LINEAR EXTENSIONS OF FINITE POSETS

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Abstract. We give a broad survey of inequalities for the number of linear extensions of finite posets. We review many examples, discuss open problems, and present recent results on the subject. We emphasize the bounds, the equality conditions of the inequalities, and the computational complexity aspects of the results.

1. Introduction

1.1. **Foreword.** The world of linear extensions of finite posets is a microcosm of contradictions. Although the counting problem is #P-hard in general, it is polynomially easy in many special cases (see §12.1). Although posets themselves do not have a geometric structure, the number of linear extensions is given by the volume of two different polytopes with distinct geometric applications (see §14.5 and §14.6). Finally, although there is a large number of correlation inequalities that hold in full generality, direct injective proofs are rare and difficult to construct, sometimes provably so.

We straddle the boundary between inequalities which hold for general posets and specialized inequalities for classes of posets, with the emphasis on the former. There is a dearth of powerful tools for study of linear extensions, and yet the sheer volume of results continues to astonish us. This is hardly reflected in our presentation style which continues the dry tradition of stating results and letting the reader judge for themselves. But please be assured — many results in this survey are extremely surprising and worthy of elaboration, reflection and contemplation.

1.2. **Content.** In this paper we give a broad survey of inequalities for the *number of linear extensions* of finite posets, both as a function of the poset and when closely related posets are compared. We also discuss several closely related inequalities for the *order polynomial*, and *correlation inequalities* for probabilities of various events associated with linear extensions.

Of the ocean of inequalities for linear extensions, we single out several key inequalities which are treated in separate sections along with their many extensions and variations:

- *Sidorenko's inequality*, see Section 5,
- *Björner–Wachs inequality*, see Section 6,
- *Fishburn's inequality*, see Section 7,
- *XYZ inequality*, see Section 8,
- *Stanley's inequality*, see Sections 9, 10 and 15.

We emphasise the equality conditions of the inequalities, so e.g. Section 10 is dedicated to equality conditions of Stanley type inequalities. Examples of various families of posets are given in Section 11. We discuss various computational complexity aspects around the numbers of linear extension in Section 12.

In our presentation we aim to be as complete as possible, but for reasons of space and readability we abandoned the hope to include any proofs (with few exceptions, see Section 15). Instead, we include Section 14 with some vague ideas on what is going on, and some additional references. We conclude with final remarks in Section 16.

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2. Definitions and notation

We start by introducing basic definitions and notations which appear throughout the survey. We also include many references to monographs, surveys and other background reading on the subject.

2.1. **General notation.** Let $[n] = \{1,\ldots,n\}$, $\mathbb{N} = \{0,1,2,\ldots\}$ and $\mathbb{R}_+ = \{x \geq 0\}$. For a subset $S \subseteq X$ and element $x \in X$, we write $S + x := S \cup \{x\}$ and $S - x := S \smallsetminus \{x\}$.

For a sequence $\boldsymbol{a} = (a_1,\ldots,a_m)$, denote $|\boldsymbol{a}| := a_1 + \ldots + a_m$. This sequence is *unimodal* if $a_1 \leq a_2 \leq \ldots \leq a_\ell \geq a_{\ell+1} \geq \ldots a_m$ for some ℓ. Sequence $\boldsymbol{a}$ is *log-concave* if $a_i^2 \geq a_{i-1}a_{i+1}$ for all $1 < i < m$. We refer to [Brä15, Bre89, Sta89b] for many examples of log-concave sequences in Algebraic and Enumerative Combinatorics.

We use the *q-analogues* $(n)_q := 1 + q + \ldots + q^{n-1}$, $n!_q := (1)_q \cdots (n)_q$ and

$$\binom{n}{k}_q := \frac{n!_q}{k!_q\,(n-k)!_q}\,.$$

These can be viewed either polynomials in $\mathbb{N}[q]$, or as real numbers for a fixed $q \in \mathbb{R}$.

For an inequality $f \geqslant g$, the difference $(f-g)$ is called the *defect*. For polynomials $f, g \in \mathbb{R}[z_1,\ldots,z_n]$, we write $f(\boldsymbol{z}) \geq g(\boldsymbol{z})$ for the inequality between the values at a given $\boldsymbol{z} = (z_1,\ldots,z_n) \in \mathbb{R}^n$, and $f \geqslant_{\boldsymbol{z}} g$ for the stronger inequality which holds coefficient-wise.

2.2. **Posets.** Suppose $P = (X, \prec)$ and $Q = (Y, \prec')$ are two posets on sets $X \subseteq Y$, such that $x \prec' y$ implies $x \prec y$. Then we say that P is a *subposet* of Q. If $x \prec' y \Leftrightarrow x \prec y$, we say that P is an *induced subposet* of Q. For a poset $P = (X, \prec)$ and a subset $A \subseteq X$, denote by $P|_A := (A, \prec)$ the induced subposet of P.

We use $(P - z)$ to denote a subposet $P|_{X-z}$, where $z \in X$. Element $x \in X$ is *minimal* in P, if there exists no element $y \in X - x$ such that $y \prec x$. Define *maximal* elements similarly. Denote by $\min(P)$ and $\max(P)$ the set of minimal and maximal elements in P, respectively.

When P has a unique minimal element, we use $\widehat{0}$ to denote it. Similarly, when P has a unique maximal element, we use $\widehat{1}$ to denote it. Element $x \in X$ is said to *cover* $y \in X$, if $y \prec x$ and there are no elements $z \in X$ such that $y \prec z \prec x$. For two subsets $A, B \subseteq X$, $A \cap B = \varnothing$, we write $A \prec B$ if $x \prec y$ for all $x \in A$ and $y \in B$.

A subset $A \subseteq X$ is an *upper ideal* if $x \in A$ and $y \succ x$ implies $y \in A$. Similarly, a subset $A \subseteq X$ is a *lower ideal* if $x \in A$ and $y \prec x$ implies $y \in A$. Denote by $x\!\downarrow := \{y \in X \,:\, y \preccurlyeq x\}$ and $x\!\uparrow := \{y \in X \,:\, y \succcurlyeq x\}$ the *lower* and *upper order ideals* generated by x, respectively. Similarly, for a subset $B \subseteq X$, denote by $B\!\downarrow := \cup_{b\in B}\, b\!\downarrow$ and $B\!\uparrow := \cup_{b \in B}\, b\!\uparrow$ the *lower* and *upper closure* of B, respectively. We use $\alpha(x) := |x\!\downarrow|$, $\beta(x) := |x\!\uparrow|$, $\alpha(B) := |B\!\downarrow|$ and $\beta(x) := |B\!\uparrow|$, to denote their sizes.

In a poset $P = (X, \prec)$, elements $x, y \in X$ are called *parallel* or *incomparable* if $x \not\prec y$ and $y \not\prec x$. We write $x \parallel y$ in this case. Denote by $\operatorname{comp}(x) := \{y \in X \,:\, x \prec y \text{ or } x \succ y\}$ the set of elements $y \in X$ comparable to x. *Comparability graph* $\Gamma(P) = (X, E)$ is a graph on X with edges $E = \{(x,y) \,:\, x \prec y \text{ or } x \succ y\}$.

A *chain* is a subset $C \subset X$ of pairwise comparable elements. Let $\mathcal{C}(P)$ denote the set of chains in P. The *height* of poset $P = (X, \prec)$ is the maximum size $|C|$ of a chain $C \in \mathcal{C}(P)$. Chain $C \in \mathcal{C}(P)$ is called *maximal* if there is no chain $C' \in \mathcal{C}(P)$ such that $C \subset C'$.

An *antichain* is a subset $A \subset X$ of pairwise incomparable elements. Let $\mathcal{A}(P)$ denote the set of antichains in P. The *width* of poset $P = (X, \prec)$ is the maximal size $|A|$ of a antichain $A \in \mathcal{A}(P)$. Antichain A is called *maximal* if there is no antichain $A' \in \mathcal{C}(P)$ such that $A \subset A'$.

A *dual poset* is a poset $P^* = (X, \prec^*)$, where $x \prec^* y$ if and only if $y \prec x$. A *product* $P \times Q$ of posets $P = (X, \prec)$ and $Q = (Y, \prec^\circ)$ is a poset $(X \times Y, \prec^\diamond)$, where the relation $(x,y) \preccurlyeq^\diamond (x',y')$ if and only if $x \preccurlyeq x'$ and $y \preccurlyeq^\circ y'$, for all $x, x' \in X$ and $y, y' \in Y$.

A *disjoint sum* $P+Q$ of posets $P=(X,\prec)$ and $Q=(Y,\prec')$ is a poset $(X\cup Y,\prec^\diamond)$, where the relation $\prec^\diamond$ coincides with $\prec$ and $\prec'$ on X and Y, and $x\,\|\,y$ for all $x\in X$, $y\in Y$. A *linear sum* $P\oplus Q$ of posets $P=(X,\prec)$ and $Q=(Y,\prec')$ is a poset $(X\cup Y,\prec^\diamond)$, where the relation $\prec^\diamond$ coincides with $\prec$ and $\prec'$ on X and Y, and $x\prec^\diamond y$ for all $x\in X$, $y\in Y$.

Posets constructed from one-element posets by recursively taking disjoint and linear sums are called *series-parallel*. Both *n-chain* C_n and *n-antichain* A_n are examples of series-parallel posets. These posets can be characterized by not having poset $N=\{x\prec y\succ z\prec w\}$ as induced subposet (thus they are also called *N-free*). *Forest* is a series-parallel poset formed by recursively taking disjoint sums (as before), and linear sums with one element: $C_1\oplus P$.

Let $P=(X,\prec)$ and $P'=(X,\prec')$ be two posets on the same ground set. We say that P and P' are *consistent* if there are no elements $x,y\in X$, s.t. $x\prec y$ and $y\prec' x$. For consistent posets, the *intersection* $P\cap P'=(X,\prec^\diamond)$ is the poset with the union of the relations: $x\prec y$ if $x\prec y$ or $x\prec' y$.

Poset $P=(X,\prec)$ is *graded* if there is a rank function $\rho:X\to\mathbb{N}$ such that $\rho(x)\le\rho(y)$ for all $x\prec y$, and $\rho(y)=\rho(x)+1$ for all covers y of x. If P and Q are graded, then so are $P+Q$, $P\oplus Q$ and $P\times Q$. *Boolean algebra* $B_m:=C_2\times\cdots\times C_2$ (m times) and the *grid poset* $G_{k\ell}:=C_k\times C_\ell$ are examples of graded posets.

Let $P=(X,\prec)$ be a poset. For elements $x,y\in X$, the *greatest lower bound* $x\wedge y$ is an element s.t. $z\prec x, z\prec y\,\Rightarrow\, z\prec(x\wedge y)$. Similarly, the *least upper bound* $x\vee y$ is an element s.t. $z\succ x, z\succ y\,\Rightarrow\, z\succ(x\wedge y)$. Poset P is a *lattice* if $x\wedge y$ and $x\vee y$ are well defined for all $x,y\in X$. We use $\mathcal{L}=(X,\vee,\wedge)$ notation in this case. Lattice $\mathcal{L}$ is *distributive* if $x\wedge(y\vee z)=(x\wedge y)\vee(x\wedge z)$ for all $x,y,z\in X$ (note that the dual identity follows). Both B_m and $G_{k\ell}$ are examples of distributive lattices.

We refer to [Sta99, Ch. 3] and [West21, Ch. 12] for accessible textbook introductions to posets, to surveys [BW00, Tro95] for further definitions and standard results, and to [CLM12] for a recent monograph.[1]

2.3. **Linear extensions and order polynomial.** Let $P=(X,\prec)$ be a poset with $|X|=n$ elements. Denote $[n]:=\{1,\ldots,n\}$. A *linear extension* of P is a bijection $f:X\to[n]$, such that $f(x)<f(y)$ for all $x\prec y$. Denote by $\mathcal{E}(P)$ the set of linear extensions of P, and let $e(P):=|\mathcal{E}(P)|$.

For an integer $t\ge 1$, denote by $\Omega(P,t)$ the number of order preserving maps $g:X\to[t]$, i.e. maps which satisfy $g(x)\le g(y)$ for all $x\prec y$. This is the *order polynomial* corresponding to poset P.

Let $P=(X,\prec)$, where $X=\{x_1,\ldots,x_n\}$. We will always assume that X has a *natural labeling*, i.e. $f:x_i\to i$ is a linear extension. A *P-partition* is an order preserving map $h:X\to\mathbb{N}$, i.e. maps which satisfies $h(x)\le h(y)$ for all $x\prec y$. Denote by $\mathcal{P}(P)$ the set of P-partitions and let $\mathcal{P}(P,t)$ be the set of P-partitions with values at most t.[2]

Let

$$\Omega_q(\mathcal{P}) \,:=\, \sum_{h\in\mathcal{P}(P)} q^{h(x_1)\,+\,\ldots\,+\,h(x_n)} \quad\text{and}\quad \Omega_q(\mathcal{P},t) \,:=\, \sum_{h\in\mathcal{P}(P,t)} q^{h(x_1)\,+\,\ldots\,+\,h(x_n)}\,. \tag{2.1}$$

Stanley showed, see [Sta99, Thm 3.15.7], that there is a statistics $\operatorname{maj}:\mathcal{E}(\mathcal{P})\to\mathbb{N}$, such that

$$\Omega_q(P) \;=\; \frac{e_q(P)}{(1-q)(1-q^2)\cdots(1-q^n)}\,, \tag{2.2}$$

[1] See also w.wiki/7Yy6 for a quick guide to the terminology.

[2] In [Sta72, Sta99], Stanley uses P-partitions to denote order-reversing rather than order-preserving maps.

where

$$e_q(P) := \sum_{f\in\mathcal{E}(P)} q^{\operatorname{maj}(f)}. \tag{2.3}$$

More generally, let

$$\Omega_{\boldsymbol{q}}(P,t) := \sum_{h\in\mathcal{P}(P,t)} q_1^{h(x_1)}\cdots q_n^{h(x_n)}. \tag{2.4}$$

We call this GF the *multivariate order polynomial*. Note that Stanley gave a generalization of (2.2) and (2.3) for $\Omega_{\boldsymbol{q}}(\mathcal{P})$, see [Sta99, Thm 3.15.5]. Finally, for integer $t \geq 0$, define

$$\Phi_{\boldsymbol{z}}(P,t) := \sum_{h\in\mathcal{P}(P,t)} z_0^{m_0(h)}\, z_1^{m_1(h)} \cdots z_t^{m_t(h)}, \tag{2.5}$$

where $m_i(h) := |h^{-1}(i)|$ is the number of values i in the P-partition h. Clearly,

$$\Phi_{\boldsymbol{z}}(P,t) = \Omega_q(P,t), \quad \text{where} \quad \boldsymbol{z} = (1,q,q^2,\ldots,q^t). \tag{2.6}$$

Finally, for a fixed linear ordering $X = \{x_1,\ldots,x_n\}$, denote by $\operatorname{sign}(f)$ the sign of $f \in \mathcal{E}(P)$ viewed as a permutation in S_n. Define the *sign-imbalance*:

$$\operatorname{SI}(P) := |\Sigma(P)|, \quad \text{where} \quad \Sigma(P) := \sum_{f\in\mathcal{E}(P)} \operatorname{sign}(f). \tag{2.7}$$

Clearly, the sign-imbalance $\#\mathsf{P}(P)$ is independent of the ordering of X. Poset P is called *sign-balanced* if $\#\mathsf{P}(P) = 0$. We refer to [Sta05] for the introduction to sign-(im)balance.

2.4. **Poset polytopes.** Let $P = (X,\prec)$ be a poset with $|X| = n$ elements. The *order polytope* $\mathcal{O}_P \subset \mathbb{R}^n$ is defined as

$$0 \leq \alpha_x \leq 1 \quad \text{for all} \quad x \in X, \qquad \alpha_x \leq \alpha_y \quad \text{for all} \quad x \prec y, \quad x,y \in X. \tag{2.8}$$

Similarly, the *chain polytope* (also known as the *stable set polytope*) $\mathcal{S}_P \subset \mathbb{R}^n$ is defined as

$$\beta_x \geq 0 \quad \text{for all} \quad x \in X\,, \qquad \beta_x + \beta_y + \ldots \leq 1 \quad \text{for all} \quad C = \{x \prec y \prec \cdots\} \in \mathcal{C}(P). \tag{2.9}$$

In [Sta86], Stanley computed the volume of both polytopes:

$$\operatorname{Vol}(\mathcal{O}_P) = \operatorname{Vol}(\mathcal{S}_P) = \frac{e(P)}{n!}. \tag{2.10}$$

This connection is the key to many applications of geometry to poset theory and vice versa. Stanley also computed the *Ehrhart polynomial*

$$\bigl|t\cdot\mathcal{O}_P \cap \mathbb{Z}^n\bigr| = \bigl|t\cdot\mathcal{S}_P\cap\mathbb{Z}^n\bigr| = \Omega(P,t+1). \tag{2.11}$$

Note that (2.10) and (2.11) give:

$$\Omega(P,t) \sim \frac{e(P)\,t^n}{n!} \quad \text{as} \quad t\to\infty. \tag{2.12}$$

2.5. **Complexity.** We assume that the reader is familiar with basic notions and results in computational complexity and only recall a few definitions. We use standard complexity classes: P, FP, $\mathsf{NP}, \mathsf{coNP}$, $\#\mathsf{P}$, Σ^{p}_m and PH. The notation $\{a =^? b\}$ is used to denote the decision problem whether $a = b$. We use the *oracle notation* R^{S} for two complexity classes R, $\mathsf{S} \subseteq \mathsf{PH}$, and the polynomial closure $\langle \mathrm{A} \rangle$ for a problem $\mathrm{A} \in \mathsf{PSPACE}$. We will also use less common classes

$$\mathsf{GapP} := \{f - g \mid f, g \in \#\mathsf{P}\} \quad \text{and} \quad \mathsf{C}_{=}\mathsf{P} := \{f(x) =^? g(y) \mid f, g \in \#\mathsf{P}\}.$$

Note that $\mathsf{coNP} \subseteq \mathsf{C}_{=}\mathsf{P}$.

We also assume that the reader is familiar with standard decision and counting problems: 3SAT, #3SAT and PERMANENT. Denote by #LE the problem of computing the number $e(P)$ of linear extensions. For a counting function $f \in \#\mathsf{P}$, the *coincidence problem* is defined as:

$$\mathrm{C}_f \ := \ \big\{ f(x) =^? f(y) \big\}.$$

Note the difference with the *equality verification problem*:

$$\mathrm{E}_{f-g} \ := \ \big\{ f(x) =^? g(x) \big\},$$

where $f, g \in \#\mathsf{P}$ are counting functions and $x \in X$ is an input. Clearly, we have both $\mathrm{E}_{f-g} \in \mathsf{C}_{=}\mathsf{P}$ and $\mathrm{C}_f \in \mathsf{C}_{=}\mathsf{P}$. Note also that $\mathrm{C}_{\#3\mathrm{SAT}}$ is both $\mathsf{C}_{=}\mathsf{P}$-complete and coNP-hard.

The distinction between *binary* and *unary* presentation will also be important. We refer to [GJ78] and [GJ79, §4.2] for the corresponding notions of NP-completeness and *strong* NP-completeness. Unless stated otherwise, we use the word "*reduction*" to mean the polynomial Turing reduction.

We refer to [AB09, Gol08, Pap94b] for definitions and standard results in computational complexity, and to [Aar16, Wig19] for extensive surveys of computational complexity applications in mathematics. See [GJ79] for the classical introduction and a long list of NP-complete problems. See also [Pak22, §13] for a recent overview of $\#\mathsf{P}$-complete problems in combinatorics.

3. Basic inequalities for the numbers linear extensions

Here by basic inequalities we mean inequalities for $e(P)$ in terms of various poset parameters. The inequalities themselves range from elementary to highly nontrivial.

3.1. **Induced subsets.** Let $P = (X, \prec)$ be a finite poset. Denote by $k(P)$ the number of incomparable pairs of elements: $x \| y$, where $x, y \in X$. Equivalently, $k(P)$ is the number of induced subposets isomorphic to A_2. Similarly, denote by $\ell(P)$ and $m(P)$ the number of induced subposets isomorphic to A_3 and $(C_2 + C_1)$, respectively.

Proposition 3.1 (Ewacha, Rival and Zaguia [ERZ97]). *Let $P = (X, \prec)$ be a finite poset, and let $k = k(P)$, $\ell = \ell(P)$ and $m = m(P)$ defined as above. Then:*

$$2^k \left(\tfrac{3}{4}\right)^{\ell+m} \ \le \ e(P) \ \le \ 2^k. \tag{3.1}$$

The upper bound is trivial and tight for linear sums of A_2 and C_1. The lower bound is tight for linear sums of A_3 and $(C_2 + C_1)$ and is weak when $(\ell + m)$ is large. The authors have conjectured further inequalities of this type.

3.2. **Partitions into chains and antichains.** Suppose $P = (X, \prec)$ is a subposet of $Q = (X, \prec')$, i.e., we have $x \prec' y$ implies $x \prec y$. Clearly, we have $e(P) \leq e(Q)$. The following easy consequence is especially notable:

Proposition 3.2. *Let $P = (X, \prec)$ be a poset with $|X| = n$ elements, and let $X = C_1 \cup \ldots \cup C_\ell$ be a partition into disjoint chains of sizes $c_1, \ldots, c_\ell$. Then we have:*

$$e(P) \, \leq \, \tbinom{n}{c_1, \ldots, c_\ell}. \tag{3.2}$$

We also have the following antichain version, straight by definition.

Proposition 3.3. *Let $P = (X, \prec)$, and let $X = A_1 \cup \ldots \cup A_m$ be a partition into disjoint antichains of sizes $a_1, \ldots, a_m$, such that $A_1 \prec \ldots \prec A_m$. Then we have:*

$$e(P) \, \geq \, a_1! \, \cdots \, a_m! \tag{3.3}$$

The following is a surprising generalization that extends this to all partitions into antichains.

Theorem 3.4 (Bochkov and Petrov [BP21, Cor. 6])**.** *Let $P = (X, \prec)$ be a poset, and let $X = A_1 \cup \ldots \cup A_m$ be a partition into disjoint antichains of sizes $a_1, \ldots, a_m$. Then we have:*

$$e(P) \, \geq \, a_1! \, \cdots \, a_m! \tag{3.4}$$

In [BP21, Thm 1], both the upper bound (3.2) and the lower bound (3.3) are extended to the *Greene–Kleitman–Fomin parameters*, see e.g. [BF01, GK78].

3.3. **Recursion over antichains.** The following result follows directly from the definition:

Proposition 3.5. *Let $\min(P)$ be the set of minimal elements of $P = (X, \prec)$. We have:*

$$e(P) \, = \sum_{x \in \min(P)} e(P - x). \tag{3.5}$$

By induction, this gives the following upper bound:

Corollary 3.6. *Let $P = (X, \prec)$ be a poset of width w with $|X| = n$ elements. Then we have: $e(P) \leq w^n$.*

The following inequality is a surprising generalization of the proposition:

Theorem 3.7 (Edelman–Hibi–Stanley [EHS89] and Stachowiak [Sta89a])**.** *Let A be an antichain in $P = (X, \prec)$. We have:*

$$e(P) \, \geq \sum_{x \in A} e(P - x). \tag{3.6}$$

Moreover, this inequality is an equality if and only if A intersects every maximal chain, i.e. $|C \cap A| = 1$ for all $C \in \mathcal{C}(P)$. Additionally, the defect of (3.6) is in $\#\mathsf{P}$.

Clearly, the set of minimal elements $\min(P)$ satisfies contains an element from every maximal chain, so Theorem 3.7 implies Proposition 3.5. Note that not every maximal antichain satisfies this condition. For example, take chains C_2 on $\{1, 2\}$ and C_3 on $\{1', 2', 3'\}$, and let $P = C_2 \times C_3$. Consider an antichain $A = \big\{(1, 3'), (2, 1')\big\}$ in P. Now note that a maximal chain $C = \big\{(1, 1'), (1, 2'), (2, 2'), (2, 3')\big\}$ does not contain elements in A.

