recently in binary matroids<sup>4</sup> [KN23]. We also note that the volumes of both order polytopes and zonotopes are  $\#P$ -hard, see [BW91, §3] and [DGH98, Thm 1]. See §15.4 for further results and applications.

<sup>4</sup>There is a mild controversy over priority of this result, see a short discussion in [CP24d, §6.3].



Finally, in a major breakthrough, Anari, Liu, Oveis Gharan and Vintant [ALOV19], used Lorentzian polynomials to prove that the basis exchange random walk mixes in polynomial time. This gave a FPRAS for the number of bases of a matroid, making a fast probabilistic algorithm for approximate counting of bases. This resolved an open problem by Mihail and Vazirani (1989). Previously, FPRAS for the number of bases was known for regular matroids [FM92], paving matroids [CW96], and bicircular matroids [GJ21].

### 3. NOTATIONS AND DEFINITIONS

**3.1. Basic notation.** Let  $\mathbb{N} = \{0, 1, 2, \dots\}$  and  $[n] = \{1, \dots, n\}$ . For a set  $A$  and an element  $x \notin A$ , we write  $A + x := A \cup \{x\}$ . Similarly, for an element  $x \in A$ , we write  $A - x := A \setminus \{x\}$ .

We use bold letters  $\mathbf{a} = (a_1, a_2, \dots)$  to denote a sequence of integers, and  $\mathbf{A} = (A_1, \dots, A_n)$  to denote a sequence of sets. We write  $\mathbf{v} = (v_1, \dots, v_d) \in \mathbb{k}^d$  to denote vectors over the field  $\mathbb{k}$ . Let  $\mathbf{e}_1, \dots, \mathbf{e}_d$  denote the standard basis in  $\mathbb{k}^d$ , and let  $\mathbf{0} = (0, \dots, 0) \in \mathbb{k}^d$ . Let  $\mathbb{F}_q$  to denote the finite field with  $q$  elements.

We say that  $\mathbf{v} \in \mathbb{R}^d$  is *strictly positive* if  $v_i > 0$  for all  $1 \leq i \leq d$ . For  $\mathbf{a} = (a_1, \dots, a_d) \in \mathbb{N}^d$ , denote  $|\mathbf{a}| := a_1 + \dots + a_d$ . The support of a vector  $\mathbf{v} = (v_1, \dots, v_d) \in \mathbb{R}^d$  is the set of indices  $i \in [d]$  such that  $v_i \neq 0$ . The support of a symmetric vector  $\mathbf{M} = (M_{i,j})_{i,j \in [d]}$  is the set of indices  $i \in [d]$  such that  $M_{ij} \neq 0$  for some  $j \in [d]$ .

**3.2. Matroids.** A (finite) *matroid*  $\mathcal{M}$  is a pair  $(X, \mathcal{I})$  of a *ground set*  $X$  with  $|X| = n$  elements, and a nonempty collection of *independent sets*  $\mathcal{I} \subseteq 2^X$  that satisfies the following:

- (*hereditary property*)  $A \subset B, B \in \mathcal{I} \Rightarrow A \in \mathcal{I}$ , and
- (*exchange property*)  $A, B \in \mathcal{I}, |A| < |B| \Rightarrow \exists x \in B \setminus A$  s.t.  $A + x \in \mathcal{I}$ .

The *rank* of a matroid is the maximal size of the independent set, i.e.,  $\text{rk}(\mathcal{M}) := \max_{A \in \mathcal{I}} |A|$ . More generally, the *rank*  $\text{rk}(A)$  of a subset  $A \subseteq X$  is the size of the largest independent set contained in  $A$ . A *basis* of  $\mathcal{M}$  is a maximal independent set of  $\mathcal{M}$ , or equivalently an independent set with size  $\text{rk}(\mathcal{M})$ . We denote by  $\mathcal{B}(\mathcal{M})$  the set of bases of  $\mathcal{M}$ .

An element  $x \in X$  is a *loop* if  $\{x\} \notin \mathcal{I}$ , and is a *non-loop* otherwise. Matroid without loops is called *loopless*.<sup>5</sup> We denote by  $\text{NL}(\mathcal{M})$  the set of non-loops of  $\mathcal{M}$ . Two non-loops  $x, y \in \text{NL}(\mathcal{M})$  are *parallel* if  $\{x, y\} \notin \mathcal{I}$ . Note that the parallelship relation between non-loops is an equivalence relation (see e.g. [CP24a, Prop 4.1]). The equivalence classes of this relation are called *parallel classes*.

Given matroids  $\mathcal{M} = (X, \mathcal{I})$ ,  $\mathcal{M}' = (X', \mathcal{I}')$ , the *direct sum*  $\mathcal{M} \oplus \mathcal{M}' := (Y, \mathcal{J})$  is a matroid with ground set  $Y = X \sqcup X'$ , and whose independent sets  $A \in \mathcal{J}$  are disjoint unions of independent sets:  $A = I \sqcup I', I \in \mathcal{I}, I' \in \mathcal{I}'$ .

Let  $x \in \text{NL}(\mathcal{M})$ . The *deletion*  $\mathcal{M} - x$  is the matroid with ground set  $X$  and with independent sets  $\{A \subseteq X - x : A \in \mathcal{I}\}$ . The *contraction*  $\mathcal{M}/x$  is the matroid with ground set  $X$  and with independent sets  $\{A \subseteq X - x : A + x \in \mathcal{I}\}$ . Note that both  $\mathcal{M}/x$  and  $\mathcal{M} - x$  share the same ground set as  $\mathcal{M}$ . This is slightly different than the usual convention, and is adopted here for technical reasons that will be apparent in Section 10.

More generally, for  $B \subseteq \text{NL}(\mathcal{M})$ , the *contraction*  $\mathcal{M}/B$  is the matroid with ground set  $X$  and with independent sets  $\{A \subseteq X - B : A \cup B \in \mathcal{I}\}$ . Recall the *deletion-contraction recurrence* for the number of bases of matroids:

$$\mathcal{B}(\mathcal{M}) = \mathcal{B}(\mathcal{M} - x) + \mathcal{B}(\mathcal{M}/x).$$

A *representation* of a matroid  $\mathcal{M}$  over the field  $\mathbb{k}$  is a map  $\phi : X \rightarrow \mathbb{k}^d$ , such that

$$A \in \mathcal{I} \iff \phi(x_1), \dots, \phi(x_m) \text{ are linearly independent over } \mathbb{k},$$

for every subset  $A = \{x_1, \dots, x_m\} \subseteq X$ . A matroid is *binary* if it has a representation over  $\mathbb{F}_2$ . Matroid is *rational* if it has a representation over  $\mathbb{Q}$ .

<sup>5</sup>Unless stated otherwise, we allow matroids to have loops and parallel elements.

A matroid is *regular* (also called *unimodular*), if it has a representation over every field  $\mathbb{k}$ . Representation  $\phi : X \rightarrow \mathbb{Z}^d$  is called *unimodular* if  $\det(\phi(x_1), \dots, \phi(x_r)) = \pm 1$ , for every basis  $\{x_1, \dots, x_r\} \in \mathcal{B}(\mathcal{M})$ . Regular matroids are known to have a unimodular representation (see e.g. [Ox11, Lem. 2.2.21]).

Let  $G = (V, E)$  be a finite connected graph, and let  $\mathcal{F}$  be the set of *forests* in  $G$  (subsets  $F \subseteq E$  with no cycles). Then  $\mathcal{M}_G = (E, \mathcal{F})$  is a *graphical matroid* corresponding to  $G$ . Bases of the graphical matroid  $\mathcal{M}_G$  are the spanning trees in  $G$ , so  $B(\mathcal{M}_G) = \tau(G)$ . Recall that graphical matroids are regular.

**3.3. Complexity.** We refer to [AB09, Gol08, Pap94] for definitions and standard results in computational complexity, and to [Aar16, Wig19] for a modern overview.

We assume that the reader is familiar with basic notions and results in computational complexity and only recall a few definitions. We use standard complexity classes: P, FP, NP, coNP, #P,  $\Sigma_m^P$ , PH and PSPACE.

The notation  $\{a =^? b\}$  is used to denote the decision problem whether  $a = b$ . We use the *oracle notation*  $R^S$  for two complexity classes  $R, S \subseteq \text{PH}$ , and the polynomial closure  $\langle A \rangle$  for a problem  $A \in \text{PSPACE}$ .

For a counting function  $f \in \#P$ , the *coincidence problem* is defined as  $\{f(x) =^? f(y)\}$ . Note the difference with the *equality verification problem*  $\{f(x) =^? g(x)\}$ . Unless stated otherwise, we use *reduction* to mean a polynomial Turing reduction.

#### 4. REDUCTION FROM COINCIDENCES

**4.1. Setup.** Let  $\mathcal{M} = (X, \mathcal{I})$  be a binary matroid, let  $x \in X$  be a non-loop:  $x \in \text{NL}(\mathcal{M})$ . Define the *basis ratio*

$$\rho(\mathcal{M}, x) := \frac{B(\mathcal{M} - x)}{B(\mathcal{M}/x)}.$$

Denote by #BASES the problem of computing the number of bases  $B(\mathcal{M})$  in  $\mathcal{M}$ . Similarly, denote by #BASISRATIO the problem of computing the basis ratio  $\rho(\mathcal{M}, x)$ .

Let  $\mathcal{M} = (X, \mathcal{I})$  and  $\mathcal{N} = (Y, \mathcal{I}')$  be binary matroids, let  $x \in X$  and  $y \in Y$  be non-loop elements:  $x \in \text{NL}(\mathcal{M})$ ,  $y \in \text{NL}(\mathcal{N})$ . Consider the following decision problem:

$$\text{BASISRATIOCOINCIDENCE} := \{\rho(\mathcal{M}, x) =^? \rho(\mathcal{N}, y)\}.$$

The following is the main technical lemma in the proof.

**Lemma 4.1.** BASISRATIOCOINCIDENCE *reduces to* EQUALITYSY<sub>1</sub>.

The lemma follows from two parsimonious reductions presented below.

**4.2. Deletion-contraction coincidences.** Let  $\mathcal{M} = (X, \mathcal{I})$  be a binary matroid, and let  $x, y \in X$  be non-parallel and non-loop elements. Consider the following decision problem:

$$\text{COINCIDENCEDC} := \{B(\mathcal{M}/x - y) =^? B(\mathcal{M}/y - x)\}.$$

**Lemma 4.2.** COINCIDENCEDC *reduces to* EQUALITYSY<sub>1</sub>.

*Proof.* Let  $\phi : X \rightarrow \mathbb{F}_2^d$  be a binary representation of  $\mathcal{M}$ . Let  $X' := X \cup \{u, v\}$ , where  $u, v$  are two new elements. Consider a matroid  $\mathcal{M}' = (X', \mathcal{I}')$  defined by its binary representation  $\phi' : X' \rightarrow \mathbb{F}_2^{d+1}$ , where

$$\phi'(z) := \begin{cases} (\phi(z), 0) & \text{for } z \in X, \\ (\mathbf{0}, 1) & \text{for } z \in \{u, v\}. \end{cases}$$

That is, we append a zero to the vector representation of all  $z \in X$ , and we represent  $u, v$  by the basis vector  $\mathbf{e}_{d+1}$ .

Let  $r := \text{rk}(\mathcal{M})$  be the rank of  $\mathcal{M}$ , and let  $n := |X|$  be the number of elements. Note that  $\mathcal{M}'$  is a matroid of rank  $r + 1$  and with  $n + 2$  elements. Note also that the bases of  $\mathcal{M}'$  are of the form  $A + u$  and  $A + v$ , where  $A \in \mathcal{B}(\mathcal{M})$  is a basis of  $\mathcal{M}$ .

To define the reduction in the lemma, let

$$\begin{aligned} R &:= \{x, u\}, & a &:= 1, \\ S &:= X - \{x, y\}, & c &:= r - 1. \end{aligned}$$

It then follows that

$$B_{Sc}(\mathcal{M}', R, a + 1) = B(\mathcal{M}/x - y).$$

Indeed,  $B_{Sc}(\mathcal{M}', R, a + 1)$  are subsets  $A \subseteq X \cup \{u, v\}$  that are of the form

$$A \cap R = \{x, u\}, \quad A \cap \{y, v\} = \emptyset, \quad A - \{u\} \in \mathcal{B}(\mathcal{M}).$$

It then follows that  $A - \{x, u\}$  is a basis of  $\mathcal{M}/x - y$ , and that this correspondence is a bijection, proving our claim. By the argument as above, we have:

$$\begin{aligned} B_{Sc}(\mathcal{M}', R, a) &= B(\mathcal{M}/x - y) + B(\mathcal{M}/y - x), \\ B_{Sc}(\mathcal{M}', R, a - 1) &= B(\mathcal{M}/y - x). \end{aligned}$$

We have:

$$\begin{aligned} &P_{Sc}(\mathcal{M}', R, a)^2 - P_{Sc}(\mathcal{M}', R, a + 1) P_{Sc}(\mathcal{M}', R, a - 1) \\ &= \frac{1}{r^2(r+1)^2} [B_{Sc}(\mathcal{M}', R, a)^2 - 4 B_{Sc}(\mathcal{M}', R, a + 1) B_{Sc}(\mathcal{M}', R, a - 1)] \\ &= \frac{1}{r^2(r+1)^2} [(B(\mathcal{M}/x - y) + B(\mathcal{M}/y - x))^2 - 4 B(\mathcal{M}/x - y) B(\mathcal{M}/y - x)] \\ &= \frac{1}{r^2(r+1)^2} (B(\mathcal{M}/x - y) - B(\mathcal{M}/y - x))^2. \end{aligned}$$

Therefore, we have:

$$P_{Sc}(\mathcal{M}', R, a)^2 = P_{Sc}(\mathcal{M}', R, a + 1) P_{Sc}(\mathcal{M}', R, a - 1) \iff B(\mathcal{M}/y - x) = B(\mathcal{M}/y - x),$$

which completes the proof of the reduction.  $\square$

**4.3. Back to ratio coincidences.** Lemma 4.1 now follows from the following reduction.

**Lemma 4.3.** BASISRATIOCOINCIDENCE *reduces to* COINCIDENCEDC.

*Proof.* Let  $\mathcal{M}, \mathcal{N}, x, y$  be the input of BASISRATIOCOINCIDENCE. Let  $\mathcal{M}' := \mathcal{M} \oplus \mathcal{N}$  be the direct sum of matroids  $\mathcal{M}$  and  $\mathcal{N}$ . Note that  $\mathcal{M}'$  is also binary. We have:

$$B(\mathcal{M}'/x - y) = B(\mathcal{M}/x) B(\mathcal{N} - y) \quad \text{and} \quad B(\mathcal{M}'/y - x) = B(\mathcal{M} - x) B(\mathcal{N}/y),$$

which proves the reduction.  $\square$

## 5. PLANAR GRAPHS AND CONTINUED FRACTIONS

**5.1. Graph theoretic definitions.** Throughout this paper  $G = (V(G), E(G))$  will be a graph with vertex set  $V(G)$  and edge set  $E(G)$ , possibly with loops and parallel edges. We will write  $V$  and  $E$  when the underlying graph  $G$  is clear from the context.

For an edge  $e = (v, w) \in E$ , the *deletion*  $G - e$  is the graph obtained by deleting the edge  $e$  from the graph, and the *contraction*  $G/e$  is the graph obtained by identifying  $v$  and  $w$ , and removing the resulting loops. Recall that  $\tau(G)$  denotes the number of spanning trees in  $G$ . Note that  $\tau(G)$  satisfies the *deletion-contraction* recurrence for every non-loop  $e \in E$ :

$$\tau(G) = \tau(G - e) + \tau(G/e).$$

Let  $G = (V, E)$  be a planar graph. For every planar embedding of  $G$ , the *dual graph*  $G^* = (V^*, E)$  is the graph where vertices of  $G^*$  are faces of  $G$ , and each edge is incident to faces of  $G$  that are separated from each other by the edge in the planar embedding. While the dual graph

$G^*$  can depend on the given planar embedding of  $G$ , we will not emphasize that as our proof is constructive and the embedding will be clear from the context.

Note that deletion and contraction for dual graphs swap their meaning. Formally, for an edge  $e \in E$  that is neither a bridge nor a loop, we have:

$$(5.1) \quad \tau(G - e) = \tau(G^*/e) \quad \text{and} \quad \tau(G/e) = \tau(G^* - e).$$

Therefore,  $\rho(G^*, e) = \rho(G, e)^{-1}$ .

**5.2. Continued fraction representation.** Given  $a_0 \geq 0$ ,  $a_1, \dots, a_s \geq 1$ , where  $s \geq 0$ , the corresponding *continued fraction* is defined as follows:

$$[a_0; a_1, \dots, a_s] := a_0 + \frac{1}{a_1 + \frac{1}{\ddots + \frac{1}{a_s}}}.$$

Integers  $a_i$  are called *quotients* or *partial quotients*, see e.g. [HW08, §10.1]. We refer to [Knu98, §4.5.3] for a detailed asymptotic analysis of the quotients in connection with the Euclidean algorithm, and further references.

