

KIRSZBRAUN-TYPE THEOREMS FOR GRAPHS

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ABSTRACT. The classical *Kirszbraun theorem* says that all 1-Lipschitz functions $f : A \rightarrow \mathbb{R}^n$, $A \subset \mathbb{R}^n$, with the Euclidean metric have a 1-Lipschitz extension to \mathbb{R}^n . For metric spaces X, Y we say that Y is *X-Kirszbraun* if all 1-Lipschitz functions $f : A \rightarrow Y$, $A \subset X$, have a 1-Lipschitz extension to X . We analyze the case when X and Y are graphs with the usual path metric. We prove that \mathbb{Z}^d -Kirszbraun graphs are exactly graphs that satisfies a certain *Helly's property*. We also consider complexity aspects of these properties.

1. INTRODUCTION

Discretizing results in metric geometry is important for many applications, ranging from discrete differential geometry to numerical methods. The discrete results are stronger as they typically imply the continuous results in the limit. Unfortunately, more often than not, straightforward discretizations fall apart; new tools and ideas are needed to even formulate these extensions; see e.g. [BS08, Lin02] and [Pak09, §21–§24].

In this paper we introduce a new notion of *G-Kirszbraun graphs*, where G is vertex-transitive graph. The idea is to discretize the classical *Kirszbraun theorem* in metric geometry [Kir34] (see also [BL00, §1.2]). Our main goal is to explain the variational principle for the height functions of tilings introduced by the third author in [Tas14] and further developed in [PST16, MT16] (see Section 2); we also aim to lay a proper foundation for the future work.

Our second goal is to clarify the connection to the *Helly theorem*, a foundational result in convex and discrete geometry [Hel23] (see also [DGK63, Mat02]). Graphs that satisfy the *Helly's property* has been intensely studied in recent years [BC08], and we establish a connection between two areas. Roughly, we show that \mathbb{Z}^d -Kirszbraun graphs are somewhat rare, and are exactly the graphs that satisfy the Helly's property with certain parameters.

1.1. Main results. Let ℓ_2 denote the usual Euclidean metric on \mathbb{R}^n for all n . Given a metric space X and a subset A , we write $A \subset X$ to mean that the subset A is endowed with the restricted metric from X . The *Kirszbraun theorem* says that for all $A \subset (\mathbb{R}^n, \ell_2)$, and all Lipschitz functions $f : A \rightarrow (\mathbb{R}^n, \ell_2)$, there is an extension to a Lipschitz function on \mathbb{R}^n with the same Lipschitz constant.

Recall now the *Helly theorem*: Suppose a collection of convex sets B_1, B_2, \dots, B_k satisfies the property that every $(n + 1)$ -subcollection has a nonempty intersection, then $\cap B_i \neq \emptyset$. Valentine in [Val45] famously showed how the Helly theorem can be used to obtain the Kirszbraun theorem. The connection between these two theorems is the key motivation behind this paper.

Given metric spaces X and Y , we say that Y is *X-Kirszbraun* if all $A \subset X$, every 1-Lipschitz maps $f : A \rightarrow Y$ has a 1-Lipschitz extension from A to X . In this notation, the Kirszbraun theorem says that (\mathbb{R}^n, ℓ_2) is (\mathbb{R}^n, ℓ_2) -Kirszbraun.

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Let $m \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{\infty\}$, $n > m$. Metric space X is said to have (n, m) -Helly's property if for every collection of closed balls B_1, B_2, \dots, B_n of radius ≥ 1 whenever every m -subcollection has a nonempty intersection, we have $\bigcap_{i=1}^n B_i \neq \emptyset$. Since balls in \mathbb{R}^n with the Euclidean metric are convex, the Helly theorem can be restated to say that (\mathbb{R}^n, ℓ_2) is $(\infty, n + 1)$ -Helly. Note that the metric is important here, i.e. $(\mathbb{R}^n, \ell_\infty)$ is $(\infty, 2)$ -Helly, see e.g. [DGK63, Pak09].

Given a graph H , we endow the set of vertices (also denoted by H) with the path metric. By \mathbb{Z}^d we mean the Cayley graph of the group \mathbb{Z}^d with respect to standard generators. All graphs in this paper are nonempty, connected and simple (no loops and multiple edges). The following is the main result of this paper.

Theorem 1.1 (Main theorem). *Graph H is \mathbb{Z}^d -Kirszbraun if and only if H is $(2d, 2)$ -Helly.*

Let K_n denote the complete graph on n -vertices. Clearly, K_n is G -Kirszbraun for all graphs G , since all maps $f : G \rightarrow K_n$ are 1-Lipschitz. On the other hand, \mathbb{Z}^2 is not \mathbb{Z}^2 -Kirszbraun, see Figure 1. This example can be modified to satisfy a certain extendability property, see §4.4.