Note that the original proof of Theorem 3.7 in [EHS89] is combinatorial and uses promotion operators (see §14.1). We refer to [CPP23b, Cor. 8.2] for a proof using Sidorenko's flow (see §14.5). The proof in [Sta89a] uses a simple induction, and the theorem is applied to obtain the following result:

Theorem 3.8 (Stachowiak [Sta89a]). *Let $P = (X, \prec)$ and $Q = (X, \prec')$ be two posets on the same ground set, such that their comparability graphs satisfy: $\Gamma(P) \supseteq \Gamma(Q)$. Then $e(P) \leq e(Q)$. In particular, $e(P)$ depends only on $\Gamma(P)$. Additionally, if $e(P) = e(Q)$, then $P = Q$.*

For posets of height two, Theorem 3.8 was proved in an earlier paper [Sta88]. That $e(P)$ depends only on the comparability graph $\Gamma(P)$ was also proved in [EHS89], and extended to the order polynomial. Note that Theorem 3.8 follows easily from the volume formula (2.10) and geometric description of vertices of the chains polytope $\mathcal{C}_P(P)$, see below. We refer to [Sta86] for the introduction, and to [Iri17] for a recent exploration of the connection between $e(P)$ and orientations of $\Gamma(P)$.

Corollary 3.9. *Let $P = (X, \prec)$ be a poset with $|X| = n$ elements, and let $x \in X$. Then: $e(P) \leq n\, e(P - x)$.*

This follows immediately from Theorem 3.8 by taking $Q := (P - x) + C_1$ and noting that $\Gamma(P) \supseteq \Gamma(Q)$, $e(Q) = n\, e(P - x)$.

3.4. **Weighted chains.** There is a better way to give an upper bound for $e(P)$, by assigning weights to elements of antichains.

Proposition 3.10. *Let $\xi : X \to \mathbb{R}_{>0}$ be a positive function s.t.*

$$\sum_{x \in A} \xi(x) \,\leq\, 1 \quad \text{for all} \quad A \in \mathcal{A}(P). \tag{3.7}$$

Then:

$$e(P) \,\leq\, \prod_{x \in X} \frac{1}{\xi(x)}\,. \tag{3.8}$$

For example, $\xi(x) := 1/w$ gives Corollary 3.6. The following construction shows how this bound can be improved. For $x \in X$, denote by $\mathcal{C}_x(P) \subseteq \mathcal{C}(P)$ the subset of chains in P which contain x. Let $v : \mathcal{C}(P) \to \mathbb{R}_{\geq 0}$ be a probability distribution on chains in P. Denote

$$c_x \,:=\, \sum_{C \in \mathcal{C}_x(P)} v(C)\,.$$

Corollary 3.11. *Suppose $c_x > 0$ for all $x \in X$. Then:*

$$e(P) \,\leq\, \prod_{x \in X} \frac{1}{c_x}\,. \tag{3.9}$$

Proof. Take $\xi(x) := c_x$. Observe that condition (3.7) holds trivially for every chain $C \in \mathcal{C}(P)$, and thus for every probability distribution v on $\mathcal{C}(P)$. Thus, (3.9) follows from (3.8). $\square$

3.5. **LYM property.** Let $P = (X, \prec)$ be graded poset with the rank function $\rho : X \to \{0, \ldots, \ell\}$. Denote by X_r the set of elements of rank r, i.e. $X_r = \rho^{-1}(r)$, and let $n_r := |X_r|$ for all $1 \leq r \leq \ell$. Clearly, $n_r \leq w$, where w is the width of P.

We say that P has *LYM property* if for every antichain $A \in \mathcal{A}(P)$ we have:

$$\sum_{x \in A} \frac{1}{n_{\rho(x)}} \,\leq\, 1\,. \tag{LYM}$$

In particular, the width $w = \max_r n_r$. Such posets are called *LYM posets*.

Let G be a subgroup of $\mathrm{Aut}(P)$ which acts transitively on the set $\mathcal{C}_{\max}$ of maximal chains in P. Let v be a uniform distribution on $\mathcal{C}_{\max}$. By transitivity, $c_x = \frac{1}{n_r}$ where $r = \rho(x)$. Since $\mathcal{C}_x \cap \mathcal{C}_y = \varnothing$ for all $x, y \in A$, we have (LYM). We can now use (3.9) to conclude:

Corollary 3.12 ([SK87])**.** *Let $P = (X, \prec)$ be a graded poset with n_r elements of rank r, and with a transitive action on the set of maximal chains in P. Then P is a LYM poset, and*

$$e(P) \leq \prod_{r=0}^{\ell} (n_r)^{n_r}. \tag{3.10}$$

Note the sequence of implications here: transitive action implies (LYM) and the upper bound in (3.10). The following result proves the implication directly.

Theorem 3.13 (Brightwell–Tetali [BT03, Thm 5.1])**.** *Let $P = (X, \prec)$ be a graded LYM poset with n_r elements of rank r, $0 \leq r \leq \ell$. Then:*

$$\prod_{r=0}^{\ell} (n_r)! \leq e(P) \leq \prod_{r=0}^{\ell} (n_r)^{n_r}. \tag{3.11}$$

Here the lower bound is given by (3.3). The upper bound is an improvement over the $e(P) \leq w^n$ bound in Corollary 3.6. Corollary 3.12 was extended to all graded posets with LYM property. See also [Sha98] for the same result with slightly stronger assumptions. We refer to [GK76, GK78, Kle74] for equivalent definitions of LYM posets, and to [West21, §12.2] for further applications of (LYM).

Example 3.14 (Boolean algebra)**.** Let B_k be the poset of all subsets of $[k]$ by inclusion, so $n = 2^k$ and $\ell = k$. Observe that the symmetric group S_k acts transitively on $\mathcal{C}(B_k)$, and thus B_k has the LYM property. In particular, this implies *Sperner's theorem*:

$$w := \text{width}(B_k) = \tbinom{k}{\lfloor k/2 \rfloor}.$$

Now both Sha–Kleitman inequality (3.10) and Brightwell–Tetali inequality (3.11) give:

$$\prod_{r=0}^{k} \tbinom{k}{r}! \leq e(B_k) \leq \prod_{r=0}^{k} \tbinom{k}{r}^{\binom{k}{r}} \leq \tbinom{k}{\lfloor k/2 \rfloor}^{2^k}. \tag{3.12}$$

Lower bound and either upper bound give correct two leading terms of the asymptotics:

$$\frac{\log_2 e(B_k)}{2^k} = \log_2 \tbinom{k}{\lfloor k/2 \rfloor} + \Theta(1) = k - \tfrac{1}{2}\log_2 k + \Theta(1),$$

see e.g. [Coo09] for a careful calculation. In [BT03], it was shown that the lower bound in (3.12) gives the exact value of the constant implied by the $\Theta(1)$ notation.

Example 3.15 (Products of LYM posets)**.** Let P, Q be LYM posets with log-concave rank numbers. It was shown in [Har74, HK73], that the product $P \times Q$ satisfies the same properties. In particular, this implies that the product of chains $C_p \times C_q \times \ldots$ are also LYM posets. Thus, the bound in Theorem 3.13 applies and generalized the bounds in (3.12). Note that there is no transitive action on maximal chains in this case.

3.6. **Entropy bounds.** For a poset $P = (X, \prec)$ on $|X| = n$ elements, define the *entropy*

$$\mathrm{H}(P) := \min_{\boldsymbol{\beta} \in \mathcal{S}_P} \Big(-\frac{1}{n} \sum_{x \in X} \log \beta_x \Big), \tag{3.13}$$

where the minimum is over all vectors $\boldsymbol{\beta} : X \to \mathbb{R}_{>0}$ in the chain polytope $\mathcal{S}_P$.

Theorem 3.16 (Kahn–Kim [KK95])**.** *We have:*

$$n \log_2 n - n \cdot \mathrm{H}(P) \geq \log_2 e(P) \geq 0.09\big(n \log_2 n - n \cdot \mathrm{H}(P)\big). \tag{3.14}$$

Additionally,

$$\log_2 e(P) \geq \log_2 n! - n \cdot \mathrm{H}(P) \geq n \log_2 n - (\log_2 e)n - n \cdot \mathrm{H}(P). \tag{3.15}$$

Since the entropy on convex bodies can be approximated in polynomial time, this result can be viewed as a deterministic approximation algorithm for $e(P)$.

3.7. **Height two posets.** Let $P = (X, \prec)$ be a poset of height two with $n = |X|$ elements. Let $X = Y \cup Z$ be a partition into two antichains $Y, Z \in \mathcal{A}(P)$ corresponding to rank 0 and 1, respectively.

Theorem 3.17 (Brightwell–Tetali [BT03, Thm 1.4])**.** *Suppose there exist integers $a, b \in \mathbb{N}$, such that $\alpha(z) = a$ and $\beta(y) = b$, for all $y \in Y$ and $z \in Z$. Then:*

$$e(P) \,\le\, n!\, \binom{a+b}{a}^{-n/(a+b)}. \tag{3.16}$$

This inequality is sharp for $k := n/(a+b) \in \mathbb{N}$, as can be seen for a disjoint sum of k copies of poset $K_{ab} := A_b \oplus A_a$. Curiously, (3.16) fails if we instead use $\alpha(x) \ge a$ and $\beta(y) \ge b$, see [BT03]. The authors deduce Theorem 3.17 form the following result and the asymptotic formula (2.12).

Theorem 3.18 (Brightwell–Tetali [BT03, Thm 3.2])**.** *In conditions of Theorem 3.17, for all $t \ge 1$ we have:*

$$\Omega(P, t) \,\le\, \Omega(K_{ab}, t)^{n/(a+b)}. \tag{3.17}$$

The authors prove the result using technical entropy computations.

4. Basic inequalities for order polynomials

Order polynomial is just as fundamental object as the number of linear extensions, and in many cases easier to work with. Additionally, it has a clear geometric interpretation as the Ehrhart polynomial for poset polytopes, see (2.11).

4.1. **Explicit lower bound.** The following inequality extends the asymptotic formula (2.12):

Theorem 4.1 ([CPP23b, Thm 1.4, Cor 6.3])**.** *Let $P = (X, \prec)$ be a poset with $|X| = n$ elements. Then, for all integer $t \ge 1$, we have:*

$$\Omega(P, t) \,\ge\, \frac{e(P)\, t^n}{n!}\,. \tag{4.1}$$

Moreover, the equality holds for a given $t \ge 1$ if and only if $P = A_n$ is an antichain. Additionally, we have:

$$\Omega(P, t) \cdot n! \,-\, e(P)\, t^n \,\in\, \#\mathsf{P}.$$

Note that (4.1) improves upon a straightforward inequality $\Omega(P, t) \ge e(P)\binom{t}{n}$, where $t \ge n$. The authors prove the inequality by an explicit injection.

4.2. **Log-concavity.** Evaluations of the order polynomial have additional properties:

Theorem 4.2 (*log-concavity*, Brenti [Bre89, Thm 7.6.5])**.** *Let $P = (X, \prec)$ be a poset with $|X| = n$ elements. Then, for all integer $t \ge 2$, we have:*

$$\Omega(P, t)^2 \,\ge\, \Omega(P, t+1)\ \Omega(P, t-1). \tag{4.2}$$

In other words, (4.2) gives log-concavity of values of the order polynomial. This inequality is always strict:

Theorem 4.3 ([CPP23b, Thm 4.8])**.** *Let $P = (X, \prec)$ be a poset with $|X| = n$ elements. Then, for all integer $t \ge 2$, we have:*

$$\Omega(P, t)^2 \,\ge\, \left(1 + \frac{1}{(t+1)^{n+1}}\right) \Omega(P, t+1)\ \Omega(P, t-1). \tag{4.3}$$

Note that the $\frac{1}{(t+1)^{n+1}}$ term is far from optimal, see [CPP23b, Rem. 4.11]. We refer to [FH23] for the background on log-concavity of the Ehrhart polynomials of integral polytopes. Let us emphasize that although the original proof of Theorem 4.2 is via direct injection, this approach does not extend to Theorem 4.3 which is proved using the FKG inequality (see §14.4). Note that another direct combinatorial proof of (4.2) is given in [DDP84, Thm 5] (see also [Day84, §4.4]).

Theorem 4.4 (*q-log-concavity* [CPP23b, Thm 1.5]). *Let $P = (X, \prec)$ be a poset with $|X| = n$ elements. Then, for every integer $t \geq 2$, we have:*

$$\Omega_q(P,t)^2 \;\geqslant_q\; \Omega_q(P,t+1) \cdot \Omega_q(P,t-1), \tag{4.4}$$

where the inequality holds coefficient-wise as a polynomial in q.

We also have a multivariate version of this result:

Theorem 4.5 (*$\boldsymbol{q}$-log-concavity* [CP23a, Cor. 9.5]). *Let $P = (X, \prec)$ be a poset with $|X| = n$ elements. Then, for every integer $t \geq 2$, we have:*

$$\Omega_{\boldsymbol{q}}(P,t)^2 \;\geqslant_{\boldsymbol{q}}\; \Omega_{\boldsymbol{q}}(P,t+1) \cdot \Omega_{\boldsymbol{q}}(P,t-1), \tag{4.5}$$

where the inequality holds coefficient-wise as a polynomial in $\boldsymbol{q} = (q_1, \ldots, q_n)$.

Again, the proof of both theorems uses the FKG inequality (see §14.4). We conclude with a special case of an open problem by Ferroni and Higashitani stated in the language of Ehrhart polynomials of integral polytopes [FH23, Question 5.10], which asks if negative values of Ehrhart polynomials of integral polytopes are log-concave.

Theorem 4.6 (*negative log-concavity* [DDP84, Thm 3]). *Let $P = (X, \prec)$ be a poset with $|X| = n$ elements. Then, for all integer $t \leq -2$, we have:*

$$\Omega(P,t)^2 \;\geq\; \Omega(P,t+1)\;\Omega(P,t-1). \tag{4.6}$$

Note that for negative $t \in \mathbb{Z}$, the number $|\Omega(P,t)|$ counts the number of integral points in the relative interior of the expansion of the order polytope $\mathcal{O}(P)$. We refer to [BS18] and [Sta99, §4.6] for an extensive discussion of this connection.

4.3. **Monotonicity.** The following conjecture is mentioned in the solution to Exc. 3.163(b) in [Sta99], see also [CPP23b, Conj. 4.12].

Conjecture 4.7 (*Kahn–Saks monotonicity conjecture*). *Let $P = (X, \prec)$ be a poset with $|X| = n$ elements. Then, for all integer $t \geq 1$, we have:*

$$\frac{\Omega(P,t)}{t^n} \;\geq\; \frac{\Omega(P,t+1)}{(t+1)^n}\,. \tag{4.7}$$

The conjecture holds trivially when $\Omega(P,t)$ has positive coefficients. We refer to [LT19] for some explicit examples of order polynomials with negative coefficients. Stanley noted that the conjecture holds for t large enough, since the coefficient $[t^{n-1}]\,\Omega(P,t) > 0$. The proof is based on an elegant direct injection. The following result lend further support of the conjecture:

Proposition 4.8 ([CPP23b, Prop. 4.14]). *Let $P = (X, \prec)$ be a poset with $|X| = n$ elements. Then, for all integer $k, t \geq 1$, we have:*

$$\frac{1}{t^n}\;\Omega(P,t) \;\geq\; \frac{1}{(kt)^n}\;\Omega(P,kt).$$

Moreover, we have:

$$\Omega(P,t)\,k^n \;-\; \Omega(P,kt) \;\in\; \#\mathsf{P}.$$

Here the proof is elementary, via direct injection, and does not extend to $k \notin \mathbb{N}$. One way to approach the Kahn–Saks monotonicity conjecture is to prove the following inductive inequality:

Conjecture 4.9 ([CPP23b, Conj. 4.17])**.** *Let $P = (X, \prec)$ be a finite poset, and let $t \geq k \geq 1$ be positive integers. Then there exists $x \in X$, such that*

$$\frac{\Omega(P,k)}{\Omega(P,t)} \ \geq \ \frac{k\,\Omega(P-x,k)}{t\,\Omega(P-x,t)}\,. \tag{4.8}$$

Proposition 4.10 ([CPP23b, Prop. 4.18])**.** *Conjecture 4.9 implies Conjecture 4.7.*

We conclude with a curious counterpart of (4.7):

Theorem 4.11 ([CPP23b, Thm 4.8])**.** *Let $P = (X, \prec)$ be a finite poset of width w. Then, for all integer $t \geq 1$, we have:*

$$\frac{\Omega(P,t)}{t^w} \ \leq \ \frac{\Omega(P,t+1)}{(t+1)^w}\,. \tag{4.9}$$

This is asymptotically trivial, but not obvious for small t and large $w \leq n$. When $w = n$, we have $P = \mathrm{A}_n$, $\Omega(P,t) = t^n$, and both (4.7) and (4.9) are equalities. We note that proofs of both Proposition 4.10 and Theorem 4.11 use the FKG inequality (see §14.4).

Conjecture 4.12 (Chan–Panova, 2023)**.** *Let $P = (X, \prec)$ be a finite poset that is not a chain. Then, there exists elements $x, y \in X$, s.t. y covers x, and for all positive integers $t \geq k \geq 1$, we have:*

$$\frac{\Omega(Q,k)}{\Omega(P,k)} \ \leq \ \frac{\Omega(Q,t)}{\Omega(P,t)}\,. \tag{4.10}$$

where $Q = (X, \prec')$ is a poset obtained from P by removing $\{x \prec y\}$.

By analogy with Proposition 4.10, Conjecture 4.12 implies Conjecture 4.7.

5. Sidorenko type inequalities

5.1. **Sidorenko inequality.** The following result is a poset theoretic version of polyhedral duality.

Theorem 5.1 (*Sidorenko inequality* [Sid91, Thm 11])**.** *Let $P = (X, \prec)$ and $Q = (X, \prec')$ be two posets on the same set with $|X| = n$ elements. Suppose*

$$\bigl|C \cap C'\bigr| \leq 1 \quad \text{for all} \ \ C \in \mathcal{C}(P), \ C' \in \mathcal{C}(Q). \tag{5.1}$$

Then:

$$e(P)\,e(Q) \ \geq \ n! \tag{Sid}$$

Moreover, (Sid) *is an equality* <u>*if and only if*</u> *P is series-parallel.*

The assumption (5.1) can also be written in terms of comparability graphs: $\Gamma(P) \subseteq \overline{\Gamma(Q)}$, see §11.5 for examples. Note that testing if a poset is series-parallel is in P since they are N-free, see also [VTL82]. There are several proofs of Theorem 5.1. The original proof uses combinatorial optimization (see §14.5). More recent proofs use direct surjection [GG20] (cf. [MPP18c]), and injection [CPP23b, GG22]. In particular, we have the following:

Theorem 5.2 ([CPP23b, Thm 1.14] and [GG22, Thm 3.8])**.** *The defect of the Sidorenko inequality* (Sid) *is in* $\#\mathsf{P}$.

Remark 5.3 ([BBS99])**.** In condition of Theorem 5.1, suppose $\Gamma(P)=\overline{\Gamma(Q)}$. Then one can view Sidorenko's inequality (Sid) as a negative correlation result for uniform bijections $g: X \to [n]$. Indeed, note that $e(P\cap Q)=1$. Thus, we have:

$$\mathbf{P}\big(g\in\mathcal{E}(P)\cap\mathcal{E}(Q)\big) \ = \ \frac{1}{n!} \ \le \ \frac{e(P)\cdot e(Q)}{(n!)^2} \ = \ \mathbf{P}\big(g\in\mathcal{E}(P)\big)\,\cdot\,\mathbf{P}\big(g\in\mathcal{E}(Q)\big).$$

5.2. **Generalizations of the Sidorenko inequality.** The assumption (5.1) in the theorem can be relaxed to give the following result.

Theorem 5.4 ([CPP23b, Thm 1.7])**.** *Let $P=(X,\prec)$ and $Q=(X,\prec')$ be two posets on the same set with $|X|=n$ elements. Suppose*

$$\big|C\cap C'\big| \ \le \ k \quad \textit{for all} \ \ C\in\mathcal{C}(P), \ C'\in\mathcal{C}(Q).$$

Then:

$$e(P)\,e(Q) \ \ge \ \frac{n!}{k^{n-k}\,k!}\,. \tag{5.2}$$

The following result is a natural generalization of the Sidorenko inequality.

Theorem 5.5 (Sidorenko [Sid91, Thm 14])**.** *Let $P_1=(X,\prec_1)$, ..., $P_k=(X,\prec_k)$ and $Q=(X,\prec')$ be posets on the same set. Suppose*

$$\bigcap_{i=1}^{k}\Gamma(P_i) \ \subseteq \ \Gamma(Q). \tag{5.3}$$

Then:

$$e(P_1)\,\cdots\,e(P_k) \ \ge \ e(Q). \tag{5.4}$$

For example, take posets P_1,P_2 which satisfy $\Gamma(P_1)\subseteq\overline{\Gamma(P_2)}$. and let $Q\leftarrow A_n$. Then (5.3) gives $e(P_1)\,e(P_2)\ge e(Q)=n!$. In other words, Theorem 5.5 implies Theorem 5.1.

5.3. **Reverse Sidorenko inequality.** It may come as a surprise that the lower bound in the Sidorenko inequality is always sharp up to a simple exponential factor. Formally, we have the following:

Theorem 5.6 (*Reverse Sidorenko inequality* [BBS99])**.** *Let $P=(X,\prec)$ and $Q=(X,\prec')$ be two posets on the same set with $|X|=n$ elements which satisfy $\Gamma(P)=\overline{\Gamma(Q)}$. Denote by $\omega_n:=\operatorname{vol}(B_n)$ be the volume of a unit ball in $\mathbb{R}^n$. Then:*

$$e(P)\,e(Q) \ \le \ \frac{(n!\,\omega_n)^2}{4^n}\,. \tag{5.5}$$

The Stirling formula and the asymptotics for ω_n show that the Sidorenko inequality is asymptotically sharp:

$$n! \ \le \ e(P)\,e(Q) \ \le \ n!\,\big(\tfrac{\pi}{2}\big)^n\,O\big(\tfrac{1}{\sqrt{n}}\big). \tag{5.6}$$

Question 5.7. Denote by $\mu(n)$ the maximal value of the product $e(P)\,e(Q)$ over all posets P,Q on n elements which satisfy (5.1). In [BBS99], the authors ask to determine

$$\varkappa \ := \ \limsup_{n\to\infty}\left(\frac{\mu(n)}{n!}\right)^{1/n}.$$

They observe that $\varkappa>1.123$ and conjecture that $\varkappa<1.2$. The upper bound $\varkappa\le\frac{\pi}{2}\approx 1.571$ given by (5.6), remains the best known asymptotic upper bound.

Remark 5.8. *Mixed Sidorenko inequality* is another generalization of Sidorenko's inequality to *double posets* is given in [AASS20, Thm 6.2]. It would be interesting to see if this inequality has a direct injective proof.

6. Björner–Wachs type inequalities

6.1. **Björner–Wachs inequality.** The following inequality is elementary, but surprisingly rich in generalizations and applications:

Theorem 6.1 (*Björner–Wachs inequality* [BW89, Thm 6.3])**.** *Let $P = (X, \prec)$ be a poset with $|X| = n$ elements. We have:*

$$e(P) \ \geq \ n! \cdot \prod_{x \in X} \frac{1}{\beta(x)} \,. \tag{BW}$$

Moreover, the equality holds <u>if and only if</u> P is a forest. Additionally, the defect of the (BW) *is in* $\#\mathsf{P}$.

The original proof uses a direct injection. This inequality was popularized by Stanley, who stated it without proof or a reference in [Sta99, Exc. 3.57].[3] Unaware of the provenance, in [HP08], Hammett and Pittel gave a laborious proof in the language of geometric probability. Note that (BW) is asymmetric, i.e. not invariant under poset duality, leading to the following:

Corollary 6.2. *Let $P = (X, \prec)$ be a forest. Then:*

$$\prod_{x \in X} \alpha(x) \ \geq \ \prod_{x \in X} \beta(x) \,. \tag{6.1}$$

This inequality follows immediately from Theorem 6.1, since $\alpha(x)$ and $\beta(x)$ switch role in dual posets:

$$\prod_{x \in X} \beta(x) \ = \ \frac{n!}{e(P)} \ = \ \frac{n!}{e(P^*)} \ \leq \ \prod_{x \in X} \alpha(x) \,.$$

The corollary was also proved combinatorially and generalized in [PPS20]. The proof uses *Karamata's inequality*, which does not lead to an injection (cf. [IP22, §7.5]).