The following result gives a connection between spanning trees and continued fractions. It is inspired by a similar construction for perfect matchings given in [Sch19, Thm 3.2].

**Theorem 5.1.** *Let  $a_0, \dots, a_s \geq 1$ . Then there exists a connected loopless bridgeless planar graph  $G = (V, E)$  and an edge  $e \in E$ , such that*

$$\frac{\tau(G - e)}{\tau(G/e)} = [a_0; a_1, \dots, a_s]$$

and  $|E| = a_0 + \dots + a_s + 1$ .

We start with the following lemma.

**Lemma 5.2.** *Let  $G = (V, E)$  be a connected loopless bridgeless planar graph, and let  $e \in E$ . Then there exists a connected loopless bridgeless planar graph  $G' = (V', E')$  and  $e' \in E'$  such that*

$$\frac{\tau(G' - e')}{\tau(G'/e')} = 1 + \frac{\tau(G - e)}{\tau(G/e)}$$

and  $|E'| = |E| + 1$ .

*Proof.* Let  $G'$  be obtained from  $G$  by adding an edge  $e'$  that is parallel to  $e$ . Note that  $G' - e'$  is isomorphic to  $G$ , and  $G'/e'$  is isomorphic to  $G/e$ , and it follows that

$$\tau(G'/e') = \tau(G/e) \quad \text{and} \quad \tau(G' - e') = \tau(G) = \tau(G - e) + \tau(G/e).$$

This implies the lemma.  $\square$

**5.3. Proof of Theorem 5.1.** We use induction on  $s$ . For  $s = 0$ , let  $H$  be the graph with two vertices and with  $a_0 + 1$  parallel edges connecting the two vertices, and let  $f$  be any edge of  $H$ . Note that  $H - f$  is the same graph but with  $a_0$  edges instead, while  $H/f$  is the graph with one vertex and  $a_0$  loops. Thus we have

$$\tau(H - f) = a_0 \quad \text{and} \quad \tau(H/f) = 1.$$

We also have  $|E(H)| = a_0 + 1$ . It then follows that

$$\frac{\tau(H - f)}{\tau(H/f)} = a_0,$$

and the claim follows by taking  $G \leftarrow H$  and  $e \leftarrow f$ .

For  $s \geq 1$ , by induction there exists a connected loopless bridgeless planar graph  $H$  and  $f \in E(H)$  such that

$$\frac{\tau(H-f)}{\tau(H/f)} = [a_1; a_2, \dots, a_s],$$

and with  $|E(H)| = a_1 + \dots + a_s + 1$ .

Now, by applying Lemma 5.2  $a_0$  many times to  $H^*$ , there exists a graph  $G$  and a  $e \in E(G)$  such that

$$\frac{\tau(G-e)}{\tau(G/e)} = a_0 + \frac{\tau(H^*-f)}{\tau(H^*/f)} = a_0 + \frac{\tau(H/f)}{\tau(H-f)} = a_0 + \frac{1}{[a_1; a_2, \dots, a_s]} = [a_0; a_1, \dots, a_s],$$

and with  $|E(G)| = a_0 + |E(H^*)| = a_0 + \dots + a_s + 1$ . This completes the proof.  $\square$

**5.4. Sums of continued fractions.** We now extend Theorem 5.1 to sums of two continued fractions:

**Theorem 5.3.** *Let  $a_0, \dots, a_s, b_0, \dots, b_t \geq 1$ . Then there exists a connected loopless bridgeless planar graph  $G = (V, E)$  and an edge  $e \in E$ , such that*

$$\frac{\tau(G-e)}{\tau(G/e)} = \frac{1}{[a_0; a_1, \dots, a_s]} + \frac{1}{[b_0; b_1, \dots, b_t]}$$

and  $|E| = a_0 + \dots + a_s + b_0 + \dots + b_t + 1$ .

We start with the following lemma.

**Lemma 5.4.** *Let  $G, H$  be connected loopless bridgeless planar graphs, and let  $e \in E(G)$ ,  $f \in E(H)$ . Then there exists a connected loopless bridgeless planar graph  $G'$  and an edge  $e' \in E(G')$ , such that*

$$\frac{\tau(G'-e')}{\tau(G'/e')} = \frac{\tau(G-e)}{\tau(G/e)} + \frac{\tau(H-f)}{\tau(H/f)}$$

and  $|E(G')| = |E(G)| + |E(H)| - 1$ .

*Proof.* Let  $e = (x, y) \in E(G)$  and let  $f = (u, v) \in E(H)$ . Consider the graph

$$G' := G \oplus H / (x, u), (y, v)$$

obtained by taking the disjoint union of  $G$  and  $H$ , then identifying  $e$  and  $f$ . Denote by  $e' \in E(G')$  the edge resulted from identifying  $e$  and  $f$ .

First, note that

$$(5.2) \quad \tau(G'/e') = \tau(G/e) \tau(H/f).$$

This is because  $G'/e' = (G/e \oplus H/f) / (x, u)$ , i.e. can be obtained by identifying  $x$  with  $u$  in the disjoint union of  $G/e$  and  $H/f$ .

Second, note that

$$(5.3) \quad \tau(G'-e') = \tau(G-e) \tau(H/f) + \tau(G/e) \tau(H-f).$$

Indeed, let  $T$  be a spanning tree of  $G'-e'$ . There are two possibilities. First,  $x$  and  $y$  are connected in  $T$  through a path in  $G$ . Then, restricting  $T$  to edges of  $G$  gives us a spanning tree in  $G-e$ , while restricting  $T$  to edges of  $H$  gives us a spanning tree of  $H/f$ . This bijection gives us the first term in the RHS of (5.3).

Second, suppose that  $x$  and  $y$  are connected in  $T$  through a path in  $H$ . Then, restricting  $T$  to edges of  $G$  gives us a spanning tree in  $G/e$ , while restricting  $T$  to edges of  $H$  gives us a spanning tree of  $H-f$ . This bijection gives us the second term in the RHS of (5.3).

The lemma now follows by combining (5.2) and (5.3).  $\square$



**5.5. Proof of Theorem 5.3.** By Theorem 5.1, there exists connected loopless bridgeless planar graphs  $G, H$  and  $e \in E(G)$ ,  $f \in E(H)$  such that

$$\frac{\tau(G - e)}{\tau(G/e)} = [a_0; a_1, \dots, a_s], \quad \frac{\tau(H - f)}{\tau(H/f)} = [b_0; b_1, \dots, b_t],$$

and with  $|E(G)| = a_0 + \dots + a_s + 1$ ,  $|E(H)| = b_0 + \dots + b_t + 1$ . Applying Lemma 5.4 to  $(G^*, e)$  and  $(H^*, f)$ , gives a planar graph  $G'$  and  $e' \in E(G')$ , such that

$$\begin{aligned} \frac{\tau(G' - e')}{\tau(G'/e')} &= \frac{\tau(G^* - e)}{\tau(G^*/e)} + \frac{\tau(H^* - f)}{\tau(H^*/f)} = \frac{\tau(G/e)}{\tau(G - e)} + \frac{\tau(H/f)}{\tau(H - f)} \\ &= \frac{1}{[a_0; a_1, \dots, a_s]} + \frac{1}{[b_0; b_1, \dots, b_t]} \end{aligned}$$

and  $E(G') = E(G^*) + E(H^*) - 1 = a_0 + \dots + a_s + b_0 + \dots + b_t + 1$ , as desired.  $\square$

## 6. COUNTING SPANNING TREES

In this section, we prove Theorem 1.13 and Lemma 1.14.

**6.1. Proof of Theorem 1.13.** For  $\alpha \in \mathbb{Q}_{>0}$ , consider the sum of the quotients of  $\alpha$  :

$$s(\alpha) := a_0 + \dots + a_s \quad \text{where} \quad \alpha = [a_0; a_1, \dots, a_s].$$

We will need the following theorem from number theory.

**Theorem 6.1** (Larcher [Lar86, Cor. 2]). *For  $m \geq 9$  and  $L \geq 2$ , the set*

$$\left\{ d \in [m] : \gcd(d, m) = 1 \text{ and } s\left(\frac{d}{m}\right) \leq L \frac{m}{\phi(m)} \log m \log \log m \right\}$$

*contains at least  $\left(1 - \frac{16}{\sqrt{L}}\right) \phi(m)$  many elements, where  $\phi$  is the Euler's totient function.*

First, assume that  $N$  is prime and note that  $\phi(N) = N - 1$ . By Larcher's Theorem 6.1, there exists  $d < N$  such that

$$s\left(\frac{d}{N}\right) \leq C \log N \log \log N \quad \text{for some } C > 0.$$

By Theorem 5.1 and planar duality (5.1), there exists a planar graph  $G = (V, E)$  and edge  $e \in E$ , such that

$$\frac{\tau(G - e)}{\tau(G/e)} = \frac{N}{d} \quad \text{and} \quad |E(G)| \leq 1 + C \log N \log \log N.$$

The conclusion follows by taking  $(G - e)$ .

In full generality, let  $N = p_1^{b_1} \dots p_\ell^{b_\ell}$  be the prime factorization of  $N$ . Let  $G_i = (V_i, E_i)$ ,  $1 \leq i \leq \ell$ , be the planar graphs constructed above:

$$\tau(G_i) = p_i \quad \text{and} \quad |E_i| \leq C \log p_i \log \log p_i.$$

Finally, let  $G = (V, E)$  be a union of  $b_i$  copies of  $G_i$  attached at vertices, so that  $G$  is planar and connected. Clearly,  $\tau(G) = N$  and

$$|E| \leq \sum_{i=1}^{\ell} b_i C \log p_i \log \log p_i \leq \left( \sum_{i=1}^{\ell} b_i \log p_i \right) C \log \log N = C (\log N) \log \log N,$$

as desired.  $\square$

**6.2. Number theoretic estimates.** We start with the following number theoretic estimates that is based on Larcher's Theorem 6.1.

**Proposition 6.2.** *There exists constants  $C, K > 0$ , such that for all coprime integers  $A, B$  which satisfy  $C < B \leq A \leq 2B$ , there exists a positive integer  $m := m(A, B)$  such that  $m < B$ , and*

$$s\left(\frac{m}{A}\right) \leq K(\log A)(\log \log A)^2 \quad \text{and} \quad s\left(\frac{B-m}{A}\right) \leq K(\log A)(\log \log A)^2.$$

*Proof of Proposition 6.2.* Define

$$\zeta(A, B) := \left| \left\{ m \in [B] : s\left(\frac{m}{A}\right) \leq K(\log A)(\log \log A)^2, s\left(\frac{B-m}{A}\right) \leq K(\log A)(\log \log A)^2 \right\} \right|.$$

We will prove a stronger claim, that  $\zeta(A, B) = \Omega(B)$  as  $C \rightarrow \infty$ .

It follows from the inclusion-exclusion, that

$$\begin{aligned} \zeta(A, B) &\geq B - \left| \left\{ m \in [B] : s\left(\frac{m}{A}\right) > K(\log A)(\log \log A)^2 \right\} \right| \\ &\quad - \left| \left\{ m \in [B] : s\left(\frac{B-m}{A}\right) > K(\log A)(\log \log A)^2 \right\} \right|. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &\left| \left\{ m \in [B] : s\left(\frac{m}{A}\right) > K(\log A)(\log \log A)^2 \right\} \right| \\ &\leq \left| \left\{ m \in [A] : s\left(\frac{m}{A}\right) > K(\log A)(\log \log A)^2 \right\} \right| \\ &\leq \left| \left\{ m \in [A] : s\left(\frac{m}{A}\right) > \frac{K}{2} \frac{A}{\phi(A)} \log A \log \log A \right\} \right| \leq 0.2A, \end{aligned}$$

where the second inequality is because  $\frac{A}{\phi(A)} < 2 \log \log A$  for sufficiently large  $A$ , and the third inequality is because of Larcher's Theorem 6.1. Similarly, we have

$$\left| \left\{ m \in [B] : s\left(\frac{B-m}{A}\right) > K \log A (\log \log A)^2 \right\} \right| \leq 0.2A.$$

Combining these inequalities, we get

$$\zeta(A, B) \geq B - 0.4A \geq 0.2B,$$

and the result follows.  $\square$

**6.3. Proof of Lemma 1.14.** It follows from Proposition 6.2, that there exists fixed  $K > 0$  and an integer  $m < B$ , such that

$$s\left(\frac{m}{A}\right) \leq K(\log A)(\log \log A)^2 \quad \text{and} \quad s\left(\frac{B-m}{A}\right) \leq K(\log A)(\log \log A)^2.$$

Let  $[a_0, \dots, a_s]$  and  $[b_0, \dots, b_t]$  be a continued fraction representation of  $A/m$  and  $A/(B-m)$ , respectively. By Theorem 5.3, there exists a connected loopless bridgeless planar graph  $G$  and an edge  $e \in E(G)$ , such that

$$\frac{\tau(G-e)}{\tau(G/e)} = \frac{1}{[a_0; a_1, \dots, a_s]} + \frac{1}{[b_0; b_1, \dots, b_t]} = \frac{B}{A}$$

and

$$|E(G)| = s\left(\frac{m}{A}\right) + s\left(\frac{B-m}{A}\right) + 1 \leq 2K(\log A)(\log \log A)^2 + 1 = O((\log N)(\log \log N)^2).$$

Taking the dual graph  $G^*$  gives the result.  $\square$

## 7. VERIFICATION OF MATROID BASIS RATIOS

Throughout this and the next section, we assume that all matroids are binary and given by their binary representations.

**7.1. Setup.** Let  $\mathcal{M} = (X, \mathcal{I})$  be a binary matroid, let  $x \in \text{NL}(\mathcal{M})$ , and let  $A, B \in \mathbb{N}$ , where  $B > 0$ . Consider the following decision problem:

$$\text{BASISRATIOVERIFICATION} := \left\{ \rho(\mathcal{M}, x) = ? \frac{A}{B} \right\}.$$

**Lemma 7.1** (verification lemma).  $\text{NP}^{\langle \text{BASISRATIOVERIFICATION} \rangle} \subseteq \text{NP}^{\langle \text{BASISRATIOCOINCIDENCE} \rangle}$ .

The proof is broadly similar to that in [CP23a], but with major technical differences. We start with the following simple result.

**Lemma 7.2.** Let  $\mathcal{M} = (X, \mathcal{I})$  be a matroid on  $n = |X|$  elements, and let  $x \in X$  be a non-loop of  $\mathcal{M}$ . Then  $\rho(\mathcal{M}, x) \leq n$ .

*Proof.* To prove that

$$\rho(\mathcal{M}, x) = \frac{B(\mathcal{M} - x)}{B(\mathcal{M}/x)} \leq n,$$

we construct an explicit injection  $\gamma : \mathcal{B}(\mathcal{M} - x) \rightarrow \mathcal{B}(\mathcal{M}/x) \times X$ . Fix a basis  $A \in \mathcal{B}(\mathcal{M})$  such that  $x \in A$ . Such basis  $A$  exists since  $x$  is not a loop. By the symmetric basis exchange property, for every basis  $B \in \mathcal{B}(\mathcal{M})$  such that  $x \notin B$ , there exists  $y \in B$ , such that  $B' := B - y + x$  is a basis of  $\mathcal{M}$ . Now take the lex-smallest of such  $y$ , and define  $\gamma(B) := (B', y)$ . Note that map  $\gamma$  is an injection because  $B$  can be recovered from  $(B', y)$  by taking  $B = B' - x + y$ . This completes the proof.  $\square$

**7.2. Proof of Lemma 7.1.** We now simulate  $\text{BASISRATIOVERIFICATION}$  with an oracle for  $\text{BASISRATIOCOINCIDENCE}$  as follows.

Let  $\mathcal{M} = (X, \mathcal{I})$  be a binary matroid of rank  $\text{rk}(\mathcal{M}) = r$  on  $n = |X|$  elements. Let  $x \in \text{NL}(\mathcal{M})$  and  $A, B \in \mathbb{N}$ , where  $B > 0$ . We can assume that  $A \geq 1$ , as otherwise  $A = 0$  and  $\text{BASISRATIOVERIFICATION}$  is then equivalent with checking if  $B(\mathcal{M} - x) = 0$ , i.e. checking if  $\mathcal{M} - x$  has rank  $(r - 1)$ .