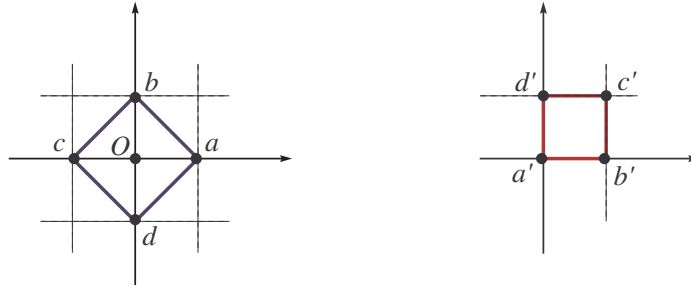


FIGURE 1. Here $A = \{a, b, c, d\} \subset \mathbb{Z}^2$. Define $f : a \rightarrow a', b \rightarrow b', c \rightarrow c', d \rightarrow d'$. Then $f : A \rightarrow \mathbb{Z}^2$ is 1-Lipschitz but not extendable to $\{O, a, b, c, d\}$.

1.2. Structure of the paper. We begin with a short non-technical Section 2 describing some background results and ideas. In essence, it is a remark which is too long to be in the introduction. It reflects the authors' different points of view on the subject, which includes ergodic theory, geometric combinatorics and discrete probability. While in principle this section it can be skipped, we do recommend reading it as it motivates other parts of the paper.

We then proceed to prove Theorem 1.1 in Section 3. In Section 4, we present several extensions and applications of the main theorem. These section is a mixed bag: we include a continuous analogue of the main theorem (Theorem 4.1), the extension to larger integral Lipschitz constants (Theorem 4.3), and the bipartite extension useful for domino tilings (Theorem 4.6). In a short Section 5, we discuss computational aspects of the \mathbb{Z}^d -Kirszbraun property, motivated entirely by applications to tilings. We conclude with final remarks and open problems in Section 6.

2. MOTIVATION AND BACKGROUND

A *graph homomorphism* from a graph G to a graph H is an adjacency preserving map between the respective vertices. Let $\text{Hom}(G, H)$ denote the set of all graph homomorphisms from G to H . We refer to [HN04] for background on graph homomorphisms and connections to coloring and complexity problems.

Our motivation comes from two very distinct sources:

- (1) Finding ‘fast’ algorithms to determine whether a given graph homomorphism on the boundary of a box in \mathbb{Z}^d to H extends to the entire box.
- (2) Finding a natural parametrization of the so-called ergodic Gibbs measures on space of graph homomorphisms $\text{Hom}(\mathbb{Z}^d, H)$ (see [She05, MT16]).

For (1), roughly, suppose we are given a certain simple set of tiles T , such as dominoes or more generally bars $\{k \times 1, 1 \times \ell\}$. It turns out, that T -tileability of a simply-connected region Γ corresponds to existence of a graph homomorphism with given boundary conditions on $\partial\Gamma$. We refer to [Pak03, Thu90] for the background, and to [PST16, Tas14] for further details. Our Theorem 5.2 is motivated by these problems.

For both these problems, the \mathbb{Z}^d -Kirszbraun property of the graph H (or a related graph) is critical and motivates this line of research; the space of 1-Lipschitz maps is the same as the space of graph homomorphisms if and only if H is *reflexive*, that is, every vertex has a self-loop. The study of Kirszbraun-type theorems among metric spaces and its relationship to Helly-like properties is an old one and goes back to the original paper by Kirszbraun [Kir34]. A short and readable proof is given in [Fed69, p. 201]. This was later rediscovered in [Val45] where it was generalized to the cases where the domain and the range are spheres in the Euclidean space or Hilbert spaces. The effort of understanding which metric spaces satisfy Kirszbraun properties culminated in the theorem by Lang and Schroeder [LS97] that identified the right curvature assumptions on the underlying spaces for which the theorem holds.

In the metric graph theory, the research has focused largely on a certain universality property. Formally, a graph is called *Helly* if it is $(\infty, 2)$ -Helly. An easy deduction, for instance following the discussion in [DGK63, Page 153], shows that H is Helly if and only if for all graphs G , H is G -Kirszbraun. Some nice characterizations of Helly graphs can be found in the survey [BC08, §3]. However, we are not aware of any other study of G -Kirszbraun graphs for fixed G .

3. PROOF OF THE MAIN THEOREM

3.1. Geodesic extensions. Let d_H denote the path metric on the graph H . A *walk* γ in the graph H of *length* k , is a sequence of $k + 1$ vertices (v_0, v_1, \dots, v_k) , s.t. $d_H(v_i, v_{i+1}) \leq 1$, for all $0 \leq i \leq k - 1$. We say that γ starts at v_0 and ends at v_k . A *geodesic* from vertex v to w in a graph G is a walk γ from v to w of the shortest length.