Remark 6.3. Theorem 6.1 is a correlation inequality in the following sense. Let $\Sigma(X)$ denote the set of bijections $\sigma : X \to [n]$. By definition, we have:

$$\mathbf{P}\big(\sigma(x) < \sigma(y) \ \ \forall \, x, y \in X, \, x \prec y\big) \ = \ \frac{e(P)}{n!} \,,$$

where $\mathbf{P}$ is a uniform measure on $\Sigma(X)$. Denote by $\mathcal{A}_x \subseteq \Sigma(X)$ the event that $\sigma(x) \leq \sigma(y)$ for all $x \prec y$. Then (BW) says that every collection of $\mathcal{A}_x$ is mutually positively correlated. The second part implies that for forests these events are mutually independent.

[3] Richard Stanley informed us that he indeed took it from [BW89] (personal communication, March 27, 2022).

6.2. **Reiner's inequality.** The following result is a natural q-analogue of the Björner–Wachs inequality (BW), but was discovered only recently:

Theorem 6.4 (*Reiner's inequality* [CPP23b, Thm 5.1] and [BW89, Thm 6.2])**.** *Let $P = (X, \prec)$ be a poset with $|X| = n$ elements. Then:*

$$\Omega_q(P) \ \geqslant_q \ \prod_{x \in X} \frac{1}{1 - q^{\beta(x)}}\,, \tag{6.2}$$

where the inequality between two power series is coefficient-wise. Moreover, this inequality is an equality if and only if P is a forest. Additionally, the coefficient $[q^m]$ of the defect of this inequality is in $\#\mathsf{P}$, where m is given in binary.

Reiner's inequality (6.2) was proved by Reiner by remarkably short and direct proof, see below. It was published by the authors in [CPP23b]. The equality part was proved in the original Björner–Wachs paper. Multiplying both sides of (6.2) by $(1-q)(1-q^2)\cdots(1-q^n)$ and using the equality (2.2), we conclude:

Corollary 6.5. *For all $0 < q < 1$, we have:*

$$e_q(P) \ \geq \ (1-q)(1-q^2)\cdots(1-q^n) \prod_{x \in X} \frac{1}{1 - q^{\beta(x)}}\,. \tag{6.3}$$

Taking the limit $q \to 1-$, gives the Björner–Wachs inequality (BW). This is probably the shortest and the most conceptual proof of (BW).

Proof of Theorem 6.4. Interpret the RHS of (6.2) as the GF for maps $g \in \mathcal{P}(P)$ which are obtained as a nonnegative integer linear combination of characteristic functions of upper order ideals in poset P:

$$g \ = \ \sum_{x \in X} m(x)\, \chi(x\!\uparrow)\,, \quad \text{where} \ \ m(x) \in \mathbb{N} \ \text{ for all } \ x \in X.$$

Note that characteristic functions $\chi(x\!\uparrow)$ are linearly independent because in the standard basis $\{\chi(y) \, : \, y \in X\}$, the transition matrix is unitriangular. Since $\sum_{x\in X} g(x) = \sum_{x \in X} m(x)\,\beta(x)$, the result follows immediately from (2.1). □

6.3. **Order polynomials version.** The following is the extension of the Björner–Wachs inequality for order polytopes:

Theorem 6.6 ([CPP23b, Thm 1.2])**.** *Let $P = (X, \prec)$ be a poset with $|X| = n$ elements, and let $r = |\min(P)|$ be the number of minimal elements. Then, for all $t \in \mathbb{N}$, we have:*

$$\Omega(P,t) \ \geq \ t^r\,(t+1)^{n-r} \prod_{x \in X} \frac{1}{\beta(x)}\,. \tag{6.4}$$

By (2.12), the inequality (6.4) implies (BW). The following result shows that (6.4) can be slightly improved if Conjecture 4.7 holds:

Theorem 6.7 ([CPP23b, Thm 4.13])**.** *Let $P = (X, \prec)$, let $\min(P) \subseteq X$ be the subset of maximal elements, and let $r := |\min(P)|$ be the number of maximal elements. If Conjecture 4.7 holds, then we have:*

$$\Omega(P,t) \ \geq \ t^r \prod_{x \in X \smallsetminus \min(P)} \left(\frac{t}{\beta(x)} + \frac{1}{2} \right). \tag{6.5}$$

It would be interesting to find an unconditional proof of this inequality. Both Theorems 6.6 and 6.7 were proved using the FKG inequality (see §14.4).

7. Fishburn type inequalities

7.1. **Two minimal elements.** We start with the following special case which is already interesting and hard to prove.

Theorem 7.1 (see [CP22b, Thm 1.1])**.** *Let $P = (X, \prec)$ be a poset with $|X| = n > 2$ elements. Let $x, y \in \min(X)$ be distinct minimal elements of P. Then:*

$$\frac{n}{n-1} \ \leq \ \frac{e(P) \cdot e(P-x-y)}{e(P-x) \cdot e(P-y)} \ \leq \ 2. \tag{7.1}$$

This correlation inequality is the most natural and the simplest to state. The lower bound in (7.1) is a special case of the Fishburn's inequality (7.3) below, while the upper bound is a special case of (8.13) below. Note that the lower bound is tight for $P = A_n$ and the upper bounds is tight for the linear sum $P = A_2 \oplus C_{n-2}$.

Remark 7.2. Correlation inequalities are best understood in probabilistic notations. The inequality (7.1) can be rewritten as

$$\frac{n}{n-1} \ \leq \ \frac{\mathbf{P}[f(x)=1, f(y)=2]}{\mathbf{P}[f(x)=1] \cdot \mathbf{P}[f(y)=1]} \ \leq \ 2, \tag{7.2}$$

where the probability $\mathbf{P}$ is over the uniform random linear extension $f \in \mathcal{E}(P)$. The asymmetry in the numerator is an illusion, since $\mathbf{P}[f(x)=1, f(y)=2] = \mathbf{P}[f(x)=2, f(y)=1]$.

Open Problem 7.3. *For many examples of large posets, the lower bound in* (7.1) *is tight. Can one improve the upper bound for a large natural class of posets?*

7.2. **Fishburn's inequality.** Let $P = (X, \prec)$ be a poset with $|X| = n$ elements. Denote

$$\varepsilon(P) \ := \ \frac{e(P)}{n!} \ = \ \mathbf{P}\big(f \in \mathcal{E}(P)\big),$$

where the probability is over uniform bijections $f : X \to [n]$.

Let $A \subseteq X$ be a subset of the ground set. By a small abuse of notation, denote by $e(A)$ the number of linear extensions of the induced subposet $P|_A = (A, \prec)$.

Theorem 7.4 (*Fishburn's inequality* [Fis84, Lemma, p. 130])**.** *Let $P = (X, \prec)$ be a finite poset, and let $A, B \subseteq X$ be lower ideals of P. Then:*

$$\varepsilon(A \cup B) \cdot \varepsilon(A \cap B) \ \geq \ \varepsilon(A) \cdot \varepsilon(B). \tag{7.3}$$

Taking $A := X - x$ and $B := X - y$ gives (7.1). The original proof of Fishburn's inequality uses the AD inequality (see §14.4). The following is a self-dual generalization.

Theorem 7.5 (*generalized Fishburn's inequality* [CP23a, Thm 3.4])**.** *Let $P = (X, \prec)$ be a finite poset. Let $A, B \subseteq X$ be lower ideals, and let $C, D \subseteq X$ be upper ideals of P, such that $A \cap C = B \cap D = \varnothing$. Then:*

$$\varepsilon(X - V) \cdot \varepsilon(X - W) \ \geq \ \varepsilon(X - A - C) \cdot \varepsilon(X - B - D), \tag{7.4}$$

where $V := (A \cap B) \cup (C \cup D)$ and $W := (A \cup B) \cup (C \cap D)$.

The proof of this generalization also uses the AD inequality.

7.3. **Lam–Pylyavskyy extension.** To simplify the notation, denote by $\Omega(A,t)$ the order polynomial of the induced subposet $P|_A$. Define $\Omega_q(A,t)$, $\Omega_{\boldsymbol{q}}(A,t)$ and $\Phi_{\boldsymbol{z}}(A,t)$ in a similar way.

Theorem 7.6 (Lam–Pylyavskyy [LP07, Thm 3.6])**.** *Let $P=(X,\prec)$ be a finite poset, and let $A,B\subseteq X$ be lower ideals of P. Then, for all integer $t\geq 1$, we have:*

$$\Omega(A\cup B,t)\,\cdot\,\Omega(A\cap B,t)\;\geq\;\Omega(A,t)\,\cdot\,\Omega(B,t). \tag{7.5}$$

Additionally, the defect of this inequality is in $\#\mathsf{P}$.

The authors proved this result by an explicit injection, which they generalize in several different ways. Some of these generalization are natural from the algebraic combinatorics point of view, but some are natural and apply to all posets.

Theorem 7.7 (Lam–Pylyavskyy [LP07, Prop. 3.7])**.** *Let $P=(X,\prec)$ be a finite poset, and let $A,B\subseteq X$ be lower ideals of P. Then, for all integer $t\geq 1$, we have:*

$$\Omega_q(A\cup B,t)\,\cdot\,\Omega_q(A\cap B,t)\;\geqslant_q\;\Omega_q(A,t)\,\cdot\,\Omega_q(B,t), \tag{7.6}$$

where the inequality holds coefficient-wise as a polynomial in q. Moreover, we have:

$$\Phi_{\boldsymbol{z}}(A\cup B,t)\,\cdot\,\Phi_{\boldsymbol{z}}(A\cap B,t)\;\geqslant_{\boldsymbol{z}}\;\Phi_{\boldsymbol{z}}(A,t)\,\cdot\,\Phi_{\boldsymbol{z}}(B,t), \tag{7.7}$$

where the inequality holds coefficient-wise as a polynomial in $\boldsymbol{z}=(z_0,z_1,\ldots)$. Additionally, the defect of this inequality is in $\#\mathsf{P}$.

Here the second part (7.7) implies the first part (7.6) by the substitution in (2.6). Letting $t\to\infty$, using the equality (2.2), and multiplying both sides by an appropriate product of the type $(1-q)(1-q^2)\cdots$ gives a q-analogue of Fishburn's inequality in the style of Corollary 6.5:

Corollary 7.8 (*q-Fishburn's inequality*)**.** *Let $P=(X,\prec)$ be a finite poset, and let $A,B\subseteq X$ be lower ideals of P. Then, for all $0<q<1$, we have:*

$$\frac{e_q(A\cup B)\cdot e_q(A\cap B)}{e_q(A)\cdot e_q(B)}\;\geq\;\frac{|A\cup B|!_q\cdot|A\cap B|!_q}{|A|!_q\cdot|B|!_q}\,. \tag{7.8}$$

Taking the limit $q\to 1-$ gives back Fishburn's inequality (7.3). The original proof of Theorem 7.7 uses an explicit injection. The proof in [CP23a] uses generalizations of the AD inequality (see §14.4).

7.4. **Self-dual extension of Fishburn's inequality.** Note that Fishburn's inequality is defined to be asymmetric up to duality. The following generalization is self-dual.

Theorem 7.9 ([CP23a, Thm 3.4])**.** *Let $P=(X,\prec)$ be a finite poset. Let $A,B\subseteq X$ be lower ideals, and let $C,D\subseteq X$ be upper ideals of P, such that $A\cap C=B\cap D=\varnothing$. Then:*

$$\varepsilon(X-V)\,\cdot\,\varepsilon(X-W)\;\geq\;\varepsilon(X-A-C)\,\cdot\,\varepsilon(X-B-D)\,. \tag{7.9}$$

where $V:=(A\cap B)\cup(C\cup D)$ and $W:=(A\cup B)\cup(C\cap D)$.

Fishburn's inequality (7.3) is a special case of the theorem when $C=D=\varnothing$. Curiously, we are able to prove both multivariate analogues in this setting:

Theorem 7.10 ([CP23a, Thms 4.9 and 4.10])**.** *Let $\mathcal{P}=(X,\prec)$ be a finite poset. Let $A,B\subseteq X$ be lower ideals, and let $C,D\subseteq X$ be upper ideals of $\mathcal{P}$, such that $A\cap C=B\cap D=\varnothing$. Fix and integer $t\geq 1$. Then:*

$$\Omega_{\boldsymbol{q}}(X-V,t)\,\cdot\,\Omega_{\boldsymbol{q}}(X-W,t)\;\geqslant_{\boldsymbol{q}}\;\Omega_{\boldsymbol{q}}(X-A-C,t)\,\cdot\,\Omega_{\boldsymbol{q}}(X-B-D,t), \tag{7.10}$$

where $V := (A \cap B) \cup (C \cup D)$ *and* $W := (A \cup B) \cup (C \cap D)$, *and the inequality holds coefficient-wise as a polynomial in* $\boldsymbol{q} = (q_1, q_2, \ldots)$. *Similarly, we have:*

$$\Phi_{\boldsymbol{z}}(X - V, t) \cdot \Phi_{\boldsymbol{z}}(X - W, t) \ \geqslant_{\boldsymbol{z}} \ \Phi_{\boldsymbol{z}}(X - A - C, t) \cdot \Phi_{\boldsymbol{z}}(X - B - D, t), \tag{7.11}$$

and the inequality holds coefficient-wise as a polynomial in $\boldsymbol{z} = (z_0, z_1, \ldots)$.

The proof of Theorems 7.9 and 7.10 also uses generalizations of the AD inequality (see §14.4), and thus fundamentally non-injective. This leaves open whether Lam–Pylyavskyy injective arguments can be modified to answer the following:

Question 7.11. *Is the deficit of inequalities* (7.10) *and* (7.11) *in* #P*?*

8. Correlation inequalities

8.1. **GYY inequality.** Let $P = C + C'$ be a disjoint sum of two chains with ℓ and $(n - \ell)$ elements, respectively, where $C = \{u_1 \prec \ldots \prec u_\ell\}$ and $C' = \{v_1 \prec \ldots \prec v_{n-\ell}\}$.

Denote $\Lambda := [\ell] \times [n - \ell]$. For all $S \subseteq \Lambda$, let $A_S := (X, \prec_S)$ be a poset with the relations $u_i \prec_S v_j$ for all $(i, j) \in S$. Note that posets A_S, A_T are consistent with each other and with P, for all $S, T \subseteq \Lambda$.

Theorem 8.1 (*Graham–Yao–Yao inequality* [GYY80, Thm 1])**.** *Let* $S, T \subseteq \Lambda$. *Then:*

$$e(P \cap A_S \cap A_T)\, e(P) \ \geq \ e(P \cap A_S)\, e(P \cap A_T). \tag{GYY}$$

Additionally, the defect of this inequality is in #P.

The theorem was originally proved by Graham, Yao and Yao in [GYY80] using a lattice paths argument (see §14.3). A proof using the FKG inequality was given in [KS81], and soon after in [She80], see the generalization below.

Remark 8.2. The result simplifies in a probabilistic setting. Denote by $\mathcal{A}_S \subseteq \mathcal{E}(P)$ the event that a linear extension $f \in \mathcal{E}(P)$ satisfies relations in $\prec_S$. Then (GYY) can be rewritten as a positive correlation:

$$\mathbf{P}(\mathcal{A}_S \cap \mathcal{A}_T) \ \geq \ \mathbf{P}(\mathcal{A}_S)\, \mathbf{P}(\mathcal{A}_T), \tag{8.1}$$

where $\mathbf{P}$ is the uniform measure on $\mathcal{E}(P)$. Note that $\mathbf{P}(\mathcal{A}_S) = 0$ if $\mathcal{A}_S$ and P are inconsistent.

8.2. **Shepp's inequality.** Let $P = Q + Q'$, where $Q = (U, \prec)$ and $Q = (V, \prec')$. Denote by $U = \{u_1, \ldots, u_\ell\}$ and $V = \{v_1, \ldots, v_{n-\ell}\}$ the elements in these two posets.

Theorem 8.3 (*Shepp's inequality* [She80, Thm 2])**.** *Let* $S, T \subseteq \Lambda$. *Then:*

$$e(P \cap A_S \cap A_T)\, e(P) \ \geq \ e(P \cap A_S)\, e(P \cap A_T). \tag{8.2}$$

Similarly,

$$e(P \cap A_S \cap A_T^*)\, e(P) \ \leq \ e(P \cap A_S)\, e(P \cap A_T^*), \tag{8.3}$$

where we use the notation $e(P \cap R) = 0$ *if posets* P *and* R *are not consistent.*

This inequality was conjectured in [GYY80, p. 252], which also mentioned that it becomes false if P has even one relation of the form $u_i \prec v_j$. In a note added in proof, the authors wrote that Theorem 8.3 was proved by Shepp [She80] using "an ingenious application of the FKG inequalities" [GYY80, p. 258].

Open Problem 8.4. *Prove of disprove: the defect of inequality* (8.2) *is in* #P.

If true, this would extend the second part of Theorem 8.1.

Theorem 8.5 (*Shepp's inequality for order polynomials* [She80, Eq. (2.12)]). *Let $S, T \subseteq \Lambda$. Then, for every integer $t \geq 1$, we have:*

$$\Omega(P \cap A_S \cap A_T, t) \cdot \Omega(P, t) \ \geq \ \Omega(P \cap A_S, t) \cdot \Omega(P \cap A_T, t). \tag{8.4}$$

Similarly,

$$\Omega(P \cap A_S \cap A_T^*, t) \cdot \Omega(P, t) \ \leq \ \Omega(P \cap A_S, t) \cdot \Omega(P \cap A_T, t), \tag{8.5}$$

where we use the notation $\Omega(P \cap R, t) = 0$ if posets P and R are not consistent.

By (2.12), this theorem implies Shepp's inequality (Theorem 8.3). The following q-analogue is the most general result in this direction.

Theorem 8.6 (*q-analogue of Shepp's inequality* [CPP23b, Thm 5.4]). *Let $S, T \subseteq \Lambda$. Then we have:*

$$\Omega_q(P \cap A_S \cap A_T) \cdot \Omega_q(P) \ \geqslant_q \ \Omega_q(P \cap A_S) \cdot \Omega_q(P \cap A_T), \tag{8.6}$$

where the inequality holds coefficient-wise as a polynomial in q. More generally, for every integer $t \geq 1$, we have:

$$\Omega_q(P \cap A_S \cap A_T, t) \cdot \Omega_q(P, t) \ \geqslant_q \ \Omega_q(P \cap A_S, t) \cdot \Omega_q(P \cap A_T, t). \tag{8.7}$$

The negative correlation version can be obtained in a similar way. Theorem 8.6 is proved using Björner's q-FKG inequality (see §14.4).

8.3. **XYZ inequality.** The following result is perhaps the most celebrated correlation inequality for linear extensions:

Theorem 8.7 (*XYZ inequality* [She82]). *Let $P = (X, \prec)$ be a finite poset, and let $x, y, z \in X$ be incomparable elements. Denote $P_{xy} := P \cap \{x \prec y\}$, $P_{xz} := P \cap \{x \prec z\}$ and $P_{xyz} := P \cap \{x \prec y, x \prec z\}$. Then:*

$$e(P)\, e(P_{xyz}) \ \geq \ e(P_{xy})\, e(P_{xz}). \tag{XYZ}$$

Moreover, for all integer $t \geq 1$, we have:

$$\Omega(P, t) \cdot \Omega(P_{xyz}, t) \ \geq \ \Omega(P_{xy}, t) \cdot \Omega(P_{xz}, t). \tag{8.8}$$

Inequality (XYZ) was first conjectured by Ivan Rival and Bill Sands [Riv82, p. 806]. Shepp's original proof of (XYZ) used the FKG inequality and goes through (8.8). It was proved by Fishburn [Fis84], that (XYZ) is always strict. A combinatorial (but not fully injective) argument was given in [BT02].

Conjecture 8.8 ([Pak22, Conj. 6.4]). *The defect of (XYZ) is not in $\#\mathsf{P}$.*

Remark 8.9. As with other correlation inequalities, the XYZ inequality is easier to understand in terms of uniform random linear extensions $f \in \mathcal{E}(P)$. For incomparable elements $u, v \in X$, denote $\mathcal{E}_{uv} := \mathcal{E}(P_{uv}) \subset \mathcal{E}(P)$. Then:

$$\mathbf{P}(\mathcal{E}_{xy} \cap \mathcal{E}_{xz}) \geq \mathbf{P}(\mathcal{E}_{xy}) \cdot \mathbf{P}(\mathcal{E}_{xz}). \tag{8.9}$$

To simplify the notation, we write $\mathcal{A} \vartriangle \mathcal{B}$ if the events $\mathcal{A}$ and $\mathcal{B}$ have *positive correlation*: $\mathbf{P}(\mathcal{A} \cap \mathcal{B}) \geq \mathbf{P}(\mathcal{A}) \cdot \mathbf{P}(\mathcal{B})$. Similarly, we write $\mathcal{A} \triangledown \mathcal{B}$ if these events have *negative correlation*: $\mathbf{P}(\mathcal{A} \cap \mathcal{B}) \leq \mathbf{P}(\mathcal{A}) \cdot \mathbf{P}(\mathcal{B})$. In this notation, (8.9) can be written as $(\mathcal{E}_{xy}) \vartriangle (\mathcal{E}_{xz})$, or, equivalently, as $(\mathcal{E}_{xy}) \triangledown (\mathcal{E}_{yz})$.

In [Bri85], Brightwell described all collections of inequalities for which we have the analogue of (8.9). Typical examples include:

$$(\mathcal{E}_{xy} \cap \mathcal{E}_{uv}) \vartriangle (\mathcal{E}_{xv} \cap \mathcal{E}_{uy}), \quad (\mathcal{E}_{xz} \cap \mathcal{E}_{yz}) \triangledown (\mathcal{E}_{zu} \cap \mathcal{E}_{zv}) \quad \text{and} \quad (\mathcal{E}_{xw} \cap \mathcal{E}_{yw} \cap \mathcal{E}_{zu} \cap \mathcal{E}_{zv}) \triangledown (\mathcal{E}_{wz}).$$

This resolved Colin McDiarmid's question and negatively resolved a conjecture of Kahn and Saks, see e.g. [Win86, p. 168]. For the GYY inequality (GYY), the corresponding result was obtained by Winkler in [Win83]. See also an extensive discussion in [Bri85, Day84, Fis92].

8.4. **Average height.** The *average height* of an element x in poset P, is defined as

$$h(P,x) \, := \, \mathbf{E}[f(x)] \, = \, \frac{1}{e(P)} \sum_{f \in \mathcal{E}(P)} f(x).$$

Theorem 8.10 (Winkler [Win82]). *Let $P = (X, \prec)$ be a poset, and let $x, y \in X$ be incomparable elements. Then:*

$$h(P,x) \, \geq \, h(P_{xy}, x). \tag{8.10}$$

Moreover, for all $k \in \mathbb{N}$, we have:

$$\mathbf{P}\big(f(x) > k\big) \, \geq \, \mathbf{P}\big(f(x) > k \mid f(x) < f(y)\big). \tag{8.11}$$

The proof follows easily from Shepp's proof of the XYZ inequality (Theorem 8.7). Now, for a subset $S \subseteq X$, denote

$$h(S) \, := \, \mathbf{E}\big[\min_{x \in S} f(x)\big].$$

In particular, $h(S) = h(P,x)$ for $S = \{x\}$, and $h(X) = 1$.

Theorem 8.11 (Winkler [Win82, Thm 4]). *Let $P = (X, \prec)$ be a poset, and let $U, V \subset X$ such that $U \cup V = X$. Then:*

$$h(U) \cdot h(V) \, \leq \, h(U) \, + \, h(V). \tag{8.12}$$

The proof of (8.12) is another elementary probabilistic application of the XYZ inequality.

Corollary 8.12. *Let $x, y \in X$ be the only two minimal elements in P. Then $h(P,x) \cdot h(P,y) \leq h(P,x) + h(P,y)$. In particular, either $h(P,x) \leq 2$ or $h(P,y) \leq 2$.*

Note that this is tight, since for $P = C_\ell + C_\ell$ and $x \in \min(P)$, we have $h(P,x) \to 2$ as $\ell \to \infty$.

Proof. Let $U := x\!\uparrow$ and $V := y\!\uparrow$. Observe that $U \cup V = X$. By definition, we also have $h(P,x) = h(U)$ and $h(P,y) = h(V)$. The result now follows from (8.12). □

8.5. **Deletion correlations.** Let $P = (X, \prec)$ be a poset with $|X| = n$ elements, let $z \in X$ and $a \in [n]$. Let $\mathcal{E}(P, z, a)$ be the set of linear extensions $f \in \mathcal{E}(P)$, such that $f(z) = a$. Denote by $\mathrm{N}(P, z, a) := \big|\mathcal{E}(P, z, a)\big|$ the number of such linear extensions.