Without loss of generality we can assume that integers  $A$  and  $B$  are coprime. Since we have  $B(\mathcal{M} - x) \leq \binom{n}{r}$  and  $B(\mathcal{M}/x) \leq \binom{n}{r}$ , we can also assume that

$$(7.1) \quad 1 \leq A, B \leq \binom{n}{r},$$

as otherwise  $\text{BASISRATIOVERIFICATION}$  fails. Similarly, by Lemma 7.2 we can also assume that

$$(7.2) \quad \frac{A}{B} \leq n.$$

Let  $A'$  be the positive integer given by

$$A' := B + A - \left\lfloor \frac{A}{B} \right\rfloor B.$$

Note that  $B \leq A' \leq 2B$ . From this point on we proceed following the proof of Lemma 1.14.

It follows from Proposition 6.2 that there exists fixed  $K > 0$  and an integer  $m < B$  such that

$$s\left(\frac{m}{A'}\right) \leq K \log A' (\log \log A')^2 \quad \text{and} \quad s\left(\frac{B-m}{A'}\right) \leq K \log A' (\log \log A')^2.$$

At this point we *guess* such  $m$ . Since computing the quotients of  $m/A'$  can be done in polynomial time, we can verify in polynomial time that  $m$  satisfies the inequalities above.

Let  $[a_0, \dots, a_s]$  and  $[b_0, \dots, b_t]$  be a continued fraction representation of  $A'/m$  and  $A'/(B-m)$ , respectively. By Theorem 5.3, there exists a connected loopless bridgeless planar graph  $G$  and an edge  $e \in E(G)$  such that

$$\frac{\tau(G - e)}{\tau(G/e)} = \frac{1}{[a_0; a_1, \dots, a_s]} + \frac{1}{[b_0; b_1, \dots, b_t]} = \frac{B}{A'}$$

and

$$\begin{aligned} |E(G)| &= s\left(\frac{m}{A'}\right) + s\left(\frac{B-m}{A'}\right) + 1 \leq 2K \log A' (\log \log A')^2 + 1 \\ &\stackrel{(7.1)}{\leq} 2K \log 2 \binom{n}{r} \left( \log \log 2 \binom{n}{r} \right)^2 + 1 = O(n(\log n)^2). \end{aligned}$$

Let  $G'$  and  $e'$  be the graph and edge obtained by applying Lemma 5.2 for  $\lfloor A/B \rfloor - 1$  many times to the planar dual  $G^*$  of  $G$ . Then we have

$$\frac{\tau(G' - e')}{\tau(G'/e')} = \left\lfloor \frac{A}{B} \right\rfloor - 1 + \frac{\tau(G^* - e)}{\tau(G^*/e)} = \left\lfloor \frac{A}{B} \right\rfloor - 1 + \frac{A'}{B} = \frac{A}{B}$$

and

$$E(G') = \left\lfloor \frac{A}{B} \right\rfloor - 1 + |E(G)| \stackrel{(7.2)}{\leq} n - 1 + |E(G)| = O(n(\log n)^2).$$

Now, let  $\mathcal{N} := (E(G'), \mathcal{J})$  be the graphical matroid corresponding to  $G'$ , where  $\mathcal{J}$  are spanning forests in  $G'$ , and let  $y = e$ . Then we have

$$\rho(\mathcal{N}, y) = \rho(G', e') = \frac{\tau(G' - e')}{\tau(G'/e')} = \frac{A}{B}.$$

Thus, the decision problem BASISRATIOVERIFICATION with input  $\mathcal{M}, x, A, B$  can be simulated by BASISRATIOCOINCIDENCE with input  $\mathcal{M}, x, \mathcal{N}, y$ . This completes the proof.  $\square$

## 8. PROOF OF THEOREM 1.2

**8.1. Two more reductions.** We also need two minor technical lemmas:

**Lemma 8.1.** *For all  $k > \ell$ , EQUALITYSY $_{\ell}$  reduces to EQUALITYSY $_k$ .*

*Proof of Lemma 8.1.* Let  $\mathcal{M}, R, a, \mathbf{S} = (S_1, \dots, S_{\ell}), \mathbf{c} = (c_1, \dots, c_{\ell})$  be an input EQUALITYSY $_{\ell}$ . Let  $S_{\ell+1} = \dots = S_k = \emptyset$  and  $c_{\ell+1} = \dots = c_k = 0$ . Let  $\mathbf{S}' := (S_1, \dots, S_k)$  and  $\mathbf{c}' := (c_1, \dots, c_k)$ . It then follows that

$$P_{\mathbf{S}'\mathbf{c}'}(\mathcal{M}, R, a) = P_{\mathbf{S}\mathbf{c}}(\mathcal{M}, R, a).$$

We conclude that the decision problem EQUALITYSY $_{\ell}$  with input  $\mathcal{M}, R, a, \mathbf{S}, \mathbf{c}$ , is equivalent to the decision problem EQUALITYSY $_k$  with input  $\mathcal{M}, R, a, \mathbf{S}', \mathbf{c}'$ .  $\square$

**Lemma 8.2.** *#BASES is polynomial time equivalent to #BASISRATIO.*

*Proof of Lemma 8.2.* Note that #BASISRATIO reduces to #BASES by definition. In the opposite direction, let  $\mathcal{M}$  be a binary matroid of rank  $r = \text{rk}(\mathcal{M})$ . Compute a basis  $\{x_1, \dots, x_r\}$  of  $\mathcal{M}$  by a greedy algorithm. Denote by  $\mathcal{M}_i$  the contraction of  $\mathcal{M}$  by  $\{x_1, \dots, x_i\}$ . We have

$$B(\mathcal{M}) = \frac{B(\mathcal{M}_0)}{B(\mathcal{M}_1)} \cdot \frac{B(\mathcal{M}_1)}{B(\mathcal{M}_2)} \cdots = \left( 1 + \frac{B(\mathcal{M}_0 - x_1)}{B(\mathcal{M}_0/x_1)} \right) \left( 1 + \frac{B(\mathcal{M}_1 - x_2)}{B(\mathcal{M}_1/x_2)} \right) \cdots,$$

which gives the desired reduction.  $\square$

**8.2. Putting everything together.** First, we need the following recent result:

**Theorem 8.3** (Knapp–Noble [KN23, Thm 53]). *#BASES is #P-complete for binary matroids.*

By Lemma 8.2, we conclude that #BASISRATIO is #P-hard. We then have:

$$(8.1) \quad \text{PH} \subseteq \text{P}^{\#P} \subseteq \text{P}^{\langle \text{\#BASISRATIO} \rangle} \subseteq \text{NP}^{\langle \text{BASISRATIOVERIFICATION} \rangle},$$

where the first inclusion is Toda's theorem [Toda91], the second inclusion is because #BASISRATIO is #P-hard, and the third inclusion is because one can simulate #BASISRATIO by first guessing and then verifying the answer.

We now have:

$$(8.2) \quad \begin{aligned} \text{NP}^{\langle \text{BASISRATIOVERIFICATION} \rangle} &\subseteq \text{NP}^{\langle \text{BASISRATIOCOINCIDENCE} \rangle} \\ &\subseteq \text{NP}^{\langle \text{EQUALITYSY}_1 \rangle} \subseteq \text{NP}^{\langle \text{EQUALITYSY}_k \rangle} \end{aligned}$$

where the first inclusion is the Verification Lemma 7.1, the second inclusion is Lemma 4.1, and the third inclusion is Lemma 8.1.

Now, suppose  $\text{EQUALITYSY}_k \in \text{PH}$ . Then  $\text{EQUALITYSY}_k \in \Sigma_m^p$  for some  $m$ . Combining (8.1) and (8.2), this implies:

$$(8.3) \quad \text{PH} \subseteq \text{NP}^{\langle \text{EQUALITYSY}_1 \rangle} \subseteq \text{NP}^{\Sigma_m^p} \subseteq \Sigma_{m+1}^p,$$

as desired.  $\square$

## 9. COMBINATORIAL ATLAS

In this section we give a brief review of the theory of combinatorial atlas. This is the main tool used to prove Theorem 1.3. We will be concise in our explanation of this tool for the sake of brevity and refer the reader to [CP22a, §3, §4] for more detailed introduction, and to [CP24a] for an even more in-depth discussion on this topic.

**9.1. Informal overview.** On a basic level, an inequality  $a^2 \geq bc$  can be viewed as a result that

$$\det(M) \leq 0 \quad \text{where} \quad M := \begin{pmatrix} a & b \\ c & a \end{pmatrix}.$$

In other words, the inequality says that the eigenvalues of  $M$  can be zero, but cannot be of the same sign. In a similar vein, log-concavity of a sequence can be reformulated as a claim that a certain matrix has one positive eigenvalue (OPE). Constructing such a matrix is rather technical, and for the sequence  $\{P(\mathcal{M}, R, a)\}$  this will be done in the next section.

Proving that a matrix has (OPE) is difficult, and is done essentially by a strong induction. In this section, we present the setup (combinatorial atlas) which allows us to formalize this induction and obtain (OPE) in Theorem 9.3 (proved in our prior work). An inelegant but unavoidable feature of the setup is the one-parameter deformation which allows us to prove the result for matrices with no zero eigenvalues, and taking the limit later.

Note that we do not need to reprove the Stanley–Yan inequality (SY), even though our construction does give an independent proof, see Proposition 10.5. An important feature of the combinatorial atlas is its ability to also give equality conditions in certain cases. This requires further assumptions and a technical Theorem 9.6 (also proved in our prior work). We include this setup in this section before using it to prove Theorem 1.3 in the next section.

**9.2. The setup.** Let  $\Gamma = (\Omega, \Theta)$  be a (possibly infinite) acyclic digraph. We denote by  $\Omega^0 \subseteq \Omega$  the set of *sink vertices* in  $\Gamma$  (i.e. vertices without outgoing edges). Similarly, denote by  $\Omega^+ := \Omega \setminus \Omega^0$  the *non-sink vertices*. We denote by  $D_v$  the set of *out-neighbors* of  $v$ , i.e. vertices  $v' \in \Omega$  such that  $(v, v') \in \Theta$ <sup>6</sup>.

**Definition 9.1.** Let  $d$  be a positive integer. A *combinatorial atlas*  $\mathbb{A}$  of dimension  $d$  is an acyclic digraph  $\Gamma := (\Omega, \Theta)$  with an additional structure:

- Each vertex  $v \in \Omega$  is associated with a pair  $(\mathbf{M}_v, \mathbf{h}_v)$ , where  $\mathbf{M}_v$  is a nonnegative symmetric  $d \times d$  matrix, and  $\mathbf{h}_v \in \mathbb{R}_{\geq 0}^d$  is a nonnegative vector.
- Each vertex  $v \in \Omega^+$  has out-degree equal to  $d$ , and the  $i$ -th edge is labeled  $e^{(i)} = (v, v^{(i)})$  for  $1 \leq i \leq d$ .
- Each edge  $e^{(i)}$  is associated to a linear transformation  $\mathbf{T}_v^{(i)} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ .

<sup>6</sup>The set  $D_v$  was denoted by  $v^*$  in [CP22a, CP24a].



We call  $\mathbf{M}_v = (M_{ij})_{i,j \in [d]}$  the *associated matrix* of  $v$ , and  $\mathbf{h} = \mathbf{h}_v = (h_i)_{i \in [d]}$  the *associated vector* of  $v$ . In notation above, we have  $v^{(i)} \in D_v$ , for all  $1 \leq i \leq d$ .

A common objective of the setup above is to demonstrate that the matrices in the atlas satisfy the following property. A matrix  $\mathbf{M}$  is called *hyperbolic*, if

$$(\text{Hyp}) \quad \langle \mathbf{v}, \mathbf{M} \mathbf{w} \rangle^2 \geq \langle \mathbf{v}, \mathbf{M} \mathbf{v} \rangle \langle \mathbf{w}, \mathbf{M} \mathbf{w} \rangle \quad \text{for every } \mathbf{v}, \mathbf{w} \in \mathbb{R}^d, \text{ such that } \langle \mathbf{w}, \mathbf{M} \mathbf{w} \rangle > 0.$$

For the atlas  $\mathbb{A}$ , we say that  $v \in \Omega$  is *hyperbolic*, if the associated matrix  $\mathbf{M}_v$  is hyperbolic, i.e. it satisfies (Hyp). We say that  $\mathbb{A}$  satisfies *hyperbolic property* if every  $v \in \Omega$  is hyperbolic.

Property (Hyp) is equivalent to the following property:

$$(\text{OPE}) \quad \mathbf{M} \text{ has at most one positive eigenvalue (counting multiplicity).}$$

The equivalence between these two properties is well-known in the literature, see e.g. [Gre81], [COSW04, Thm 5.3], [SvH19, Lem. 2.9] and [BH20, Lem. 2.5].

**Lemma 9.2** ([CP24a, Lem. 5.3]). *Let  $\mathbf{M}$  be a symmetric matrix. Then:*

$$\mathbf{M} \text{ satisfies (Hyp)} \iff \mathbf{M} \text{ satisfies (OPE)}.$$

**9.3. Sufficient conditions for hyperbolic property.** In practice, verifying that an atlas satisfies the hyperbolic property can be a nontrivial task. In this subsection we present a set of conditions that are sufficient to imply the hyperbolic property.

We say that the atlas  $\mathbb{A}$  satisfies *inheritance property* if for every non-sink vertex  $v \in \Omega^+$ , we have:

$$(\text{Inh}) \quad (\mathbf{M} \mathbf{v})_i = \langle \mathbf{T}^{(i)} \mathbf{v}, \mathbf{M}^{(i)} \mathbf{T}^{(i)} \mathbf{h} \rangle \quad \text{for every } i \in \text{supp}(\mathbf{M}) \text{ and } \mathbf{v} \in \mathbb{R}^d,$$

where  $\mathbf{M} := \mathbf{M}_v$ ,  $\mathbf{T}^{(i)} = \mathbf{T}_v^{(i)}$ ,  $\mathbf{h} = \mathbf{h}_v$  and  $\mathbf{M}^{(i)} := \mathbf{M}_{v^{(i)}}$  is the matrix associated with  $v^{(i)}$ . Note that for the remainder of this section, we omit the subscript  $v$  from some notations to prevent cluttering the equations.

We say that  $\mathbb{A}$  satisfies *pullback property* if for every non-sink vertex  $v \in \Omega^+$ , we have:

$$(\text{Pull}) \quad \sum_{i \in \text{supp}(\mathbf{M})} h_i \langle \mathbf{T}^{(i)} \mathbf{v}, \mathbf{M}^{(i)} \mathbf{T}^{(i)} \mathbf{v} \rangle \geq \langle \mathbf{v}, \mathbf{M} \mathbf{v} \rangle \quad \text{for every } \mathbf{v} \in \mathbb{R}^d.$$

We say that  $\mathbb{A}$  satisfies *pullback equality property* if for every non-sink vertex  $v \in \Omega^+$ , we have:

$$(\text{PullEq}) \quad \sum_{i \in \text{supp}(\mathbf{M})} h_i \langle \mathbf{T}^{(i)} \mathbf{v}, \mathbf{M}^{(i)} \mathbf{T}^{(i)} \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{M} \mathbf{v} \rangle \quad \text{for every } \mathbf{v} \in \mathbb{R}^d.$$

Clearly (PullEq) implies (Pull).

We say that a non-sink vertex  $v \in \Omega^+$  is *regular* if the following positivity conditions are satisfied<sup>7</sup>:

$$(\text{Irr}) \quad \text{The associated matrix } \mathbf{M}_v \text{ restricted to its support is irreducible.}$$

$$(\text{h-Pos}) \quad \text{Vector } \mathbf{h}_v \text{ is strictly positive when restricted to the support of } \mathbf{M}_v.$$

The following theorem provides a method to establish the hyperbolic property for atlases by reducing the problem to checking the hyperbolic property only for the sink vertices of the atlases.

**Theorem 9.3** (local–global principle, see [CP24a, Thm 5.2], [CP22a, Thm 3.4]). *Let  $\mathbb{A}$  be a combinatorial atlas that satisfies properties (Inh) and (Pull), and let  $v \in \Omega^+$  be a non-sink regular vertex of  $\Gamma$ . Suppose every out-neighbor of  $v$  is hyperbolic. Then  $v$  is also hyperbolic.*

<sup>7</sup>In [CP22a], there was an additional assumption that  $\mathbf{M}_v \mathbf{h}_v$  given in (h-Pos) is strictly positive when restricted to the support of  $\mathbf{M}_v$ . Note that this additional assumption is redundant here because we assume that  $\mathbf{M}_v$  is a nonnegative matrix.

In our applications, the pullback property ([PullEq](#)) is more involved than the inheritance property ([Inh](#)). Below we give sufficient conditions for ([PullEq](#)) that are easier to establish.