Consider a graph G , a subset $A \subset G$ and $b \in G \setminus A$. Define the *geodesic extension of A with respect to b* as the following set:

$$\text{Ext}(A, b) := \{a \in A : \text{there does not exist } a' \in A \setminus \{a\} \text{ s.t. there is a geodesic } \gamma \text{ from } a \text{ to } b \text{ which passes through } a'\}.$$

For example, let $A \subset \mathbb{Z}^2 \setminus \{(0, 0)\}$. If $(i, j), (k, l) \in \text{Ext}(A, \vec{0})$ are elements of the same quadrant, then $|i| > |k|$ if and only if $|j| < |l|$.

Remark 3.1. If $A \subset \mathbb{Z}^d$ be contained in the coordinate axes then $|\text{Ext}(A, \vec{0})| \leq 2d$.

The notion of geodesic extension allows us to prove that certain 1-Lipschitz maps can be extended:

Proposition 3.2. *Let $A \subset G$, map $f : A \rightarrow H$ be 1-Lipschitz, and let $b \in G \setminus A$. The map f has a 1-Lipschitz extension to $A \cup \{b\}$ if and only if $f|_{\text{Ext}(A, b)}$ has a 1-Lipschitz extension to $\text{Ext}(A, b) \cup \{b\}$.*

Proof. The forward direction of the proof is immediate because $\text{Ext}(A, b) \subset A$. For the backwards direction let $\tilde{f} : \text{Ext}(A, b) \cup \{b\} \subset G \rightarrow H$ be a 1-Lipschitz extension of $f|_{\text{Ext}(A, b)}$ and consider the

map $\hat{f} : A \cup \{b\} \subset G \rightarrow H$ given by

$$\hat{f}(a) := \begin{cases} f(a) & \text{if } a \in A \\ \tilde{f}(b) & \text{if } a = b. \end{cases}$$

To prove that \hat{f} is 1-Lipschitz we need to verify that for all $a \in A$, $d_H(\hat{f}(a), \hat{f}(b)) \leq d_G(a, b)$. From the hypothesis it follows for $a \in \text{Ext}(A, b)$. Now suppose $a \in A \setminus \text{Ext}(A, b)$. Then there exists $a' \in \text{Ext}(A, b)$ such that there exists a geodesic from a to b passing through a' . This implies that $d_G(a, b) = d_G(a, a') + d_G(a', b)$. But

$$d_G(a, a') \geq d_H(\hat{f}(a), \hat{f}(a')) = d_H(f(a), f(a')) \text{ because } f \text{ is 1-Lipschitz}$$

$$d_G(a', b) \geq d_H(\hat{f}(a'), \hat{f}(b)) = d_H(\tilde{f}(a'), \tilde{f}(b)) \text{ because } \tilde{f} \text{ is 1-Lipschitz.}$$

By the triangle inequality, the proof is complete. \square

3.2. Helly's property. Given a graph H , a vertex $v \in H$ and $n \in \mathbb{N}$ denote by $B_n^H(v)$, the ball of radius n in H centered at v . We will now interpret the $(n, 2)$ -Helly's property in a different light.

Proposition 3.3. *Let H be a graph satisfying the $(n, 2)$ -Helly's property. For all 1-Lipschitz maps $f : A \subset G \rightarrow H$ and $b \in G \setminus A$ such that $|\text{Ext}(A, b)| \leq n$, there exists a 1-Lipschitz extension of f to $A \cup \{b\}$.*

Proof. Consider the extension \tilde{f} of f to the set $A \cup \{b\}$, where $\tilde{f}(b)$ is any vertex in

$$\bigcap_{b' \in \text{Ext}(A, b)} B_{d_G(b, b')}^H(f(b'));$$

the intersection is nonempty because $|\text{Ext}(A, b)| \leq n$ and for all $a, a' \in \text{Ext}(A, b)$, we have:

$$d_H(f(a), f(a')) \leq d_G(a, a') \leq d_G(a, b) + d_G(b, a')$$

which implies

$$B_{d_G(a, b)}^H(f(a)) \cap B_{d_G(b, a')}^H(f(a')) \neq \emptyset.$$

The function $\tilde{f}|_{\text{Ext}(A, b) \cup \{b\}}$ is 1-Lipschitz, so Proposition 3.2 completes the proof. \square

3.3. Examples. Let C_n and P_n denote the *cycle graph* and the *path graph* with n vertices, respectively.

Corollary 3.4. *All connected graphs are P_n -, C_n - and \mathbb{Z} -Kirszbraun.*

In the case when $G = P_n, C_n$ or \mathbb{Z} we have for all $A \subset G$ and $b \in G \setminus A$, $|\text{Ext}(A, b)| \leq 2$; the corollary follows from Proposition 3.3 and the fact that all graphs are $(2, 2)$ -Helly.