Theorem 8.13 ([CP22b, Thm 6.3]). *Let $P = (X, \prec)$ be a poset with $|X| = n > 2$ elements. Fix an element $z \in X$ and integer $1 \leq a \leq n-2$. Then, for all distinct minimal elements $x, y \in \min(X - z)$, we have:*

$$\mathrm{N}(P, z, a) \cdot \mathrm{N}(P - x - y, z, a) \, \leq \, 2\,\mathrm{N}(P - x, z, a) \cdot \mathrm{N}(P - y, z, a). \tag{8.13}$$

Taking a disjoint sum $P \leftarrow P + z$ and $a = 1$, we get a special case of the upper bound in (7.1), a counterpart to the corollary of Fishburn's inequality. The original proof uses the combinatorial atlas (see §14.7). The same holds for results for the rest of the section.

8.6. **Subsets.** Fix a nonempty subset $A \subseteq X$. For a linear extension $f \in \mathcal{E}(P)$, define

$$f(A) := \big\{ f(x) : x \in A \big\} \quad \text{and} \quad f_{\min}(A) := \min f(A). \tag{8.14}$$

Note that $f_{\min}(A) = f(x)$ for all singletons $A = \{x\}$, where $x \in X$. The following result is more natural in probabilistic notation:

Theorem 8.14. *Let $P = (X, \prec)$ be a poset on $|X| \geq 2$ elements. Fix a nonempty subset $A \subseteq X$. Then:*

$$\mathbf{P}[1, 2 \notin f(A)] \;\leq\; \mathbf{P}[1 \notin f(A)]^2 \qquad \textit{and} \tag{8.15}$$

$$\mathbf{P}[1 \in f(A)] \cdot \mathbf{P}[1 \notin f(A)] \;\leq\; \mathbf{P}[1 \notin f(A), 2 \in f(A)], \tag{8.16}$$

We return to this result in §9.9. The following corollary is an easy consequence of Theorem 8.14.

Corollary 8.15 ([CP22b, Cor. 1.7])**.** *Let $P = (X, \prec)$ be a poset on $|X| \geq 2$ elements, and let $A \subseteq X$ be a nonempty subset of elements. Then:*

$$\mathbf{P}[1, 2 \in f(A)] \cdot \mathbf{P}[1, 2 \notin f(A)] \;\leq\; \mathbf{P}[1 \in f(A), 2 \notin f(A)]^2. \tag{8.17}$$

Proof. Multiply (8.15) for subsets A and $X \smallsetminus A$. Then use (8.16). $\square$

Note that A is an *arbitrary* nonempty subset of the ground set X. For a subset $\mathrm{W} \subseteq \mathcal{E}(P)$, we write $\mathrm{N}\big(P, z, a \,|\, \mathrm{W}\big)$ to denote the number of linear extensions $f \in \mathrm{N}(P, z, a)$ which satisfy condition W. The following result is a generalization of Corollary 8.15.

Theorem 8.16 ([CP22b, Lem. 6.4])**.** *Let $P = (X, \prec)$ be a poset on $|X| = n \geq 3$ elements, let $z \in X$, $a \in \{3, \ldots, n\}$, and let $A \subseteq X - z$ be a nonempty subset. Then:*

$$\begin{aligned} &\mathrm{N}\big(P, z, a \,|\, 1 \in f(A), 2 \in f(A{\uparrow})\big) \cdot \mathrm{N}(P, z, a \,|\, 1, 2 \notin f(A)) \\ &\qquad \leq\; \mathrm{N}\big(P, z, a \,|\, 1 \in f(A), 2 \notin f(A)\big)^2. \end{aligned} \tag{8.18}$$

Taking a disjoint sum $P \leftarrow P + z$ and $a = n$ implies (8.17). Note also that

$$\mathrm{N}\big(P, z, a \,|\, 1 \in f(A), 2 \in f(A)\big) \;\leq\; \mathrm{N}\big(P, z, a \,|\, 1 \in f(A), 2 \in f(A{\uparrow})\big),$$

so (8.18) gives a stronger inequality.

8.7. **Covariance inequalities.** The following theorem gives a similar upper bound for the covariances:

Theorem 8.17 ([CP22b, Thm 1.2])**.** *Let $P = (X, \prec)$ be a finite poset, and let $x, y \in X$ be fixed poset elements. Then:*

$$\frac{\mathbf{E}[f(x) f(y)] \,+\, \mathbf{E}\big[\min\{f(x), f(y)\}\big]}{\mathbf{E}[f(x)] \cdot \mathbf{E}[f(y)]} \;\leq\; 2. \tag{8.19}$$

The following result generalized this to subsets:

Theorem 8.18 ([CP22b, Thm 1.8])**.** *Let $P = (X, \prec)$ be a finite poset, and let $A, B \subseteq X$ be nonempty subsets. Then:*

$$\frac{\mathbf{E}\big[f_{\min}(A)\, f_{\min}(B)\big] \,+\, \mathbf{E}\big[f_{\min}(A \cup B)\big]}{\mathbf{E}\big[f_{\min}(A)\big] \cdot \mathbf{E}\big[f_{\min}(B)\big]} \;\leq\; 2. \tag{8.20}$$

Let us emphasize that here A and B are *arbitrary* subsets of the ground set X. Recall that $B{\uparrow} := \cup_{b \in B}\, b{\uparrow}$ denotes the *upper closure* of a subset $B \subseteq X$. The following is a symmetric generalization of Theorem 8.16 to two disjoint subsets of minimal elements:

Theorem 8.19 ([CP22b, Thm 1.9]). *Let $P = (X, \prec)$ be a finite poset, and let $A, B \subset \min(P)$ be disjoint nonempty subsets of minimal elements. Then:*

$$\mathbf{P}\big[1 \in f(A), 2 \in f(A\!\uparrow)\big] \cdot \mathbf{P}\big[1 \in f(B), 2 \in f(B\!\uparrow)\big] \ \leq \ \mathbf{P}\big[1 \in f(A), 2 \in f(B)\big]^2. \tag{8.21}$$

See also [CP22b, Thm 1.10], for a three element generalization of this inequality.

8.8. **Unique covers.** Let $P = (X, \prec)$ be a poset, and let $x, y \in X$. Recall that element y *covers* x, if $x \prec y$, and there is no $v \in X$ s.t. $x \prec v \prec y$. For elements $x \prec y$ in X, we say that y is a *unique cover* of x, if y covers x and does not cover any other elements in X.

Theorem 8.20 ([CP22b, Cor. 3.10]). *Let $P = (X, \prec)$ be a finite poset, and let $x, y \in \min(P)$ be distinct minimal elements. Suppose element $v \in X$ is a unique cover of x, and $w \in X$ is a unique cover of y. Then:*

$$e(P - x - y)^2 \ \geq \ e(P - x - v) \cdot e(P - y - w). \tag{8.22}$$

This inequality is derived from (8.21) for $A = \{x\}$ and $B = \{y\}$. We conclude with the following four element inequality.

Theorem 8.21 ([CP22b, Cor. 3.11]). *Let $P = (X, \prec)$ be a finite poset, and let $x, y, z \in \min(P)$ be distinct minimal elements. Suppose element $u \in X$ is a unique cover of z. Then:*

$$e(P - u - z) \cdot e(P - x - y) \ \leq \ 2\, e(P - x - z) \cdot e(P - y - z). \tag{8.23}$$

This is a direct corollary of a three element generalization of Theorem 8.19 mentioned above.

9. Stanley type inequalities

In this section we present a collection of Stanley type inequalities. In the next section, we discuss various equality conditions for some of these inequalities.

9.1. **Stanley inequality.** Let $P = (X, \prec)$ be a poset with $|X| = n$ elements, let $x \in X$ and $a \in [n]$. Recall that $\mathcal{E}(P, x, a)$ denotes the set of linear extensions $f \in \mathcal{E}(P)$, such that $f(x) = a$, and that $\mathrm{N}(P, x, a) := \big|\mathcal{E}_{\boldsymbol{z}\mathbf{c}}(P, x, a)\big|$. *Stanley's inequality* states that $\big\{\mathrm{N}(P, x, a)\big\}$ is log-concave:

Theorem 9.1 (*Stanley's inequality* [Sta81, Thm 3.2]). *We have:*

$$\mathrm{N}(P, x, a)^2 \ \geq \ \mathrm{N}(P, x, a+1) \cdot \mathrm{N}(P, x, a-1). \tag{Sta}$$

The unimodality of $\big\{\mathrm{N}(P, x, a)\big\}$ was conjectured by Kislitsyn [Kis68, §4.4] and later independently by Rivest. The log-concavity was conjectured by Chung, Fishburn and Graham [CFG80], who established both conjectures for posets of width two. The authors of [CFG80] called Rivest's conjecture "tantalizing" and add a note characterizing Stanley's then forthcoming proof using the *Alexandrov–Fenchel inequality* as "very ingenious" (see §14.6).

Conjecture 9.2 ([Pak22, Conj. 6.3]). *The defect of Stanley's inequality* (Sta) *is not in* $\#\mathsf{P}$.

In [CPP23b, §9.12], we wrote "At this point, it is even hard to guess which way the answer would go. While some of us believe the answer should be negative, others disagree." We have stronger convictions now.

9.2. **Kahn–Saks inequality.** Let $P = (X, \prec)$ be a poset with $|X| = n$ elements, let $x, y \in X$ and $a \in [n]$. Denote by $\mathcal{F}(P, x, y, a)$ the set of linear extensions $f \in \mathcal{E}(P)$, such that $f(y) - f(x) = a$. Let $\mathrm{F}(P, x, y, a) := \big|\mathcal{F}(P, x, y, a)\big|$.

Theorem 9.3 (*Kahn–Saks inequality* [KS84, Thm 2.5])**.** *We have:*

$$\mathrm{F}(P, x, y, a)^2 \,\geq\, \mathrm{F}(P, x, y, a+1) \cdot \mathrm{F}(P, x, y, a-1). \tag{KS}$$

It is easy to see that (KS) implies (Sta) by taking $x \leftarrow \widehat{0}$, $y \leftarrow x$, and $a \leftarrow a+1$. The theorem is proved using the Alexandrov–Fenchel inequality again (see §14.6).

9.3. **q-Stanley and q-KS inequalities.** Let $P = (X, \prec)$ be a poset of width two with $|X| = n$ elements. Fix a partition $X = C \sqcup C'$ into two chains, where $C = \{u_1 \prec \ldots \prec u_\ell\}$, and $C' = \{v_1 \prec \ldots \prec v_{n-\ell}\}$.

Let $\boldsymbol{q} := (q_1, \ldots, q_\ell)$ be formal variables. Define the *$\boldsymbol{q}$-weight* of $\mathrm{N}(P, x, a)$ and $\mathrm{F}(P, x, y, a)$ as follows:

$$\mathrm{N}_{\boldsymbol{q}}(a) \,:=\, \sum_{f \in \mathcal{E}(P,x,a)} \boldsymbol{q}^f \qquad \text{and} \qquad \mathrm{F}_{\boldsymbol{q}}(a) \,:=\, \sum_{f \in \mathcal{F}(P,x,y,a)} \boldsymbol{q}^f,$$

where $\boldsymbol{q}^f \,:=\, q_1^{f(u_1)} \cdots q_\ell^{f(u_\ell)}$.

Theorem 9.4 (*$\boldsymbol{q}$-Stanley inequality* [CPP23a, Thm 7.1])**.** *In notation above, let $x \in C$ and $a \in [n]$. Then:*

$$\mathrm{N}_{\boldsymbol{q}}(P, x, a)^2 \;\geqslant_{\boldsymbol{q}}\; \mathrm{N}_{\boldsymbol{q}}(P, x, a+1) \cdot \mathrm{N}_{\boldsymbol{q}}(P, x, a-1), \tag{9.1}$$

where the inequality between polynomials is coefficient-wise.

More generally, we have:

Theorem 9.5 (*$\boldsymbol{q}$-KS inequality* [CPP23a, Thm 7.2])**.** *In notation above, let $x, y \in C$ be distinct elements, and let $a \in [n]$. Then:*

$$\mathrm{F}_{\boldsymbol{q}}(P, x, y, a)^2 \;\geqslant_{\boldsymbol{q}}\; \mathrm{F}_{\boldsymbol{q}}(P, x, y, a+1) \cdot \mathrm{F}_{\boldsymbol{q}}(P, x, y, a-1), \tag{9.2}$$

where the inequality between polynomials is coefficient-wise.

Note that (9.2) implies (9.1) in a similar way that (KS) implies (Sta). Taking all $q_i \leftarrow 1$ in these two inequalities gives (Sta) and (KS), respectively. Explicit equality conditions for both inequalities are given in [CPP23a, Thm 1.6] and [CPP23a, Thm 1.7]. Theorems 9.4 and 9.5 are proved by an explicit injection.

9.4. **Weighted Stanley inequality.** Let $\omega : X \to \mathbb{R}_{>0}$ be a positive weight function on X. We say that ω is *order-reversing* if it satisfies

$$u \preccurlyeq v \quad \Rightarrow \quad \omega(u) \,\geq\, \omega(v), \tag{9.3}$$

for all $u, v \in X$. Define

$$\mathrm{N}_\omega(P, x, a) \,:=\, \sum_{f \in \mathcal{E}(P,x,a)} \omega(f, x)\,, \quad \text{where} \quad \omega(f, x) \,:=\, \prod_{y \in X \,:\, f(y) < f(x)} \omega(y)\,. \tag{9.4}$$

Theorem 9.6 (*weighted Stanley inequality*, [CP21, Thm 1.35])**.** *For every order-reversing weight function ω, we have:*

$$\mathrm{N}_\omega(P, x, a)^2 \,\geq\, \mathrm{N}_\omega(P, x, a) \cdot \mathrm{N}_\omega(P, x, a)\,, \tag{9.5}$$

where $\mathrm{N}_\omega(P, x, a)$ is defined by (9.4).

Taking all $\omega(x) \leftarrow 1$ in the inequality (9.5) gives (Sta). Explicit equality conditions for (9.5) are given in [CP21, Thm 1.40], generalizing Theorem 10.2. Theorem 9.6 is proved by using a combinatorial atlas (see §14.7).

9.5. **Generalized Stanley inequality.** Let $x, z_1, \ldots, z_k \in X$ and $a, c_1, \ldots, c_k \in [n]$; we write $\boldsymbol{z} = (z_1, \ldots, z_k)$ and $\mathbf{c} = (c_1, \ldots, c_k)$, and assume that $c_1 < \cdots < c_k$.

Let $\mathcal{E}_{\boldsymbol{z}\mathbf{c}}(P)$ be the set of linear extensions $f \in \mathcal{E}(P)$, such that $f(z_i) = c_i$ for all $1 \le i \le k$. Similarly, let $\mathcal{E}_{\boldsymbol{z}\mathbf{c}}(P, x, a)$ be the set of linear extensions $f \in \mathcal{E}_{\boldsymbol{z}\mathbf{c}}(P)$, such that $f(x) = a$. Denote by $\mathrm{N}_{\boldsymbol{z}\mathbf{c}}(P) := \left|\mathcal{E}_{\boldsymbol{z}\mathbf{c}}(P)\right|$ and $\mathrm{N}_{\boldsymbol{z}\mathbf{c}}(P, x, a) := \left|\mathcal{E}_{\boldsymbol{z}\mathbf{c}}(P, x, a)\right|$ the number of such linear extensions. The following result states that the sequence $\left\{\mathrm{N}_{\boldsymbol{z}\boldsymbol{c}}(P, x, a), a \in [n]\right\}$ is log-concave:

Theorem 9.7 (*generalized Stanley inequality* [Sta81, Thm 3.2])**.** *In notation above, for all $k \ge 0$, we have:*

$$\mathrm{N}_{\boldsymbol{z}\boldsymbol{c}}(P, x, a)^2 \ \ge\ \mathrm{N}_{\boldsymbol{z}\boldsymbol{c}}(P, x, a+1) \cdot \mathrm{N}_{\boldsymbol{z}\boldsymbol{c}}(P, x, a-1). \tag{9.6}$$

The theorem is proved using the Alexandrov–Fenchel inequality (see §14.6).

Open Problem 9.8. *Find a weighted version of Theorem 9.7, i.e. a common generalization of Theorems 9.4 and 9.7.*

9.6. **Order polynomial version of Stanley inequality.** The following is a natural generalization of Brenti's log-concavity for the order polynomial (4.2) to the setting of Stanley inequality (Sta).

Theorem 9.9 (*Daykin–Daykin–Paterson inequality* [DDP84, Thm 4])**.** *Let $P = (X, \prec)$ be a finite poset, and let $x \in X$. Denote by $\Omega(P, t; x, a)$ the number of order preserving maps $h : X \to [t]$, such that $h(x) = a$. Then, for all integer $t > a > 1$, we have:*

$$\Omega(P, t; x, a)^2 \ \ge\ \Omega(P, t; x, a+1) \cdot \Omega(P, t; x, a-1). \tag{9.7}$$

Additionally, the defect of this inequality is in $\#\mathsf{P}$.

The inequality (9.7) was conjectured by Graham [Gra83, p. 129], by analogy with Stanley's inequality (Sta). The proof in [DDP84] uses an explicit injection. The authors prove, in fact, a stronger result, in the style of the generalized Stanley inequality (9.6).

Theorem 9.10 (*generalized DDP inequality* [DDP84, Thm 4])**.** *Let $P = (X, \prec)$ be a finite poset, let $x \in X$. Fix $k \in \mathbb{N}$ and let $\boldsymbol{z} \in X^k$. Denote by $\Omega(P, t; \boldsymbol{z}, \boldsymbol{c}; x, a)$ the number of order preserving maps $h : X \to [t]$, such that $h(x) = a$, and $h(z_i) = c_i$ for all $1 \le i \le k$. Then, for all integer $t > a > 1$, we have:*

$$\Omega(P, t; \boldsymbol{z}, \boldsymbol{c}; x, a)^2 \ \ge\ \Omega(P, t; \boldsymbol{z}, \boldsymbol{c}; x, a+1) \cdot \Omega(P, t; \boldsymbol{z}, \boldsymbol{c}; x, a-1). \tag{9.8}$$

Additionally, the defect of this inequality is in $\#\mathsf{P}$.

Graham believed that there should exist a proof based on the FKG or AD inequalities. He lamented: "such a proof has up to now successfully eluded all attempts to find it" [Gra83, p. 129]. Such proof was given in [CP22b], which also gave a generalization of the q-log-concavity (4.4) and $\boldsymbol{q}$-log-concavity (4.5) to this setting:

Theorem 9.11 (*$\boldsymbol{q}$–DDP inequality* [CP22b, Thm 9.3])**.** *Let $P = (X, \prec)$ be a finite poset, let $t \in \mathbb{N}$, and let $x \in X$. Then, for every $t > a > 1$, we have:*

$$\Omega_{\boldsymbol{q}}(P, t; x, a)^2 \ \geqslant_{\boldsymbol{q}}\ \Omega_{\boldsymbol{q}}(P, t; x, a+1) \cdot \Omega_{\boldsymbol{q}}(P, t; x, a-1), \tag{9.9}$$

where the inequality holds coefficient-wise as a polynomial in $\boldsymbol{q} = (q_1, \ldots, q_n)$.

9.7. **Cross-product conjecture.** Let $P = (X, \prec)$ be a poset with $|X| = n$ elements. Fix distinct elements $x, y, z \in X$. For $a, b \geq 1$, let $\mathcal{F}_{xyz}(P, a, b) := \mathcal{F}(P, x, y, a) \cap \mathcal{F}(P, y, z, b)$. Equivalently,

$$\mathcal{F}_{xyz}(P, a, b) \ := \ \big\{ f \in \mathcal{E}(P) \, : \, f(y) - f(x) = a, \, f(z) - f(y) = b \big\}.$$

Denote $\mathrm{F}_{xyz}(P, a, b) := \big|\mathcal{F}_{xyz}(P, a, b)\big|$. By Theorem 10.9, we have $\big\{\mathrm{F}_{xyz}(P, a, b) =^? 0\big\} \in \mathsf{P}$, since there are at most n choices for $f(x)$, which then determine $f(y)$ and $f(z)$.

Conjecture 9.12 (*Cross–product conjecture*, Felsner–Trotter [FT93, Conj. 8.3])**.** *We have:*

$$\text{(CPC)} \qquad \mathrm{F}_{xyz}(P, a+1, b) \cdot \mathrm{F}_{xyz}(P, a, b+1) \ \geq \ \mathrm{F}_{xyz}(P, a, b) \cdot \mathrm{F}_{xyz}(P, a+1, b+1) \, .$$

The following result give a summary of known special cases

Theorem 9.13. *Conjecture 9.12 holds in the following cases:*

(1) $a = b = 1$*, see* [BFT95, Thm 3.2],
(2) $\mathrm{width}(P) = 2$*, see* [CPP22a, Thm 1.4],
(3) $\mathrm{F}_{xyz}(P, a, b+2) = \mathrm{F}_{xyz}(P, a+2, b) = 0$*,* $\mathrm{F}_{xyz}(P, a, b) > 0$ *and* $\mathrm{F}_{xyz}(P, a+1, b+1) > 0$*, see* [CPP23c, Thm 1.2].

The proof of (1) is based on the AD inequality (see §14.4). The authors lamented: "something more powerful seems to be needed" to prove the general form of (CPC).

Note that (CPC) easily implies (KS), by taking $y \leftarrow z$ and $P \leftarrow P + y$, see e.g. [CPP22a, §3.1]. In fact, (GYY) also follows from (CPC), by a more involved argument (ibid., §3.4). Of course, the value of these implications is low given that (CPC) remains an open problem.

For posets of width two, the q-analogue of (CPC) and the equality conditions are given in [CPP22a, Thm 1.7 and 1.8]. In fact, a stronger inequality holds in this case:

$$\text{(9.10)} \quad \mathrm{F}_{xyz}(P, a, b) \cdot \mathrm{F}_{xyz}(P, c, d) \ \leq \ \mathrm{F}_{xyz}(P, c, b) \cdot \mathrm{F}_{xyz}(P, a, d), \quad \text{for all } \ a \leq c \text{ and } b \leq d,$$

see [CPP22a, Thm 1.6]. For $c = a + 1$ and $d = b + 1$, where $a, b \geq 1$, this gives (CPC). The inequality (9.10) fails already for posets of width three [CPP23c, Thm 1.6].

When $\mathrm{F}_{xyz}(P, a, b) \cdot \mathrm{F}_{xyz}(P, a+1, b+1) = 0$, the inequality (CPC) holds trivially. Note that this assumption as well as the assumptions in (3) can be verified in polynomial time. For the remaining possible cases, we have the following weak version of Conjecture 9.12.

Theorem 9.14 ([CPP23c, Thm 1.2])**.** *Let $P = (X, \prec)$ be a poset on $|X| = n$ elements. Fix distinct elements $x, y, z \in X$. Suppose that $\mathrm{F}_{xyz}(P, a, b+2) \, \mathrm{F}_{xyz}(P, a+2, b) > 0$. Then:*

$$\text{(9.11)} \quad \mathrm{F}_{xyz}(P, a+1, b) \, \mathrm{F}_{xyz}(P, a, b+1) \ \geq \ \left(\tfrac{1}{2} \, + \, \tfrac{1}{4n\sqrt{ab}} \right) \mathrm{F}_{xyz}(P, a, b) \, \mathrm{F}_{xyz}(P, a+1, b+1).$$

Alternatively, suppose that $\mathrm{F}_{xyz}(P, a, b+2) = 0$ and $\mathrm{F}_{xyz}(P, a+2, b) > 0$. Then:

$$\text{(9.12)} \quad \mathrm{F}_{xyz}(P, a+1, b) \, \mathrm{F}_{xyz}(P, a, b+1) \ \geq \ \left(\tfrac{1}{2} \, + \, \tfrac{1}{16nab^2} \right) \mathrm{F}_{xyz}(P, a, b) \, \mathrm{F}_{xyz}(P, a+1, b+1).$$

Finally, suppose that $\mathrm{F}_{xyz}(P, a, b+2) \, \mathrm{F}_{xyz}(P, a+2, b) = 0$. Then:

$$\text{(9.13)} \qquad \mathrm{F}_{xyz}(P, a, b) \, \mathrm{F}_{xyz}(P, a+1, b+1) \ = \ 0.$$

Note that (9.13) implies part (3) in Theorem 9.13. The theorem is proved using geometric inequalities (see §14.6).