We say that the atlas  $\mathbb{A}$  satisfies the *identity property*, if for every non-sink vertex  $v \in \Omega^+$  and every  $i \in \text{supp}(\mathbf{M})$ , we have:

$$(\text{Iden}) \quad \mathbf{T}^{(i)} : \mathbb{R}^d \rightarrow \mathbb{R}^d \text{ is the identity mapping.}$$

We say that  $\mathbb{A}$  satisfies the *transposition-invariant property*, if for every non-sink vertex  $v \in \Omega^+$ ,

$$(\text{T-Inv}) \quad M_{jk}^{(i)} = M_{ki}^{(j)} = M_{ij}^{(k)} \quad \text{for every } i, j, k \in \text{supp}(\mathbf{M}),$$

where  $M_{jk}^{(i)}$  is the  $(j, k)$ -th entry of the matrix  $\mathbf{M}^{(i)}$ .

We say that  $\mathbb{A}$  has the *decreasing support property*, if for every non-sink vertex  $v \in \Omega^+$ ,

$$(\text{DecSupp}) \quad \text{supp}(\mathbf{M}) \supseteq \text{supp}(\mathbf{M}^{(i)}) \quad \text{for every } i \in \text{supp}(\mathbf{M}).$$

**Theorem 9.4** (cf. [[CP24a](#), Thm 6.1], [[CP22a](#), Thm 3.8]). *Let  $\mathbb{A}$  be a combinatorial atlas that satisfies ([Inh](#)), ([Iden](#)), ([T-Inv](#)) and ([DecSupp](#)). Then  $\mathbb{A}$  also satisfies ([PullEq](#)).*

**9.4. Equality conditions for hyperbolic inequalities.** In this subsection, we discuss a strengthening of the hyperbolic property for a given matrix, where the inequality in ([Hyp](#)) is replaced with equality.

A *global pair*  $\mathbf{f}, \mathbf{g} \in \mathbb{R}^d$  is a pair of nonnegative vectors, such that

$$(\text{Glob-Pos}) \quad \mathbf{f} + \mathbf{g} \text{ is a strictly positive vector.}$$

Here  $\mathbf{f}$  and  $\mathbf{g}$  are global in a sense that they are the same for all vertices  $v \in \Omega$ .

Fix a number  $s > 0$ . We say that a vertex  $v \in \Omega$  satisfies ([s-Equ](#)), if

$$(\text{s-Equ}) \quad \langle \mathbf{f}, \mathbf{M}\mathbf{f} \rangle = s \langle \mathbf{g}, \mathbf{M}\mathbf{f} \rangle = s^2 \langle \mathbf{g}, \mathbf{M}\mathbf{g} \rangle,$$

where  $\mathbf{M} = \mathbf{M}_v$  is the matrix associated to  $v$ . Observe that ([s-Equ](#)) implies that equality occurs in ([Hyp](#)) for substitutions  $\mathbf{v} \leftarrow \mathbf{g}$  and  $\mathbf{w} \leftarrow \mathbf{f}$ , since

$$\langle \mathbf{g}, \mathbf{M}\mathbf{f} \rangle^2 = s \langle \mathbf{g}, \mathbf{M}\mathbf{g} \rangle s^{-1} \langle \mathbf{f}, \mathbf{M}\mathbf{f} \rangle = \langle \mathbf{g}, \mathbf{M}\mathbf{g} \rangle \langle \mathbf{f}, \mathbf{M}\mathbf{f} \rangle.$$

The following lemma gives another equivalent condition to check ([s-Equ](#)).

**Lemma 9.5** ([[CP24a](#), Lem 7.2]). *Let  $\mathbf{M}$  be a nonnegative symmetric hyperbolic  $d \times d$  matrix. Let  $\mathbf{f}, \mathbf{g} \in \mathbb{R}^d$  be nonnegative vectors, let  $s > 0$ , and let  $\mathbf{z} := \mathbf{f} - s\mathbf{g}$ . Then ([s-Equ](#)) holds if and only if  $\mathbf{M}\mathbf{z} = 0$ .*

A common objective of setting up an atlas is to identify vertices  $v \in \Omega$  that satisfy ([s-Equ](#)). As it turns out, the property ([s-Equ](#)) for a given vertex  $v \in \Omega$  is equivalent to the statement that *some*, but not *all*, of the out-neighbors of  $v$ , satisfy ([s-Equ](#)). The focus then shifts to identifying neighbors that inherit ([s-Equ](#)), and we describe these vertices as follows.

A vertex  $v \in \Omega^+$  is called a *functional source* if the following conditions are satisfied:

$$(\text{Glob-Proj}) \quad f_j = (\mathbf{T}^{(i)} \mathbf{f})_j \quad \text{and} \quad g_j = (\mathbf{T}^{(i)} \mathbf{g})_j \quad \forall i \in \text{supp}(\mathbf{M}), j \in \text{supp}(\mathbf{M}^{(i)}),$$

$$(\text{h-Glob}) \quad \mathbf{f} = \mathbf{h}_v.$$

Here condition ([Glob-Proj](#)) means that  $\mathbf{f}, \mathbf{g}$  are fixed points of the projection  $\mathbf{T}^{(i)}$  when restricted to the support.

We say that an edge  $e^{(i)} = (v, v^{(i)}) \in \Theta$  is *functional* if  $v$  is a functional source and  $i \in \text{supp}(\mathbf{M}) \cap \text{supp}(\mathbf{h})$ . A vertex  $w \in \Omega$  is a *functional target* of  $v$ , if there exists a directed path  $v \rightarrow w$  in  $\Gamma$  consisting of only functional edges. Note that a functional target is not necessarily a functional source.

The following theorem is the main result in this section and is a key to the proof of Theorem 1.3 we give in the next section.

**Theorem 9.6** (local–global equality principle, [CP24a, Thm 7.1]). *Let  $\mathbb{A}$  be a combinatorial atlas that satisfies properties (Inh), (Pull). Suppose also  $\mathbb{A}$  satisfies property (Hyp) for every vertex  $v \in \Omega$ . Let  $\mathbf{f}, \mathbf{g}$  be a global pair of  $\mathbb{A}$ . Suppose a non-sink vertex  $v \in \Omega^+$  satisfies (s-Equ) with constant  $s > 0$ . Then every functional target of  $v$  also satisfies (s-Equ) with the same constant  $s$ .*

## 10. PROOF OF THEOREM 1.3

**10.1. Combinatorial atlas construction.** In this subsection, we construct a combinatorial atlas, which will be used to prove Theorem 1.3.

Let  $\mathcal{M} := (X, \mathcal{I})$  be a matroid with rank  $r \geq 2$  on  $n := |X|$  elements. Denote by  $X^*$  the set of finite words in the alphabet  $X$ . A word is called *simple* if it contains each letter at most once; we consider only simple words in this paper. For a word  $\alpha \in X^*$ , the *length*  $|\alpha|$  of  $\alpha$  is the number of letters in  $\alpha$ . For two words  $\alpha, \beta \in X^*$ , we denote by  $\alpha\beta \in X^*$  the concatenation of  $\alpha$  and  $\beta$ .

Let  $a \in \{0, \dots, r\}$  and let  $R \subseteq X$ . A word  $\gamma = x_1 \dots x_r \in X^*$  of length  $r$  is called *compatible* with a triple  $(\mathcal{M}, R, a)$ , if

- $\{x_1, \dots, x_r\}$  forms a basis of  $\mathcal{M}$ , and
- $x_1, \dots, x_a \in R$  and  $x_{a+1}, \dots, x_r \in X - R$ .

We denote by  $\text{Comp}(\mathcal{M}, R, a)$  the set of words compatible with  $(\mathcal{M}, R, a)$ . Note that every such word  $\gamma \in \text{Comp}(\mathcal{M}, R, a)$  is simple. It also follows that

$$(10.1) \quad |\text{Comp}(\mathcal{M}, R, a)| = r! P(\mathcal{M}, R, a).$$

For every  $a \in [r - 1]$ , we denote by  $\mathbf{C}(\mathcal{M}, R, a) := (C_{xy})_{x, y \in X}$  the symmetric  $n \times n$  matrix given by

$$(10.2) \quad C_{xy} := \begin{cases} |\text{Comp}(\mathcal{M}/\{x, y\}, R, a - 1)|, & \text{if } x \neq y \text{ and } \{x, y\} \in \mathcal{I}, \\ 0, & \text{if } x = y \text{ or } \{x, y\} \notin \mathcal{I}. \end{cases}$$

Equivalently,  $C_{x, y}$  is given by

$$(10.3) \quad C_{xy} := \begin{cases} |\{\gamma : x\gamma y \in \text{Comp}(\mathcal{M}, R, a)\}| & \text{for } x \in R, y \in X - R, \\ |\{\gamma : xy\gamma \in \text{Comp}(\mathcal{M}, R, a + 1)\}| & \text{for } x, y \in R, \\ |\{\gamma : \gamma xy \in \text{Comp}(\mathcal{M}, R, a - 1)\}| & \text{for } x, y \in X - R. \end{cases}$$

Both definitions will be frequently used throughout this section. It follows from the definition that  $\mathbf{C}(\mathcal{M}, R, a)$  is a nonnegative symmetric matrix, and the diagonal entries of  $\mathbf{C}(\mathcal{M}, R, a)$  are equal to 0. Note that the Stanley–Yan inequality (Theorem 1.1) is a direct consequence of the preceding matrix satisfying (Hyp), as explained below.

Let  $\mathbf{f}, \mathbf{g} \in \mathbb{R}^n$  be the indicator vector of  $R$  and  $X - R$ , respectively. It follows from (DefC-2) and (10.1) that

$$(10.4) \quad \begin{aligned} \langle \mathbf{f}, \mathbf{C}(\mathcal{M}, R, a) \mathbf{g} \rangle &= r! P(\mathcal{M}, R, a), & \langle \mathbf{f}, \mathbf{C}(\mathcal{M}, R, a) \mathbf{f} \rangle &= r! P(\mathcal{M}, R, a + 1), \\ \langle \mathbf{g}, \mathbf{C}(\mathcal{M}, R, a) \mathbf{g} \rangle &= r! P(\mathcal{M}, R, a - 1). \end{aligned}$$

Therefore, Stanley–Yan inequality will follow once we demonstrate that  $\mathbf{C}(\mathcal{M}, R, a)$  satisfies (Hyp) (which we will prove in Proposition 10.5).

We define the combinatorial atlas  $\mathbb{A} = \mathbb{A}(\mathcal{M}, R, a)$  of dimension  $d = n$ , corresponding to the matroid  $\mathcal{M}$ , the subset  $R \subseteq X$ , and the integer  $a \in 2, \dots, r - 1$ , as the acyclic graph and the linear algebraic data defined below.

Let  $\Gamma := \Gamma(\mathcal{M}, R, a) := (\Omega, \Theta)$  be the acyclic graph with  $\Omega = \Omega^0 \cup \Omega^1$ , where

$$\Omega^1 := \{t \in \mathbb{R} \mid 0 \leq t \leq 1\}, \quad \Omega^0 := X.$$

For a non-sink vertex  $v = t \in \Omega^1$  and  $x \in X$ , the corresponding outneighbor in  $\Omega^0$  is  $v^{(x)} := x$ .

For each vertex  $v = x \in \Omega^0$ , the associated matrix is

$$\mathbf{M}_v := \mathbf{C}(\mathcal{M}/x, R, a-1) \quad \text{if } x \text{ is a non-loop of } \mathcal{M},$$

and is equal to the zero matrix otherwise. Note that the ground set of  $\mathcal{M}/x$  is still  $X$  (instead of  $X - x$ ) under our convention. For each vertex  $v = t \in \Omega^1$ , the associated matrix is

$$\mathbf{M}_v := t \mathbf{C}(\mathcal{M}, R, a) + (1-t) \mathbf{C}(\mathcal{M}, R, a-1),$$

and the associated vector  $\mathbf{h}_v := (\mathbf{h}_x)_{x \in V} \in \mathbb{R}^d$  is defined to have coordinates

$$\mathbf{h}_x := \begin{cases} t & \text{if } x \in R, \\ 1-t & \text{if } x \in X - R. \end{cases}$$

Finally, let the linear transformation  $\mathbf{T}_v^{(x)} : \mathbb{R}^d \rightarrow \mathbb{R}^d$  associated to the edge  $(v, v^{(x)})$  to be the identity map.

**10.2. Properties of the constructed matrices.** In this subsection we gather properties of the matrix  $\mathbf{C}(\mathcal{M}, R, a)$  that will be used in the proof. Recall that  $\text{NL}(\mathcal{M})$  is the set of non-loops of  $\mathcal{M}$ .

**Lemma 10.1.** *Let  $\mathcal{M}$  be a matroid of rank  $r \geq 2$ , let  $R \subseteq X$ , and let  $a \in [r-1]$  such that  $P(\mathcal{M}, R, a) > 0$ . Then we have:*

- *the support of  $\mathbf{C}(\mathcal{M}, R, a)$  is equal to  $\text{NL}(\mathcal{M})$ , and*
- *matrix  $\mathbf{C}(\mathcal{M}, R, a)$  is irreducible.*

*Proof.* Since  $P(\mathcal{M}, R, a) > 0$ , there exists a basis  $B$  of  $\mathcal{M}$  such that  $|B \cap R| = a$ . Since  $a \in [r-1]$ , this implies that there exists  $x \in R$  and  $y \in X - R$  such that  $x, y \in B$ . It also follows from (DefC-1) that  $x$  and  $y$  are contained in the same irreducible component of the matrix  $\mathbf{C}(\mathcal{M}, R, a)$ . Now, let  $z$  be an arbitrary non-loop of  $\mathcal{M}$ . For the first claim it suffices to show that  $z \in \text{supp}(\mathbf{C}(\mathcal{M}, R, a))$ , and for the second claim it suffices to show that  $z$  is contained in the same irreducible component as  $x$  and  $y$ . We will without loss of generality assume that  $z \in R$ , as the proof of the other case is analogous.

There are now two possibilities. First suppose that  $z \in B$ . Then  $B' := B - y - z$  is a basis of  $\mathcal{M}/\{y, z\}$  such that  $|B' \cap R| = a-1$ . This implies that  $|\text{Comp}(\mathcal{M}/\{y, z\}, R, a-1)| > 0$ , so it follows from (DefC-1) that  $z$  is contained in the support of  $\mathbf{C}(\mathcal{M}, R, a)$  and  $z$  is contained in the same irreducible component as  $y$ .

Now suppose that  $z \notin B$ . By the symmetric basis exchange property, there exists  $z' \in B$  such that  $A := B - z' + z$  is a basis of  $\mathcal{M}$ . Now, if  $|A \cap R| = a$  (i.e.  $z' \in R$ ), then  $A' := A - y - z$  is a basis of  $\mathcal{M}/\{y, z\}$  satisfying  $|A' \cap R| = a-1$ . This implies that  $|\text{Comp}(\mathcal{M}/\{y, z\}, R, a-1)| > 0$ , so it follows from (DefC-1) that  $z$  is contained in the support of  $\mathbf{C}(\mathcal{M}, R, a)$  and  $z$  is contained in the same irreducible component as  $y$ . On the other hand, if  $|A \cap R| = a+1$  (i.e.  $z' \notin R$ ), then  $A' := A - x - z$  is a basis of  $\mathcal{M}/\{x, z\}$  satisfying  $|A' \cap R| = a-1$ . This implies that  $|\text{Comp}(\mathcal{M}/\{x, z\}, R, a-1)| > 0$ , so it follows from (DefC-1) that  $z$  is contained in the support of  $\mathbf{C}(\mathcal{M}, R, a)$  and  $z$  is contained in the same irreducible component as  $x$ . This completes the proof.  $\square$

**Lemma 10.2.** *Let  $\mathcal{M}$  be a matroid of rank  $r \geq 2$ , let  $R \subseteq X$ , and let  $a \in \{2, \dots, r-1\}$  such that  $P(\mathcal{M}, R, a) > 0$  and  $P(\mathcal{M}, R, a-1) > 0$ . Then, for every  $x \in X$  that is not a loop,*

$$P(\mathcal{M}/x, R, a-1) > 0.$$

*Proof.* We will without loss of generality assume that  $x \in R$ , as the proof of the other case is analogous. By the assumption, there exists a basis  $A$  and  $B$  of  $\mathcal{M}$ , such that  $|A \cap R| = a$  and  $|B \cap R| = a-1$ . Applying the symmetric basis exchange property to  $x$  and  $A$ , we get that there exists a basis  $A'$  of  $\mathcal{M}$  such that  $x \in A'$  and  $|A' \cap R| \in \{a, a+1\}$ . Similarly, by applying the

symmetric basis exchange property to  $x$  and  $B$ , there exists a basis  $B'$  of  $\mathcal{M}$  such that  $x \in B'$  and  $|B' \cap R| \in \{a-1, a\}$ .

