Hidden symmetries of weighted lozenge tilings

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Abstract

We study the weighted partition function for lozenge tilings, with weights given by multivariate rational functions originally defined in [MPP3] in the context of the factorial Schur functions. We prove that this partition function is symmetric for large families of regions. We employ both combinatorial and algebraic proofs.

Mathematics Subject Classifications: 05A19, 05E05

1 Introduction

Hidden symmetries are pervasive across the natural sciences, but are always a delight whenever discovered. In Combinatorics, they are especially fascinating, as they point towards both advantages and limitations of the tools, cf. §5.1. Roughly speaking, the combinatorial approach strips away much of the structure, be it algebraic, geometric, etc., while allowing a direct investigation often resulting in an explicit resolution of a problem. But this process comes at a cost — when the underlying structure is lost, some symmetries become invisible, or "hidden".

Occasionally this process runs in reverse. When a hidden symmetry is discovered for a well-known combinatorial structure, it is as surprising as it is puzzling, since this points to a rich structure which yet to be understood (sometimes uncovered many years later). This is the situation of this paper.

We enumerate the (weighted) lozenge tilings of regions on a triangular lattice. These tiling problems appear in a number of interrelated areas: from general tiling literature [Thu] to combinatorics of plane partitions [Kra], to statistical physics of the dimer model [Gor]. First studied by MacMahon, Kasteleyn and Temperley–Fisher in other settings, these lozenge tilings are now extremely well understood by tools of the determinant calculus,

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algebraic combinatorics and integrable probability (see Section 5). Yet our hidden symmetries appear to be new (see, however, §5.2).

The results of this paper are somewhat technical, but the backstory is quite interesting. We start with a classical result of MacMahon: the number P_{abc} of plane partitions which fit into $[a \times b \times c]$ box is given by a product formula:

$$P_{abc} = \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2}, \tag{1.1}$$

If you think of these *boxed plane partitions* as 3-dimensional objects and squint your eyes, you see that they are in natural bijection with lozenge tilings of the $\langle a \times b \times c \times a \times b \times c \rangle$ hexagon $\mathrm{H}\langle a, b, c \rangle$, see Figure 1.

There are numerous extensions and generalizations of (1.1), and it is key to many recent probabilistic studies. On a combinatorial side, there is a classical q-analogue $P_{abc}(q)$ by the "volume" of the tilings, which corresponds to the size of the plane partition. If one views (1.1) as an evaluation of the Schur function, this q-analogue is given by

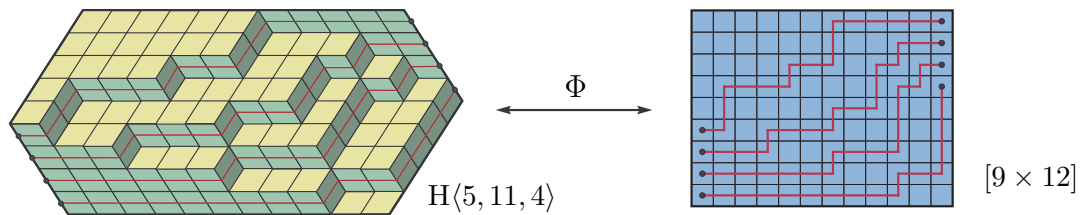$$P_{abc}(q) = q^{-a(a+1)b/2} \cdot s_{(b^a)}(1, q, \ldots, q^{c-1}).$$

Figure 1: A lozenge tiling of $\mathrm{H}\langle 5, 11, 4 \rangle$ and the corresponding collection of non-intersecting paths in $[9 \times 12]$.

When the bottom rectangle (b^a) is replaced by a Young diagram λ, there is *Stanley's hook-content formula* for $s_\lambda(1, q, \ldots, q^{c-1})$. There are many other exact product formulas for various further extensions, some related to other root systems and symmetry classes recently surveyed in [Kra], some with surprising coincidences and hidden symmetries [Ste].

On a probabilistic side, there is a celebrated *Arctic circle* phenomenon first discovered in [CLP] for $\mathrm{H}\langle n, n, n \rangle$, and then extended to general regions in [CKP]. This work led to an incredible wealth of results on the *limit shapes* and *random surfaces*, most of which goes outside the scope of this paper, see an extensive survey [Gor]. Let us single out [BGR] which gives a 5-parameter elliptic deformations (with one relation) of $P_{abc}(q)$, and computed the exact asymptotic formulas for the limit shape.

Our approach to a multivariate deformation of P_{abc} is based on the recent work [MPP3] in Algebraic Combinatorics, in turn inspired by the extensive study of the (equivariant) cohomology of the Grassmannian. To set this up, recall that the lozenge tilings of $\mathrm{H}\langle a, b, c \rangle$

are in bijection with collections of non-intersecting paths in the rectangle, see Figure 1. These lattice paths are in bijection with the *excited diagrams*, thus giving a connection to the *Naruse hook-length formula* [MPP1, MPP2] the number of standard Young tableaux of skew shapes.

In [MPP3], the authors introduce a multivariate deformation $F_{abc}(x_1, x_2, \ldots \mid y_1, y_2, \ldots)$ of P_{abc} with two sets of variables which play a superficially similar role:

$$F_{abc}(1, 1, \ldots \mid 0, 0, \ldots) = F_{abc}(0, 0, \ldots \mid 1, 1, \ldots) = P_{a(b-1)c}.$$

The key technical result in [MPP3] is the symmetry of F_{abc}.

Formally, the Morales–Pak–Panova (MPP–) Theorem 2.1, shows that $F_{abc}(\boldsymbol{x} \mid \boldsymbol{y})$ is symmetric in the first set of variables $\boldsymbol{x} = (x_1, x_2, \ldots)$, with the second set $\boldsymbol{y} = (y_1, y_2, \ldots)$ as parameters, and vice versa (see §5.6). This result is derived from the algebraic properties of the *factorial symmetric functions* defined by Macdonald in one of his "variations" [Mac]. These symmetric functions were later studied by Molev–Sagan [MS], Ikeda–Naruse [IN], and others, in connection with the *equivariant Schubert calculus*. The authors use a special case of this hidden symmetry to give product formulas for the number of standard Young tableaux $\mathrm{SYT}(\lambda/\mu)$, for a 6-parameter family $\{\lambda/\mu\}$ of skew Young diagrams.