Let $\mathbf{r} = (r_1, r_2, \dots, r_n) \in \mathbb{N}^n$. Denote by $T_{\mathbf{r}}$ the *star-shaped tree* with a central vertex b_0 and n disjoint walks of lengths r_1, \dots, r_n emanating from it and ending in vertices b_1, b_2, \dots, b_n .

Corollary 3.5. *Graph H has the $(n, 2)$ -Helly's property if and only if H is $T_{\mathbf{r}}$ -Kirszbraun, for all $\mathbf{r} \in \mathbb{N}^n$.*

Proof. For all $\mathbf{r} \in \mathbb{N}^n$, $A \subset T_{\mathbf{r}}$ and $b \in T_{\mathbf{r}} \setminus \{A\}$, we have $|\text{Ext}(A, b)| \leq n$. Thus by Proposition 3.3 if H has the $(n, 2)$ -Helly's property then H is $T_{\mathbf{r}}$ -Kirszbraun. For the other direction, let H be $T_{\mathbf{r}}$ -Kirszbraun for all $\mathbf{r} \in \mathbb{N}^n$. Suppose that

$$B_{r_1}^H(v_1), B_{r_2}^H(v_2), \dots, B_{r_n}^H(v_n)$$

are balls in H such that $B_{r_i}^H(v_i) \cap B_{r_j}^H(v_j) \neq \emptyset$ for all $1 \leq i, j \leq n$. Then $f : \{b_1, b_2, \dots, b_n\} \subset T_{\mathbf{r}} \rightarrow H$ given by $f(b_i) := v_i$ is 1-Lipschitz with a 1-Lipschitz extension $\tilde{f} : T_{\mathbf{r}} \rightarrow H$. It follows that $\tilde{f}(b_0) \in \bigcap_{i=1}^n B_{r_i}^H(v_i)$ proving that H has the $(n, 2)$ -Helly's property. \square

3.4. Proof of Theorem 1.1. We will first prove the “only if” direction. Let H be a graph which is \mathbb{Z}^d -Kirszbraun. For all $\mathbf{r} \in \mathbb{N}^{2d}$ there is an isometry from $T_{\mathbf{r}}$ to \mathbb{Z}^d mapping the walks emanating from the central vertex to the coordinate axes. Hence H is $T_{\mathbf{r}}$ -Kirszbraun for all $\mathbf{r} \in \mathbb{N}^{2d}$. By Corollary 3.5, we have proved the $(2d, 2)$ -Helly’s property for H .

In the “if” direction, suppose H has the $(2d, 2)$ -Helly’s property. We need to prove that for all $A \subset \mathbb{Z}^d$, every 1-Lipschitz maps $f : A \rightarrow H$ has a 1-Lipschitz extension. It is sufficient to prove this for finite subsets A . We proceed by induction on $|A|$. Namely, we prove the following property $\text{St}(n)$:

Let $f : A \subset \mathbb{Z}^d \rightarrow H$ be 1-Lipschitz with $|A| = n$. Let $b \in \mathbb{Z}^d \setminus A$. Then the function f has a 1-Lipschitz extension to $A \cup \{b\}$.

We know $\text{St}(n)$ for $n \leq 2d$ by the $(2d, 2)$ -Helly’s property. Let us assume $\text{St}(n)$ for some $n \geq 2d$; we want to prove $\text{St}(n+1)$. Let $f : A \rightarrow H$, $A \subset \mathbb{Z}^d$, be 1-Lipschitz with $|A| = n+1$ and $b \in \mathbb{Z}^d \setminus A$. Without loss of generality assume that $b = \vec{0}$. Also assume that $\text{Ext}(A, \vec{0}) = A$; otherwise we can use the induction hypothesis and Proposition 3.2 to obtain the required extension to $A \cup \{\vec{0}\}$.

We will prove that there exists a set $\tilde{A} \subset \mathbb{Z}^d$ and a 1-Lipschitz function $\tilde{f} : \tilde{A} \rightarrow H$, such that

- (1) If \tilde{f} has an extension to $\tilde{A} \cup \{\vec{0}\}$ then f has an extension to $A \cup \{\vec{0}\}$.
- (2) Either the set \tilde{A} is contained in the coordinate axes of \mathbb{Z}^d or $|\tilde{A}| \leq 2d$.

By Remark 3.1, if A is contained in the coordinate axis then $|\text{Ext}(A, \vec{0})| \leq 2d$. Since H has the $(2d, 2)$ -Helly’s property, by Proposition 3.3 it follows that \tilde{f} has an extension to $\tilde{A} \cup \{\vec{0}\}$ which completes the proof.