If either  $|A' \cap R| = a$  or  $|B' \cap R| = a$  then we are done, since  $A' - x$  (resp.  $B' - x$ ) is then a basis of  $\mathcal{M}/x$  for which  $|(A' - x) \cap R| = a-1$  (resp.  $|(B' - x) \cap R| = a-1$ ). So we assume that  $|A' \cap R| = a-1$  and  $|B' \cap R| = a+1$ . Then by applying the basis exchange properties to  $A'$  and  $B'$  (possibly more than once), there exists a basis  $C'$  of  $\mathcal{M}$  such that  $|C' \cap R| = a$  and  $x \in C'$ , and the claim follows by the same argument as before.  $\square$

**10.3. Properties of the atlas.** In this subsection, we show that the atlas  $\mathbb{A}(\mathcal{M}, R, a)$ , constructed in Section 10.1, satisfies the properties outlined in Section 9.3. This, in turn, enables us to apply tools and methods associated to atlases as described in Section 9.

**Lemma 10.3.** *Let  $\mathcal{M}$  be a matroid of rank  $r \geq 2$ , let  $R \subseteq X$ , and let  $a \in \{2, \dots, r-1\}$ . Then the atlas  $\mathbb{A}(\mathcal{M}, R, a)$  satisfies (Inh), (Iden), (T-Inv).*

*Proof.* The condition (Iden) follows directly from the definition. For (Inh), let  $v := t \in \Omega^1$ , let  $x \in X$ , and let  $\mathbf{M} := \mathbf{M}_v$  be the associated matrix of  $v$ . By linearity, it suffices to prove that, for every  $y \in X$ ,

$$M_{xy} = \langle \mathbf{T}^{(x)} \mathbf{e}_y, \mathbf{M}^{(x)} \mathbf{T}^{(x)} \mathbf{h} \rangle.$$

Now we have

$$\langle \mathbf{T}^{(x)} \mathbf{e}_y, \mathbf{M}^{(x)} \mathbf{T}^{(x)} \mathbf{h} \rangle = \langle \mathbf{e}_y, \mathbf{M}^{(x)} \mathbf{h} \rangle = t \sum_{z \in R} M_{yz}^{(x)} + (1-t) \sum_{z \in X-R} M_{yz}^{(x)}.$$

Now note that, if either  $x$  or  $y$  is a loop of  $\mathcal{M}$ , then it follows from the definition (DefC-1) that the sum above is 0. Also note that in this case we also have  $M_{xy} = 0$  by definition (DefC-1). Hence it suffices to consider the case when both  $x$  and  $y$  are non-loops of  $\mathcal{M}$ . Now, continuing the equation above,

$$\begin{aligned} \langle \mathbf{T}^{(x)} \mathbf{e}_y, \mathbf{M}^{(x)} \mathbf{T}^{(x)} \mathbf{h} \rangle &=_{(\text{DefC-2})} t \sum_{z \in R} |\{\gamma : z\gamma \in \text{Comp}(\mathcal{M}/\{x, y\}, R, a-1)\}| \\ &\quad + (1-t) \sum_{z \in S} |\{\gamma : \gamma z \in \text{Comp}(\mathcal{M}/\{x, y\}, R, a-2)\}| \\ &= t |\text{Comp}(\mathcal{M}/\{x, y\}, R, a-1)| + (1-t) |\text{Comp}(\mathcal{M}/\{x, y\}, R, a-2)| \\ &=_{(\text{DefC-1})} M_{xy}. \end{aligned}$$

This completes the proof of (Inh).

For (T-Inv), it suffices to show that

$$M_{yz}^{(x)} = M_{zx}^{(y)} = M_{xy}^{(z)}$$

holds for all  $x, y, z \in X$ . Note that all three numbers are equal to 0 if either one of  $x$ ,  $y$ , or  $z$  is a loop of  $\mathcal{M}$ , so we assume that  $x, y, z \in \text{NL}(\mathcal{M})$ . In this case, it follows from (DefC-1) that

$$M_{yz}^{(x)} = M_{zx}^{(y)} = M_{xy}^{(z)} = |\text{Comp}(\mathcal{M}/\{x, y, z\}, R, a-2)|.$$

This completes the proof of (T-Inv) and finishes the proof of the lemma.  $\square$

**Lemma 10.4.** *Let  $\mathcal{M}$  be a matroid of rank  $r \geq 2$ , let  $R \subseteq X$ , and let  $a \in \{2, \dots, r-1\}$ , such that  $P(\mathcal{M}, R, a) > 0$  and  $P(\mathcal{M}, R, a-1) > 0$ . Then the atlas  $\mathbb{A}(\mathcal{M}, R, a)$  satisfies (DecSupp).*

*Proof.* Let  $v = t \in \Omega^1$ . In the notation above, we have:

$$\mathbf{M} := \mathbf{M}_v = t \mathbf{C}(\mathcal{M}, R, a) + (1-t) \mathbf{C}(\mathcal{M}, R, a-1).$$



It follows from Lemma 10.1, that the support of  $\mathbf{M}$  is equal to the set  $\text{NL}(\mathcal{M})$  of non-loop elements of  $\mathcal{M}$ . Let  $x$  be an arbitrary element of  $X$ . If  $x$  is a loop of  $\mathcal{M}$ , then

$$\text{supp}(\mathbf{M}^{(x)}) = \emptyset \subseteq \text{supp}(\mathbf{M}).$$

If  $x$  is not a loop of  $\mathcal{M}$ , then

$$\text{supp}(\mathbf{M}^{(x)}) \subseteq \text{NL}(\mathcal{M}/x) \subseteq \text{NL}(\mathcal{M}) = \text{supp}(\mathbf{M}),$$

as desired.  $\square$

**10.4. Hyperbolicity of the constructed atlas.** The following proposition is the main technical result we need in the proof of Theorem 1.3. Also note that this proposition directly implies Stanley–Yan inequality (Theorem 1.1) as previously discussed in Section 10.1.

**Proposition 10.5.** *Let  $\mathcal{M}$  be a matroid of rank  $r \geq 2$ , let  $R \subseteq X$ , and let  $a \in [r - 1]$ , such that  $P(\mathcal{M}, R, a) > 0$ . Then the matrix  $\mathbf{C}(\mathcal{M}, R, a)$  satisfies (Hyp).*

*Proof.* We prove the claim by induction on the rank  $r$  of  $\mathcal{M}$ . First suppose that  $r = 2$ . Note that this implies  $a = 1$ . Write  $(C_{xy})_{x,y \in X} := \mathbf{C}(\mathcal{M}, R, a)$ . It then follows from (DefC-1) that

$$(10.2) \quad C_{x,y} = \begin{cases} 1 & \text{if } x \neq y \text{ and } \{x, y\} \in \mathcal{I}, \\ 0 & \text{if } x = y \text{ or } \{x, y\} \notin \mathcal{I}. \end{cases}$$

In particular, this shows that the  $x$ -row (respectively,  $x$ -column) of  $\mathbf{C}$  is identical to the  $y$ -row (respectively,  $y$ -column) of  $\mathbf{C}$  whenever  $x, y$  are non-loops in the same parallel class. In this case, deduct the  $y$ -row and  $y$ -column of  $\mathbf{C}$  by the  $x$ -row and  $x$ -column of  $\mathbf{C}$ . It then follows that the resulting matrix has  $y$ -row and  $y$ -column is equal to zero, and note that (Hyp) is preserved under this transformation.

Now, apply the above linear transformation repeatedly and remove the zero rows and columns, and let  $\mathbf{C}'$  be the resulting matrix. It suffices to prove that  $\mathbf{C}'$  satisfies (Hyp). Note that  $\mathbf{C}'$  is a  $p \times p$  matrix (where  $p$  is the number of parallel classes of  $\mathcal{M}$ ), with 0s at the diagonal entries and 1s as the non-diagonal entries. It follows from direct calculations that  $(p - 1)$  is the only positive eigenvalue of  $\mathbf{C}'$ , and it follows that  $\mathbf{C}'$  (and thus  $\mathbf{C}$ ) satisfies (Hyp). This proves the base case of the induction.

We now assume that  $r \geq 3$ , and that the claim holds for matroids of rank  $(r - 1)$ . First, suppose that we have  $P(\mathcal{M}, R, a + 1) = P(\mathcal{M}, R, a - 1) = 0$ . Then  $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$  is a direct sum of matroids  $\mathcal{M}_1$  and  $\mathcal{M}_2$ , where  $\mathcal{M}_1$  (resp.  $\mathcal{M}_2$ ) is the matroid obtained from  $\mathcal{M}$  by restricting the ground set to  $R$  (resp.  $S$ ). It then follows that

$$C_{x,y} = \begin{cases} B(\mathcal{M}_1/x)B(\mathcal{M}_2/y) & \text{if } x \in \text{NL}(\mathcal{M}) \cap R \text{ and } y \in \text{NL}(\mathcal{M}) \cap S, \\ 0 & \text{otherwise.} \end{cases}$$

Rescale the rows and columns of  $x \in R$  by  $B(\mathcal{M}_1/x)$ , and the rows and columns of  $y \in S$  by  $B(\mathcal{M}_2/y)$ , and note that these rescalings preserve hyperbolicity. Then  $\mathbf{C}$  becomes a special case of the matrix in (10.2), which was already shown to satisfy (Hyp). So we can assume that either  $P(\mathcal{M}, R, a + 1) > 0$  or  $P(\mathcal{M}, R, a - 1) > 0$ . By the symmetry, we can without loss of generality assume that  $P(\mathcal{M}, R, a - 1) > 0$ .

We split the proof into three parts. First assume that  $a \geq 2$ . Let  $\mathbf{A}(\mathcal{M}, R, a)$  be the atlas defined in §10.1. It follows from Lemma 10.3 and Lemma 10.4 (note that these lemmas require  $a \geq 2$ ), that this atlas satisfies (Inh), (Iden), (T-Inv), and (DecSupp). We now show that, for every sink vertex  $v = x \in \Omega^0$ , the matrix  $\mathbf{M}_v$  satisfies (Hyp). If  $x$  is a loop of  $\mathcal{M}$ , then  $\mathbf{M}_v$  is equal to the zero matrix, which satisfies (Hyp). If  $x$  is a non-loop of  $\mathcal{M}$ , then by definition  $\mathbf{M}_v$  is equal to  $\mathbf{C}(\mathcal{M}/x, R, a - 1)$ . Also note that  $P(\mathcal{M}/x, R, a - 1) > 0$  by Lemma 10.2. It then follows from the induction assumption that  $\mathbf{M}_v$  satisfies (Hyp).

Now every condition in Theorem 9.3 has been verified in the paragraph above, so it follows that every non-sink regular vertex in  $\Gamma$  is hyperbolic. On the other hand, we have from Lemma 10.1 that the vertex  $v := t \in \Omega^1$  is regular if and only if  $t \in (0, 1)$ . Hence this implies that, for every  $t \in (0, 1)$ , the matrix

$$\mathbf{M}_v = t \mathbf{C}(\mathcal{M}, R, a) + (1 - t) \mathbf{C}(\mathcal{M}, R, a - 1),$$

satisfies (Hyp). By taking the limit  $t \rightarrow 0$  and  $t \rightarrow 1$ , it then follows that  $\mathbf{C}(\mathcal{M}, R, a)$  and  $\mathbf{C}(\mathcal{M}, R, a - 1)$  also satisfies (Hyp).

For the second case, assume that  $a = 1$  and  $P(\mathcal{M}, R, a + 1) > 0$ . Then let  $a' := a + 1 = 2$ . Note that we have  $P(\mathcal{M}, R, a') > 0$ ,  $P(\mathcal{M}, R, a' - 1) > 0$ . By the same argument as before, we conclude that  $\mathbf{C}(\mathcal{M}, R, a) = \mathbf{C}(\mathcal{M}, R, a' - 1)$  satisfies (Hyp).

For the third case, assume that  $a = 1$  and  $P(\mathcal{M}, R, a + 1) = 0$ . Since  $P(\mathcal{M}, R, 1) > 0$ , there exists  $A \in \mathcal{B}(\mathcal{M})$  such that  $|A \cap R| = 1$ . Since  $|A| = r \geq 2$ , there exists  $y \in X - R$  such that  $y \in A$ . Let  $\mathcal{M}'$  be the matroid obtained by adding an element  $x'$  that is parallel to  $y$ , and let  $R' := R + x'$ . Observe that  $\mathcal{M}'$  has the same rank as  $\mathcal{M}$ . Note also that  $P(\mathcal{M}', R', 2) > 0$  because  $A' := A - y + x'$  is a basis of  $\mathcal{M}'$  and satisfies  $|A' \cap R'| = 2$ .

Finally, note that  $\mathbf{C}(\mathcal{M}, R, 1)$  can be obtained from  $\mathbf{C}(\mathcal{M}', R', 1)$  by removing the row and column corresponding to  $x'$ . By the same argument as the second case, we conclude that  $\mathbf{C}(\mathcal{M}', R', a)$  satisfies (Hyp). Since (Hyp) is a property that is preserved under restricting to principal submatrices, it then follows that  $\mathbf{C}(\mathcal{M}, R, a)$  also satisfies (Hyp), and the proof is complete.  $\square$

**10.5. Proof of Theorem 1.3.** We will first prove Theorem 1.3 under the assumption that the rank  $r = \text{rk}(\mathcal{M}) = 2$ . Recall that  $\text{Par}_{\mathcal{M}}(x)$  denotes the set of elements of  $\mathcal{M}$  that are parallel to  $x$ .

**Lemma 10.6.** *Let  $\mathcal{M} := (X, \mathcal{I})$  be a matroid of rank 2, and let  $R \subseteq X$  such that  $P(\mathcal{M}, R, 1) > 0$ . Let  $s > 0$  be a positive real number. Then*

$$(10.3) \quad P(\mathcal{M}, R, 2) = s P(\mathcal{M}, R, 1) = s^2 P(\mathcal{M}, R, 0)$$

*if and only if, for every non-loop  $x$  of  $\mathcal{M}$ ,*

$$(10.4) \quad |\text{Par}_{\mathcal{M}}(x) \cap R| = s |\text{Par}_{\mathcal{M}}(x) \cap (X - R)|.$$

*Proof.* Let  $\mathbf{M} := \mathbf{C}(\mathcal{M}, R, 1)$ , and recall that  $\mathbf{f}, \mathbf{g} \in \mathbb{R}^n$  are the indicator vectors of  $R$  and  $X - R$  respectively. It follows from (Cfg) that (10.3) is equivalent to

$$(10.5) \quad \langle \mathbf{f}, \mathbf{M}\mathbf{f} \rangle = s \langle \mathbf{f}, \mathbf{M}\mathbf{g} \rangle = s^2 \langle \mathbf{g}, \mathbf{M}\mathbf{g} \rangle.$$

i.e. (s-Equ) holds. It then follows from Lemma 9.5 that (10.3) is equivalent to  $\mathbf{z} := \mathbf{f} - s\mathbf{g}$  is contained in the kernel of  $\mathbf{M}$ . On the other hand, the matrix  $\mathbf{M}$  is described by

$$M_{x,y} = \begin{cases} 1 & \text{if } x \neq y \text{ and } \{x, y\} \in \mathcal{I}, \\ 0 & \text{if } x = y \text{ or } \{x, y\} \notin \mathcal{I}. \end{cases}$$

It then follows that the kernel of  $\mathbf{M}$  is the set of vectors  $\mathbf{v} \in \mathbb{R}^n$  such that, for every non-loop  $x$  of  $\mathcal{M}$ ,

$$\sum_{y \in \text{Par}_{\mathcal{M}}(x)} v_y = 0.$$

Substituting  $\mathbf{v} \leftarrow \mathbf{z}$ , the equation above is equivalent to

$$|\text{Par}_{\mathcal{M}}(x) \cap R| - s |\text{Par}_{\mathcal{M}}(x) \cap (X - R)| = 0,$$

and the lemma follows.  $\square$

We now give an intermediate lemma which takes us halfway towards Theorem 1.3.

**Lemma 10.7.** *Let  $\mathcal{M} := (X, \mathcal{I})$  be a matroid of rank  $r \geq 3$ , let  $R \subseteq X$ , and let  $a \in \{2, \dots, r-1\}$ , such that  $P(\mathcal{M}, R, a) > 0$ . Finally, let  $s > 0$ . Then*

$$(10.6) \quad P(\mathcal{M}, R, a+1) = s P(\mathcal{M}, R, a) = s^2 P(\mathcal{M}, R, a-1)$$

*holds if and only if for every  $x \in R \cap \text{NL}(\mathcal{M})$ , we have:*

$$(10.7) \quad P(\mathcal{M}/x, R, a) = s P(\mathcal{M}/x, R, a-1) = s^2 P(\mathcal{M}/x, R, a-2) > 0.$$

*Proof.* We first prove the  $\Leftarrow$  direction. Note that

$$\begin{aligned} P(\mathcal{M}, R, a+1) &= \binom{r}{a+1}^{-1} |\{B \in \mathcal{B}(\mathcal{M}) : |B \cap R| = a+1\}|, \\ &= \binom{r}{a+1}^{-1} \frac{1}{a+1} \sum_{x \in R \cap \text{NL}(\mathcal{M})} |\{B \in \mathcal{B}(\mathcal{M}) : |B \cap R| = a+1, x \in B\}| \\ &= \binom{r}{a+1}^{-1} \frac{1}{a+1} \sum_{x \in R \cap \text{NL}(\mathcal{M})} |\{B' \in \mathcal{B}(\mathcal{M}/x) : |B' \cap R| = a\}| \\ &= \frac{1}{r} \sum_{x \in R \cap \text{NL}(\mathcal{M})} P(\mathcal{M}/x, R, a). \end{aligned}$$

Applying (10.7) to the RHS, we get

$$P(\mathcal{M}, R, a+1) = \frac{1}{r} \sum_{x \in R \cap \text{NL}(\mathcal{M})} s P(\mathcal{M}/x, R, a-1) = s P(\mathcal{M}, R, a).$$

By the same argument, we also get  $P(\mathcal{M}, R, a) = s P(\mathcal{M}, R, a-1)$ , as desired.