We obtain two generalizations and refinements of the MPP–theorem, to:
  (1) *trapezoid* (sawtooth) *regions* obtained from $\mathrm{H}(a, b, c)$ by horizontal cuts,
  (2) *parallelogram regions* obtained from $\mathrm{H}(a, b, c)$ by two vertical cuts.
Formally, for general regions Γ, we define a multivariate partition function $F(\boldsymbol{x} \mid \boldsymbol{y})$ by summing over all lozenge tilings of Γ. In case (1), we show that $F(\boldsymbol{x} \mid \boldsymbol{y})$ is symmetric in $\boldsymbol{x}$, and in case (2) we show that $F(\boldsymbol{x} \mid \boldsymbol{y})$ is symmetric in $\boldsymbol{y}$. Both results generalize (two parts of) the MPP–theorem, which until now had only a technical proof based on the properties of factorial Schur functions. We then obtain a common generalization Main Theorem 3.2. We leave open the problem of finding probabilistic and enumerative applications of these general hidden symmetries.

The rest of the paper is structured as follows. We start by stating both the background and the results in Section 2, followed by their lozenge tilings interpretation and quick pointers to the literature. In the following two sections (Section 3 and 4), we give completely independent combinatorial and algebraic proofs of the results, including the proof of Main Theorem 3.2. We conclude with final remarks and open problems in Section 5.

2 Main results

2.1 Known results

We start with the MPP–theorem mentioned in the introduction:

Theorem 2.1 (Morales–Pak–Panova [MPP3, Thm 3.10]). Define the following multivariate rational function:

$$F_{abc}\big(x_1, \ldots, x_{a+c} \mid y_1, \ldots, y_{b+c}\big) := \sum_{\substack{\Upsilon = (\gamma_1, \ldots, \gamma_c) \\ \gamma_k : (a+k, 1) \to (k, b+c)}} \prod_{k=1}^{c} \prod_{(i,j) \in \gamma_k} \frac{1}{x_i + y_j}, \qquad (2.1)$$

where the sum is over all collections Υ of non-intersecting lattice paths in the $[(a+c)\times(b+c)]$ rectangle (see Figure 2). Then $F_{abc}(\boldsymbol{x}\,|\,\boldsymbol{y})$ is symmetric in $\boldsymbol{x}=(x_1,\dots,x_{a+c})$ and in $\boldsymbol{y}=(y_1,\dots,y_{b+c})$.

Strictly speaking, Theorem 2.1 follows from the proof of Thm 3.10 in [MPP3], but not from the statement.
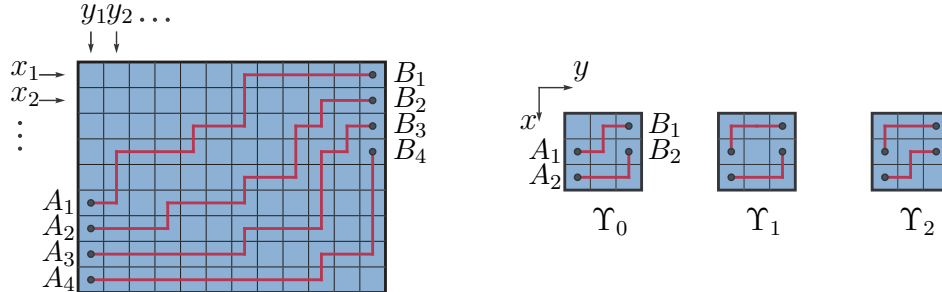

Figure 2: <u>Left</u>: An example of a collection Υ of c paths as in Theorem 2.1, where $a=5$, $b=8$, and $c=4$. <u>Right</u>: An example of all three possible paths $\Upsilon_0, \Upsilon_1, \Upsilon_2$ for $a=1$, $b=1$, and $c=2$.

Here and everywhere below we adopt the coordinate system that is standard for matrices: the first coordinate x is increasing downwards and the second coordinate y is increasing from left to right (see Figure 2).

Example 2.2. Let $a=1$, $b=1$ and $c=2$. We have $A_1=(2,1)$, $A_2=(3,1)$, $B_1=(1,3)$ and $B_2=(2,3)$, and $\Upsilon=(\gamma_1,\gamma_2)$ are non-intersecting paths $\gamma_1:A_1\to B_1$ and $\gamma_2:A_2\to B_2$ inside a 3×3 square. There are three such Υ avoiding either $(1,1)$, or $(2,2)$, or $(3,3)$, see Figure 2 (Right). For example, $\Upsilon_0=(\gamma_1,\gamma_2)$, where $\gamma_1:A_1=(2,1)\to(2,2)\to(1,2)\to(1,3)=B_1$, and $\gamma_2:A_2=(3,1)\to(3,2)\to(3,3)\to(2,3)=B_2$, i.e., Υ_0 is avoiding $(1,1)$. We have:

$$F_{132}(\boldsymbol{x}\,|\,\boldsymbol{y}) = w(\Upsilon_0)+w(\Upsilon_1)+w(\Upsilon_2) = \Big[(x_1+y_1)+(x_2+y_2)+(x_3+y_3)\Big]\prod_{i=1}^{3}\prod_{j=1}^{3}\frac{1}{x_i+y_j},$$

which is symmetric in $\boldsymbol{x}$ and in $\boldsymbol{y}$ (but not in *both* $\boldsymbol{x}$ and $\boldsymbol{y}$).

Let us emphasize that although the symmetry in both sets of variables may seem to play the same role, the result is not symmetric under the transposition giving $\boldsymbol{x}\leftrightarrow\boldsymbol{y}$. In fact, these are fundamentally different symmetries: the one in $\boldsymbol{x}$ is both difficult and interesting, while the one in $\boldsymbol{y}$ is relatively straightforward. As we mentioned in the introduction, the two generalizations we present each retain only one of these symmetries.

2.2 New results

There is a natural way to generalize the setting of Theorem 2.1. Let $[m\times n]=\{(p,q)\in\mathbb{N}^2, 1\leqslant p\leqslant m, 1\leqslant q\leqslant n\}$, $\mathcal{A}=(A_1,\dots,A_k)$, $\mathcal{B}=(B_1,\dots,B_k)$ be two k-tuples of points

in $[m \times n]$. Denote by $\Upsilon : \mathcal{A} \to \mathcal{B}$ a collection $(\gamma_1, \ldots, \gamma_k)$ of non-intersecting lattice paths $\gamma_i : A_i \to B_i$, and let $N(\mathcal{A}, \mathcal{B}) := \#\{\Upsilon : \mathcal{A} \to \mathcal{B}\}$ be the number of such collections. Throughout the paper, unless stated otherwise, all paths will use only Up and Right steps, where the coordinates are arranged as in Figure 2 (see also §5.9).