Since $|A| \geq n+1 > 2d$, there exists $\vec{i}, \vec{j} \in A$ and a coordinate $1 \leq k \leq d$ such that i_k, j_k are non-zero and have the same sign. Suppose $i_k \leq j_k$. Then there is a geodesic from \vec{j} to $\vec{i} - i_k \vec{e}_k$ which passes through \vec{i} . Since $A = \text{Ext}(A, \vec{0})$ we have that

$$\vec{i} - i_k \vec{e}_k \notin \{\vec{0}\} \cup A.$$

Thus $\vec{j} \notin \text{Ext}(A, \vec{i} - i_k \vec{e}_k)$ and hence $|\text{Ext}(A, \vec{i} - i_k \vec{e}_k)| \leq n$. By $\text{St}(n)$ there exists a 1-Lipschitz extension of $f|_{\text{Ext}(A, \vec{i} - i_k \vec{e}_k)}$ to $\text{Ext}(A, \vec{i} - i_k \vec{e}_k) \cup \{\vec{i} - i_k \vec{e}_k\}$. By Proposition 3.2 there is a 1-Lipschitz extension of f to $f' : A \cup \{\vec{i} - i_k \vec{e}_k\} \rightarrow H$. But there is a geodesic from \vec{i} to $\vec{0}$ which passes through $\vec{i} - i_k \vec{e}_k$. Thus

$$\text{Ext}(A \cup \{\vec{i} - i_k \vec{e}_k\}, \vec{0}) \subset (A \setminus \{\vec{i}\}) \cup \{\vec{i} - i_k \vec{e}_k\}.$$

Set $A' := (A \setminus \{\vec{i}\}) \cup \{\vec{i} - i_k \vec{e}_k\}$. By Proposition 3.2, map f' has a 1-Lipschitz extension to $A \cup \{\vec{i} - i_k \vec{e}_k\} \cup \{\vec{0}\}$ if and only if $f'|_{A'}$ has a 1-Lipschitz extension to $A' \cup \{\vec{0}\}$.

Thus we have obtained a set A' and a 1-Lipschitz map $f' : A' \subset \mathbb{Z}^d \rightarrow H$ for which

- (1) If f' has an extension to $A' \cup \{\vec{0}\}$ then f has an extension to $A \cup \{\vec{0}\}$.
- (2) The sum of the number of non-zero coordinates of elements of A' is less than the sum of the number of non-zero coordinates of elements of A .

By repeating this procedure (formally this is another induction) we get the required set $\tilde{A} \subset \mathbb{Z}^d$ and 1-Lipschitz map $\tilde{f} : \tilde{A} \rightarrow H$. This completes the proof.

4. APPLICATIONS OF THE MAIN THEOREM

4.1. Back to continuous setting. The techniques involved in the proof of Theorem 1.1 extend to the continuous case with only minor modifications. The following result might be of interest in metric geometry.

A metric space (X, \mathbf{m}) is *geodesically complete* if for all $x, y \in X$ there exists a continuous function $f : [0, 1] \rightarrow X$, such that

$$\mathbf{m}(x, f(t)) = t\mathbf{m}(x, y) \quad \text{and} \quad \mathbf{m}(f(t), y) = (1 - t)\mathbf{m}(x, y).$$

Theorem 4.1. *Let Y be a metric space such that every closed ball in Y is compact. Then Y is (\mathbb{R}^d, ℓ_1) -Kirszbraun if and only if Y is geodesically complete and $(2d, 2)$ -Helly.*

First, we need the following result.

Lemma 4.2. *Let (X, \mathbf{m}) be separable and the closed balls in (Y, \mathbf{m}') be compact. Then (Y, \mathbf{m}') is (X, \mathbf{m}) -Kirszbraun if and only if all finite sets $A \subset X$ and 1-Lipschitz maps $f : A \rightarrow Y$, $y \in Y \setminus A$ have a 1-Lipschitz extension to $A \cup \{y\}$.*

Proof. The “only if” part is obvious. For the “if” part, let $A \subset X$ and $f : A \rightarrow Y$ be 1-Lipschitz. Since X is separable, A is also separable. Let

$$\{x_i : i \in \mathbb{N}\} \subset X \quad \text{and} \quad \{a_i : i \in \mathbb{N}\} \subset A$$

be countable dense sets. By the hypothesis, we have

$$\bigcap_{1 \leq i \leq n} B_{\mathbf{m}(x_1, a_i)}(f(a_i)) \neq \emptyset.$$

Since closed balls in Y are compact it follows that

$$\bigcap_{i \in \mathbb{N}} B_{\mathbf{m}(x_1, a_i)}(f(a_i)) \neq \emptyset.$$

Thus f has a 1-Lipschitz extension to $A \cup \{x_1\}$. By induction we get that f has a 1-Lipschitz extension $\tilde{f} : A \cup \{x_i : i \in \mathbb{N}\} \rightarrow Y$. Let $g : X \rightarrow Y$ be the map given by

$$g(x) := \lim_{j \rightarrow \infty} \tilde{f}(x_{i_j}) \quad \text{for all } x \in X,$$

where x_{i_j} is some sequence such that $\lim_{j \rightarrow \infty} x_{i_j} = x$. The limit above exists since closed balls in Y are compact, and hence Y is a complete metric space. By the continuity of \tilde{f} it follows that $g|_A = f$ and by the continuity of the distance function it follows that g is 1-Lipschitz. \square

For the proof of Theorem 4.1, note that the main property of \mathbb{Z}^d exploited in proof of Theorem 1.1 is that the graph metric is same as the ℓ_1 metric. From the lemma above, the proof proceeds analogously. We omit the details.