We now prove the  $\Rightarrow$  direction. Let  $\mathbb{A}(\mathcal{M}, R, a)$  be the combinatorial atlas defined in §10.1. It follows from the assumption, that  $P(\mathcal{M}, R, a) > 0$  and  $P(\mathcal{M}, R, a-1) > 0$ . It follows from Lemma 10.3 and Lemma 10.4 that this atlas satisfies (Inh), (Iden), (T-Inv), and (DecSupp). It also follows from Proposition 10.5, that this atlas satisfies (Hyp). Recall that  $\mathbf{f}, \mathbf{g} \in \mathbb{R}^n$  is the indicator vector of the subset  $R$  and  $X - R$ , respectively. It follows from the definition that  $\mathbf{f}, \mathbf{g}$  is a global pair for this atlas, and that the edge  $(1, x)$  is a functional edge for every  $x \in R$ .

Let  $\mathbf{M} := \mathbf{C}(\mathcal{M}, R, a)$  be the matrix associated to  $v = 1 \in \Omega^1$ . It follows from (Cfg) that (10.6) is equivalent to  $v = 1$  satisfying (s-Equ). By Theorem 9.6, this implies that every vertex  $x \in \Omega^0$  contained in  $R$  also satisfies (s-Equ) with the same constant  $s$ . In other words, for every  $x \in R$ , we have:

$$(10.8) \quad \langle \mathbf{f}, \mathbf{M}^{(x)} \mathbf{f} \rangle = s \langle \mathbf{f}, \mathbf{M}^{(x)} \mathbf{g} \rangle = s^2 \langle \mathbf{f}, \mathbf{M}^{(x)} \mathbf{g} \rangle.$$

On the other hand, for every  $x \in R$  that is a non-loop of  $\mathcal{M}$ , we have  $\mathbf{M}^{(x)}$  is equal to  $\mathbf{C}(\mathcal{M}/x, R, a-1)$  by definition, so it follows from (Cfg) that

$$\begin{aligned} \langle \mathbf{f}, \mathbf{M}^{(x)} \mathbf{f} \rangle &= (r-1)! P(\mathcal{M}/x, R, a), & \langle \mathbf{f}, \mathbf{M}^{(x)} \mathbf{g} \rangle &= (r-1)! P(\mathcal{M}/x, R, a-1), \\ \langle \mathbf{g}, \mathbf{M}^{(x)} \mathbf{g} \rangle &= (r-1)! P(\mathcal{M}/x, R, a-2). \end{aligned}$$

Finally note that  $P(\mathcal{M}/x, R, a) > 0$  by Lemma 10.2. This completes the proof.  $\square$

*Proof of Theorem 1.3.* Note that (1.1) is equivalent to

$$(10.9) \quad P(\mathcal{M}, R, a+1) = s P(\mathcal{M}, R, a) = s^2 P(\mathcal{M}, R, a-1) > 0,$$

for some positive  $s > 0$ . It then suffices to show that (10.9) is equivalent to (1.2) with the same  $s > 0$ .

By applying Lemma 10.7  $a-1$  many times, we have that (10.9) is equivalent to

$$(10.10) \quad P(\mathcal{M}/A, R, 2) = s P(\mathcal{M}/A, R, 1) = s^2 P(\mathcal{M}/A, R, 0) > 0,$$

for every  $A \subseteq R$  that is independent in  $\mathcal{M}$ , and such that  $|A| = a - 1$ . Now note that (10.10) is equivalent to

$$(10.11) \quad P(\mathcal{M}/A, X - R, r - a - 1) = s P(\mathcal{M}/A, X - R, r - a) = s^2 P(\mathcal{M}/A, X - R, r - a + 1) > 0,$$

for every  $A \subseteq R$  that is independent in  $\mathcal{M}$  and such that  $|A| = a - 1$ . By applying Lemma 10.7  $r - a - 1$  many times, it then follows that (10.11) is equivalent to

$$(10.12) \quad P(\mathcal{M}/B, X - R, 0) = s P(\mathcal{M}/B, X - R, 1) = s^2 P(\mathcal{M}/B, X - R, 2) > 0,$$

for every  $B \subseteq R$  that is independent in  $\mathcal{M}$  and such that  $|B| = r - 2$  and  $|B \cap R| = a - 1$ . Noting that  $\mathcal{M}/B$  is a matroid of rank 2, it then follows that (10.12) is equivalent to

$$(10.13) \quad P(\mathcal{M}/B, R, 2) = s P(\mathcal{M}/B, R, 1) = s^2 P(\mathcal{M}/B, R, 0) > 0,$$

for every  $B \subseteq R$  that is independent in  $\mathcal{M}$  and such that  $|B| = r - 2$  and  $|B \cap R| = a - 1$ . The theorem now follows by applying Lemma 10.6 to (10.13).  $\square$

## 11. VANISHING CONDITIONS

**11.1. Setup.** A *discrete polymatroid*  $\mathcal{D}$  is a pair  $([n], \mathcal{J})$  of a ground set  $[n] := \{1, \dots, n\}$  and a nonempty finite collection  $\mathcal{J}$  of integer points  $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{N}^n$  that satisfy the following:

- $\mathbf{a} \in \mathcal{J}$ ,  $\mathbf{b} \in \mathbb{N}^n$  s.t.  $\mathbf{b} \leq \mathbf{a} \Rightarrow \mathbf{b} \in \mathcal{J}$ , and
- $\mathbf{a}, \mathbf{b} \in \mathcal{J}$ ,  $|\mathbf{a}| < |\mathbf{b}| \Rightarrow \exists i \in [n]$  s.t.  $a_i < b_i$  and  $\mathbf{a} + \mathbf{e}_i \in \mathcal{J}$ .

Here  $\mathbf{b} \leq \mathbf{a}$  is a componentwise inequality,  $|\mathbf{a}| := a_1 + \dots + a_n$ , and  $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$  is a standard linear basis in  $\mathbb{R}^n$ . When  $\mathcal{J} \subseteq \{0, 1\}^n$ , discrete polymatroid  $\mathcal{D}$  is a matroid. The role of bases in discrete polymatroids is played by *maximal elements* with respect to the order “ $\leq$ ”. These are also called *M-convex sets* in [Mur03, §1.4] and [BH20, §2]. We refer to [HH02] and [Mur03] for further details on discrete polymatroids.

**11.2. Proof of Theorem 1.6.** Consider  $\mathcal{D}_1 := ([\ell], \mathcal{J}_1)$  defined by

$$(11.1) \quad \mathcal{J}_1 := \{ \mathbf{c} \in \mathbb{N}^\ell : \exists A \in \mathcal{I} \text{ such that } |A \cap S_i| = c_i \text{ for all } i \in [\ell] \}.$$

It follows from the matroid exchange property that  $\mathcal{D}_1$  is a discrete polymatroid. Similarly, consider  $\mathcal{D}_2 := ([\ell], \mathcal{J}_2)$  defined by

$$(11.2) \quad \mathcal{J}_2 := \left\{ \mathbf{c} \in \mathbb{N}^\ell : \sum_{i \in L} c_i \leq \text{rk}(\cup_{i \in L} S_i) \text{ for all } L \in 2^{[\ell]} \right\}.$$

It follows from [HH02, Thm 8.1], that  $\mathcal{D}_2$  is a discrete polymatroid.

The theorem claims that  $\mathcal{J}_1 = \mathcal{J}_2$ . We prove the claim by induction on  $\ell$ . The case  $\ell = 1$  is trivial. We now assume that  $\ell > 1$ , and that the claim holds for smaller values. Note that  $\mathcal{J}_1 \subseteq \mathcal{J}_2$  by definition, so it suffices to show that  $\mathcal{J}_2 \subseteq \mathcal{J}_1$ .

Let  $\mathcal{P}_1, \mathcal{P}_2 \subset \mathbb{R}_+^\ell$  be convex hulls of  $\mathcal{J}_1, \mathcal{J}_2 \subset \mathbb{N}^\ell$ , respectively. Note that  $\mathcal{P}_i$  are convex polytopes with vertices in  $\mathbb{N}^\ell$ , and with  $\mathcal{P}_i \cap \mathbb{N}^\ell = \mathcal{J}_i$ , see [HH02, Thm 3.4]. Hence the theorem follows by showing that all vertices of  $\mathcal{P}_2$  belong to  $\mathcal{J}_1$ . In fact, because  $\mathcal{P}_2$  is closed downward under  $\leq$ , it suffices to prove the claim for every vertex  $\mathbf{c}$  of  $\mathcal{P}_2$  satisfying  $|\mathbf{c}| = \text{rk}(X)$ .

First suppose that  $c_i = 0$  for some  $i \in [\ell]$ . Then it follows from induction that  $\mathbf{c} \in \mathcal{J}_1$  by applying the theorem to the matroid  $\mathcal{M}$  restricted to the ground set  $X \setminus S_i$ . So we assume that  $c_i \geq 1$  for all  $i \in [\ell]$ . Since  $\mathbf{c}$  is a vertex of  $\mathcal{P}_2$ , there exists a non-empty  $L \subsetneq 2^{[\ell]}$ , such that

$$\sum_{i \in L} c_i = \text{rk}(\cup_{i \in L} S_i).$$

Let  $S := \cup_{i \in L} S_i$ . On one hand, it follows from induction that there exists an independent set  $A_1$  of the matroid  $\mathcal{M}$  restricted to the ground  $S$ , that satisfies

$$|A_1 \cap S_i| = c_i \text{ for all } i \in L.$$

On the other hand, it again follows from induction that there exists an independent set  $A_2$  of the matroid  $\mathcal{M}/S$  that satisfies

$$|A_2 \cap S_i| = c_i \quad \text{for all } i \notin L.$$

Let  $A := A_1 \cup A_2$ . Since  $\text{rk}(S) = \sum_{i \in L} c_i$ , it follows that  $A$  is an independent set of  $\mathcal{M}$  satisfying

$$|A \cap S_i| = c_i \quad \text{for all } i \in [\ell].$$

This implies that  $\mathbf{c} \in \mathcal{J}_1$ , which completes the proof.  $\square$

## 12. TOTAL EQUALITY CASES

**12.1. Proof of Corollary 1.8.** To simplify the notation, denote  $p_a := P(\mathcal{M}, R, a)$ . By Proposition 1.4, we have  $p_a > 0$  for all  $0 \leq a \leq r$ . Writing (SY) for all  $1 \leq a < r$ , we get:

$$(12.1) \quad \frac{p_1}{p_0} \geq \frac{p_2}{p_1} \geq \frac{p_3}{p_2} \geq \dots \geq \frac{p_r}{p_{r-1}} > 0.$$

This gives:

$$(12.2) \quad \left( \frac{p_1}{p_0} \right)^r \geq \frac{p_1 \cdot p_2 \cdots p_r}{p_0 \cdot p_1 \cdots p_{r-1}} = \frac{p_r}{p_0},$$

and proves the first part. For the second part, observe that all inequalities in (12.1) must be equalities, which implies the second part.  $\square$

**12.2. Proof of Theorem 1.11, part (1).** As described in the introduction, it remains to show the (ii)  $\Rightarrow$  (iv) implication. It follows from Theorem 1.3, that

$$(12.3) \quad |\text{Par}_{\mathcal{M}/A}(x) \cap R| = s |\text{Par}_{\mathcal{M}/A}(x) \cap (X - R)|,$$

for every independent set  $A \in \mathcal{I}$  of size  $|A| = r - 2$ , and every  $x \in \text{NL}(\mathcal{M}/A)$ . It thus suffices to show that (12.3) implies (iv) for the same value of  $s$ .

We use induction on  $r$ . For  $r = 2$ , the claim follows immediately by applying (12.3), since  $\mathcal{M}$  is loopless and we must have  $A = \emptyset$  in this case.

For  $r > 2$ , suppose the claim holds for all matroids of rank  $(r - 1)$ . Then, for all  $x \in X$ , it follows from applying the claim to  $\mathcal{M}/x$ , that

$$(12.4) \quad |\text{Par}_{\mathcal{M}/x}(y) \cap R| = s |\text{Par}_{\mathcal{M}/x}(y) \cap (X - R)|,$$

for  $y \in \text{NL}(\mathcal{M}/x)$ . In particular, it then follows from (12.4) that

$$(12.5) \quad |\text{NL}(\mathcal{M}/x) \cap R| = s |\text{NL}(\mathcal{M}/x) \cap (X - R)|.$$

On the other hand, it follows from  $\text{NL}(\mathcal{M}) = X$ , that

$$(12.6) \quad \text{NL}(\mathcal{M}/x) = X \setminus \text{Par}_{\mathcal{M}}(x).$$

By combining (12.5) and (12.6), we conclude:

$$(12.7) \quad |\text{Par}_{\mathcal{M}}(x) \cap R| - s |\text{Par}_{\mathcal{M}}(x) \cap (X - R)| = |R| - s |X - R|,$$

for all  $x \in X$ . Summing the equation above over all parallel classes of  $\mathcal{M}$ , we then have

$$|R| - s |X - R| = p (|R| - s |X - R|),$$

where  $p \geq r \geq 3$  is the number of parallel classes of  $\mathcal{M}$ . This implies  $|R| = s |X - R|$ . Together with (12.7) this gives

$$|\text{Par}_{\mathcal{M}}(x) \cap R| = s |\text{Par}_{\mathcal{M}}(x) \cap (X - R)| \quad \text{for all } x \in X,$$

as desired.  $\square$



## 13. EXAMPLES AND COUNTEREXAMPLES

**13.1. Double matroid.** Let  $\mathcal{M}$  be a loopless matroid with a ground set  $X$  that is given by a representation  $\phi : X \rightarrow \mathbb{R}^r$ . Denote by  $X'$  the second copy of  $X$ . Define a matroid  $\mathcal{M}^2$  with the ground set  $Y := X \sqcup X'$ , that is given by a representation  $\psi : Y \rightarrow \mathbb{R}^r$ , where  $\psi(x) := \phi(x)$  and  $\psi(x') := -\phi(x)$ .

Now let  $R \leftarrow X$  and  $S \leftarrow X'$ . By the symmetry, observe that

$$(13.1) \quad |\text{Par}_{\mathcal{M}^2}(y) \cap R| = |\text{Par}_{\mathcal{M}^2}(y) \cap S|,$$

for all  $y \in Y$ . In other words, matroid  $\mathcal{M}^2$  satisfies condition (iv) in Theorem 1.9 with  $s = 1$ . By Theorem 1.11, we conclude that  $\mathcal{M}^2$  has total equality in the Stanley–Yan inequality, i.e. condition (ii) in Theorem 1.9.

Note that (13.1) is a special case of (1.2) with  $s = 1$ . In fact, it is easy to modify this example to make the ratio to be any rational number. Indeed, to get the ratio  $s = a/b$ , take  $a$  copies of  $R$  and  $b$  of  $S$ . One can make all vectors to be distinct by taking different multiples (over  $\mathbb{R}$ ). We omit the easy details.

**13.2. Linear matroid.** Fix  $r \geq 3$ . Let  $X = \mathbb{F}_2^r$  and let  $\mathcal{M}$  be a binary matroid with a ground set  $X$  in its natural representation. Let  $R \subset X$  be a subspace of dimension  $(r - 1)$ . Note that  $\mathbf{0}$  is the only loop in  $\mathcal{M}$ .

Take an independent set of vectors  $A \subset X$  such that  $|A| = r - 2$  and  $|A \cap R| = 0$ . Since  $A \neq \emptyset$ , it is easy to see that for every non-loop  $x \in \text{NL}(\mathcal{M}/A)$ , we have:

$$(13.2) \quad |\text{Par}_{\mathcal{M}/A}(x) \cap R| = |\text{Par}_{\mathcal{M}/A}(x) \cap (X - R)|.$$

This is (1.2) for  $a = 1$ , with  $s = 1$ . By Theorem 1.3, this implies that (1.1) also holds for  $a = 1$ .

Finally, note that  $P(\mathcal{M}, R, 0) = 0$  in this case, which is why this is not an example of total equality condition (ii) in Theorem 1.9 (and why the theorem is inapplicable in any event).