Note that for fixed $\mathcal{A}, \mathcal{B} \subset \mathbb{N}^2$, the set $\{\Upsilon : \mathcal{A} \to \mathcal{B}\}$ is a classical combinatorial object which generalizes Dyck paths, plane partitions, Young tableaux, etc. [GJ, Ch. 5]. Under mild conditions, the number $N(\mathcal{A}, \mathcal{B})$ has a determinant formula via the *Lindström–Gessel–Viennot* (LGV–) *lemma* (see §4.2). As we discussed in the introduction, for $\mathcal{A}, \mathcal{B}$ as in Theorem 2.1, the number $N(\mathcal{A}, \mathcal{B})$ of non-intersecting collections of paths $\Upsilon : \mathcal{A} \to \mathcal{B}$ is equal to P_{abc} given by (1.1).

Define the *weight* of Υ as

$$w(\Upsilon) := \prod_{i=1}^{k} w(\gamma_i), \quad \text{where} \quad w(\gamma) := \prod_{(i,j) \in \gamma} \frac{1}{x_i + y_j}.$$

Let

$$F_{\mathcal{A},\mathcal{B}}(x_1, \ldots, x_m \mid y_1, \ldots, y_n) := \sum_{\Upsilon : \mathcal{A} \to \mathcal{B}} w(\Upsilon). \tag{2.2}$$

Note that F is not symmetric for general $\mathcal{A}, \mathcal{B}$. For example, let $k = 2$, $A_1 = B_1 = (1, 1)$, $A_2 = B_2 = (2, 2)$. Then $N(\mathcal{A}, \mathcal{B}) = 1$, and

$$F_{\mathcal{A},\mathcal{B}}(x_1, x_2 \mid y_1, y_2) = \frac{1}{(x_1 + y_1)(x_2 + y_2)},$$

which is not symmetric in either set of variables. Since there is no apparent action of either symmetric group on the paths collections Υ in Theorem 2.1, the theorem represents a *hidden symmetry*, and raises the following general question:

Question 1. Are there other sets $\mathcal{A}, \mathcal{B} \subset [m \times n]$, for which the multivariate generating function $F_{\mathcal{A},\mathcal{B}}(\boldsymbol{x} \mid \boldsymbol{y})$ is symmetric in $(x_1, \ldots, x_m)$?

We give two positive answers to this question, refining both symmetries in Theorem 2.1 :

Theorem 2.3 (Horizontal cut). Let $m = a + k$, $A_1 = (a + 1, 1)$, $\ldots$, $A_k = (m, 1)$, and $\mathcal{A} = (A_1, \ldots, A_k)$. Similarly, let $B_1 = (1, b_1)$, $\ldots$, $B_k = (1, b_k)$, for some $1 \leqslant b_1 < b_2 < \ldots < b_k \leqslant n$, and $\mathcal{B} = (B_1, \ldots, B_k)$. Then the multivariate function

$$F_{\mathcal{A},\mathcal{B}}(x_1, \ldots, x_m \mid y_1, \ldots, y_n)$$

defined in (2.2), is symmetric in $\boldsymbol{x} = (x_1, \ldots, x_m)$.

See Figure 3 for the explanation of the *horizontal cut* in the title. Let us show that Theorem 2.3 implies the $\boldsymbol{x}$-symmetry part of Theorem 2.1, for $a \geqslant c$. Apply Theorem 2.3 to two adjacent cuts: above and below row $(a + 1)$, including the latter into both parts. Of course, to apply Theorem 2.3 to the top part, rotate it 180 degrees. We obtain that F_{abc} is symmetric in both $(x_1, \ldots, x_{a+1})$ and in $(x_{a+1}, \ldots, x_{a+c})$, implying the symmetry in $(x_1, \ldots, x_{a+c})$, for every fixed start/end points $\mathcal{C}$ in row $(a + 1)$. Summing over all such $\mathcal{C}$, we obtain Theorem 2.1.
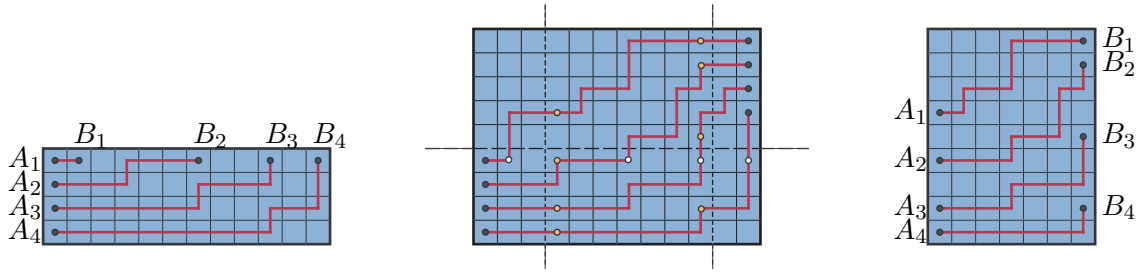
Figure 3: Examples of paths collections in Theorems 2.3 and 2.4 and how they refine Theorem 2.1.

Theorem 2.4 (Vertical double cut). Let $A_1 = (a_1, 1)$, ..., $A_k = (a_k, 1)$, for some $1 \leqslant a_1 < a_2 < \ldots < a_k \leqslant k + \ell$, and $\mathcal{A} = (A_1, \ldots, A_k)$. Similarly, let $m \geqslant 1$, $B_1 = (b_1, m)$, ..., $B_k = (b_k, m)$, for some $1 \leqslant b_1 < b_2 < \ldots < b_k \leqslant k + \ell$, and $\mathcal{B} = (B_1, \ldots, B_k)$. Then the multivariate function

$$F_{\mathcal{A},\mathcal{B}}\big(x_1, \ldots, x_{k+\ell} \,|\, y_1, \ldots, y_m\big),$$

defined in (2.2), is symmetric in $\boldsymbol{y} = (y_1, \ldots, y_m)$.