4.2. Lipschitz constants. The following extension deals with other Lipschitz constants. In the continuous case this is trivial; however it is more delicate in the discrete setting. Since we are interested in Lipschitz maps between graphs we restrict our attention to integral Lipschitz constants.

Theorem 4.3. *Let $t \in \mathbb{N}$ and H be a connected graph. Then every t -Lipschitz map $f : A \rightarrow H$, $A \subset \mathbb{Z}^d$, has a t -Lipschitz extension to \mathbb{Z}^d if and only if*

$$\text{for all balls } B_1, B_2, \dots, B_{2d} \text{ of radii multiples of } t \text{ mutually intersect} \implies \bigcap B_i \neq \emptyset.$$

The proof of Theorem 4.3 follow verbatim the proof of Theorem 1.1; we omit the details.

Let H be a \mathbb{Z}^d -Kirszbraun graph. The theorem implies that all t -Lipschitz maps $f : A \rightarrow H$, $A \subset \mathbb{Z}^d$, have a t -Lipschitz extension. On the other hand, it is easy to construct graphs G and H for which H is G -Kirszbraun but there exists a 2-Lipschitz map $f : A \rightarrow H$, $A \subset G$, which does not have a 2-Lipschitz extension. First, we need the following result.

Proposition 4.4. *Let G be a finite graph with diameter n and H be a connected graph such that $B_n^H(v)$ is G -Kirszbraun for all $v \in H$. Then H is a G -Kirszbraun graph.*

Proof. Let $f : A \subset G \rightarrow H$ be 1-Lipschitz and pick $a \in A$. Then $\text{Image}(f) \subset B_n^H(f(a))$. Since $B_n^H(f(a))$ is G -Kirszbraun the result follows. \square

Since trees are Helly graphs, we have as an immediate application of the above that C_n is G -Kirszbraun if $\text{diam}(G) \leq n - 1$. For instance, let $\mathbf{r} = (1, 1, 1, 1, 1, 1) \in \mathbb{N}^6$ and consider the *star* $T_{\mathbf{r}}$. We obtain that C_6 is $T_{\mathbf{r}}$ -Kirszbraun. Now label the leaves of $T_{\mathbf{r}}$ as b_i , $1 \leq i \leq 6$, respectively. For $A = \{b_1, \dots, b_6\} \subset T_{\mathbf{r}}$, consider the map

$$f : A \rightarrow C_6 \quad \text{given by} \quad f(b_i) = i.$$

The function f is 2-Lipschitz but it has no 2-Lipschitz extension to $T_{\mathbf{r}}$.

4.3. Hyperoctahedron graphs. The *hyperoctahedron graph* O_d is the graph obtained by removing a perfect matching from the complete graph K_{2d} . Theorem 1.1 combined with the following proposition implies that O_d are \mathbb{Z}^{d-1} -Kirszbraun but not \mathbb{Z}^d -Kirszbraun. When $d = 2$ this is the example in the introduction (see Figure 1).

Proposition 4.5. *Graph O_d is $(2d - 1, 2)$ -Helly but not $(2d, 2)$ -Helly.*

Proof. Let $B_1, B_2, \dots, B_{2d-1}$ be balls of radius ≥ 1 . Then, for all $1 \leq i \leq 2d - 1$, $B_i \supset O_d \setminus \{j_i\}$ for some $j_i \in O_d$. Thus:

$$\bigcap B_i \supseteq O_d \setminus \{j_i : 1 \leq i \leq 2d - 1\} \neq \emptyset.$$

This implies that O_d is $(2d - 1, 2)$ -Helly.

In the opposite direction, let B_1, B_1, \dots, B_{2d} be distinct balls of radius one in O_d . It is easy to see that they intersect pairwise, but $\bigcap B_i = \emptyset$. Thus, graph O_d is not $(2d, 2)$ -Helly, and Theorem 1.1 proves the claim. \square

Let us mention that the hyperoctahedron graph $O_d \simeq K_{2, \dots, 2}$ is a well-known obstruction to the Helly's property, see e.g. [BC08].

4.4. Bipartite version. In the study of Helly graphs it is well-known (see e.g. [BC08, §3.2]) that results which are true with regard to 1-Lipschitz extensions usually carry forward to graph homomorphisms in the bipartite case after some small technical modifications. This is also true in our case.