9.8. **Order polynomial version of CPC.** The following is a natural generalization of the DDP inequality (9.7) to the setting of CPC (Sta).

Let $P = (X, \prec)$ be a poset on $|X| = n$ elements, and let $X = \{x_1, \ldots, x_n\}$. Fix $t \geq 0$ and distinct elements $x, y, z \in X$. For integers $a, b \geq 0$, let

$$\mathcal{P}(P, t; x, y, z; a, b) \;:=\; \big\{h \in \mathcal{P}(P, t) \;:\; h(y) - h(x) = a \;\text{ and }\; h(z) - h(y) = b\big\}.$$

Denote

$$\Lambda(P, t; x, y, z; a, b) \;:=\; \big|\mathcal{P}(P, t; x, y, z; a, b)\big|, \quad \text{and}$$
$$\Lambda_{\boldsymbol{q}}(P, t; x, y, z; a, b) \;:=\; \sum_{f \in \mathcal{P}(P, t\,;\, x, y, z\,;\, a, b)} q_1^{f(x_1)} \cdots q_n^{f(x_n)}.$$

Theorem 9.15 (*Cross-product inequality for P-partitions* [CP23a, Thm 9.3])**.** *Let $P = (X, \prec)$ be a finite poset, let $x, y, z \in \mathcal{P}$, and let $t \geq 1$ be a positive integer. Then, for every $a, b \geq 0$, we have:*

$$\begin{aligned}\Lambda(P, t; x, y, z; a+1, b) \cdot \Lambda(P, t; x, y, z; a, b+1)\\ \geq\; \Lambda(P, t; x, y, z; a, b) \cdot \Lambda(P, t; x, y, z; a+1, b+1).\end{aligned} \tag{9.14}$$

More generally:

$$\begin{aligned}\Lambda_{\boldsymbol{q}}(P, t; x, y, z; a+1, b) \cdot \Lambda_{\boldsymbol{q}}(P, t; x, y, z; a, b+1)\\ \geqslant_{\boldsymbol{q}}\; \Lambda_{\boldsymbol{q}}(P, t; x, y, z; a, b) \cdot \Lambda_{\boldsymbol{q}}(P, t; x, y, z; a+1, b+1).\end{aligned} \tag{9.15}$$

The proof uses a generalization of the AD inequality (see §14.4).

9.9. **Conjectural generalization.** Recall (8.14), that $f_{\min}(A) := \min\{f(x) : x \in A\}$. The following is the natural generalization of Stanley's inequality (Sta).

Conjecture 9.16 (*extended Stanley inequality* [CP22b, Conj. 1.5])**.** *Let $P = (X, \prec)$ be a poset with $|X| = n$ elements. Fix a nonempty subset $A \subseteq X$, and let $2 \leq k \leq n-1$. Then:*

$$\mathbf{P}[f_{\min}(A) = k]^2 \;\geq\; \mathbf{P}[f_{\min}(A) = k+1] \cdot \mathbf{P}[f_{\min}(A) = k-1]. \tag{9.16}$$

This conjecture also implies Theorem 8.14, see [CP22b, §7.1].

9.10. **Second moment conjecture.** In Theorem 8.17, letting $y = x$ gives the following curious bound on the second moment:

Corollary 9.17 (*second moment inequality* [CP22b, Cor. 3.5])**.** *Let $P = (X, \prec)$ be a finite poset, and let $x \in X$ be a fixed element. Then:*

$$1 \;\leq\; \frac{\mathbf{E}[f(x)^2]}{\mathbf{E}[f(x)]^2} \;<\; 2. \tag{9.17}$$

The lower bound is trivial and holds for every random variable. The (non-strict) upper bound also follows from Stanley's inequality, since for every log-concave random variable Z, we have: $\mathbf{E}[Z^2] \leq 2\mathbf{E}[Z]^2$, see [CP22b, Prop. 3.7]. The following conjecture improves upon the upper bound in (9.17).

Conjecture 9.18 (*second moment conjecture* [CP22b, Conj. 3.8])**.** *Let $P = (X, \prec)$ be a finite poset, and let $x \in X$ be a fixed element. Then:*

$$\frac{\mathbf{E}[f(x)^2]}{\mathbf{E}[f(x)]^2} \;\leq\; \frac{4}{3}\,. \tag{9.18}$$

In fact, the inequality is probably always strict as the following example suggests.

Example 9.19. Let $P := C_{n-1} + \{x\}$ be a disjoint sum of two chains. We have:

$$\frac{\mathbf{E}[f(x)^2]}{\mathbf{E}[f(x)]^2} \ = \ \frac{\frac{1}{n}\sum_{k=1}^{n} k^2}{\left(\frac{1}{n}\sum_{k=1}^{n} k\right)^2} \ = \ \frac{4n+2}{3n+3} \ \to \ \frac{4}{3} \qquad \text{as} \qquad n \to \infty.$$

Thus, the constant in the upper bound (9.18) must be at least 4/3.

10. Equality conditions of Stanley type inequalities

10.1. **Stanley inequality.** Recall that $\alpha(x) := |x\!\downarrow\!|$ and $\beta(x) := |x\!\uparrow\!|$ denote the sizes of the lower and upper order ideals, respectively.

Theorem 10.1 (*vanishing conditions*, Daykin and Daykin [DD85, Thm 8.2])**.** *Let $P = (X, \prec)$ be a poset with $|X| = n$ elements, let $x \in X$ and $a \in [n]$. Then $\mathrm{N}(P, x, a) > 0$ if and only if*

$$\alpha(x) \le a \quad \text{and} \quad \beta(x) \le n - a + 1.$$

Moreover, if $\mathrm{N}(P, x, a) > 0$, then a linear extension $f \in \mathcal{E}(P, x, a)$ can be found in polynomial time.

The original proof uses promotion/demotion operators (under a different name, cf. §14.2), This results was rediscovered in [SvH23, Lem. 15.2]. By Theorem 9.1, Stanley's inequality (Sta) is an equality whenever $\mathrm{N}(P, x, a) = 0$. The other equality cases are given by the following result:

Theorem 10.2 (*equality conditions*, Shenfeld and van Handel [SvH23, Thm 15.3])**.** *Let $P = (X, \prec)$ be a poset with $|X| = n$ elements, let $x \in X$ and $a \in [n]$. Suppose that $\mathrm{N}(P, x, a) > 0$. The following are equivalent:*

(1) $\mathrm{N}(P, x, a)^2 \ = \ \mathrm{N}(P, x, a+1) \cdot \mathrm{N}(P, x, a-1),$
(2) $\mathrm{N}(P, x, a+1) \ = \ \mathrm{N}(P, x, a) \ = \ \mathrm{N}(P, x, a-1),$
(3) *we have $\alpha(y) > a$ for all $y \succ x$, and $\beta(y) > n - a + 1$ for all $y \prec x$.*

The original proof uses a technical geometric argument (see §14.6). The result was reproved in [CP21, Thm 1.39] using the combinatorial atlas technology, and extended to equality conditions of the weighted Stanley inequality (9.5).

Corollary 10.3. *The equality of Stanley's inequality (Sta) can be verified in polynomial time:*

$$\left\{\mathrm{N}(P, x, a)^2 =^{?} \mathrm{N}(P, x, a+1) \cdot \mathrm{N}(P, x, a-1)\right\} \in \mathsf{P}.$$

This follows from Theorem 10.1 in the vanishing case, since the equality always holds, and from $(1) \Leftrightarrow (3)$ in Theorem 10.2 in the nonvanishing cases.

10.2. **CPC implies equality conditions.** We start with the following surprising inequality:

Conjecture 10.4. *Let $P = (X, \prec)$ be a poset on $|X| = n$ elements. Fix an element $z \in X$. Then, for all integer $a, i \ge 1$, we have:*

$$\begin{aligned} &(a+i-1)\,\mathrm{N}(P, z, a+i-1) \cdot \mathrm{N}(P, z, a) \\ &\quad \ge \ (a-1)\,\mathrm{N}(P, z, a-1) \cdot \mathrm{N}(P, z, a+i) \ + \ i\,\mathrm{N}(P, z, a+i) \cdot \mathrm{N}(P, z, a). \end{aligned} \tag{10.1}$$

The following results were left on the cutting floor from [CPP23c]:

Theorem 10.5 (Chan–Pak–Panova, see §15.1)**.** *The Cross–product Conjecture 9.12 implies Conjecture 10.4.*

Proposition 10.6 (Chan–Pak–Panova, see §15.1)**.** *Inequality (10.1) implies that (1) ⇔ (2) in Theorem 10.2. Additionally, inequality (10.1) implies Stanley's inequality (Sta).*

Combined, these two result show that Conjecture 9.12 implies the first part of the equality conditions of Stanley's inequality (Sta) given in Theorem 10.2. Since the only known proofs of the latter are rather difficult (using either convex geometry or the combinatorial atlas technology), this suggests that Conjecture 9.12 is also very difficult to prove. Another possibility is that Conjecture 9.12 is false for posets of large width (cf. §16.3).

10.3. **Kahn–Saks inequality.** Denote $\gamma(u,v) := \#\{y \in X, \text{ s.t. } u \prec y \prec v\}$.

Theorem 10.7 (*vanishing conditions* [CPP23b, Thm 8.5])**.** *Let $P = (X, \prec)$ be a poset with $|X| = n$ elements, let $x, y \in X$ and $a \in [n]$. We have: $\mathrm{F}(P,x,y,a) > 0$ if and only if*

$$\gamma(x,y) \, < \, a \, < \, n \, - \, \alpha(x) \, - \, \beta(y).$$

Moreover, if $\mathrm{F}(P,x,y,a) > 0$, then a linear extension $f \in \mathcal{F}(P,x,y,a)$ can be found in polynomial time.

The original proof uses a variation on promotion/demotion operators (see §14.2). See also [vHYZ23] for an alternative proof of Theorem 10.7. The equality conditions for the Kahn–Saks inequality (KS) were completely resolved in [vHYZ23], but too cumbersome to state here. The following theorem is a compilation of several results in that paper.

Theorem 10.8 (*equality conditions*, van Handel, Yan and Zeng [vHYZ23])**.** *Let $P = (X, \prec)$ be a poset with $|X| = n$ elements, let $x, y \in X$ and $a \in [n]$. Suppose that $\mathrm{F}(P,x,y,a) > 0$. Then*

$$\mathrm{F}(P,x,y,a)^2 \, = \, \mathrm{F}(P,x,y,a+1) \, \cdot \, \mathrm{F}(P,x,y,a-1) \tag{10.2}$$

if and only if

$$\begin{aligned} \textit{either} \quad & \mathrm{F}(P,x,y,a+1) \, = \, \mathrm{F}(P,x,y,a) \, = \, \mathrm{F}(P,x,y,a-1) \\ \textit{or} \quad & \mathrm{F}(P,x,y,a+1) \, = \, 2 \cdot \mathrm{F}(P,x,y,a) \, = \, 4 \cdot \mathrm{F}(P,x,y,a-1). \end{aligned}$$

Additionally, the equality (10.2) *can be verified in polynomial time.*

Note the asymmetric structure of the three-term geometric progression with ratio 2, a phenomenon which does not occur for the Stanley inequality (Theorem 10.2). Nor does it occur for posets of width two when the equality conditions are especially simple, see [CPP23a, Thm 1.7 and §8.4]. Two equivalent conjectural characterizations of the first part (the complete equality) were given in [CPP23a, Conj. 8.7 and Thm 8.9].

10.4. **Vanishing conditions.** The following result generalizes Theorem 10.1 to vanishing conditions of the generalized Stanley inequality (9.6).

Theorem 10.9 (*vanishing conditions*, Daykin and Daykin [DD85, Thm 8.2])**.** *Let $P = (X, \prec)$ be a poset with $|X| = n$ elements. Let $\boldsymbol{z} = (z_1, \ldots, z_k) \in X^k$, $\boldsymbol{c} = (c_1, \ldots, c_k) \in [n]^k$, and assume that $c_1 < \cdots < c_k$. Let $\mathcal{E}_{\boldsymbol{zc}}(P)$ denotes the set of linear extensions $f \in \mathcal{E}(P)$, s.t. $f(z_i) = c_i$ for all $1 \le i \le k$. We have $\mathrm{N}_{\boldsymbol{zc}}(P) > 0$ if and only if*

$$\begin{aligned} & \alpha(z_i) \, \le \, c_i \, , \quad \beta(z_i) \le n - c_i + 1 \, , \qquad \textit{for all} \quad 1 \, \le \, i \, \le \, k, \quad \textit{and} \\ & c_j \, - \, c_i \, > \, \gamma(z_i, z_j) \qquad \textit{for all} \quad 1 \, \le \, i \, < \, j \, \le \, k \, . \end{aligned} \tag{10.3}$$

Consequently, the vanishing problem $\big\{\mathrm{N}_{\boldsymbol{zc}}(P) =^? 0\big\} \in \mathsf{P}$. Moreover, if $\mathrm{N}_{\boldsymbol{zc}}(P) > 0$, then a linear extension $f \in \mathcal{E}_{\boldsymbol{zc}}(P)$ can be found in polynomial time.

The original proof used promotion/demotion operators (see §14.2). This result was rediscovered in [CPP23b, Thm 1.11] and [MS22, Thm 5.3], where the latter used a geometric argument.

10.5. **Uniqueness conditions.** The uniqueness conditions of the generalized Stanley inequality (9.6) provide another special case of a polynomial time decision problem.

Let $v_i := f^{-1}(c_i - 1)$ and $w_i := f^{-1}(c_i + 1)$ for $1 \le i \le k$. We adopt the convention that $v_1 = \widehat{0}$ if $c_1 = 1$, and $w_k = \widehat{1}$ if $c_k = n$. For $1 \le i \le j \le n$, let

$$f^{-1}[i,j] \;:=\; \big\{ f^{-1}(i)\,,\,\ldots\,,f^{-1}(j) \big\}.$$

Theorem 10.10 (*uniqueness conditions*, [CPP23b, Thm 7.5])**.** *Let $f \in \mathcal{E}_{\boldsymbol{zc}}(P)$. Then we have $\mathrm{N}_{\boldsymbol{zc}}(P) = 1$ <u>if and only if</u> the following conditions hold:*

(1) *$f^{-1}[c_i + 1, c_{i+1} - 1]$ forms a chain in P for every $1 \le i \le k$, and*

(2) *there are no $1 \le i \le j \le k$, such that $\{v_i, w_j\} \parallel f^{-1}[c_i, c_j]$.*

Consequently, the uniqueness problem $\big\{\mathrm{N}_{\boldsymbol{zc}}(P) =^? 1\big\} \in \mathsf{P}$.

The first part is proved using promotion/demotion operators (see §14.2). The last part follows form the first part of the theorem, since by Theorem 10.9, a linear extension $f \in \mathcal{E}_{\boldsymbol{zc}}(P)$ can be found in polynomial time.

Conjecture 10.11. *For every fixed integer $m \in \mathbb{N}$, the decision problem $\big\{\mathrm{N}_{\boldsymbol{zc}}(P) =^? m\big\} \in \mathsf{P}$.*

10.6. **Equality conditions, positive results.** In contrast with equality conditions for the Kahn–Saks inequality (Theorem 10.8), the equivalence of (1) ⇔ (2) in Theorem 10.2 generalizes to all $k \ge 0$.

Theorem 10.12 (*complete equality property*, Ma–Shenfeld [MS22, Thm 1.3 and 1.5])**.** *Let $P = (X, \prec)$ be a poset with $|X| = n$ elements, let $x, z_1, \ldots, z_k \in X$ and $a, c_1, \ldots, c_k \in [n]$. We have*

$$\mathrm{N}_{\boldsymbol{zc}}(P,x,a)^2 \;=\; \mathrm{N}_{\boldsymbol{zc}}(P,x,a+1)\,\cdot\,\mathrm{N}_{\boldsymbol{zc}}(P,x,a-1) \tag{10.4}$$

<u>*if and only if*</u>

$$\mathrm{N}_{\boldsymbol{zc}}(P,x,a+1) \;=\; \mathrm{N}_{\boldsymbol{zc}}(P,x,a) \;=\; \mathrm{N}_{\boldsymbol{zc}}(P,x,a-1). \tag{10.5}$$

For $k = 1$, the equality cases of (10.4) are given by the following result:

Theorem 10.13 ([CP23c, §9.1])**.** *Let $P = (X, \prec)$ be a poset on $n = |X|$ elements, let $x, z \in X$, and $a, c \in [n]$. Then*

$$\mathrm{N}_{zc}(P,x,a)^2 \;=\; \mathrm{N}_{zc}(P,x,a+1)\cdot\mathrm{N}_{zc}(P,x,a-1).$$

<u>*if and only if*</u>

$$\mathrm{N}_{zy,cb'}(P,x,a') \;=\; 0 \quad \text{for all} \quad y \in \mathrm{comp}(x) \quad \text{and} \quad a', b' \in \{a-1, a, a+1\}. \tag{10.6}$$

Since checking (10.6) can be done in polynomial time by Theorem 10.9, we easily have:

Corollary 10.14 ([CP23c, Thm 1.4])**.** *For $k = 1$, the equality verification of the generalized Stanley inequality (10.4) can be done in polynomial time:*

$$\big\{\mathrm{N}_{zc}(P,x,a)^2 =^? \mathrm{N}_{zc}(P,x,a+1)\cdot\mathrm{N}_{zc}(P,x,a-1)\big\} \in \mathsf{P}.$$

Corollary 10.14 trivially implies Corollary 10.3.

10.7. **Equality conditions, Ma–Shenfeld theory.** Let $x, z_1, \ldots, z_k \in X$ and $a, c_1, \ldots, c_k \in [n]$; we write $\boldsymbol{z} = (z_1, \ldots, z_k)$ and $\mathbf{c} = (c_1, \ldots, c_k)$. Throughout the section, we assume that $x \prec z_1 \prec \ldots \prec z_k$ and $c_1 < \cdots < c_k$. Suppose poset P has elements $z_0 = \widehat{0}$ and $z_{k+1} = \widehat{1}$. Denote

$$\Lambda \ := \ \big\{ (r,s) \, : \, 0 \le r < s \le k+1, \, c_r < a < c_s \, , \, (r,s) \ne (0, k+1) \big\}.$$

Pairs $(r,s) \in \Lambda$ are called *splitting pairs*.[4]

Definition/Lemma 10.15. *Suppose that*

$$\mathrm{N}_{\boldsymbol{z}\boldsymbol{c}}(P,x,a-1), \ \mathrm{N}_{\boldsymbol{z}\boldsymbol{c}}(P,x,a), \ \mathrm{N}_{\boldsymbol{z}\boldsymbol{c}}(P,x,a+1) \ > \ 0 \quad \textit{and} \quad n > k+3. \tag{10.7}$$

We say that $(P, x, a, \boldsymbol{z}, \boldsymbol{c})$ is subcritical, critical, and supercritical <u>if and only if</u> for every splitting pair $(r,s) \in \Lambda$, we have, respectively:

$$\gamma(z_r, z_s) \ \le \ c_s \, - \, c_r \, - \, 1, \tag{subcrit}$$

$$\gamma(z_r, z_s) \ \le \ c_s \, - \, c_r \, - \, 2, \tag{crit}$$

$$\gamma(z_r, z_s) \ \le \ c_s \, - \, c_r \, - \, 3. \tag{supercrit}$$

In particular, each membership problem is in P.

The definition in [MS22] is cumbersome, so we use this definition which is more transparent. We prove that it is equivalent to the original definition in §15.2. By Definition 10.15, we have:

$$\{\text{subcritical}\} \quad \supseteq \quad \{\text{critical}\} \quad \supseteq \quad \{\text{supercritical}\}.$$

Note that (10.7) implies (subcrit), since for every $f \in \mathrm{N}_{\boldsymbol{z}\mathbf{c}}(P,x,a)$ and $(r,s) \in \Lambda$, we have:

$$\gamma(z_r, z_s) \ \le \ \big| \big\{ f^{-1}(i) \, : \, c_r < i < c_s \big\} \big| \ = \ c_s \, - \, c_r \, - \, 1. \tag{10.8}$$

Note also that for $k = 0$, every quintuple $(P, x, a, \boldsymbol{z}, \mathbf{c})$ is supercritical, since $\Lambda = \varnothing$.

Theorem 10.16 ([MS22, Thm 1.3, Rem. 1.6])**.** *Suppose the positivity conditions (10.7) hold and that quintuple $(P, x, a, \mathbf{z}, \mathbf{c})$ is supercritical. Then the equality (10.4) holds <u>if and only if</u>*

$$\begin{cases} \forall \, y \in x\!\uparrow \ \exists \, r \in \{0, \ldots, k+1\} & \textit{s.t.} \quad y \succ z_r \ \textit{ and } \ \gamma(z_r, y) > a - c_r \\ \forall \, y \in x\!\downarrow \ \exists \, s \in \{0, \ldots, k+1\} & \textit{s.t.} \quad y \prec z_s \ \textit{ and } \ \gamma(y, z_s) > c_s - a \end{cases} \tag{10.9}$$

Additionally, verifying (10.9) is in P.

Theorem 10.16 shows that deciding whether a supercritical quintuple gives the equality case for the generalized Stanley inequality (9.6) can be done is polynomial time. Note that this generalizes Theorem 10.2, but not Theorem 10.13 since there are critical cases for $k = 1$ as the following example shows.

Example 10.17. Let $k = 1$. Consider a poset $P = (X, \prec)$, where $X := \{x, y_1, \ldots, y_{n-2}, z\}$, and $z \prec x$, $z \prec y_{n-2}$ are the only relations. Let $a := n-1$, $b := n-3$. We have:

$$\mathrm{N}_{zb}(P,x,a) \ = \ \mathrm{N}_{zb}(P,x,a+1) \ = \ \mathrm{N}_{zb}(P,x,a-1) \ = \ 2\,(n-3)!,$$

so (9.6) is an equality. Note that (P, z, a, z, b) is critical but not supercritical. Indeed, in this case we have $\Lambda = \{(1,2)\}$, $c_1 = b$, $c_2 = n+1$, and

$$\gamma(z, \widehat{1}) \ = \ \big| \{x, y_{n-2}\} \big| \ = \ 2 \ = \ (n+1) \, - \, (n-3) \, - \, 2 \ = \ c_2 \, - \, c_1 \, - \, 2.$$

[4] In [MS22, Def. 5.2], these are called *ℓ-splitting pairs*, which are instead written as $(r+1, s)$.

10.8. **Equality conditions, negative results.** Corollary 10.14 is in sharp contrast with the following result:

Theorem 10.18 ([CP23c, Thm 1.3]). *Fix $k \ge 2$. Then the equality verification of the generalized Stanley inequality* (9.6) *is not in the polynomial hierarchy unless the polynomial hierarchy collapses to a finite level:*

$$\big\{\mathrm{N}_{\boldsymbol{zc}}(P,x,a)^2 =^? \mathrm{N}_{\boldsymbol{zc}}(P,x,a+1)\cdot \mathrm{N}_{\boldsymbol{zc}}(P,x,a-1)\big\} \in \mathsf{PH} \quad\Longrightarrow\quad \mathsf{PH} = \Sigma^{\mathrm{p}}_m \quad \textit{for some } m.$$

The proof is based on the following result of independent interest.

Theorem 10.19. $\big\{\mathrm{N}(P,x,a) =^? \mathrm{N}(P,x,a+1)\big\} \in \mathsf{PH} \quad\Longrightarrow\quad \mathsf{PH} = \Sigma^{\mathrm{p}}_m \quad$ *for some m.*

Compare this with the following consequence of (2) $\Leftrightarrow$ (3) in Theorem 10.2:

Corollary 10.20. *Deciding whether the following holds is in* P:

$$\mathrm{N}(P,x,a-1) = \mathrm{N}(P,x,a) = \mathrm{N}(P,x,a+1).$$

In particular, deciding if the distribution $\big\{\mathrm{N}(P,x,a)\big\}$ has a unique mode is not in PH, unless PH collapses.