In the theorem, one can assume that $a_i \geqslant b_i$ for all $i = 1, \ldots, k$, since otherwise there are no collections of `Up-Right` paths Υ, and the claim is vacuously true (cf. §5.9). We should mention that this generalization of the $\boldsymbol{y}$-symmetry part of Theorem 2.1 is conceptually more straightforward, as it both contains it as a special case and refines it, see Figure 3.

Remark 2.5. Darij Grinberg (private communication) suggested the following way to deduce Theorem 2.3 from Theorem 2.4. Denote $C_i = (m, i)$ for $i = 1, \ldots, k$, $\mathcal{C} = (C_1, \ldots, C_k)$. There is a natural weight-preserving bijection between collections of paths $\Upsilon_a : \mathcal{A} \to \mathcal{B}$ and $\Upsilon_c : \mathcal{C} \to \mathcal{B}$: replace the horizontal initial segments $[A_i, (a + i, i)]$ in Υ_a to the vertical initial segments $[C_i, (a + i, i)]$ in Υ_c.

2.3 Lozenge tilings formulation

Let us recall the bijection Φ in Figure 1 which allows us to translate the lattice paths results into statements about lozenge tilings. Start with $\Upsilon = (\gamma_1, \ldots, \gamma_c)$ in the rectangle $S := [(a + c) \times (b + c)]$. Place points in the middle of edges of the opposite c edges in $H = H\langle a, b - 1, c\rangle$ as in the Figure 1. Think of paths γ_i in S in as a union of edges. Start with vertices in the lower left edge of H as in the Figure. For every `Right` edge in γ_i, make a `Right` edge through a light green lozenge in H. Similarly, for every `Up` edge in γ_i make a `Up-Right` (diagonal) edge through a dark green lozenge in H. When all of Υ is mapped onto H, we obtain a partial tiling of the hexagon with light and dark green lozenges. Fill the remaining space with yellow lozenges. This completes the construction of Φ.

We refer to [MPP3, §7] for more details and properties of this bijection, reformulation of Theorem 2.1 into the lozenge language and several applications. We should also mention that our deformation $F_{abc}(\boldsymbol{x}\,|\,\boldsymbol{y})$ for $x_i = q^i$, $y_j = -q^{-j}$, is well-known as a q-Racah special case studied in [BGR], see [MPP3, §9.6] for a detailed explanation.

Now, consider the *trapezoid* (sawtooth) *region* $\Gamma(c_1, \ldots, c_k)$ defined as in Figure 4. This region corresponds to Theorem 2.3 with $a = 0$ and $b_1 = 1 + c_1$, $b_2 = 1 + c_1 + c_2$, ...,
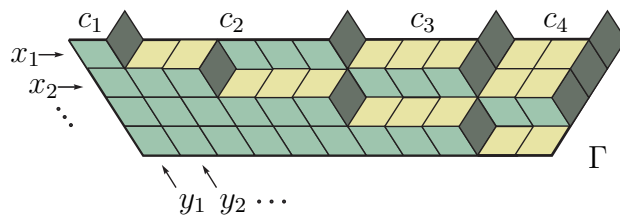

Figure 4: Lozenge tiling of a trapezoid region $\Gamma = \Gamma(1, 5, 3, 2)$ for $k = 4$.

$b_k = 1 + c_1 + \ldots + c_k$. For the example in Figure 4 the region $\Gamma(1, 5, 3, 2)$ corresponds to $\mathcal{A} = \{(1, 1), (1, 2), (1, 3), (1, 4)\}$ and $\mathcal{B} = \{(1, 2), (1, 7), (1, 10), (1, 12)\}$, as in Figure 3 (left).

In fact, the lozenge tilings of regions $\Gamma(\boldsymbol{c})$ are heavily studied in integrable probability, see [Nov, Pet]. The total number $N(\lambda)$ of such tilings is given by the formula

$$N(\lambda) = s_\lambda(1^k) = \prod_{1 \leqslant i < j \leqslant k} \frac{b_j - b_i}{j - i},$$

where $\lambda = (\lambda_1, \ldots, \lambda_k)$, and $\lambda_i = b_{k+1-i} - k + i$ for all $1 \leqslant i \leqslant k$. We refer to [Gor, §19] for an interesting discussion of this special case, further results and references.

Theorem 2.3 thus gives a multivariate deformation of $N(\lambda)$. The weights $1/(x_i + y_j)$ are assigned to light green lozenges and bottom halves of dark green lozenges as shown in Figure 4. Yellow lozenges get weight 1. The weight of a tiling is then a product of weights of all lozenges. The resulting partition function is then the sum of all weights of lozenge tilings of fixed Γ as above. By Theorem 2.3, this function is symmetric.
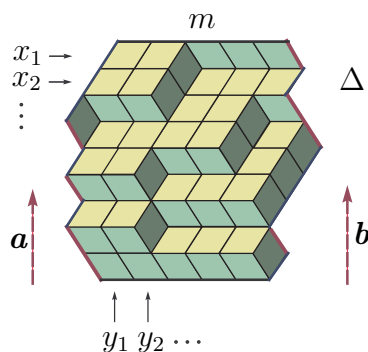

Figure 5: Lozenge tiling of a parallelogram region $\Delta = \Delta(\boldsymbol{a}, \boldsymbol{b}, m)$, for $k = 4$, $\ell = m = 5$.

Note that in every simply connected region tileable with lozenges, the boundary has $2k, 2\ell$ and $2m$ edges in each of the three directions. A *parallelogram region* Δ is defined to

have two intervals of m consecutive (say, horizontal) edges. This condition automatically implies that between the horizontal edges there are $(k+\ell)$ edges on each side, see Figure 5. The region is thus encoded $\Delta = \Delta(\boldsymbol{a}, \boldsymbol{b}, m)$ by two increasing sequences $\boldsymbol{a} = (a_1, \ldots, a_k)$ and $\boldsymbol{b} = (b_1, \ldots, b_k)$, where $1 \leqslant a_i, b_i \leqslant k + \ell$. For example, for the region in the figure, we have $k = 4$, $\ell = m = 5$, and the sequences are $\boldsymbol{a} = (1, 2, 3, 5)$, and $\boldsymbol{b} = (2, 6, 7, 9)$.