A bipartite graph H is called *bipartite (n, m) -Helly* if for balls $B_1, B_2, B_3, \dots, B_n$ (if $n \neq \infty$ and any finite collection otherwise) and partite class H_1 , we have that any subcollection of size m among $B_1 \cap H_1, B_2 \cap H_1, \dots, B_n \cap H_1$ has a nonempty intersection implies

$$\bigcap_{i=1}^n B_i \cap H_1 \neq \emptyset.$$

Let G, H be bipartite graphs with partite classes G_1, G_2 and H_1, H_2 respectively. The graph H is called *bipartite G -Kirszbraun* if for all 1-Lipschitz maps $f : A \subset G \rightarrow H$ for which $f(A \cap G_1) \subset H_1$ and $f(A \cap G_2) \subset H_2$ there exists $\tilde{f} \in \text{Hom}(G, H)$ extending it.

Theorem 4.6. *Graph H is bipartite \mathbb{Z}^d -Kirszbraun if and only if H is bipartite $(2d, 2)$ -Helly.*

As noted in the introduction, Theorem 1.1 implies that graph \mathbb{Z}^2 is not $(4, 2)$ -Helly. However it is bipartite $(\infty, 2)$ -Helly (see below).

Given a graph H we say that $v \sim_H w$ to mean that (v, w) form an edge in the graph. Let H_1, H_2 be graphs with vertex sets V_1, V_2 respectively. Define:

(1) *Strong product* $H_1 \boxtimes H_2$ as the graph with the vertex set $V_1 \times V_2$, and edges given by

$$\begin{aligned} (v_1, v_2) \sim_{H_1 \boxtimes H_2} (w_1, w_2) \quad & \text{if } v_1 = w_1 \text{ and } v_2 \sim_{H_2} w_2, \\ & \text{or } v_1 \sim_{H_1} w_1 \text{ and } v_2 = w_2, \\ & \text{or } v_1 \sim_{H_1} w_1 \text{ and } v_2 \sim_{H_2} w_2. \end{aligned}$$

(2) *Tensor product* $H_1 \times H_2$ as the graph with the vertex set $V_1 \times V_2$, and edges given by

$$(v_1, v_2) \sim_{H_1 \times H_2} (w_1, w_2) \quad \text{if } v_1 \sim_{H_1} w_1 \text{ and } v_2 \sim_{H_2} w_2.$$

Proposition 4.7. *If for a graph G , graphs H_1 and H_2 are G -Kirszbraun then $H_1 \boxtimes H_2$ is G -Kirszbraun. If for a bipartite graph G , bipartite graphs H'_1 and H'_2 are bipartite G -Kirszbraun then the connected components of $H'_1 \times H'_2$ are bipartite G -Kirszbraun.*

Proof. We will prove this in the non-bipartite case; the bipartite case follows similarly. Let $f := (f_1, f_2) : A \subset G \rightarrow H_1 \boxtimes H_2$ be 1-Lipschitz. It follows that the functions f_1 and f_2 are 1-Lipschitz as well; hence they have 1-Lipschitz extensions $\tilde{f}_1 : G \rightarrow H_1$ and $\tilde{f}_2 : G \rightarrow H_2$. Thus $(\tilde{f}_1, \tilde{f}_2) : G \rightarrow H_1 \boxtimes H_2$ is 1-Lipschitz and extends f . \square

Corollary 4.8 (cf. [BC08, §3.2]). *Graph \mathbb{Z}^2 is bipartite $(\infty, 2)$ -Helly.*

Proof. As we mentioned above, it is easy to see that all trees are Helly graphs. By Proposition 4.7, so are the connected components of $\mathbb{Z} \times \mathbb{Z}$ which are graph isomorphic to \mathbb{Z}^2 . Now Theorem 4.6 implies the result. \square

5. COMPLEXITY ASPECTS

5.1. The recognition problem. Below we give a polynomial time algorithm to decide whether a given graph is \mathbb{Z}^d -Kirszbraun. We assume that the graph is presented by its adjacency matrix.

Proposition 5.1. *For all fixed $n, m \in \mathbb{N}$, the recognition problem of (n, m) -Helly graphs and bipartite (n, m) -Helly graphs can be decided in $\text{poly}(|H|)$ time.*

For $n = \infty$ and $m = 2$, the recognition problem was solved in [BP89]; that result does not follow from the proposition.

Proof. Let us seek the algorithm in the case of (n, m) -Helly graphs; as always, the bipartite case is similar. In the following, for a function $g : \mathbb{R} \rightarrow \mathbb{R}$ by $t = O(g(|H|))$ we mean $t \leq kg(|H|)$, where k is independent of $|H|$ but might depend on m, n .