11. Examples and applications

In this section, we present a selection of poset classes for which the number of linear extensions is interesting either because it is easy to compute, or because it is provably hard to compute.

11.1. **Bounded width.** Posets $P = (X,\prec)$ of bounded width are especially elegant and have many properties that general posets do not have. In this case computing the number of linear extensions $\#\mathrm{LE} \in \mathsf{FP}$ via dynamic programming. *Young diagrams* of bounded height represent especially nice examples of posets of bounded width, see below.

For posets of width two with a fixed partition of P into two chains, the number $e(P)$ of linear extensions can be viewed as the number of certain grid walks (lattice paths) in $\mathbb{Z}^2$, see §14.3. For posets of width three, beside Young diagrams (see below), there is an interesting *Kreweras–Niederhausen poset* $(A_2 \oplus C_1) \times C_n$ [KN81, HR22]. This poset has a *Kreweras number* of linear extensions, see also [OEIS, A006335].

11.2. **Series-parallel and N-free posets.** The class of *SP posets* generalizes forests, and has an extremely easy structure. Given the *SP decomposition*, one can the following formulas to compute the number of linear extensions:

$$e\big(P \oplus Q\big) \;=\; e(P)\, e(Q) \quad \text{and} \quad e\big(P + Q\big) \;=\; \tbinom{m+n}{m}\, e(P)\, e(Q),$$

where P and Q have m and n elements, respectively. Given a SP poset, the SP decomposition can be found in polynomial time [VTL82].

Recall that SP posets can be characterized by not having poset N as an induced subposet (see §2.2). There is a closely related class of *N-free posets* whose cover graph does not contain N. There is a more general notion of decomposition in this case. We refer to [HJ85, Möh89] for the introduction to this class of posets. See [FM14] for bounds on the numbers of linear extensions and a simple dynamic programming algorithm.

11.3. **Young diagrams.** For a subset $S \subset \mathbb{N}^2$, denote by $P_S := (S, \preccurlyeq)$, where $(i,j) \preccurlyeq (i',j')$ if and only if $i \le i'$ and $j \le j'$. Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ be a *partition of* n, i.e. $\lambda_1 \ge \ldots \ge \lambda_\ell > 0$, $|\lambda| := \lambda_1 + \ldots + \lambda_\ell = n$. *Young diagram* is a subset $S_\lambda := \{(i,j) \in \mathbb{N}^2 \, : \, 1 \le i \le \ell, 1 \le j \le \lambda_i\}$. Let $P_\lambda := (S_\lambda, \preccurlyeq)$. For example, *Catalan poset* Cat_m corresponds to the partition $\lambda = (m,m)$, with $n = 2m$. Famously, $e(\mathrm{Cat}_m) = \frac{1}{m+1}\binom{2m}{m}$ in the *Catalan number*, see e.g. [Sta99, §6.2] and [Sta15].

Linear extensions of $P_{\lambda/\mu}$ are called *standard Young tableaux* of shape λ/μ. We use $e(\lambda/\mu) := e(P_{\lambda/\mu})$ to simplify the notation. The *hook-length formula* by Frame, Robinson and Thrall [FRT54], states:

$$e(\lambda) \; = \; n! \prod_{(i,j) \in \lambda} \frac{1}{h_\lambda(i,j)}\,, \tag{11.1}$$

where $h_\lambda(i,j) = \lambda_i + \lambda'_j - i - j + 1$ is the *hook-length* in λ. This implies that $\#\mathrm{LE} \in \mathsf{FP}$ for Young diagram shapes. See [Pak22, §11.2] for a list of different proofs.

More generally, *skew Young diagram* λ/μ is a pair of partitions with $S_\mu \subseteq S_\lambda$. We use $S_{\lambda/\mu} := S_\lambda \smallsetminus S_\mu$. Denote by $P_{\lambda/\mu} := (S_{\lambda/\mu}, \preccurlyeq)$ the corresponding subposet of P_λ, and let $|\lambda/\mu| := |S_{\lambda/\mu}|$. The *Aitken–Feit determinant formula* [Ait43, Feit53], states:

$$e(\lambda/\mu) \; = \; n! \det \left(\frac{1}{(\lambda_i - \mu_j - i + j)!} \right)_{i,j=1}^{\ell}. \tag{11.2}$$

This implies that $\#\mathrm{LE} \in \mathsf{FP}$ for skew Young diagram shapes as well.

There are several notable positive summation formulas for $e(P_{\lambda/\mu})$, called the *Naruse hook-length formula* (NHLF), see [Kon20, MPP17, MPP18a], *Okounkov–Olshanski formula* [MZ22, OO98], and the *flipped hook-length formula* [Pak21, §9.1]. Among the implications let us single out two inequalities:

$$e(\lambda/\mu) \; \ge \; n! \prod_{(i,j) \in \lambda} \frac{1}{h_\lambda(i,j)} \quad \text{and} \quad \prod_{(i,j)\in\lambda} h_\lambda(i,j) \; \le \; \prod_{(i,j)\in\lambda} h_{\lambda^*}(i,j), \tag{11.3}$$

where $h_{\lambda^*}(i,j) := i + j - 1$ denotes the *dual hook-length* (hook-length in a skew shape rotated 180°, corresponding to the dual poset). The first inequality in (11.3) is a strong improvement over (BW) in this case [MPP18b]. The second follows from the first, and is a variation on (6.1) in the case of Young diagrams, see [MPP18b, §12.1] and [PPS20].

Many inequalities for the numbers of linear extensions become surprising in the language of standard Young tableaux. For example, Fishburn's inequality (7.3) in this case states:

$$e(\lambda) \cdot e(\mu) \; \le \; e(\lambda \cup \mu) \cdot e(\lambda \cap \mu). \tag{11.4}$$

See [Bjö11] for a direct proof based on the HLF (11.1). Similarly, the generalized Fishburn inequality (7.4) in this case is due to Lam–Pylyavskyy [LP07], and states:

$$e(\lambda/\alpha) \cdot e(\mu/\beta) \; \le \; e\big((\lambda \cup \mu)/(\alpha \cup \beta)\big) \cdot e\big((\lambda \cap \mu)/(\alpha \cap \beta)\big). \tag{11.5}$$

Let us single out the following immediate corollary from Theorem 8.20. For a partition λ, a *conjugate partition* λ' is obtained by reflection of S_λ across the $i = j$ line. We say that λ is a *self-conjugate partition*, if $\lambda = \lambda'$.

Corollary 11.1 ([CP22b, Cor. 4.1])**.** *Let λ/μ be a skew shape, let $x, y \in S_{\lambda/\mu}$ be corners, and let $v, w \in S_{\lambda/\mu}$ be a boundary square adjacent to x and y, respectively. Then we have:*

$$e\big(\lambda/\mu - x - y\big)^2 \; \ge \; e\big(\lambda/\mu - x - v\big) \, \cdot \, e\big(\lambda/\mu - y - w\big). \tag{11.6}$$

In particular, when λ and μ are self-conjugate, $x = (i,j)$ and $y = (j,i)$, we have:

$$e\big(\lambda/\mu - x - y\big) \; \ge \; e\big(\lambda/\mu - x - v\big)\,. \tag{11.7}$$

Special cases of (11.6) when dominos (x, v) and (y, w) are in the same directions also follow from the Schur function inequalities in [LP07], but (11.7) does not extend to a Schur functions. We refer to [CP22b, §4] for further applications of poset inequalities to the number of standard Young tableaux.

Remark 11.2. We also have product formulas for the order polynomial and for the GF for P-partitions:

$$\Omega(P_\lambda, t) \, = \, \prod_{(i,j)\in\lambda} \frac{t+i-j}{h_\lambda(i,j)} \qquad \text{and} \qquad \Omega_q(P_\lambda) \, = \, \prod_{(i,j)\in\lambda} \frac{1}{1-q^{h_\lambda(i,j)}}\,. \tag{11.8}$$

Both formulas are special cases of *Stanley's hook-content formula* for $\Omega_q(P_\lambda, t)$, see [Sta99, §7.21]. We refer to [HG76, Kra99, Pak01] for bijective proofs of these formulas, and to [Hop20] for other product formulas for the order polynomial. Note that the NHLF was extended to $\Omega_q(P_{\lambda/\mu})$ in [MPP18a]. Finally, both (11.4) and (11.5) extend to inequalities for Schur functions, see [CP23a, LP07, LPP07].

Remark 11.3. Define *shifted Young diagrams* as posets given by intersections of the usual Young diagrams and cone $i \le j$. Much of the work on the (usual) Young diagrams and standard Young tableaux directly translates to this case. We refer to [Sta99, §7.20] as the starting point and further references.

11.4. **Ribbon posets.** Let $Z_m := P_{\lambda/\mu}$ be a height two poset with $n = 2m-1$ elements, corresponding to the skew Young diagram $\lambda/\mu := \delta_m/\delta_{m-2}$, where $\delta_m := (m, \ldots, 2, 1)$. These are called *zigzag posets*. Linear extensions of Z_m are in bijection with *alternating permutations* $\sigma \in S_{2m-1}$ s.t. $\sigma(1) > \sigma(2) < \sigma(3) > \sigma(4) < \ldots$ Then $e(Z_m) = E_{2m-1}$ are the *Euler numbers*, see e.g. [OEIS, A000111] and [Sta10].

In the context of Sidorenko's inequality (5.4), it is easy to see that there is a width two poset Q_m such that $\Gamma(Q_m) = \overline{\Gamma(Z_m)}$ and $e(Q_m) = F_n$ is the *Fibonacci number*. Now (5.4) gives $E_n \cdot F_n \ge n!$ in this case [MPP18c]. Moreover, this inequality was proved in [MPP18c, Lem 4] by an explicit surjection.

Fix $x := (m, 1)$. It is easy to see that triangle of numbers $a(m,k) := \mathrm{N}(Z_m, x, k)$ are *Entringer numbers* [OEIS, A008282]. Stanley's inequality (Sta) proves their log-concavity:

$$a(m,k)^2 \, \ge \, a(m,k+1) \cdot a(m,k-1) \quad \text{for} \;\; 1 \le k \le 2m-2. \tag{11.9}$$

See [CP21, Ex. 1.38] for a q-analogue of this example in the style of Theorem 9.6; see also [B+19, GHMY23] for other generalizations.

We note that zigzag posets are special cases of *ribbon posets* $P_{\lambda/\mu}$, which correspond to skew shapes λ/μ with at most one square in every diagonal. Linear extensions of such $P_{\lambda/\mu}$ are in bijection with permutations which have a given set of descents, so $e(\lambda/\mu)$ satisfies *MacMahon's determinant formula*, see e.g. [Sta99, §2.2.4]. This formula was further generalized to *mobile posets* defined in [GGMM21]. Note also that among all ribbon posets, zigzag posets maximize the number of linear extensions [Sta99, Cor. 1.6.5]. This result is due to Niven (1968) and de Bruijn (1970), further generalized in [Sta88, Iri17].

11.5. **Permutation posets.** Fix $\sigma \in S_n$. The *permutation poset* $P_\sigma = ([n], \prec)$ is defined as:

$$i \preccurlyeq j \quad \Leftrightarrow \quad i \le j \;\text{ and }\; \sigma(i) \le \sigma(j)\,.$$

Permutation posets are also called *two-dimensional posets* in the literature, see e.g. [Tro95, West21]. This class includes all posets $P_{\lambda/\mu}$ and all posets of width two. Note that the height and the width in P_σ are longest increasing and the longest decreasing subsequence in σ, which are extremely well studied, see e.g. [Rom15]. It is known that #LE is #P-complete in this case [DP18].

The (weak) *Bruhat order* $\mathcal{B}_n = (S_n, \lhd)$ is defined as follows: $\tau \unlhd \pi$ if and only if $\tau \cdot \upsilon = \pi$ for some $\upsilon \in S_n$ such that $\operatorname{inv}(\tau) + \operatorname{inv}(\upsilon) = \operatorname{inv}(\pi)$. It is known that as a subset of S_n, the set of linear extensions $\mathcal{E}(P_\sigma)$ is the lower ideal $\sigma\!\downarrow$ in $\mathcal{B}_n$, see [FW97, Lem. 5] (see also [BW91a, Reu96]).

Denote by $\overline{\sigma} := \big(\sigma(n), \ldots, \sigma(1)\big)$ the *reverse permutation*. Note that P_σ and $P_{\overline{\sigma}}$ satisfy condition (5.1) of the Sidorenko inequality (Sid), which gives:

$$e\big(P_\sigma\big) \cdot e\big(P_{\overline{\sigma}}\big) \, \geq \, n!. \tag{11.10}$$

To generalize this, denote by

$$\operatorname{INV}(\sigma) \, := \, \big\{(i,j) \, : \, \sigma(i) > \sigma(j), \, 1 \leq i < j \leq n\big\}$$

the set of inversions of σ, so $\operatorname{inv}(\sigma) = |\operatorname{INV}(\sigma)|$. Suppose that

$$\operatorname{INV}(\tau) \cap \operatorname{INV}(\upsilon) \, = \, \varnothing \quad \text{and} \quad \operatorname{INV}(\tau) \cup \operatorname{INV}(\upsilon) \, = \, \operatorname{INV}(\pi).$$

Taking $i \leftarrow 2$, $P_1 \leftarrow P_\tau$, $P_2 \leftarrow P_\upsilon$ and $Q \leftarrow P_\pi$ gives conditions (5.3) of Theorem 5.5. Then (5.2) gives

$$e\big(P_\tau\big) \cdot e\big(P_\upsilon\big) \, \geq \, e\big(P_\pi\big). \tag{11.11}$$

Inequality (11.11) can be interpreted as defining a metric on the set of permutation [Ilo08, §9.3], see also [BT06, Cor. 8.7.2].

11.6. **Interval orders and semiorders.** Let X be a collection of n intervals $I_1, \ldots, I_n \subset \mathbb{R}$. Define $P = (X, \prec)$, where $I_i \prec I_j$ if $x < y$ for all $x \in I_i$ and $y \in I_j$. Such posets are called *interval orders*. They are characterized by not having $(C_2 + C_2)$ as an induced subposet which implies that the recognition problem of interval orders is in P, see e.g. [Möh89, §6].

Additionally, if all intervals have unit lengths, such posets are called *semiorders* and *unit interval orders*, see e.g. [Sta96]. They characterized by not having $(C_2 + C_2)$ and $(C_3 + C_1)$ as induced subposets (Scott and Suppes, 1958). We refer to [Fis85] for a thorough treatment and further references.

Let $P = (X, \prec)$ be a poset on $|X| = n$ elements. Let $\gamma(P) := |\{(x,y) \, : \, x \prec y, \, x, y \in X\}|$ denote the number of comparable pairs in P. Finally, let $e(n,k)$ be the maximal number of linear extensions among all posets P on n elements with $\gamma(P) = k$.

Proposition 11.4 (Fishburn–Trotter [FT92])**.** *Let $P = (X, \prec)$ be a poset on $|X| = n$ elements such that $\gamma(P) = k$ and $e(P) = e(n,k)$. Then P is a semiorder.*

See also [Tro95, Thm 8.7] for a short proof, and [MPI18] for the asymptotic analysis of numbers $e(n,k)$. This is a rare extremal result on linear extensions of finite posets. Finally, a larger class of $(C_3 + C_1)$-free posets is of interest in both Enumerative and Algebraic Combinatorics, see e.g. [GMR14].

11.7. **Random posets.** There are several interesting models of random finite posets studied in the literature, neither of which is especially satisfactory, at least when compared to random graph models. We refer to [Bri93] for a survey with many helpful references.

First, one can consider uniform (unlabeled) posets of n elements. Kleitman and Rothschild [KR75] gave a sharp asymptotic estimate on the number $p(n)$ of such posets

$$\log_2 p(n) \, = \, \tfrac{n^2}{4} \, + \, \tfrac{3n}{2} \, + \, O(\log n), \tag{11.12}$$

see also [OEIS, A000112]. It follows from the proof that uniform random posets have height three and can be partitioned into three antichains of sizes roughly $\frac{n}{4}$, $\frac{n}{2}$ and $\frac{n}{4}$, respectively.

The number of linear extensions is very large in this case and concentrated around $(\frac{n}{2})!(\frac{n}{4})!^2$, cf. (3.3). The number of pairs $(x, y) \in X^2$ such that $\mathbf{P}\big(f(x) < f(y)\big) \sim \frac{1}{2}$ is asymptotically $\frac{3n^2}{16}$ [Kor94]. A major disadvantage of this model is the difficulty of sampling uniform posets (either labeled or unlabeled).

Next, one can consider random posets from families where the sampling is easy. These include random *bipartite posets*, defined as posets of height two with relations given by a random bipartite graph. Similarly, one can consider random permutation posets P_σ, or random semiorders (there are Catalan number of them, see e.g. [Sta15, §3.180] and references therein). A curious model is given by the transitive closure of a random subset of relations $i \prec j$ on $[n]$, where $1 \le i < j \le n$, see an extensive discussion in [Bri93].

Finally, one can consider random posets P_λ corresponding to Young diagrams of size n. The problem of determining $e(P_\lambda)$ is well-studied and is especially important in combinatorial representation theory and again closely related to the study of longest increasing subsequences, see e.g. [Rom15]. Finally, see [MPI18] for the asymptotics of random interval orders.

12. Computational aspects

12.1. **Counting complexity.** In [BW91b], Brightwell and Winkler showed that computing the number of linear extensions is #P-complete. This was refined to posets of height two, posets of dimension two, and to *incidence posets* in [DP18]. In [Sta97], Stachowiak proved that computing *sign-imbalance* of posets of height two is #P-hard, by giving a simple parsimonious reduction to computing $e(P)$.

Additionally, Dittmer showed that the parity of the number of linear extensions is $\oplus P$-complete for posets of dimension two [Dit19, Thm 1.1.2]. Combined with Stachowiak's proof above implies that the parity of $e(P)$ is $\oplus P$-complete for height two posets. This is in contrast with Soukup's theorem that deciding whether poset P is *sign-balanced* is in P, see [Sou23+].

In the opposite direction, the are several classes of posets where the number of linear extensions can be computed in polynomial time.

Theorem 12.1. #LE *is in* FP *for:*

1. *bounded width posets,*
2. *skew Young diagrams, see* (11.2)*,*
3. *series-parallel posets, see* §11.2 *and* [VTL82]*,*
4. *posets with bounded decomposition diameter* [HM87]*,*
5. *posets whose covering graphs have disjoint cycles* [Atk89] (*e.g. trees* [Atk90])*,*
6. *N-free posets with bounded activity* [FM14]*,*
7. *posets with bounded treewidth* [EGKO16]*,*
8. *mobile posets* [GGMM21].

In contrast with (6), for general N-free posets, the problem #LE is conjectured to be #P-complete. We are not aware if the number of linear extension of interval order has been studied or conjectured to be #P-complete.

For the order polynomial, the coefficient $[t^2]\,\Omega(P, t)$ is the number of (lower) order ideals in P, which is #P-complete [PB83]. We refer to [FS86] for more on complexity of computing $\Omega(P, t)$. Finally, we note that the probability $\mathbf{P}\big(f(x) < f(y)\big)$ and the average height $h(P, x)$ are #P-hard [BW91b, §5].

12.2. **Random generation and approximate counting.** The problem of *random generation* of linear extensions $f \in \mathcal{E}(P)$ is closely related to *approximate counting*, i.e. computing the $(1 \pm \epsilon)$ approximation of $e(P)$. The key is the *self-reducibility* property: if one can obtain a strong approximation of the ratio $e(P)/e(P \cap \{x \prec y\})$, taking the product of these ratios gives an approximation for $e(P)$. We refer to [Vaz01, §28] for the introduction to this technology.

The first FPRAS for #LE was obtained by Matthews [Mat91] using a geometric random walk on the order polytope $\mathcal{O}_P$, based on a Markov chain (MC) with the mixing time upper bound $\text{mix} = O\big(n^8(\log n)^3\big)$. There are now several rapidly mixing Markov chains on $\mathcal{E}(P)$ worth discussing. All Markov chains start at $f \in \mathcal{E}(P)$, but have different steps described as follows.

(1) *Choose uniform* $k \in [n-1]$*, and switch the values* $k \leftrightarrow (k+1)$ *in* f *if possible.* This MC was introduced by Karzanov and Khachiyan [KK91] who proved $\text{mix} = O(n^6 \log n)$ bound using conductance estimates. This bound was steadily improved down to $O(n^3 \log n)$ [Wil04]. Furthermore, Wilson showed (ibid.), that for some posets this bound cannot be improved.

(1′) *Choose* $k \in [n-1]$ *with probability proportional to* $k(n-k)$*. Proceed as in* (1). This modification is due to Bubley and Dyer [BD99], who proved the $\text{mix} = O(n^3 \log n)$ bound using a coupling argument.

(2) *Choose uniform* $x \in X$*, and take a partial promotion to* x *in* f*.* This MC was introduced by Ayyer, Klee and Schilling [AKS14a], who gave a bound $\text{mix} = O(n^3)$ in [AKS14b]. A better bound a $\text{mix} = O(n^2)$ was obtained in [PS18].

(3) *Choose uniform* $i < j$*. For all* $k \in \{i, \ldots, j-1\}$ *in this order, switch the values* $k \leftrightarrow (k+1)$ *in* f *if possible.* This MC was introduced by Ayyer, Schilling and Thiéry [AST17], who proved the $\text{mix} = O(n^2 \log n)$ upper bound and conjecture that $\text{mix} = O(n \log n)$.

For posets of height two, a conjectured rapidly mixing MC was given in [CRS09]. A closely related MC was analyzed in [Hub14]. Let us also mention Huber's algorithm [Hub06] for *perfect sampling* of linear extensions of general posets.

Finally, the problem of (exact) *uniform generation* is also of interest in both Combinatorics [NW78] and Theoretical Computer Science [JVV86]. For classes of posets where the counting problem is polynomial, i.e. $\#\text{LE} \in \mathsf{FP}$, the self-reducibility gives a polynomial time algorithms for the uniform generation. Faster algorithms exists for Young diagrams [GNW79, NPS97] (see also [SS17]), special skew Young diagrams [H+23, §5.6], and for series-parallel posets [BDGP17].

12.3. **Graph of linear extensions.** A common consequence of the Markov chains discussed above, is the following basic result:

Proposition 12.2 (folklore)**.** *Let* $P = (X, \prec)$ *be finite poset, and let* $f, g \in \mathcal{E}(P)$ *be linear extensions. Then* f *and* g *are connected by a sequence of* $k \leftrightarrow (k+1)$ *switches,* $1 \le k < n$*.*

The proposition is a folklore result repeatedly rediscovered in different contexts. For a brief overview of generalizations and further references, we refer to the discussion which follows [DK21, Prop. 1.2].

Consider a graph $\mathcal{G}(P)$ with vertices $\mathcal{E}(P)$, and with edges corresponding to switches, see e.g. [Mas09]. The proposition above proves connectivity of $\mathcal{G}(P)$. In fact, the distance between any two linear extensions can be computed in polynomial time, see [BW91a, §6] and [Naa00, Prop. 2.2]. On the other hand, the diameter of $\mathcal{G}(P)$ is NP-hard to compute [BM13, Thm 5].

Suppose $e(P)$ is even. Ruskey noted [Rus92], that if $\mathcal{G}(P)$ has a *Hamiltonian path*, then P is sign-balanced (see §2.3). Motivated by the problem of *listing* all linear extensions, he stated:

Conjecture 12.3 (Ruskey [Rus92, §5])**.** *Let* P *be a finite sign-balanced poset. Then* $\mathcal{G}(P)$ *has a Hamiltonian path.*

We refer to [Rus03, §5.10] for the introduction, to [CW95] for algorithmic aspects of this problem, and to a recent survey [Müt23, §5.5] for further references.

12.4. **Coincidence problem and concise functions.** Following [CP23b], consider the *coincidence problem*:

$$\mathrm{C}_e \ := \ \big\{ e(P) \ =^? \ e(Q) \big\}.$$

We conjecture that this problem is $\mathsf{C_=P}$-complete under Turing reductions.

Theorem 12.4 ([CP23b, Thm 1.4]). $\mathrm{C}_e \in \mathsf{PH} \implies \mathsf{PH} = \Sigma^{\mathrm{p}}_m$ *for some* $m \geq 1$. *Moreover, this holds when* e *is restricted to permutation posets.*

A key lemma in the proof is the following result of independent interest.