**13.3. Combination matroid.** The previous two examples illustrate different reasons for the equality (1.1) to hold for  $a = 1$ . The following matroid is a combination of the two which still gives equality for  $a = 1$ , but not for  $a > 1$ .

Fix  $r \geq 3$  and let  $V = \mathbb{F}_2^r$ . Let  $R_0 \subset V$  be a subspace of dimension  $(r - 1)$ , let  $S_0 := V \setminus R_0$  be the complement. Let  $R_1, S_1 \subset S_0$  be two copies of the same nonempty set of vectors. Finally, let  $\mathcal{M}$  be a matroid on the ground set  $X := R_0 \sqcup R_1 \sqcup S_0 \sqcup S_1$  and let  $R := R_0 \sqcup R_1$ .

Clearly,  $\text{rk}(R) = \text{rk}(X \setminus R) = r$ , so  $P(\mathcal{M}, R, 0) > 0$  and  $P(\mathcal{M}, R, r) > 0$ . We have (13.2) by a direct computation. By Theorem 1.3, this again implies that (1.1) holds for  $a = 1$ . On the other hand, one can directly check that (1.2) (and thus (1.1)) does not hold for  $a = r - 1$ . We omit the details.

In summary, this gives an example when total equality condition (ii) in Theorem 1.9 fails, even though (iii) holds. This disproves Conjecture 1.10 and proves the second part of Theorem 1.11.<sup>8</sup>

## 14. GENERALIZED MASON INEQUALITY

Let  $\mathcal{M}$  be a matroid of rank  $r = \text{rk}(\mathcal{M})$ , with a ground set  $X$  of size  $|X| = n$ . Denote by  $\mathcal{I}(\mathcal{M}) \subseteq 2^X$  the set of independent sets of  $\mathcal{M}$ . Fix integers  $k \geq 0$  and  $0 \leq a, c_1, \dots, c_k \leq r$ . Additionally, fix disjoint subsets  $S_1, \dots, S_k \subset X$ , and let  $R := X \setminus \cup_i S_i$ . Define

$$\mathcal{I}_{\text{Sc}}(\mathcal{M}, a) := \{A \in \mathcal{I}(\mathcal{M}) : |A \cap R| = a, |A \cap S_1| = c_1, \dots, |A \cap S_k| = c_k\},$$

and let  $\text{I}_{\text{Sc}}(\mathcal{M}, a) := |\mathcal{I}_{\text{Sc}}(\mathcal{M}, a)|$ . Let  $m := n - c_1 - \dots - c_k$  and denote by  $\mathcal{F}_m$  a free matroid on  $m$  elements. Substituting the direct sum  $\mathcal{M} \leftarrow \mathcal{M} \oplus \mathcal{F}_m$  into the Stanley–Yan inequality, we obtain a log-concave inequality:

$$\text{I}_{\text{Sc}}(\mathcal{M}, a)^2 \geq \text{I}_{\text{Sc}}(\mathcal{M}, a + 1) \text{I}_{\text{Sc}}(\mathcal{M}, a - 1).$$

<sup>8</sup>To be precise, matroid  $\mathcal{M}$  is not loopless. To fix this, remove  $\mathbf{0}$  from  $R_0$ .

This is the argument that was used by Stanley in [Sta81, Thm 2.9], to obtain Mason’s log-concave inequality (M1) in the case  $k = 0$ . The following ultra-log-concave inequality is a natural extension.

**Theorem 14.1** (*generalized Mason inequality*). *For all  $1 \leq a \leq \min\{r-1, m-1\}$ , we have:*

$$(14.1) \quad \text{Is}_{\mathbf{c}}(\mathcal{M}, a)^2 \geq \left(1 + \frac{1}{a}\right) \left(1 + \frac{1}{m-a}\right) \text{Is}_{\mathbf{c}}(\mathcal{M}, a+1) \text{Is}_{\mathbf{c}}(\mathcal{M}, a-1).$$

This inequality is an easy consequence of the results by Brändén and Huh [BH20]. We include a short proof for completeness.

*Proof of Theorem 14.1.* We assume that  $X = [n]$ . Let  $f_{\mathcal{M}} \in \mathbb{N}[w_0, w_1, \dots, w_n]$  be a multivariate polynomial defined by

$$f_{\mathcal{M}}(w_0, w_1, \dots, w_n) := \sum_{A \in \mathcal{I}(\mathcal{M})} w_0^{n-|A|} \prod_{i \in A} w_i.$$

It is shown in [BH20, Thm 4.10], that  $f_{\mathcal{M}}$  is Lorentzian.

Take the following substitution:  $w_0 \leftarrow y$ ,  $w_i \leftarrow x$  for  $i \in R$ , and  $w_i \leftarrow z_j$  for  $i \in S_j$ ,  $1 \leq j \leq k$ . Let  $g_{\mathcal{M}}(x, y, z_1, \dots, z_k)$  be the resulting polynomial, and let

$$h_{\mathcal{M}}(x, y) := \left. \frac{\partial^{c_1 + \dots + c_k}}{\partial z_1^{c_1} \dots \partial z_k^{c_k}} \right|_{z_1, \dots, z_k=0} g_{\mathcal{M}}(x, y, z_1, \dots, z_k).$$

Since the Lorentzian property is preserved under diagonalization, taking directional derivatives, and zero substitutions, see [BH20, §2.1], it follows that  $h_{\mathcal{M}}(x, y)$  is a Lorentzian polynomial with degree  $m$ .

Now note that the coefficients  $[x^a y^{m-a}] h_{\mathcal{M}}(x, y)$  is equal to  $\text{Is}_{\mathbf{c}}(\mathcal{M}, a) c_1! \dots c_k!$  by definition. Recall now that a bivariate homogeneous polynomial with nonnegative coefficients is Lorentzian if and only if the sequence of coefficients form an ultra-log-concave sequence with no internal zeros. This implies the result.  $\square$

## 15. FINAL REMARKS AND OPEN PROBLEMS

**15.1. Computational complexity ideas.** Looking into recent developments, one cannot help but admire Rota’s prescience and keen understanding of mathematical development:

*“Anyone who has worked with matroids has come away with the conviction that the notion of a matroid is one of the richest and most useful concepts of our day. Yet, we long, as we always do, for one idea that will allow us to see through the plethora of disparate points of view.” [Rota86]*

Arguably, the idea of *hyperbolicity* is what unites both the combinatorial Hodge theory, Lorentzian polynomials and the combinatorial atlas approaches, even if technical details vary considerably. On the other hand, our complexity theoretic approach is as “disparate” as one could imagine, leaving many mathematical and philosophical questions unanswered.<sup>9</sup>

That an open problem in the old school matroid theory was resolved using tools and ideas from computational complexity might be very surprising to anyone who had not seen theoretical computer science permeate even the most distant corners of mathematics. To those finding themselves in this predicament, we recommend a recent survey [Wig23], followed by richly detailed monograph [Wig19].

<sup>9</sup>Some of these questions related to the nature of the P vs. NP problem are addressed in [Aar16, §§1–4].

**15.2. Negative results for other matroids.** One can ask if Theorem 1.2 extends to other families of matroids given by a succinct presentation. In fact, our proof is robust enough, and extends to every family of matroids which satisfies the following:

- (1) computing the number of bases is  $\#P$ -complete, and
- (2) the family includes all (loopless, bridgeless) graphical matroids.

Notably, matroids realizable over  $\mathbb{Z}$  obviously satisfy (2), and satisfy (1) by [Sno12]. On the other hand, paving matroids based on Hamiltonian cycles considered in [Jer06, §3], easily satisfy (1), but are very far from (2).

For *bicircular matroids*, property (1) was proved in [GN06]. Unfortunately, not all graphical matroids are bicircular matroids, see [Mat77]. In fact, not all graphical matroids are necessarily *transversal* (see e.g. [Ox11, Ex. 1.6.3]), or even a *gammoid* (see e.g. [Ox11, Exc. 11(ii) in §12.3]). Nevertheless, we believe the following:

**Conjecture 15.1.** *Theorem 1.2 holds for bicircular matroids.*

Note that not all bicircular matroids are binary, see [Zas87, Cor. 5.1], so the conjecture would not imply Theorem 1.2. If one is to follow the approach in this paper, a starting point would be Conjecture 6.1 in [CP24d] which is analogous to Theorem 1.13 in this case, and needs to be obtained first. Afterwards, perhaps the proof can be extended to basis ratios as in Lemma 1.14.

**15.3. Generalized Mason inequality.** Denote by  $\text{EQUALITYMASON}_k$  the equality of (14.1) decision problem. As we mentioned earlier,  $\text{EQUALITYMASON}_0$  is in  $\text{coNP}$ . In fact, it is  $\text{coNP}$ -complete, see Corollary 15.3 below. By analogy with Theorem 1.2, it would be interesting to see what happens for general  $k$ :

**Open Problem 15.2.** *For what  $k > 0$ , is  $\text{EQUALITYMASON}_k$  in  $\text{PH}$ ?*

In particular, any explicit description of equality cases for  $\text{EQUALITYMASON}_1$  would be a large step forward and potentially very difficult. We note aside that the combinatorial atlas approach can also be used to prove (14.1). Unfortunately, the specific construction we have in mind cannot be used to describe the equality cases, at least not without major changes.

**15.4. Completeness.** As evident from this paper, the computational complexity of equality cases for matroid inequalities is very interesting and remains largely unexplored. Of course, for Mason’s log-concave inequality (M1), the equality cases are trivial since the sequence satisfies a stronger inequality (M2). On the other hand, for Mason’s ultra-log-concave inequality (M2), the equality cases have a simple combinatorial description:  $\text{girth}(\mathcal{M}) > a + 1$ , i.e. the size of the minimal circuit in the matroid has to be at least  $a + 2$ , see [MNY21] and [CP24a, §1.6] for proofs using Lorentzian polynomials and combinatorial atlases, respectively.

Now, the decision problem  $\text{GIRTH} := \{\text{girth}(\mathcal{M}) \leq a + 1\}$  is in  $\text{NP}$  for matroids with concise presentation. The problem is easily in  $\text{P}$  for graphical matroids via taking powers of adjacency matrix. Recently, it was shown to be in  $\text{P}$  for regular matroids in [FGLS18]. Famously, the problem was shown to be  $\text{NP}$ -complete for binary matroids by Vardy [Var97]. This gives:

**Corollary 15.3.**  *$\text{EQUALITYMASON}_0$  is  $\text{coNP}$ -complete for binary matroids.*

We believe that the upper bound in Corollary 1.5 is optimal:

**Conjecture 15.4.**  *$\text{EQUALITYSY}_0$  is  $\text{coNP}$ -complete for binary and for bicircular matroids.*

Note that for graphical matroids, the number of bases  $B(\mathcal{M})$  is in  $\text{FP}$  by the *matrix-tree theorem*. For regular matroids, the same linear algebraic argument applies. More generally, the number  $B_{\text{sc}}(\mathcal{M}, R, a)$  can also be computed in polynomial time via the weighted (multivariate) version of the matrix-tree theorem (see e.g. [GJ83]). This gives the following observation to contrast with Conjecture 15.4.

**Proposition 15.5.**  *$\text{EQUALITYSY}_k$  is in  $\text{P}$  for regular matroids and all fixed  $k \geq 0$ .*

We conclude with a possible approach to the proof of Conjecture 15.4. The 2-SPANNING-CIRCUIT is a problem whether a matroid has a circuit containing two given elements. For graphical matroids, this problem is in P by *Menger's theorem*. For regular matroids, this problem is in P by a result in [FGLS16] based on *Seymour's decomposition theorem*. One can modify examples in Section 13 to show the following:

**Proposition 15.6.** 2-SPANNING-CIRCUIT *reduces to*  $\neg \text{EQUALITYSY}_0$  *for binary matroids.*

By the proposition, the first part of Conjecture 15.4 follows from the following natural conjecture that would be analogous to Vardy's result for the GIRTH:

**Conjecture 15.7.** 2-SPANNING-CIRCUIT *is NP-complete for binary matroids.*

**15.5. Defect.** Denote by  $\varphi$  the *defect* of the SY inequality:

$$\varphi_{\text{Sc}}(\mathcal{M}, R, a) := P_{\text{Sc}}(\mathcal{M}, R, a)^2 - P_{\text{Sc}}(\mathcal{M}, R, a+1) P_{\text{Sc}}(\mathcal{M}, R, a-1).$$

By definition,  $\varphi$  is a rational function with denominators in FP. If the numerators were also in FP for binary matroids, we would have  $\text{EQUALITYSY}_k \in \text{P}$ , implying that PH collapses for  $k \geq 1$  by Theorem 1.2. In fact, we have a stronger result:

**Proposition 15.8.**  $\varphi$  *is #P-hard for binary matroids and all*  $k \geq 0$ .

*Proof.* For  $k = 0$ , let  $\mathcal{M}$  be a matroid of rank  $r$ , and let  $\mathcal{M}' := \mathcal{M} \oplus \mathcal{M}_1$  be a direct sum of matroid  $\mathcal{M}$  with a matroid with a single element  $v$ . Note that  $\mathcal{M}'$  has rank  $r+1$  and every basis in  $\mathcal{M}'$  must contain  $v$ . Let  $R := \{v\}$ . Observe that  $B(\mathcal{M}', R, 0) = B(\mathcal{M}', R, 2) = 0$  and  $B(\mathcal{M}', R, 1) = |\mathcal{B}(\mathcal{M})|$ . By the definition of  $P_{\text{Sc}}(\mathcal{M}, R, a)$  and Theorem 8.3, we conclude that  $\varphi$  is #P-hard. Finally, for  $k \geq 1$ , the result follows from the proof of Lemma 8.1.  $\square$

Note that the argument above does not imply Conjecture 15.4 since, in the case outlined,  $\text{EQUALITYSY}_0$  reduces to deciding the vanishing of the number of bases of a given matroid, which is trivially in P. We refer to [Pak22] for an extensive overview of complexity of combinatorial inequalities and their defect.

**15.6. Spanning trees.** Note that for simple planar graphs, Stong's Theorem 1.12 is nearly optimal since the number of spanning trees is at most exponential for planar graphs with  $n$  vertices [BS10], or even all graph with bounded average degree, see [Gri76]. This gives  $\alpha(N) = \Omega(\log N)$ , where recall that  $\alpha(N)$  is the smallest number of vertices of a simple planar graph  $G$  with exactly  $N$  spanning trees. In fact, since the number of unlabeled planar graphs with  $n$  vertices is exponential in  $n$ , see e.g. [Noy15, §6.9.2], proving the corresponding upper bound  $\alpha(N) = O(\log N)$  is likely to be very difficult.

On the other hand, it follows from the proof of Theorem 1.13, that, for the smallest number of *edges* of a *non-simple* graph with exactly  $N$  spanning trees, the upper bound of  $\log n$  will be implied by the celebrated *Zaremba's conjecture*, see a discussion and further references in [CP24c] and [CKP24].

**15.7. Understanding the results.** There are several ways to think of our results. First and most straightforward, we completely resolve a 1981 open problem by Stanley by both showing that the equality cases of (SY) cannot have a satisfactory description (from a combinatorial point of view) for  $k > 0$ , and by deriving such a description for  $k = 0$ .

Second, one can think of the results as a showcase for the tools. This includes both the computational complexity and number theoretic approach towards the proof of Theorem 1.2, and the (rather technical) combinatorial atlas approach towards the proof of Theorem 1.3. Let us emphasize that Yan was unable to obtain our Theorem 1.3 using Lorentzian polynomials (cf. [Yan23, §3.3]). While we believe that Theorem 1.3 is not attainable with Lorentzian polynomials, we lack the formal language to make this claim rigorous.

Third, one can think of Theorem 1.2 as an evidence of the strength of Lorentzian polynomials. In combinatorics, some of the most natural combinatorial inequalities are proved by a direct injection. See e.g. [CPP23a, CPP23b, DD85, DDP84] for injective proofs of variations and special cases of (Sta), and to [Mani10] for a rare injective proof of a matroid inequality. Now, if an injection and its inverse (when defined) are poly-time computable, this implies that the equality cases are in  $\text{coNP}$ . Thus, having  $\text{EQUALITYSY}_1 \notin \text{PH}$  shows that Lorentzian polynomials are *powerful*, in a sense that they can prove results beyond elementary combinatorial means.

Finally, this paper gives a rare example of *limits of what is knowable* about matroid inequalities, as opposed to realizability of matroids where various hardness and undecidability results are known, see e.g. [KY22, Sch13]. This is especially in sharp contrast with the equality cases for Mason’s inequalities, which are known to have easy descriptions.