In these notation, Theorem 2.4 proves the $\boldsymbol{x}$-symmetry of the multivariate deformation of the number $N(\boldsymbol{a}, \boldsymbol{b}, m)$ of tilings of a parallelogram region $\Delta(\boldsymbol{a}, \boldsymbol{b}, m)$ defined above. Here the weighting is similar to the previous case but somewhat more awkward, see Figure 5. While we do not know (or do not recognize) the number $N(\boldsymbol{a}, \boldsymbol{b}, m)$, let us mention that it has a determinant formula via the LGV–lemma, which is also the key to the proof of Theorem 2.4.

3 Combinatorial proofs

3.1 The 2-symmetry case

We start with a special case $a = c = 1$ in Theorem 2.1 (cf. §5.7).

Lemma 2 (**2**-symmetry). *Let $A = (2, 1)$, $B = (1, m)$, $m \geqslant 1$. Let*

$$F_m(x_1, x_2 \,|\, y_1, \ldots, y_m) := \sum_{\gamma:\, A \to B} \prod_{(i,j) \in \gamma} \frac{1}{x_i + y_j}.$$

Then F_m is symmetric in $\boldsymbol{x} = (x_1, x_2)$.

Figure 6: Five paths $\gamma : (2, 1) \to (1, 5)$ in the 2-symmetry Lemma 2.

Proof. There are m paths in this case, see Figure 6. We have:

$$F_m \;=\; G_m(x_1, x_2 | y_1, \ldots, y_m) \prod_{i=1}^{2} \prod_{j=1}^{m} \frac{1}{x_i + y_j},$$

where

$$G_m \;=\; (x_1 + y_1) \cdots (x_1 + y_{m-1}) + (x_1 + y_1) \cdots (x_1 + y_{m-2}) \cdot (x_2 + y_m) + \ldots + (x_2 + y_2) \cdots (x_2 + y_m).$$

The symmetry of G_m with respect to x_1, x_2 follows from the identity

$$(\diamond) \qquad G_m = \frac{(x_1 + y_1)(x_1 + y_2) \ldots (x_1 + y_m) - (x_2 + y_1)(x_2 + y_2) \ldots (x_2 + y_m)}{x_1 - x_2}.$$

Indeed, the identity ($\diamond$) can be proved by a telescopic cancellation:

$$
\begin{aligned}
G_m \cdot (x_1 - x_2) &= (x_1 + y_1)(x_1 + y_2) \ldots (x_1 + y_{m-1})\big[(x_1 + y_m) - (x_2 + y_m)\big] \\
&\quad + (x_1 + y_1) \cdots (x_1 + y_{m-2})(x_2 + y_m)\big[(x_1 + y_{m-1}) - (x_2 + y_{m-1})\big] \\
&\quad + \ldots + (x_2 + y_2)(x_2 + y_3) \cdots (x_2 + y_m)\big[(x_1 + y_1) - (x_2 + y_1)\big] \\
&= (x_1 + y_1)(x_1 + y_2) \ldots (x_1 + y_m) - (x_2 + y_1)(x_2 + y_2) \ldots (x_2 + y_m).
\end{aligned}
$$

Another way to prove ($\diamond$) is to note that both parts are multilinear polynomials with respect to $y_1, \ldots, y_m$ and to check that they agree when $y_i \in \{-x_1, -x_2\}$ for all i. $\qquad\square$

3.2 Proof of Theorem 2.3

It suffices to show that $F_{\mathcal{A},\mathcal{B}}$ is symmetric in (x_i, x_{i+1}), for all $1 \leqslant i < m$. Fix a collection of paths Υ and consider only rows i and $(i+1)$. Remove all columns where both squares are in Υ but not connected by a path, and those columns where both squares are empty. This results in several 2-row rectangles, each connected by a path from lower left corner to upper right corner. Apply the 2-symmetry lemma to each non-empty rectangle to conclude that the sum of all $w(\Upsilon)$ is symmetric in (x_i, x_{i+1}), as desired. $\quad\square$
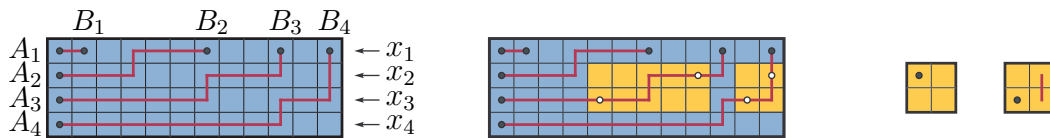

Figure 7: <u>Left</u>: Using 2-symmetry to prove the symmetry of $F_{\mathcal{A},\mathcal{B}}$ in (x_2, x_3). <u>Right</u>: Two impossible configurations.

Remark 3.1. The proof above implicitly uses the claim that $\mathcal{A}$ and $\mathcal{B}$ are as in the theorem. Indeed, otherwise we can have e.g. a rectangle with upper left square in $\mathcal{A}$ and no point of $\mathcal{A}$ below it, or a point in $\mathcal{B}$ in the bottom row without the point of $\mathcal{B}$ above it (see Figure 7).

3.3 Proof of Theorem 2.4

We follow the proof of Theorem 2.3 given above. First, switch the coordinates $\boldsymbol{x} \leftrightarrow \boldsymbol{y}$. Then $\mathcal{B}$ is as in Theorem 2.3, while $\mathcal{A}$ are on the bottom row. We need to prove the $\boldsymbol{x}$-symmetry in this case. Apply the 2-symmetries in (x_i, x_{i+1}) in exactly the same way and notice that the forbidden configuration as in the remark above do no appear. The details are straightforward. $\quad\square$

3.4 The ultimate generalization

The proofs above suggest a common generalization of Theorems 2.3 and 2.4. We chose to postpone it until this point to avoid overwhelming the reader.