Theorem 12.5 ([KS21, Thm 1.1]). *Let*

$$\mathcal{T}_e(n) \ := \ \big\{ e(P) \ : \ P = (X, \prec), \ |X| = n, \ \mathrm{width}(P) = 2 \big\}.$$

Then:

$$\mathcal{T}_e(n) \ \supseteq \ \big\{ 1, \ldots, c^{n/(\log n)} \big\} \quad \textit{for some} \ \ c > 1. \tag{12.1}$$

The authors conjecture that the upper end in (12.1) can be improved to c^n [KS21, Conj. 7.4]. The following conjecture implies that Theorem 12.4 holds for posets of height two, see [CP23b, Prop. 5.18].

Conjecture 12.6 ([CP23b, Conj 5.17]). *For all sufficiently large* m*, there is a poset* P *of height two, such that* $e(P) = m$*.*

12.5. **Combinatorial interpretation.** For every inequality $f \geqslant g$ where $f, g \in \#\mathsf{P}$ are counting functions, one can ask if the defect $(f - g)$ is in $\#\mathsf{P}$. Informally, this is a question whether $(f - g)$ has a combinatorial interpretation. For example, the proof of Theorem 12.4 implies that

$$\big(e(P) \, - \, e(Q)\big)^2 \ \notin \ \#\mathsf{P} \quad \text{unless} \quad \mathsf{PH} \, = \, \Sigma^{\mathrm{p}}_2 .$$

We refer to [Pak19] for the introduction to the problem of combinatorial interpretation, to [Sta00] for a review of open problems on combinatorial interpretation in Algebraic Combinatorics, to [IP22] for a careful treatment of polynomial inequalities, and to [Pak22] for a detailed survey.

13. Sorting probability

In this section we summarize partial results and several variations on the $\frac{1}{3} - \frac{2}{3}$ Conjecture.

13.1. **The $\frac{1}{3} - \frac{2}{3}$ Conjecture.** The following conjecture remains a major challenge in the area. It was originally stated by Kislitsyn [Kis68], and independently by Fredman [Fre75].

Conjecture 13.1 ($\frac{1}{3} - \frac{2}{3}$ *conjecture*). *In every finite poset* $P = (X, \prec)$ *that is not a chain, there exist two elements* $x, y \in X$*, such that*

$$\frac{1}{3} \ \leq \ \mathbf{P}\big[f(x) < f(y) \big] \ \leq \ \frac{2}{3}\,, \tag{13.1}$$

where the probability is over uniform random $f \in \mathcal{E}(P)$*.*

Note that the constant $\varepsilon = \frac{1}{3}$ is optimal for $P = C_2 + C_1$. A poset $P = (X, \prec)$ on n elements is called *k-thin* if $\alpha(x) + \beta(x) > n - k$, for all $x \in X$. In the opposite direction, poset P is called *(ϵ, δ)-dense* if there is $Y \subset X$, $|Y| \geq \epsilon n$, s.t. $\alpha(x) + \beta(x) < \delta n$, for all $x \in Y$.

Theorem 13.2. *Conjecture 13.1 holds for the following posets with* n *elements:*

(1) *width two posets* [Lin84],
(2) *posets with a symmetry* [GHP87],

(3) *semiorders* [Bri89] (*a concise proof was given in* [Bri99, Thm. 2.3]),
(4) *height two posets* [TGF92],
(5) *posets with* $|\min(P)| > C\sqrt{n}$ *for some* $C > 0$ [Fri93],
(6) *posets with no chains of length* $> 2\log_2 \log n - C$, *ibid.*
(7) (ϵ, δ)*-dense posets, for all* $\epsilon > 0$ *and some* $\delta = \delta(\epsilon) > 0$, *ibid.*
(8) 6*-thin posets* [Pec08] (*a weaker* 5*-thin version was given in* [BW92]),
(9) *series-parallel and* N*-free posets* [Zag12],
(10) *skew Young diagram posets* $P_{\lambda/\mu}$ [OS18],
(11) *posets whose cover graph is a forest* [Zag19].

Note that these classes are not completely disjoint. For example, for sufficiently large n, parts (5), (6) and (7) imply (4). Part (4) is proved via Komlós theorem (see below). Parts (5) – (7) are proved using geometric tools. An alternative proof of (10) given in [CPP21a, §3] uses the Naruse hook-length formula (cf. §11.3). We refer to [Bri99] for a well-written survey of early results and ideas.

13.2. **Weaker general bounds.** Currently, the best general bound is given by the following result:

Theorem 13.3 (Brightwell–Felsner–Trotter [BFT95])**.** *In every finite poset* $P = (X, \prec)$ *that is not a chain, there exist two elements* $x, y \in X$, *such that*

$$\frac{1}{2} - \frac{1}{2\sqrt{5}} \ \le\ \mathbf{P}\big[\, f(x) \,<\, f(y)\big] \ \le\ \frac{1}{2} + \frac{1}{2\sqrt{5}}\,, \tag{13.2}$$

where the probability is over uniform random $f \in \mathcal{E}(P)$.

Here the lower bound is $\varepsilon \approx 0.2764$, just shy of $\frac{1}{3}$. This is a small improvement over the first bound of the type $\varepsilon \le \mathbf{P}[f(x) < f(y)] \le 1 - \varepsilon$ proved by Kahn and Saks [KS84], with $\varepsilon = \frac{3}{11} \approx 0.2727$. The latter result uses the Kahn–Saks inequality (KS), applied to two elements $x, y \in X$ with a small difference of average heights: $|h(P, x) - h(P, y)| < 1$, cf. §8.4. The proof in [BFT95] uses the $a = b = 1$ case of the cross-product conjecture (Theorem 9.13), and builds on [KS84].

We note that an easier proof of a weaker bound with $\varepsilon = \frac{1}{2e} \approx 0.1840$, was obtained by Kahn and Linial in [KL91], using a variation on *Grünbaum's theorem* (1960), which in turn is proved using the *Brunn–Minkowski inequality* (rather than the Alexandrov–Fenchel inequality used in the proof of the Kahn–Saks inequality). We refer to [Mat02, §12.3] for a clean presentation of this proof, and to [C+13] for a survey of algorithmic applications of the sorting probability.

13.3. **Stronger specialized bounds.** For a poset $P = (X, \prec)$, the *sorting probability* is defined as

$$\delta(P) \,:=\, \min_{x,y \in X} \big|\, \mathbf{P}\big(f(x) < f(y)\big) \,-\, \mathbf{P}\big(f(x) > f(y)\big)\big|. \tag{13.3}$$

In this notation, the $\frac{1}{3} - \frac{2}{3}$ Conjecture 13.1 claims that the sorting probability $\delta(P) \le \frac{1}{3}$ unless P is a chain, while the bound (13.2) gives $\delta(P) \le \frac{1}{\sqrt{5}}$. In the cases when the conjecture is established, one can ask for better bounds. For example, Friedman showed that for all $\epsilon > 0$ and $C = C(\epsilon)$, we have $\delta(P) < \big(1 - \frac{2}{e} + \epsilon\big)$ in the cases (5) – (7) in Theorem 13.2.

Conjecture 13.4 (Kahn–Saks [KS84])**.** *We have:*

$$\delta(P) \to 0 \quad \textit{as} \quad \mathrm{width}(P) \to \infty. \tag{13.4}$$

Informally, this conjecture says that the sorting probability is small for all posets of sufficiently large width. Beside common sense, there is relatively small evidence in favor of this conjecture. Saks [Saks85] suggests that $\delta(P) \leq \frac{14}{39}$ when $\text{width}(P) \geq 3$, and gives an example where this bound is tight. Komlós [Kom90] proved that $\delta(P) \to 0$, for $|\min(P)| = n/s(n)$ and some $s(n) = \omega(1)$.

Curiously, it is known that $\delta(P) \to 0$ for several large posets of bounded width, such as several families of skew Young diagrams [CPP21a]. For example, Panova and the authors proved $\delta(P_\lambda) = O_\epsilon(1/\sqrt{n})$ for $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ with ℓ fixed and $\lambda_\ell > \epsilon n$ [CPP21a, Thm 1.3].

For posets of width two which cannot be written as a linear sum of chains and posets $(C_2 + C_1)$, Sah [Sah21] improved Linial's $\delta(P) \leq \frac{1}{3}$ upper bound to $\delta(P) < 0.3225$. Sah also conjectured in [Sah21, Conj. 5.3], that this approach extends to general posets (of any width), i.e. under mild assumptions Conjecture 13.1 can be made strict: $\delta(P) < \frac{1}{3} - \varepsilon$ for some constant $\varepsilon > 0$.

Finally, for the extremely well studied *Catalan poset* Cat_n of width two (see §11.3), Panova and the authors showed in [CPP21b], that $\delta(\text{Cat}_n) = O(n^{-5/4})$, where the constant $5/4$ is conjectured to be sharp.

13.4. **Gaps between heights.** For a poset $P = (X, \prec)$ on $|X| = n$ elements, let

$$\eta(P) := \min_{x \,|\, y} \bigl| h(P,x) - h(P,y) \bigr|$$

denote the smallest gap between heights of incomparable elements in P. As we mentioned earlier, the proof by Kahn–Saks of $\delta(P) \leq \frac{5}{11}$ follows from the observation that $\eta(P) < 1$. Improving the latter bound directly translates to a sharper for the sorting probability $\delta(P)$.

Unfortunately, there is a limit for this approach. Let

$$\vartheta := \frac{1}{4} \prod_{k=1}^{\infty} \left(1 - \frac{1}{2^k}\right)^{-1} \approx 0.8657\,.$$

Saks in [Saks85], gives a construction of posets P_n such that $\eta(P_n) \to \vartheta$. He conjectures that this example is optimal:

Conjecture 13.5 ([Saks85])**.** *For every poset P that is not a chain, we have $\eta(P) \leq \vartheta$.*

13.5. **Voting preferences.** Let $P = (X, \prec)$ be a poset on $n = |X|$ elements and let $\mathbf{P}$ be defined over uniform $f \in \mathcal{E}(P)$. One can think of the average height $h(P,x)$ a way to rank all elements in X (possibly, with ties). This is not the only natural approach, of course.

For elements $x, y \in X$, we write $x \mapsto y$ if $\mathbf{P}\bigl(f(x) < f(y)\bigr) > \frac{1}{2}$. Heuristically, this means of the random linear ordering of X which respects partial order "$\prec$", element y typically has a smaller rank than x. In [Kis68, §4.2], Kislitsyn speculated that "$\mapsto$" is transitive. This was disproved by Fishburn soon after:[5]

Proposition 13.6 (Fishburn [Fis74b])**.** *Relation "$\mapsto$" is not transitive, i.e. there exist a finite poset $P = (X, \prec)$ and three elements $x, y, z \in X$, s.t. $x \mapsto y$, $y \mapsto z$, and $z \mapsto x$.*

The original example by Fishburn has $n = 31$ elements. It was shown in [FG90], that the smallest such poset has $n = 9$ elements. See also [EFG90] for an example of a poset of height two with $n = 15$ elements. Below is a quantitative version of this problem:

[5]Fishburn was unaware of [Kis68] and independently discovered the problem motivated by voting paradoxes.

Theorem 13.7 (Fishburn [Fis86] and Kahn–Yu [KY98])**.** *Let $P = (X, \prec)$ be a finite poset, and let $x \mapsto y$, $y \mapsto z$ for some $x, y, z \in X$. Then:*

$$\mathbf{P}\big(f(x) < f(z)\big) > \tfrac{1}{4}. \tag{13.5}$$

On the other hand, the constant $\frac{1}{4}$ in the RHS cannot be replaced with $\frac{1}{e}$.

Here the second part is due to Fishburn. The bound (13.5) is due to Kahn–Yu, who write that Fishburn's $\frac{1}{e}$ is "likely to be the correct value" to appear in (13.5).

Question 13.8. *What is the optimal constant in the RHS of (13.5)?*

We conclude with a weaker notion of the preference order which happens to be transitive. For elements $x, y \in X$ and $\rho \geq \frac{1}{2}$ we write $x \mapsto_\rho y$ if $\mathbf{P}\big(f(x) < f(y)\big) > \rho$.

Theorem 13.9 (Yu [Yu98])**.** *Fix $\rho > 0.78005$. Let $P = (X, \prec)$ be a finite poset, and let $x \mapsto_\rho y$, $y \mapsto_\rho z$ for some $x, y, z \in X$. Then $x \mapsto_\rho z$.*

Let us also mention Friedman's observation [Fri93, §2], that there always exist a linear extension $g \in \mathcal{E}(P)$ such that $\mathbf{P}\big(f(x) < f(y)\big) \geq \frac{1}{2}$ for all $g(x) = k$, $g(y) = k+1$, $1 \leq k < n$. This follows from the fact that every tournament has a Hamiltonian path.

Remark 13.10. Fishburn's original motivation for Proposition 13.6 comes from voting paradoxes, see [Fis74a, Geh06]. There is also a parallel study of *intransitive dice* which exhibits similar phenomena and has been studied quantitatively in recent years, see [HMRZ20, Poly22] and references therein.

14. Tools and ideas

14.1. **Direct injections.** In contrast with many objects in enumerative and algebraic combinatorics, posets inequalities are incredibly difficult to prove by a direct combinatorial arguments. There are essentially two main tools: lattice paths for posets of width two and promotion/demotion maps for general posets (see below). Here is a quick list of ad hoc direct injections uses in the proof of results in this survey:

- Theorem 4.1 with a lower bound for the values of the order polynomials,
- Theorem 4.1 with Brenti's log-concavity for the order polynomials,
- Gaetz–Gao surjection in [GG20] proving Theorem 5.1 (but not Theorem 5.2),
- Proposition 4.8 proving a weak version of the Kahn–Saks Conjecture 4.7,
- The original proof of the Björner–Wachs inequality (Theorem 6.1),
- Reiner's proof of the q-BW inequality (Theorem 6.2),
- Lam—Pylyavskyy generalizations of Fishburn inequalities (see Theorems 7.6 and 7.7).

Let us emphasize that these injections are relatively straightforward and completely explicit, so the defect of the corresponding inequalities are all in $\#\mathsf{P}$.

14.2. **Promotion and demotion.** Let $X = (X, \prec)$ be a poset on $|X| = n$ elements. *Promotion* is a bijection $\partial : \mathcal{E}(P) \to \mathcal{E}(P)$, which we denote using the operator notation $\partial : f \to \partial f$. For $f \in \mathcal{E}(P)$, let $x_1 \prec \ldots \prec x_\ell$ be a maximal chain in P such that the sequence $f(x_1), \ldots, f(x_\ell)$ is lexicographically smallest. Equivalently, we have $f(x_1) = 1$, element x_2 is smallest cover of x_1, element x_3 is smallest cover of x_2, etc. Define $\partial f \in \mathcal{E}(P)$ as

$$\partial f(z) := \begin{cases} f(x_{i+1}) - 1 & \text{if } \ z = x_i \ \text{ for some } i < \ell, \\ n & \text{if } \ z = x_\ell, \\ f(z) - 1 & \text{otherwise.} \end{cases} \tag{14.1}$$

Partial promotion ∂_i is defined as the promotion on a poset obtained by restriction to elements with f-values $1, \ldots, i$, so that $\partial_n = \partial$ and $\partial_1 = 1$. *Demotion* and *partial demotions* are defined as inverse bijections.

These operators were introduced and initially studies by Schützenberger in [Schü72]. For *standard Young tableaux* (linear extensions of Young diagram posets), the promotion is called the *jeu-de-taquin*, and is fundamental in the whole of Algebraic Combinatorics (see e.g. [Sag01, Sta99]). It is closely related to the *Robinson–Schensted–Knuth (RSK) correspondence* (ibid.), the *Edelman–Greene bijection* [EG87], and the *NPS algorithm* [NPS97]. For *increasing trees* (connected forest posets), the enumerative applications were given in [KPP94, §6].

Partial promotion operators have algebraic relations which were investigated by Lascoux and Schützenberger for Young tableaux. In full generality, they were studied by Haiman [Hai92] and Malvenuto–Reutenauer [MR94]. See [Sta09] for an extensive survey.

The results in this survey whose proofs use explicit applications of the promotion and demotion operators include:
- Theorem 3.7 proving the summation inequality for antichains,
- Theorems 9.9 and 9.10 proving the DDP log-concave inequality for the order polynomial,
- Theorem 9.14 proving weak quantitative version of (CPC),
- Theorems 10.1, 10.7, 10.9 and 10.10, proving various vanishing and uniqueness conditions for Stanley type inequalities, and
- Theorem 10.13 proving equality conditions for the $k = 1$ case of the generalized Stanley inequality (9.6).

For the first two items, because the promotion/demotion proofs are completely explicit, the defect of the corresponding inequalities are all in $\#\mathsf{P}$. For the last three items, the corresponding decision problems are in P.

14.3. **Lattice paths.** Let $P = (X, \prec)$ be a poset of width two with $|X| = n$ elements. Fix a partition $X = C \sqcup C'$ into two chains, where $C = \{u_1 \prec \ldots \prec u_k\}$, and $C' = \{v_1 \prec \ldots \prec v_\ell\}$, where $n = k + \ell$. Denote by $h, h' \in \mathcal{E}(P)$ lexicographically minimal and maximal linear extensions in P, respectively.

For a linear extension $f \in \mathcal{E}(P)$, denote by $\gamma(f)$ a *lattice path* in $\mathbb{N}^2$, defined as follows. Let $\gamma(f) : (0,0) \to (k, \ell)$ start at $(0,0)$, end at (k, ℓ), and take Up steps for elements in C, and Right steps for elements in C'. Consider a region $\Gamma(P) \subset \mathbb{N}^2$ enclosed between paths $\gamma(h)$ and $\gamma(h')$. It is easy to see that all Up-Right paths $\gamma : (0,0) \to (k, \ell)$ are in bijection with $\mathcal{E}(P)$, i.e. $\gamma = \gamma(f)$ for some $f \in \mathcal{E}(f)$.

This connection has been frequently used to give estimates and prove inequalities for width two posets, see e.g. [BG96, CPP22a, CPP23a, CFG80, GYY80]. It is worth noting that in [CPP22b] the injective proof of Stanley's inequality (Sta) for width two posets was extended to log-concavity of hitting probabilities of general walks in the plane. These walks can be viewed as "oscillating linear extensions", generalizing *oscillating Young tableaux*, see e.g. [PP96, Sag90].

For posets of larger (bounded) width, in principle the same general approach, but direct injective arguments are harder to obtain. We are aware of only [CPP21a] which was able to combine it with anti-concentration inequalities to obtain sharp bounds for the sorting probabilities of (skew) Young diagrams (see §13.3).

The results in this survey whose proofs employ lattice paths, include:
- Theorem 8.1 proving (GYY) inequality,
- Special case of Theorem 9.1 proving (Sta) inequality for width two posets,
- Theorems 9.4 and 9.5 with q-analogues of (Sta) and (KS),
- Theorem 9.13 (2), proving (CPC) for width two posets.

Let us emphasize that because lattice walks based proofs are completely explicit, the defect of the corresponding inequalities are all in $\#\mathsf{P}$.

14.4. **Correlation inequalities.** Let $\mathcal{A}$ be a collection of subsets of $[n]$. We say that $\mathcal{A}$ is *closed upward*, if $B \in \mathcal{A}$ for every $B \supseteq A$ and $A \in \mathcal{A}$. Similarly, $\mathcal{A}$ is *closed downward*, if $B \in \mathcal{A}$ for every $B \subseteq A$ and $A \in \mathcal{A}$. The notions are easiest to understand when $\mathcal{A}$ is a *graph property*, i.e. a collection of (spanning) subgraphs of a complete graph. Then, for example, connectivity, Hamiltonicity, having all degrees at least Δ, non-planarity, non-k-colorability, or containing an r-clique K_r are upward closed properties. The negations, such as k-colorability, planarity or disconnectivity are downward closed properties.

The *Harris–Kleitman inequality* [Har60, Kle66], states that for the upward closed collections $\mathcal{A}, \mathcal{B} \subseteq 2^{[n]}$, we have positive correlation:

$$|\mathcal{A} \cap \mathcal{B}| \cdot 2^n \ \geq \ |\mathcal{A}| \cdot |\mathcal{B}|. \tag{14.2}$$

Similarly, for $\mathcal{A} \subseteq 2^{[n]}$ upward closed and $\mathcal{B} \subseteq 2^{[n]}$ downward closed, we have negative correlation:

$$|\mathcal{A} \cap \mathcal{B}| \cdot 2^n \ \leq \ |\mathcal{A}| \cdot |\mathcal{B}|.$$

These inequalities have a clear probabilistic meaning, e.g. (14.2) can be rewritten as

$$\mathbf{P}(A \in \mathcal{A} \cap \mathcal{B}) \ \geq \ \mathbf{P}(A \in \mathcal{A}) \cdot \mathbf{P}(A \in \mathcal{B}), \tag{14.3}$$

where the probability is over uniform $A \subseteq [n]$.

The Harris–Kleitman inequality had a series of generalizations, including the celebrated *FKG inequality* by Fortuin, Kasteleyn and Ginibre [FKG71]. Here we only present the AD inequality that is aptly called the *four functions theorem* in [AS16], and which implies the FKG inequality. For every $\rho : Z \to \mathbb{R}_+$ and every $X \subseteq Z$, denote

$$\rho(X) \ := \ \sum_{x \in X} \rho(x). \tag{14.4}$$

Theorem 14.1 (*Ahlswede–Daykin inequality* [AD78]). *Let $\mathcal{L} = (L, \vee, \wedge)$ be a finite distributive lattice on the ground set L, and let $\alpha, \beta, \gamma, \delta : L \to \mathbb{R}_+$ be nonnegative functions on L. Suppose we have:*

$$\alpha(x) \cdot \beta(y) \ \leq \ \gamma(x \vee y) \cdot \delta(x \wedge y) \quad \text{for every} \quad x, y \in L. \tag{14.5}$$

Then:

$$\alpha(X) \cdot \beta(Y) \ \leq \ \gamma(X \vee Y) \cdot \delta(X \wedge Y) \quad \text{for every} \quad X, Y \subseteq L. \tag{14.6}$$

The *q-FKG inequality* was introduced by Björner in [Bjö11, Thm 2.1], who used it to prove a q-analogue of (11.4). Christofides [Chr09, Thm 1.5] gave the *q-AD inequality*, while the multivariate ***q**-AD inequality* we given in [CP23a, Thm 6.1].

These correlation inequalities play a key role in many poset inequalities throughout the survey. Notably, they are used in proofs of the following results:
- Theorems 4.3, 4.4, 4.5 and 4.11 on general inequalities for the order polynomial,
- Theorems 6.6 and 6.7 on the order polynomial version of the BW inequality (BW),
- Fishburn's inequality (Theorem 7.4) and its generalizations in §7, including the most general Theorem 7.10,
- GYY inequality (Theorem 8.1) and its generalization Shepp's inequality (Theorem 8.3), obtained as a consequence of the order polynomial versions Theorems 8.5 and 8.6,
- XYZ inequality (Theorem 8.7), and its applications Theorems 8.10 and 8.11.

14.5. **Combinatorial optimization.** Both the order polytope $\mathcal{O}_P$ and the chain polytope $\mathcal{S}_P$ were defined by Stanley, see [Sta86]. The volume equality $\operatorname{vol}\mathcal{O}_p = e(P)/n!$ in (2.8) is straightforward via a simple triangulation of $\mathcal{O}_P$ into congruent *orthoschemes* (also called *path-simplices*) whose volume is $1/n!$ and which are in bijection with linear extensions $f \in \mathcal{E}(P)$. Equality between volumes and between Ehrhart polynomials of $\mathcal{O}_P$ and $\mathcal{S}_P$ in (2.11), follows from an explicit continuous piecewise-linear volume-preserving map $\xi : \mathcal{O}_P \to \mathcal{S}_P$.

Chain polytope $\mathcal{S}_P$ is especially useful in applications.

- Proposition 3.10 and Theorem 3.13 are proved using the description of the dual polytope $\mathcal{S}_P^\circ$,
- Theorem 3.16 and Theorem 3.17 are proved using the entropy functional on $\mathcal{S}_P$.