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## REFERENCES

- [Aar16] Scott Aaronson,  $\text{P} \stackrel{?}{=} \text{NP}$ , in *Open problems in mathematics*, Springer, Cham, 2016, 1–122; available at [scottaaronson.com/papers/pnp.pdf](https://scottaaronson.com/papers/pnp.pdf)
- [AHK18] Karim Adiprasito, June Huh and Eric Katz, Hodge theory for combinatorial geometries, *Annals of Math.* **188** (2018), 381–452.
- [AOV18] Nima Anari, Shayan Oveis Gharan and Cynthia Vinzant, Log-concave polynomials, entropy, and a deterministic approximation algorithm for counting bases of matroids, in *Proc. 59th FOCS*, IEEE, Los Alamitos, CA, 2018, 35–46.
- [ALOV19] Nima Anari, Kuikui Liu, Shayan Oveis Gharan and Cynthia Vinzant, Log-concave polynomials II: High-dimensional walks and an FPRAS for counting bases of a matroid, *Annals of Math.* **199** (2024), 259–299.
- [ALOV24] Nima Anari, Kuikui Liu, Shayan Oveis Gharan and Cynthia Vinzant, Log-concave polynomials III: Mason’s Ultra-log-concavity conjecture for independent sets of matroids, *Proc. AMS* **152** (2024), 1969–1981.
- [AB09] Sanjeev Arora and Boaz Barak, *Computational complexity. A modern approach*, Cambridge Univ. Press, Cambridge, UK, 2009, 579 pp.
- [AŠ13] Jernej Azarija and Riste Škrekovski, Euler’s idoneal numbers and an inequality concerning minimal graphs with a prescribed number of spanning trees, *Math. Bohem.* **138** (2013), 121–131.
- [BBL09] Julius Borcea, Petter Brändén and Thomas M. Liggett, Negative dependence and the geometry of polynomials, *Jour. AMS* **22** (2009), 521–567.
- [Brä15] Petter Brändén, Unimodality, log-concavity, real-rootedness and beyond, in *Handbook of enumerative combinatorics*, CRC Press, Boca Raton, FL, 2015, 437–483.
- [BH20] Petter Brändén and June Huh, Lorentzian polynomials, *Annals of Math.* **192** (2020), 821–891.
- [BL23] Petter Brändén and Jonathan Leake, Lorentzian polynomials on cones, preprint (2023), 34 pp.; [arXiv: 2304.13203](https://arxiv.org/abs/2304.13203).
- [Bre89] Francesco Brenti, Unimodal, log-concave and Pólya frequency sequences in combinatorics, *Mem. AMS* **81** (1989), no. 413, 106 pp.
- [BW91] Graham Brightwell and Peter Winkler, Counting linear extensions, *Order* **8** (1991), 225–247.
- [BS10] Kevin Buchin and André Schulz, On the number of spanning trees a planar graph can have, in *Proc. 18th ESA*, Springer, Berlin, 2010, 110–121.
- [BZ88] Yuri D. Burago and Victor A. Zalgaller, *Geometric inequalities*, Springer, Berlin, 1988, 331 pp.
- [CKP24] Swee Hong Chan, Alexander Kontorovich and Igor Pak, Spanning trees and continued fractions, preprint (2024), 20 pp.; [arXiv:2411.18782](https://arxiv.org/abs/2411.18782).

- [CP22a] Swee Hong Chan and Igor Pak, Introduction to the combinatorial atlas, *Expo. Math.* **40** (2022), 1014–1048.
- [CP23a] Swee Hong Chan and Igor Pak, Equality cases of the Alexandrov–Fenchel inequality are not in the polynomial hierarchy, *Forum Math. Pi* **12** (2024), Paper No. e21, 38 pp.
- [CP23b] Swee Hong Chan and Igor Pak, Linear extensions of finite posets, preprint (2023), 55 pp.; [arXiv:2311.02743](#).
- [CP24a] Swee Hong Chan and Igor Pak, Log-concave poset inequalities, *Jour. Assoc. Math. Res.* **2** (2024), 53–153.
- [CP24b] Swee Hong Chan and Igor Pak, Correlation inequalities for linear extensions, *Adv. Math.* **458** (2024), Paper No. 109954, 33 pp.
- [CP24c] Swee Hong Chan and Igor Pak, Linear extensions and continued fractions, *European J. Combin.* **122** (2024), Paper No. 104018, 14 pp.
- [CP24d] Swee Hong Chan and Igor Pak, Computational complexity of counting coincidences, *Theoret. Comput. Sci.* **1015** (2024), Paper No. 114776, 19 pp.
- [CPP23a] Swee Hong Chan, Igor Pak and Greta Panova, Effective poset inequalities, *SIAM J. Discrete Math.* **37** (2023), 1842–1880.
- [CPP23b] Swee Hong Chan, Igor Pak and Greta Panova, Extensions of the Kahn–Saks inequality for posets of width two, *Combinatorial Theory* **3** (2023), no. 1, Paper No. 8, 34 pp.
- [CW96] Laura Chávez Lomelí and Dominic Welsh, Randomised approximation of the number of bases, in *Matroid theory*, AMS, Providence, RI, 1996, 371–376.
- [COSW04] Young-bin Choe, James G. Oxley, Alan D. Sokal and David G. Wagner, Homogeneous multivariate polynomials with the half-plane property, *Adv. Appl. Math.* **32** (2004), 88–187.
- [CW06] Young-bin Choe and David G. Wagner, Rayleigh matroids, *Combin. Probab. Comput.* **15** (2006), 765–781.
- [DD85] David E. Daykin and Jacqueline W. Daykin, Order preserving maps and linear extensions of a finite poset, *SIAM J. Algebraic Discrete Methods* **6** (1985), 738–748.
- [DDP84] David E. Daykin, Jacqueline W. Daykin, and Michael S. Paterson, On log concavity for order-preserving maps of partial orders, *Discrete Math.* **50** (1984), 221–226.
- [DGH98] Martin Dyer, Peter Gritzmann and Alexander Hufnagel, On the complexity of computing mixed volumes, *SIAM J. Comput.* **27** (1998), 356–400.
- [FM92] Tomás Feder and Milena Mihail, Balanced matroids, in *Proc. 24th STOC* (1992), ACM, New York, 26–38.
- [FGLS16] Fedor V. Fomin, Petr A. Golovach, Daniel Lokshtanov and Saket Saurabh, Spanning circuits in regular matroids, *ACM Trans. Algorithms* **15** (2016), no. 4, Art. 52, 38 pp.
- [FGLS18] Fedor V. Fomin, Petr A. Golovach, Daniel Lokshtanov and Saket Saurabh, Covering vectors by spaces: regular matroids, *SIAM J. Discrete Math.* **32** (2018), 2512–2565.
- [GN06] Omer Giménez and Marc Noy, On the complexity of computing the Tutte polynomial of bicircular matroids, *Combin. Probab. Comput.* **15** (2006), 385–395.
- [God84] Christopher D. Godsil, Real graph polynomials, in *Progress in graph theory*, Academic Press, Toronto, ON, 1984, 281–293.
- [Gol08] Oded Goldreich, *Computational complexity. A conceptual perspective*, Cambridge Univ. Press, Cambridge, UK, 2008, 606 pp.
- [GJ83] Ian P. Goulden and David M. Jackson, *Combinatorial enumeration*, John Wiley, New York, 1983, 569 pp.
- [GJ21] Heng Guo and Mark Jerrum, Approximately counting bases of bicircular matroids, *Combin. Probab. Comput.* **30** (2021), 124–135.
- [Haj61] György Hajós, Über eine konstruktion nicht  $n$ -färbbarer graphen (in German), *Wiss. Zeitschrift der Martin-Luther-Univ.* **10** (1961), 116–117.
- [HH02] Jürgen Herzog and Takayuki Hibi, Discrete polymatroids, *J. Algebraic Combin.* **16** (2002), 239–268.
- [Huh18] June Huh, Combinatorial applications of the Hodge–Riemann relations, in *Proc. ICM Rio de Janeiro*, vol. IV, World Sci., Hackensack, NJ, 2018, 3093–3111.
- [HSW22] June Huh, Benjamin Schröter and Botong Wang, Correlation bounds for fields and matroids, *Jour. Eur. Math. Soc.* **24** (2022), 1335–1351.
- [IP22] Christian Ikenmeyer and Igor Pak, What is in  $\#P$  and what is not?, preprint (2022), 82 pp.; [arXiv:2204.13149](#); extended abstract in *Proc. 63rd FOCS* (2022), 860–871.
- [IPP24] Christian Ikenmeyer, Igor Pak and Greta Panova, Positivity of the symmetric group characters is as hard as the polynomial time hierarchy, *Int. Math. Res. Not.* (2024), no. 10, 8442–8458.
- [Jer94] Mark Jerrum, Counting trees in a graph is  $\#P$ -complete, *Inform. Process. Lett.* **51** (1994), no. 3, 111–116.
- [Jer06] Mark Jerrum, Two remarks concerning balanced matroids, *Combinatorica* **26** (2006), 733–742.
- [Gre81] Jiří Gregor, On quadratic Hurwitz forms, *Apl. Mat.* **26** (1981), 142–153.
- [Gri76] Geoffrey R. Grimmett, An upper bound for the number of spanning trees of a graph, *Discrete Math.* **16** (1976), 323–324.



- [HW08] Godfrey H. Hardy and Edward M. Wright, *An introduction to the theory of numbers* (sixth ed., revised), Oxford Univ. Press, Oxford, 2008, 621 pp.
- [KS84] Jeff Kahn and Michael Saks, Balancing poset extensions, *Order* **1** (1984), 113–126.
- [Kal23] Gil Kalai, The work of June Huh, in *Proc. ICM 2022*, Vol. 1, Prize lectures, EMS Press, Berlin, 2023, 50–65.
- [KN23] Christopher Knapp and Steven Noble, The complexity of the greedoid Tutte polynomial, preprint (2022), 44 pp.; [arXiv:2309.04537](https://arxiv.org/abs/2309.04537).
- [KM22] Tomer Kotek and Johann A. Makowsky, The exact complexity of the Tutte polynomial, in *Handbook of the Tutte Polynomial and Related Topics*, 2022, 175–193.
- [Knu98] Donald E. Knuth, *The art of computer programming. Vol. 2. Seminumerical algorithms* (third ed.), Addison-Wesley, Reading, MA, 1998, 762 pp.
- [KS21] Noah Kravitz and Ashwin Sah, Linear extension numbers of  $n$ -element posets, *Order* **38** (2021), 49–66.
- [KY22] Lukas Kühne and Geva Yashfe, Representability of matroids by  $c$ -arrangements is undecidable, *Israel J. Math.* **252** (2022), 95–147.
- [Lar86] Gerhard Larcher, On the distribution of sequences connected with good lattice points, *Monatsh. Math.* **101** (1986), 135–150.
- [MS24] Zhao Yu Ma and Yair Shenfeld, The extremals of Stanley’s inequalities for partially ordered sets, *Adv. Math.* **436** (2024), Paper 109404, 72 pp.
- [Mani10] Arun P. Mani, Some inequalities for Whitney–Tutte polynomials, *Combin. Probab. Comput.* **19** (2010), 425–439.
- [Mat77] Laurence R. Matthews, Bicircular matroids, *Quart. J. Math.* **28** (1977), 213–227.
- [MNY21] Satoshi Murai, Takahiro Nagaoka and Akiko Yazawa, Strictness of the log-concavity of generating polynomials of matroids, *J. Combin. Theory, Ser. A* **181** (2021), Paper 105351, 22 pp.
- [Mur03] Kazuo Murota, *Discrete convex analysis*, SIAM, Philadelphia, PA, 2003, 389 pp.
- [Noy15] Marc Noy, Graphs, in *Handbook of enumerative combinatorics*, CRC Press, Boca Raton, FL, 2015, 397–436.
- [Ox11] James Oxley, *Matroid theory* (second ed.), Oxford Univ. Press, Oxford, UK, 2011, 684 pp.
- [Pak19] Igor Pak, Combinatorial inequalities, *Notices AMS* **66** (2019), 1109–1112; an expanded version of the paper is available at [tinyurl.com/py8sv5v6](https://tinyurl.com/py8sv5v6)
- [Pak22] Igor Pak, What is a combinatorial interpretation?, in *Open Problems in Algebraic Combinatorics*, AMS, Providence, RI, 2024, 191–260.
- [Pap94] Christos H. Papadimitriou, *Computational Complexity*, Addison-Wesley, Reading, MA, 1994, 523 pp.
- [Rota86] Gian-Carlo Rota, Foreword, in Joseph P. S. Kung, *A source book in matroid theory*, Boston, MA, 1986, 413 pp.
- [Sch13] Marcus Schaefer, Realizability of graphs and linkages, in *Thirty essays on geometric graph theory*, Springer, New York, 2013, 461–482.
- [Sch19] Ralf Schiffler, Snake graphs, perfect matchings and continued fractions, *Snapshots of modern mathematics from Oberwolfach*, 2019, No. 1, 10 pp.; available at [publications.mfo.de/handle/mfo/1405](https://publications.mfo.de/handle/mfo/1405)
- [Sch85] Rolf Schneider, On the Aleksandrov–Fenchel inequality, in *Discrete geometry and convexity*, New York Acad. Sci., New York, 1985, 132–141.
- [Sch88] Rolf Schneider, On the Aleksandrov–Fenchel inequality involving zonoids, *Geom. Dedicata* **27** (1988), 113–126.
- [Sch03] Alexander Schrijver, *Combinatorial optimization. Polyhedra and efficiency*, vols. A–C, Springer, Berlin, 2003, 1881 pp.
- [Sed70] Jiří Sedláček, On the minimal graph with a given number of spanning trees, *Canad. Math. Bull.* **13** (1970), 515–517.
- [SvH19] Yair Shenfeld and Ramon van Handel, Mixed volumes and the Bochner method, *Proc. AMS* **147** (2019), 5385–5402.
- [SvH22] Yair Shenfeld and Ramon van Handel, The extremals of Minkowski’s quadratic inequality, *Duke Math. J.* **171** (2022), 957–1027.
- [SvH23] Yair Shenfeld and Ramon van Handel, The extremals of the Alexandrov–Fenchel inequality for convex polytopes, *Acta Math.* **231** (2023), 89–204.
- [Sno12] Michael Snook, Counting bases of representable matroids, *Elec. J. Combin.* **19** (2012), no. 4, Paper 41, 11 pp.
- [Sta81] Richard P. Stanley, Two combinatorial applications of the Aleksandrov–Fenchel inequalities, *J. Combin. Theory, Ser. A* **31** (1981), 56–65.
- [Sta86] Richard P. Stanley, Two poset polytopes, *Discrete Comput. Geom.* **1** (1986), no. 1, 9–23.
- [Sta89] Richard P. Stanley, Log-concave and unimodal sequences in algebra, combinatorics, and geometry, in *Graph theory and its applications*, New York Acad. Sci., New York, 1989, 500–535.
- [Sto22] Richard Stong, Minimal graphs with a prescribed number of spanning trees, *Australas. J. Combin.* **82** (2022), 182–196.

- [Toda91] Seinosuke Toda,  $\text{PP}$  is as hard as the polynomial-time hierarchy, *SIAM J. Comput.* **20** (1991), 865–877.
- [Urq97] Alasdair Urquhart, The graph constructions of Hajós and Ore, *J. Graph Theory* **26** (1997), 211–215.
- [Var97] Alexander Vardy, The intractability of computing the minimum distance of a code, *IEEE Trans. Inform. Theory* **43** (1997), 1757–1766.
- [vHYZ23] Ramon van Handel, Alan Yan and Xinneng Zeng, The extremals of the Kahn–Saks inequality, *Adv. Math.* **456** (2024), Paper No. 109892, 38 pp.
- [Wel76] Dominic J. A. Welsh, *Matroid theory*, Academic Press, London, UK, 1976, 433 pp.
- [Wel93] Dominic J. A. Welsh, *Complexity: knots, colourings and counting*, Cambridge Univ. Press, Cambridge, UK, 1993, 163 pp.
- [Wig19] Avi Wigderson, *Mathematics and computation*, Princeton Univ. Press, Princeton, NJ, 2019, 418 pp.; available at [math.ias.edu/avi/book](https://math.ias.edu/avi/book)
- [Wig23] Avi Wigderson, Interactions of computational complexity theory and mathematics, in *Proc. ICM 2022*, Vol. 2, Plenary lectures, EMS Press, Berlin, 2023, 1392–1432.
- [Yan23] Alan Yan, *Log-concavity in combinatorics*, senior thesis, Princeton University, 2023, 140 pp.; [arXiv: 2404.10284](https://arxiv.org/abs/2404.10284).
- [YK75] Andrew C. Yao and Donald E. Knuth, Analysis of the subtractive algorithm for greatest common divisors, *Proc. Nat. Acad. Sci. USA* **72** (1975), 4720–4722.
- [Zas87] Thomas Zaslavsky, The biased graphs whose matroids are binary, *J. Combin. Theory, Ser. B* **42** (1987), 337–347.

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