Theorem 3.2 (Main theorem). Let $m, n, k \geqslant 1$, $\boldsymbol{a} = (a_1, \ldots, a_n)$, $\boldsymbol{b} = (b_1, \ldots, b_n)$, where

$$a_1 + \ldots + a_n = b_1 + \ldots + b_n = k, \quad \text{where } 0 \leqslant a_i, b_i \leqslant m.$$

Let $\mathcal{A}$ be a collection of points $A_1, \ldots, A_k \in [m \times n]$, with exactly a_i points on the *bottom* of i-th column. Similarly, let $\mathcal{B}$ be a collection of points $B_1, \ldots, B_k \in [m \times n]$, with exactly b_i points on the *top* of i-th column. Here the order $\mathcal{A}$ and $\mathcal{B}$ is from left to right, and within a column from top to bottom, see Figure 8. Then the multivariate function

$$F_{\mathcal{A},\mathcal{B}}\big(x_1, \ldots, x_m \mid y_1, \ldots, y_n\big),$$

defined in (2.2), is symmetric in $\boldsymbol{x} = (x_1, \ldots, x_m)$.
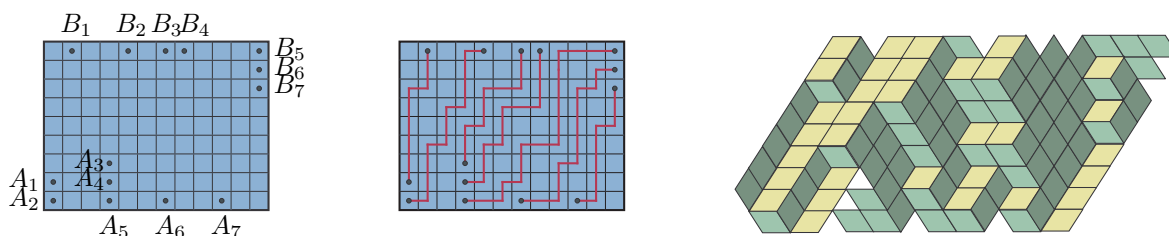

Figure 8: <u>Left</u>: Examples of a collection of points $\mathcal{A}, \mathcal{B}$, and non-intersecting paths in Theorem 3.2, with $\boldsymbol{a} = (2, 0, 0, 3, 0, 0, 1, 0, 0, 1, 0, 0)$ and $\boldsymbol{b} = (0, 1, 0, 0, 1, 0, 1, 1, 0, 0, 0, 3)$. <u>Right</u>: The corresponding lozenge tiling.

The theorem generalizes Theorem 2.3 in a straightforward way: take $\boldsymbol{a} = (k, 0, \ldots, 0)$ and $\boldsymbol{b} \in \{0, 1\}^n$, with k zeroes. It also generalizes Theorem 2.4 as follows: switch coordinates $\boldsymbol{x} \leftrightarrow \boldsymbol{y}$, and take both $\boldsymbol{a}, \boldsymbol{b} \in \{0, 1\}^n$, with k zeroes. Of course, Theorem 3.2 is much more general, even if in some cases the result is vacuously true, as there are no possible collections of non-intersecting `Up-Right` paths $\gamma_i : A_i \to B_i$.

Proof of Theorem 3.2. The proof follows verbatim the proof of Theorem 2.3 given above. We prove the $\boldsymbol{x}$-symmetry via 2-symmetries in (x_i, x_{i+1}) in exactly the same way. Indeed, notice that the forbidden configuration as in the remark above do no appear. The details are straightforward. $\qquad\square$

4 Algebraic proofs

4.1 Preliminaries

Fix $m, n \geqslant 1$ and let

$$P_k(t) := (t + y_1)(t + y_2) \cdots (t + y_k), \quad k = 0, \ldots, n.$$

For $s = 1, \ldots, m$, and $k = 1, \ldots, n$, define

$$Q_{s,k}(t) := \prod_{j=s}^{m} \frac{1}{x_j - t} \mod P_k(t).$$

Note that this expression is well defined: the polynomials $(x_j - t)$ are invertible modulo $P_k(t)$ in the ring $\mathcal{R}[t]$, where $\mathcal{R} = \mathbb{C}(\boldsymbol{x}, \boldsymbol{y})$. In other words, $\mathcal{R}[t]$ is the ring of polynomials in t with coefficients in the field of rational functions in x_i's and y_j's.

Denote

$$F_{s,k} := \sum_{\gamma:(m,1)\to(s,k)} w(\gamma).$$

We use the following description of $F_{s,k}$ which simultaneously proves a $\boldsymbol{x}$-symmetry and $\boldsymbol{y}$-symmetry of $F_{s,k}$. This is the $k = 1$ case of Theorem 2.1 generalizing Lemma 2 to all $m \geqslant 2$ (see also §5.7).

Lemma 3. *For $s = 1, \ldots, m$ and $k = 1, \ldots, n$, we have:*

$$F_{s,k} = [t^{k-1}]Q_{s,k}(t).$$

In particular $F_{s,k}$ is symmetric with respect to $(x_s, \ldots, x_m)$, and with respect to $(y_1, \ldots, y_k)$.

Proof. By definition,

$$F_{m,1} = \frac{1}{x_m + y_1} = Q_{m,1}(t).$$

Observe that

$$F_{s,k} = \frac{1}{x_s + y_k}\big(F_{s,k-1} + F_{s+1,k}\big),$$

for $s = 1, \ldots, m$, and $k = 1, \ldots, n$, such that $(s,k) \neq (m,1)$. Here we use boundary values $F_{m+1,k} = F_{s,0} = 0$. Note that

$$(t + y_k)Q_{s,k}(t) \equiv (t + y_k)Q_{s,k-1}(t) \mod P_{k-1}(t) \text{ and } \mod (t + y_k).$$

Thus, the congruence holds modulo $P_k(t)$:

$$(t + y_k)Q_{s,k}(t) \equiv (t + y_k)Q_{s,k-1}(t) \mod P_k(t).$$

Similarly,

$$(x_s - t)Q_{s,k}(t) = Q_{s+1,k}(t) \mod P_k(t).$$

Adding these two congruences, we obtain

$$(x_s + y_k)Q_{s,k}(t) \equiv Q_{s+1,k}(t) + (t + y_k)Q_{s,k-1}(t) \mod P_k(t).$$

Now observe that both the LHS and the RHS are polynomials of degree at most $(k-1)$ in t. Thus we have an equation of polynomials:

$$(x_s + y_k)Q_{s,k}(t) = Q_{s+1,k}(t) + (t + y_k)Q_{s,k-1}(t).$$

Taking the coefficients of t^{k-1}, we see that the double sequence

$$\big\{[t^{k-1}]Q_{s,k}\big\}$$

satisfies the same recurrence and initial conditions as $F_{s,k}$. This implies the result. $\square$

4.2 Non-intersecting paths

We recall the *Lindström—Gessel—Viennot lemma*:

Theorem 4.1 (LGV–lemma)*.* Let $G = (V, E)$ be a finite acyclic directed graph. Fix $k \geqslant 1$. Let $\mathcal{A} = \{A_1, \ldots, A_k\}$, $\mathcal{B} = \{B_1, \ldots, B_k\} \subset V$ be two (not necessarily disjoint) sets of vertices, such that $|\mathcal{A}| = |\mathcal{B}| = k$. Let R be a commutative ring, and let $w : E \to \mathrm{R}$ be a weight function. For a subset $S \subset E$, define a weight

$$w(S) := \prod_{e \in S} w(e), \quad \text{and} \quad w(\varnothing) := 1.$$

Consider a matrix $U = (u_{ij})_{i,j=1}^{k}$, where

$$u_{ij} := \sum_{\gamma : A_i \to B_j} w(\gamma)$$

is the sum of weights of all paths from A_i to B_j. Then:

$$\det U = \sum_{\pi \in S_k} \sum_{\substack{\Upsilon = (\gamma_1, \ldots, \gamma_k) \\ \gamma_i : A_i \to B_{\pi(i)}}} \mathrm{sign}(\pi) \cdot w(\Upsilon),$$

where the second sum is over all collections of vertex-disjoint paths γ_i from A_i to $B_{\pi(i)}$.

For the proof, see [GJ, §5.4], or [Tal] for a more general result. Below, we will use the following "vertex version" of the LGV–lemma, which easily follows from the above edge version. In this corollary, a path is defined to be a set of vertices.

Corollary 4.2 (vertex–LGV)*.* Let $G = (V, E)$ be a finite acyclic directed graph without multiple edges. Fix $k \geqslant 1$. Let $\mathcal{A} = \{A_1, \ldots, A_k\}, \mathcal{B} = \{B_1, \ldots, B_k\} \subset V$ be two (not necessarily disjoint) sets of vertices, such that $|\mathcal{A}| = |\mathcal{B}| = k$. Let R be a commutative ring, and let $w : V \to \mathrm{R}$ be a weight function. For a subset $D \subset V$, define a weight

$$w(D) := \prod_{v \in D} w(v), \quad \text{and} \quad w(\varnothing) := 1.$$

Consider a matrix $U = (u_{ij})_{i,j=1}^{k}$, where

$$u_{ij} := \sum_{\gamma : A_i \to B_j} w(\gamma)$$

is the sum of weights of all paths γ from A_i to B_j. Then

$$\det U = \sum_{\pi \in S_k} \sum_{\substack{\Upsilon = (\gamma_1, \ldots, \gamma_k) \\ \gamma_i : A_i \to B_{\pi(i)}}} \mathrm{sign}(\pi) \cdot w(\Upsilon),$$

where the sum is over all collections of vertex-disjoint paths γ_i from A_i to $B_{\pi(i)}$.

Proof. Denote by $\widehat{G} = (\widehat{V}, \widehat{E})$ the graph $G = (V, E)$ with added new vertices $C_1, \ldots, C_k$ and directed edges $(C_i A_i)$. For each edge $(XY) \in \widehat{E}$, define its weight by $w(XY) := w(Y)$. Apply Theorem 4.1 for the sets $\mathcal{C} = \{C_1, \ldots, C_k\}$ and $\mathcal{B} = \{B_1, \ldots, B_k\}$. Observe that the weight of each path $(C_i A_i X_1 X_2 \ldots X_n B_j)$ in $\widehat{G}$ is the same as the weight of the path $(A_i X_1 \ldots X_n B_j)$ in graph G. Similarly, the collections of vertex-disjoint paths from $\mathcal{C}$ to $\mathcal{B}$ in $\widehat{G}$ are in a natural correspondence with collections of vertex-disjoint paths from $\mathcal{A}$ to $\mathcal{B}$ in G. This implies the result. $\square$

In many applications of the LGV–lemma, there is a unique permutation π for which there exists a vertex-disjoint collection of paths, and this unique π is the identical permutation, and the determinant equals to the weighted sum over collections of disjoint paths from A_i to B_i. This also holds in the settings of Theorems 2.1, 2.3, 2.4 and 3.2.

4.3 Proof of Theorem 2.4

By the vertex version of the LGV–lemma in Corollary 4.2, the multivariate rational function $F_{\mathcal{A},\mathcal{B}}(\boldsymbol{x}, \boldsymbol{y})$ is a determinant of a $k \times k$ matrix U in which every entry u_{ij} is a rational function. By Lemma 3, these functions u_{ij} are $\boldsymbol{y}$-symmetric. Thus the determinant is also $\boldsymbol{y}$-symmetric, which completes the proof. $\square$

4.4 Proof of Theorem 2.3

In notation of Subsection 4.1, let $\mathcal{R} = \mathbb{C}(\boldsymbol{x}, \boldsymbol{y})$. Let $\mathcal{I}$ be the ideal generated by the polynomials $P_{b_1}(t_1), P_{b_2}(t_2), \ldots, P_{b_k}(t_k)$. Consider the ring $\mathrm{R} = \mathcal{R}[t_1, \ldots, t_k]/\mathcal{I}$; each element of this ring corresponds to a unique polynomial $H(t_1, \ldots, t_k)$ with degrees less than b_j in the variable t_j, for all $j = 1, \ldots, k$. For the elements of R, this allows us to define the coefficients of the monomials $t_1^{s_1} \ldots t_k^{s_k}$, where $0 \leqslant s_i < b_i$.

By the vertex version of the LGV–lemma in Corollary 4.2, we have:

$$F_{\mathcal{A},\mathcal{B}} = \det\big(F_{A_i,B_j}\big)_{i,j=1}^{k}.$$

Denote

$$\varphi_i(t_j) := (x_{a+i+1} - t_j)(x_{a+i+2} - t_j) \cdots (x_{a+k} - t_j),$$

and observe the Vandermonde-type determinant

$$(\circledast) \qquad \det(\varphi_i(t_j))_{i,j=1}^{k} = \prod_{1 \leqslant i < j \leqslant k} (t_j - t_i).$$

The proof of $(\circledast)$ follows the same argument as the standard proof of the (usual) Vandermonde determinant formula.