An interesting approach was introduced by Sidorenko in [Sid91], which combines combinatorial duality and network flows. Think of the comparability graph $\Gamma(P)$ as a directed network and enlarge it by adding bidirected edges for incomparable pairs. Now define the *Sidorenko flow* by sending a unit along every linear extension. Observe that the flow along these bidirected edges is equal in both directions, so these edges can be deleted. In fact, the flow can be described using promotions, see e.g. [CPP23b, §8]. The detailed analysis of this flow implies several closely related inequalities:

- Theorem 3.7 proving the summation inequality for antichains,
- Sidorenko inequality (Theorem 5.1), and its generalizations (Theorems 5.4 and 5.5).

We refer to [Schr03] for a very extensive discussion of combinatorial optimization and applications to chain polytopes.

14.6. **Geometric inequalities.** Applying known geometric inequalities to order and chain polytopes $\mathcal{O}_P$ and $\mathcal{S}_P$ gives surprisingly strong implications. These include:

- The reverse Sidorenko inequality (Theorem 5.1), proved by applying the *Saint-Raymond inequality* [StR81]. This is a special case of *Mahler's conjecture*, see e.g. [Tao08, §1.3].
- The reverse Sidorenko inequality (Theorem 5.6), proved by applying the *Blaschke–Santaló inequality* to polytope $\mathcal{S}_P$ and its dual $\mathcal{S}_P^\circ$, see e.g. [BZ88, §24.5].
- The mixed Sidorenko inequality (see Remark 5.8), proved by applying *Godberson's conjecture* established in [AASS20] for anti-blocking polytopes, including $\mathcal{S}_P$.
- Stanley inequality (Theorems 9.1 and 9.7) and Kahn–Saks inequality (Theorem 9.3), proved by applying the *Alexandrov–Fenchel inequality* to sections of $\mathcal{O}_P$.
- Special cases and weak versions of Conjecture 9.12 given in Theorem 9.13 (3) and Theorem 9.14, are proved by using *Favard's inequality*.
- Kahn–Linial's proof of a weak version of Conjecture 13.1 and parts $(5)-(7)$ of Theorem 13.2, are proved using *Mityagin's inequality*, which is a generalization of *Grünbaum's inequality*, see §13.2.
- Kahn–Yu and Yu inequalities (Theorems 13.7 and 13.9) are proved using the *Brunn–Minkowski inequality* combined with the *K. Ball inequality*.

Additionally, the geometric analysis of equality conditions of geometric inequalities can be translated to equality conditions of poset inequalities. This is a very recent direction of research pioneered by Shenfeld and van Handel [SvH23], in their study of equality cases of the Alexandrov–Fenchel inequality.

- Equality cases of Stanley inequality in Theorem 10.2,
- Equality cases of the Kahn–Saks inequality in Theorem 10.8,
- Ma–Shenfeld's proof of vanishing of the generalized Stanley inequality in Theorem 10.9,
- Equality cases of the generalized Stanley inequality in Theorem 10.13, $k = 1$ case,
- Equality conditions of the generalized Stanley inequality in Theorems 10.12 and 10.16.

In contrast with the previous list of applications, all of these results are very technical and difficult to obtain. Even the definitions can be difficult as a direct translation from the geometry can pose challenges. Streamlining one such definition is the point of our Definition/Lemma 10.15 proved in §15.2.

14.7. **Combinatorial atlas.** In [CP21], we presented a new linear algebraic approach to log-concave inequalities, based on a structure we call *combinatorial atlas*. Roughly, this is a combinatorial setup of vectors and matrices related to each other according to certain directed graphs. These matrices are associated with posets and contain counting of numbers of linear extensions. The setup allows one to prove by induction that these matrices are hyperbolic, starting with two element posets as the base of induction.

Formally, a $d \times d$ real matrix $\mathbf{M}$ is called *hyperbolic*, if

$$\text{(Hyp)} \qquad \langle \mathbf{v}, \mathbf{M}\,\mathbf{w}\rangle^2 \ \geq \ \langle \mathbf{v}, \mathbf{M}\,\mathbf{v}\rangle\langle \mathbf{w}, \mathbf{M}\,\mathbf{w}\rangle \quad \text{for all} \quad \mathbf{v}, \mathbf{w} \in \mathbb{R}^d \quad \text{s.t.} \quad \langle \mathbf{w}, \mathbf{M}\,\mathbf{w}\rangle > 0.$$

It is not hard to see that $\mathbf{M}$ is hyperbolic if and only if it has at most one positive eigenvalue (including multiplicities). While the eigenvalue conditions allows to obtain the step of induction, the inequality (Hyp) implies a number of correlation and Stanley type inequalities:

- Deletion correlations (Theorem 8.13),
- Subset correlations (Theorems 8.14 and 8.16),
- Covariance inequalities (Theorems 8.17, 8.18 and 8.19),
- Unique covers special cases (Theorem 8.20 and 8.21),
- Weighted Stanley inequality (Theorem 9.6),
- Equality conditions for the Stanley inequality (Theorem 10.2).

We refer to [CP22a] for the introduction to the combinatorial atlas technology and to [CP22b] for applications to correlation inequalities.

15. Proofs of technical results

15.1. Proof of Theorem 10.5 and Proposition 10.6.

Proof of Theorem 10.5. In notation of Conjecture 9.12, let $P := (X, \prec)$ be a poset with $|X| = n$ elements and fixed element $z \in X$. Let $Q := (Y, \prec')$ given by $Y := X \cup \{x, y, w\}$, with relations $u \prec' v \Leftrightarrow u \prec v$ for all $u, v \in X$, $x \prec' y \prec' u$ for all $u \in X$, and $x \prec' w$.

For all $a \geq 1$, we have:

$$\text{(15.1)} \qquad \mathrm{F}_{xyz}(Q, 1, a) \ = \ (a-1)\,\mathrm{N}(P, z, a-1) \ + \ (n+1-a)\,\mathrm{N}(P, z, a).$$

This follows from the observation that $\mathrm{F}_{xyz}(Q, 1, a)$ in the number of linear extensions $f \in \mathcal{E}(P)$ for which $f(x) = 1$, $f(y) = 2$, $f(z) = a+2$, while $f(w) \in \{3, \ldots, n+3\}$.

Also note that, for all $k \geq 2$,

$$\text{(15.2)} \qquad \mathrm{F}_{xyz}(Q, 2, a) \ = \ \mathrm{N}(P, z, a).$$

Indeed, this follows from the observation that $\mathrm{F}_{xyz}(Q, 2, a)$ counts those linear extensions for which $f(x) = 1$, $f(w) = 2$, $f(y) = 3$ and $f(z) = a+3$.

Now, (CPC) implies that for all $a, i \geq 1$, we have:

$$\text{(15.3)} \qquad \mathrm{F}_{xyz}(Q, 1, a) \cdot \mathrm{F}_{xyz}(Q, 2, a+i) \ \leq \ \mathrm{F}_{xyz}(Q, 1, a+i) \cdot \mathrm{F}_{xyz}(Q, 2, a).$$

By (15.1) and (15.2), the LHS of (15.3) is equal to

$$(a-1)\,\mathrm{N}(P, z, a-1) \cdot \mathrm{N}(P, z, a+i) \ + \ (n+1-a)\,\mathrm{N}(P, z, a) \cdot \mathrm{N}(P, z, a+i),$$

while the RHS of (15.3) is equal to

$$(a+i-1)\,\mathrm{N}(P, z, a+i-1) \cdot \mathrm{N}(P, z, a) \ + \ (n+1-a-i)\,\mathrm{N}(P, z, a+i) \cdot \mathrm{N}(P, z, a).$$

Thus, (10.1) follows from (15.3) and the two equations above. □

We restate Proposition 10.6 for clarity, writing out all the inequalities:

Proposition 15.1 (= Proposition 10.6)**.** *Let* $P = (X, \prec)$ *be a poset with* $|X| = n$ *elements, let* $x \in X$ *and* $a \in [n]$*. Suppose that* $\mathrm{N}(P,x,a) > 0$*. Then* (10.1) *implies:*

$$\mathrm{N}(P,x,a)^2 \,=\, \mathrm{N}(P,x,a+1)\cdot\mathrm{N}(P,x,a-1) \;\Leftrightarrow\; \mathrm{N}(P,x,a+1) = \mathrm{N}(P,x,a) = \mathrm{N}(P,x,a-1).$$

Additionally, we have Stanley's inequality:

$$\mathrm{N}(P,x,a)^2 \,\geq\, \mathrm{N}(P,x,a+1)\cdot\mathrm{N}(P,x,a-1).$$

Proof. For the first part, the $\Leftarrow$ direction is trivial. For the $\Rightarrow$ direction, combining the inequality (10.1) (with $z \leftarrow x$ and $i \leftarrow 1$) and the assumption $\mathrm{N}(P,x,a)^2 = \mathrm{N}(P,x,a+1)\cdot \mathrm{N}(P,x,a-1)$ gives

$$\mathrm{N}(P,x,a+1)\cdot\mathrm{N}(P,x,a-1) \,\geq\, \mathrm{N}(P,x,a)\,\cdot\,\mathrm{N}(P,x,a+1).$$

With the assumption $\mathrm{N}(P,x,a) > 0$ (and thus $\mathrm{N}(P,x,a+1) > 0$), this gives $\mathrm{N}(P,x,a-1) \geq \mathrm{N}(P,x,a)$. The same argument for the dual poset P^* gives $\mathrm{N}(P,x,a+1) \geq \mathrm{N}(P,x,a)$. This implies the first part.

For the second part, note that (10.1) (with $z \leftarrow x$ and $i \leftarrow 1$) can rewritten as:

$$(a-1)\big(\mathrm{N}(P,x,a)^2 - \mathrm{N}(P,x,a-1)\,\mathrm{N}(P,x,a+1)\big) \,\geq\, \mathrm{N}(P,x,a)\big(\mathrm{N}(P,x,a+1) - \mathrm{N}(P,x,a)\big).$$

Suppose $\mathrm{N}(P,x,a) < \mathrm{N}(P,x,a+1)$. Then the RHS of the above inequality is ≥ 0, and the LHS implies Stanley's inequality. Similarly, for $\mathrm{N}(P,x,a) < \mathrm{N}(P,x,a-1)$, the same argument for the dual poset P^* also implies Stanley's inequality. Therefore, we have $\mathrm{N}(P,x,a) \geq \mathrm{N}(P,x,a+1)$ and $\mathrm{N}(P,x,a) \geq \mathrm{N}(P,x,a-1)$, which immediately implies the result. □

15.2. **Proof of Lemma 10.15.** We now present the definitions as stated in [MS22], and show that they are in fact equivalent to our Definition 10.15. What follows is a technical argument which ordinarily would not fit a general survey. We include it here in order to show that the subcritical/critical/supercritical poset characterization is in P.

Assume that (10.7) holds, and that $c_{\ell-1} < a < c_\ell$. It follows from $\mathrm{N}_{\boldsymbol{zc}}(P,x,a-1) > 0$ and $\mathrm{N}_{\boldsymbol{zc}}(P,x,a+1) > 0$, that $c_{\ell-1} + 1 < a < c_\ell - 1$. Let us slightly simplify the notation:

$$\begin{aligned}(y_0, y_1, \ldots, y_k, y_{k+1}, y_{k+2}) \;&\longleftarrow\; \big(z_0 = \widehat{0}, z_1, \ldots, z_{\ell-1}, x, z_\ell, \ldots, z_k, z_{k+1} = \widehat{1}\big),\\ (d_0, d_1, \ldots, d_{k+1}, d_{k+2}) \;&\longleftarrow\; (0, c_1, \ldots, c_{\ell-1}, a, c_\ell, \ldots, c_k, n+1).\end{aligned}$$

For $0 \leq r < s \leq k+1$, let $\lambda(r,s) := \gamma(y_r, y_s)$. Denote by R_p a collection of disjoint nonempty intervals $[r_1, r_2], \ldots, [r_{2p-1}, r_{2p}]$, such that $[r_1, r_2] \neq [0, k+2]$.

Definition 15.2 (cf. [MS22, Def. 2.11])**.** We say that a quintuple $(P, x, a, \mathbf{z}, \mathbf{c})$ is[6]

- *subcritical* if for every $p \geq 1$ and every R_p as above, we have:

(subcrit-MS) $$\lambda(r_1, r_2) \,+\, \ldots \,+\, \lambda(r_{2p-1}, r_{2p}) \,\leq\, \delta \,+\, \Delta\,,$$

where $\delta := 2 - \xi - \zeta$,

$$\Delta \,:=\, (d_{r_2} - d_{r_1} - 1) \,+\, \ldots \,+\, (d_{r_{2p}} - d_{r_{2p-1}} - 1),$$

[6]This definition is more combinatorial in flavor that the one in [MS22, Def 2.11], which has a more geometric flavor. It is easy to show that these definitions are equivalent, we omit the details.

and $a := a(r_1, \ldots, r_{2p})$, $b := b(r_1, \ldots, r_{2p})$ are given by

$$\xi := \begin{cases} 1 & \text{if } \ [\ell-1,\ell] \subseteq [r_{2i-1}, r_{2i}] \ \text{ for some } \ i \in [p], \\ 0 & \text{otherwise,} \end{cases} \tag{15.4}$$
$$\zeta := \begin{cases} 1 & \text{if } \ [\ell,\ell+1] \subseteq [r_{2i-1}, r_{2i}] \ \text{ for some } \ i \in [p], \\ 0 & \text{otherwise.} \end{cases}$$

- *critical* if for every $p \geq 1$ and every R_p as above, we have:

$$\lambda(r_1, r_2) + \ldots + \lambda(r_{2p-1}, r_{2p}) \leq \delta - 1 + \Delta. \tag{crit-MS}$$

- *supercritical* if for every $p \geq 1$ and every R_p as above, we have:

$$\lambda(r_1, r_2) + \ldots + \lambda(r_{2p-1}, r_{2p}) \leq \delta - 2 + \Delta. \tag{supercrit-MS}$$

We now proceed to the proof which is separated into parts:

(subcrit) $\Leftrightarrow$ (subcrit-MS). By the argument in (10.8), the inequality (subcrit) always holds. It suffices to show the same for (subcrit-MS). By (15.4), we have $\xi, \zeta \leq 1$, so $2 - \xi - \zeta \geq 0$. Since $\mathrm{N}_{\mathbf{z}\,\mathbf{c}}(P, x, a) \geq 1$, for every $f \in \mathrm{N}_{\mathbf{z}\,\mathbf{c}}(P, x, a)$ and $i \in [p]$, we have:

$$\lambda(r_{2i-1}, r_{2i}) \leq \left|\{f^{-1}(i) : d_{r_{2i-1}} < i < d_{r_{2i}}\}\right| = d_{r_{2i}} - d_{r_{2i-1}} - 1. \tag{15.5}$$

Summing these inequalities proves the claim. □

(crit) $\Leftarrow$ (crit-MS). Let $(r, s) \in \Lambda$ be a splitting pair such that $c_r < a < c_s$. Let $r_1 := r$ and $r_2 := s + 1$. Note that $(z_r, z_s) = (y_{r_1}, y_{r_2})$, and it follows that $\xi = \zeta = 1$. We have:

$$\gamma(z_r, z_s) = \lambda(r_1, r_2) \leq_{\text{(crit-MS)}} (1 - \xi - \zeta) + (d_{r_2} - d_{r_1} - 1) = c_s - c_r - 2,$$

as desired. □

(crit) $\Rightarrow$ (crit-MS). Let R_p be as above. If $\xi + \zeta \leq 1$, then (crit-MS) follows from the fact that $\lambda(r_{2i-1}, r_{2i}) \leq d_{r_{2i}} - d_{r_{2i-1}} - 1$ by (15.5). Thus, we can assume that $\xi + \zeta = 2$. This implies that there exists $j \in [p]$ such that $[\ell-1, \ell+1] \subseteq [r_{2j-1}, r_{2j}]$, which in turn implies that

$$c_{r_{2j-1}} = d_{r_{2j-1}} \leq d_{\ell-1} < d_\ell = a < d_{\ell+1} \leq d_{r_{2j}} = c_{r_{2j}-1}.$$

It then follows that

$$\lambda(r_{2j-1}, r_{2j}) = \gamma\big(z_{r_{2j-1}}, z_{r_{2j}-1}\big) \leq_{\text{(crit)}} c_{r_{2j}-1} - c_{r_{2j-1}} - 2 = d_{r_{2j}} - d_{r_{2j-1}} - 2.$$

On the other hand, for every $i \in [p]$, $i \neq j$, we have:

$$\lambda(r_{2i-1}, r_{2i}) \leq d_{r_{2i}} - d_{r_{2i-1}} - 1.$$

Combining these two inequalities with the assumption that $\xi + \zeta = 2$, we get (crit-MS). □

(supercrit) $\Leftarrow$ (supercrit-MS). Let $(r, s) \in \Lambda$, so that $c_r < a < c_s$. Let $r_1 := r$ and $r_2 := s + 1$. Note that $(z_r, z_s) = (y_{r_1}, y_{r_2})$, and it follows that $\xi = \zeta = 1$. It then follows that

$$\gamma(z_r, z_s) = \lambda(r_1, r_2) \leq_{\text{(supercrit-MS)}} (-\xi - \zeta) + (d_{r_2} - d_{r_1} - 1) = c_s - c_r - 3,$$

as desired. □

(supercrit) $\Rightarrow$ (supercrit-MS). Let R_p be as above. We split the proof into three cases, depending on the values of $\xi + \zeta$. First, for $\xi + \zeta = 0$, the inequality (supercrit-MS) follows from $\lambda(r_{2i-1}, r_{2i}) \leq d_{r_{2i}} - d_{r_{2i-1}} - 1$ by (15.5).

Second, for $\xi+\zeta=1$, there exists $j\in[p]$ such that $\ell=r_{2j-1}$ or $\ell=r_{2j}$. Without loss of generality, we can assume that $\ell=r_{2j-1}$. For every $f\in \mathrm{N}_{\mathbf{z}\,\mathbf{c}}(P,x,a-1)$, we have:

$$\begin{aligned}\lambda(r_{2j-1},r_{2j}) \ &= \ \gamma\big(y_{r_{2j-1}},x\big) \ \leq \ \big|\{f^{-1}(i)\,:\, d_{r_{2j-1}} \,<\, i \,<\, a-1\}\big| \\ &= \ a-1-d_{r_{2j-1}}-1 \ = \ d_{r_{2j}}-d_{r_{2j-1}}-2.\end{aligned}$$

On the other hand, for every $i\in[p]$, $i\neq j$, we have:

$$\lambda(r_{2i-1},r_{2i}) \ \leq \ d_{r_{2i}}-d_{r_{2i-1}}-1.$$

Combining these two inequalities with the assumption that $\xi+\zeta=1$, we get (supercrit-MS), as desired.

Third, for $\xi+\zeta=2$, there exists $j\in[p]$, such that $[\ell-1,\ell+1]\subseteq[r_{2j-1},r_{2j}]$. This implies that

$$c_{r_{2j-1}} \ = \ d_{r_{2j-1}} \ \leq \ d_{\ell-1} \ < \ d_\ell=a \ < \ d_{\ell+1} \ \leq \ d_{r_{2j}} \ = \ c_{r_{2j}-1}\,.$$

It then follows that

$$\lambda(r_{2j-1},r_{2j}) \ = \ \gamma\big(z_{r_{2j-1}},z_{r_{2j}-1}\big) \ \leq_{(\text{supercrit})} \ c_{r_{2j}-1}-c_{r_{2j-1}}-3 \ = \ d_{r_{2j}}-d_{r_{2j-1}}-3.$$

On the other hand, for every $i\in[p]$, $i\neq j$, we have:

$$\lambda(r_{2i-1},r_{2i}) \ \leq \ d_{r_{2i}}-d_{r_{2i-1}}-1.$$

Combining the two inequalities above with the assumption that $a+b=0$, we get (supercrit-MS), as desired. $\square$

16. Final remarks and open problems

16.1. **The nature of linear extensions.** How is the number of linear extensions different from all other combinatorial counting functions? This question is worth addressing since we devoted so much space to the subject. It is also a very difficult question which has more than one answer.

First, we clarify that counting linear extensions is as much a distinct area of Poset Theory as counting colorings, spanning trees, or counting perfect matchings are distinct areas of Graph Theory. These three areas are covered by a large number of surveys and monographs, see e.g. [LP86, MR02, Moon70] for our favorites. By contrast, this is probably the first survey dedicated to linear extensions of general posets. Of course, special cases such as standard Young tableaux and increasing trees are extensively covered in the literature, see e.g. [AR15, KPP94, Sta99].

Second, from the complexity point of view, linear extensions are in the middle of the spectrum: counting is #P-complete, while the existence is trivially in P. Of course, for colorings these problems are #P-complete and NP-complete, respectively. For spanning trees, these problems are in FP and P, respectively. For perfect matchings, the counting is #P-complete and while the existence is in P, for a nontrivial reason.

Third, the $(1\pm\varepsilon)$ approximation counting is NP-hard for the number of 3-colorings and sufficiently small $\varepsilon>0$ [Vaz01], while for the number of perfect matchings there is a Markov chain based algorithm with a difficult analysis [JSV04]. This puts linear extensions in the middle of the spectrum again — it is a hard counting problem with an easy Markov chains algorithm, see the discussion in §12.2.

It is not surprising that for counting functions which are harder to compute it is easier to prove that they are hard to compute, and it is harder if not impossible to obtain good bounds. On the other hand, even when counting is easy like for the spanning trees, perfect matchings in planar graphs, or standard Young tableaux of skew shapes, the inequalities can be quite interesting, see e.g. §11.3 and [Gor21, Gri76].

In summary, there is no real pattern among these counting problems. Each of them is its own mini-universe with its unique advances and challenges. Unfortunately, counting linear extensions is the least explored of these counting problems. Hopefully, this survey will pave a way for further studies in the area.

16.2. **Early history of linear extensions.** The area of linear extensions of posets was started in 1930, with Szpilrajn's *extension theorem* [Szp30], which is nontrivial for infinite posets and gives $e(P) \geq 1$ for finite posets. Birkhoff's *fundamental theorem for finite distributive lattices* [Bir33], states that every finite distributive lattice $\mathcal{L}$ is isomorphic to a lattice $J(P)$ of lower order ideals of a finite poset P, see e.g. [Sta99, §3.4]. Stanley noticed that this implies that the number of maximal chains in $\mathcal{L}$ is $e(P)$, see [Sta72, Prop. 4.1]. For example, the number of maximal chains in the Boolean algebra B_n is $e(A_n) = n!$, cf. Example 3.14.

In Enumerative Combinatorics, enumeration of linear extensions arose naturally in the context of standard Young tableaux in MacMahon's classical work [Mac15], which led to the hook-length formula (11.1) and its generalizations, see §11.3. Building in part on Knuth's paper [Knu70], Stanley's thesis on *P-partition theory* [Sta72] generalized this work both relating and unifying it with the study of *symmetric functions*. Soon after, Schützenberger's work [Schü72] on *promotion operators* led to development of *RSK* and other combinatorial tools used to this day. For further references, see Stanley's historical notes in [Sta99, pp. 383–386].

In Computer Science, linear extensions of posets arose naturally in connection with *sorting problems*. Notably, Kislitsyn [Kis68] which stated the $\frac{1}{3} - \frac{2}{3}$ Conjecture 13.1 and the unimodality of $\{\mathrm{N}(P, x, a)\}$, which was later rediscovered by Rivest and eventually proved by Stanley [Sta81]. Soon after, Fredman [Fre75] and Schönhage [Schö76] introduced (different, but related) sorting problem in the West, and gave the information theoretic lower bounds.

16.3. **What's next?** This survey may seem extensive, but to resolve long open conjectures we would need new tools and ideas beyond those in Section 14. The latest entrants — the combinatorial atlas and the geometric approach to Alexandrov–Fenchel inequalities, led to a great deal of progress and there is hope for further advances.

Let us single out a few open problems mentioned in the survey. First, we personally find the Kahn–Saks conjecture (13.4) more important than the $\frac{1}{3} - \frac{2}{3}$ Conjecture 13.1, although both remain out of reach with existing technology. Next, we believe that the Cross Product Conjecture 9.15 is false already for width three posets. We tried to disprove it by finding a counterexample to Conjecture 10.4 (which follows from CPC), but the extensive computer experiments have yet to succeed.[7]

In a different direction, it would be interesting to make even a small improvement towards Conjecture 9.18. In fact, any constant $(2 - \epsilon)$ in the RHS of (9.18) would already be a major progress. Finally, Conjectures 8.8 and 9.2 remain a major challenge both in Combinatorics and Computational Complexity.

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