- (1) Determine the distance between the vertices of the graph. This takes $O(|H|^3)$ time.
- (2) Now make a list of all the collections of balls; each collection being of cardinality n . Since the diameter of the graph H is bounded by $|H|$; listing the centers and the radii of the balls takes time $O(|H|^{2n})$.
- (3) Find the collections for which all the subcollections of cardinality m intersect. For each collection, this step takes $O(|H|)$ time.
- (4) Check if the intersection of the balls in the collections found in the previous step is nonempty. This step again takes $O(|H|)$ time.

Thus, the total time is $O(|H|^{2n+3})$, as desired. \square

5.2. The hole filling problem. The following application is motivated by the tileability problems, see Section 2.

Fix $d \geq 2$. By a *box* B_n in \mathbb{Z}^d we mean a subgraph $\{0, 1, \dots, n\}^d$. By the *boundary* ∂_n we mean the internal vertex boundary of B_n , that is, vertices of B_n where at least one of the coordinates is either 0 or n . The *hole-filling problem* asks: Given a graph H and a graph homomorphism $f \in \text{Hom}(\partial_n, H)$, does it extend to a graph homomorphism $\tilde{f} \in \text{Hom}(B_n, H)$?

Theorem 5.2. *Fix $d \geq 1$. Let H be a finite bipartite $(2d, 2)$ -Helly graph, $B_n \subset \mathbb{Z}^d$ a box and $f \in \text{Hom}(\partial_n, H)$ be a graph homomorphism as above. Then the hole-filling problem for f can be decided in $\text{poly}(n + |H|)$ time.*

The same result holds in the context of 1-Lipschitz maps for $(2d, 2)$ -Helly graphs; the algorithm is similar. For general H , without the $(2d, 2)$ -Helly assumption, the problem is a variation on existence of graph homomorphism, see [HN04, Ch. 5]. The latter is famously NP-complete in almost all nontrivial cases, which makes the theorem above even more surprising.

Proof. In the following, for a function $g : \mathbb{R}^2 \rightarrow \mathbb{R}$, by $t = O(g(|H|, n))$ we mean $t \leq kg(|H|, n)$ where k is independent of $|H|$ and n . Let $f \in \text{Hom}(\partial_n, H)$ be given. Since H is bipartite $(2d, 2)$ -Helly graph, by Theorem 1.1, f extends to B_n if and only if f is 1-Lipschitz. Thus to decide the hole-filling problem we need to determine whether or not f is 1-Lipschitz. This can be decided in polynomial time:

- (1) Determine the distances between all pairs of vertices in H . This costs $O(|H|^3)$.
- (2) For each pair of vertices in the graph ∂_n , determine the distance between the pair and their image under f and verify the Lipschitz condition. This costs $O(n^{2d-2})$.

The total cost is $O(n^{2d-2} + |H|^3)$, which completes the proof. □

For $d = 2$ and $H = \mathbb{Z}$, the above algorithm can be modified to give a $O(n^2)$ time complexity for the hole-filling problem of an $[n \times n]$ box. This algorithm can be improved to the nearly linear time $O(n \log n)$, by using the tools in [PST16]. We omit the details.

6. FINAL REMARKS AND OPEN PROBLEMS

6.1. In the view of our motivation, we focus on the \mathbb{Z}^d -Kirszbraun property throughout the paper. It would be interesting to find characterizations for other bipartite domain graphs such as the hexagonal and the square-octahedral lattice (cf. [Ken04, Thu90]). Similarly, it would be interesting to obtain a sharper time bound on the recognition problem as in Proposition 5.1, to obtain applications similar to [PST16].

Note that a vast majority of tileability problems are computationally hard, which makes the search for tractable sets of tiles even more interesting (see [Pak03]). The results in this paper, especially the bipartite versions, give a guideline for such a search.

6.2. As we mention in Section 2, there are intrinsic curvature properties of the underlying spaces, which allow for the Helly-type theorems [LS97]. In fact, there are more general local hyperbolic properties which can also be used in this setting, see [CE07, CDV17, Lang99]. See also a curious “local-to-global” characterization of Helly graphs in [C+], and a generalization of Helly’s properties to hypergraphs [BD75, DPS14].

6.3. In literature, there are other “discrete Kirszbraun theorems” stated in different contexts. For example, papers [AT08, Bre81] give a PL-version of the result for finite $A, B \subset \mathbb{R}^d$ and 1-Lipschitz $f : A \rightarrow B$. Such results are related to the classical *Margulis napkin problem* and other isometric embedding/immersion problems, see e.g. [Pak09, §38–§40] and references therein.

In a different direction, two more related problems are worth mentioning. First, the *carpenter’s rule problem* can be viewed as a problem of finding a “discrete 1-Lipschitz homotopy”, see [CD04, CDR03]. This is also (less directly) related to the well-known *Kneser–Poulsen conjecture*, see e.g. [Bez10].

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