The elements $(t_i - x_j)$ are invertible in R, and by Lemma 3 we have:

$$F_{A_i,B_j} = \left[t_j^{b_j-1} \right] \prod_{s=1}^{a+i} \frac{1}{x_s - t_j} = \left[t_j^{b_j-1} \right] \varphi_i(t_j) \prod_{s=1}^{m} \frac{1}{x_s - t_j}.$$

Interchanging the coefficients-evaluating functional and the determinant sign and applying (⊛), we obtain:

$$F_{\mathcal{A},\mathcal{B}} = \left[t_1^{b_1-1}\ldots t_k^{b_k-1}\right]\prod_{j=1}^{k}\prod_{s=1}^{m}\frac{1}{x_s-t_j}\,\det\!\left(\varphi_i(t_j)\right)_{i,j=1}^{k}$$

$$= \left[t_1^{b_1-1}\ldots t_k^{b_k-1}\right]\prod_{1\leqslant i<j\leqslant k}(t_j-t_i)\prod_{j=1}^{k}\prod_{s=1}^{m}\frac{1}{x_s-t_j}\,.$$

The RHS is certainly symmetric in $\boldsymbol{x}=(x_1,\ldots,x_m)$. This completes the proof of the theorem. $\square$

5 Final remarks

5.1 Many hidden symmetries

As we mentioned in the introduction, hidden symmetries are a staple in Algebraic and Enumerative Combinatorics. Without aiming to review even a fraction of the literature, let us mention a few notable examples. First, the *Littlewood–Richardson coefficients* have a number of hidden symmetries not reflected in their classical combinatorial interpretation. While the *BZ-triangles* [BZ] combined with bijections in [PV1] explained some of the symmetries, others remain unexplained, see [PV2, §6.6].

Another major appearance of the hidden symmetries is in connection with the *alternating sign matrices*, which led to a conceptual proof by Kuperberg [Kup]. Further symmetries of ASMs were discovered by Razumov–Stroganov [RS] (see also [Wie]), and eventually proved by a technical argument in [CS].

Finally, in a fascinating study (completely unrelated to this work), Coxeter used the symmetry of regular solids in $\mathbb{R}^4$ to evaluate special values of the *dilogarithm* [Cox]. The following amazing identity coming from the *600-cell* is a testament to the power of hidden symmetries:

$$\sum_{n=1}^{\infty}\frac{\phi^n}{n^2}\cos\left(\frac{2\pi n}{5}\right)=\frac{\pi^2}{100}\,,\quad\text{where}\quad\phi=\frac{\sqrt{5}-1}{2}\,.$$

5.2 Yang–Baxter equations

Closer to the subject, Borodin in [Bor] initiated the study of symmetric rational functions for the *six-vertex model* which are proved via the *Yang–Baxter equations*, see [Bax]. These results were greatly extended in [BP2] (see also a survey [BP1]). These functions have multiple families of parameters, but they do not specialize to our functions $F_{\mathcal{A},\mathcal{B}}(\boldsymbol{x},\boldsymbol{y})$. To see this, note that in our setting, the intersections are not allowed, making it a *five-vertex model*, implying degeneration of many parameters.

In a parallel investigation, Bump, McNamara and Nakasuji [BMN] realized that the factorial Schur functions can be expressed as the partition function of a six-vertex model with certain particular multivariate parameters. When $t=-1$, the deformation in §4

in their paper gives new solutions of the Yang–Baxter equations exactly with the same parameters as are implicit in this paper. In particular, this gives a new proof of the 2-symmetry in Lemma 2, the fourth proof counting two proofs in this paper and one in [MPP3], but perhaps the most conceptual one. We learned about [BMN] only after this paper was written.

We should emphasize that a solution of the Yang–Baxter equations is not enough to establish the symmetry, as one needs to check the boundary conditions. This is what makes our Main Theorem 3.2 so surprising – it gives the most unusual boundary conditions for which the symmetry holds.

5.3 Further symmetries

Let us mention some recent progress in this setting, the *shift invariance* for the six-vertex model and *polymers*, discovered recently in [BGW]. It can be viewed as the new fundamental (multivariate) hidden symmetry for the number of certain lattice path configurations. This shift invariance found a surprising application in [BGR1] to certain properties of multi-particle generalization of TASEP, in turn related to the number of reduced factorizations of certain permutations in S_n. Most recently, [Gal] established a more general type of symmetries called *flip invariance*, and gave them a conceptual algebraic explanation. In a different direction, curious combinatorial implications of this and related symmetries were found in [Dau].

5.4 Factorial Schur functions

In notation of §2.3, when $a = 0$ as in Figure 4, one can think of $F_{\mathcal{A},\mathcal{B}}(\boldsymbol{x}, |\boldsymbol{y})$ in Theorem 2.3 the multivariate deformation of $N(\lambda)$. This deformation is different, but curiously similar to the $\boldsymbol{x}$-symmetric and $\boldsymbol{y}$-parametrized factorial Schur functions $s_\lambda(\boldsymbol{x}| - \boldsymbol{y})$, which forms a basis in symmetric polynomials of $\boldsymbol{x}$, see [Mac, §6]. This should not come as a surprise as the proof in [MPP3] is based on combinatorics and algebra of factorial Schur functions. It would be interesting to establish a formal connection in full generality.

5.5 Selberg integral

In [KO], the authors proved some of the corollaries of [MPP3]. The results follow from the *Selberg integral*, another yet to be fully understood hidden symmetry, see [MPP3, §9.3-9.4].

5.6 Identities

Theorem 2.1 is stated in [MPP3, Thm 3.10] in a weaker form, but the result follows from the proof. Of course, we both reprove and generalize it in this paper. Note, however, that Thm 3.12 in the same paper gives a different hidden symmetry which does not follow from this paper.

5.7 Evaluations

As suggested by both our combinatorial and algebraic proofs, Theorem 2.1 is not obvious already for $k = 1$. Even the special case $k = 1$, $x_i = i$ and $y_j = b - j + 1$, is already quite

interesting [MPP3, Cor. 3.11].

5.8   Generalizations

The combinatorial proof in Section 3 may appear to be more flexible, as it leads to the proof of our Main Theorem 3.2. However, the algebraic proofs tend to be more powerful and amenable to generalizations of different kind. For example, it would be interesting if the results generalize to three and higher dimensions as we seem to have exhausted the planar version. In a different direction, the determinant style proofs as in Section 4, suggest possibility of non-commutative generalization, cf. [GR]. Finding a proper q-analogue (or quantum analogue?) would be especially interesting.

5.9   Up-Right condition

Theorem 2.4 remains true even when the assumption that all paths are required to be `Up-Right` is removed. This leads to a somewhat stronger but less natural result. We leave the proof to the reader. Let us note, however, that while the `Up-Right` condition is vacuous for Theorem 2.3, it is necessary for our Main Theorem 3.2.

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