EFFECTIVE RESISTANCE IN PLANAR GRAPHS AND CONTINUED FRACTIONS

SWEE HONG CHAN, ALEX KONTOROVICH, AND IGOR PAK

Abstract. For a simple graph $G=(V,E)$ and edge $e\in E$, the *effective resistance* is defined as a ratio $\frac{\tau(G/e)}{\tau(G)}$, where $\tau(G)$ denotes the number of spanning trees in G. We resolve the *inverse problem* for the effective resistance for planar graphs. Namely, we determine (up to a constant) the smallest size of a simple planar graph with a given effective resistance. The results are motivated and closely related to our previous work [CKP24] on Sedláček's inverse problem for the number of spanning trees.

1. Introduction

1.1. **Spanning trees and effective resistance.** Let $G=(V,E)$ be a connected graph without loops. For an edge $e\in E$, denote by G/e the graph obtained by contracting the edge e, where all resulting loops are removed.

Let $\tau(G)$ denote the *number of spanning trees* of G. The *effective resistance* is defined as

$$\rho(G,e) \;:=\; \frac{\tau(G/e)}{\tau(G)}\,. \tag{1.1}$$

This notion goes back to Kirchhoff (1847) in the context of *electrical networks*, as it measures the current through the edge $e=(x,y)$, as a fraction of the current between nodes x and y. Effective resistance is one of the main graph invariants, with applications across mathematics and the sciences. Notably, it was proved by Nash-Williams [Nas59], that $\rho(G,e)$ is the probability that a simple random walk which starts at x and exits at y, traverses e at some point. We refer to [DS84, Lov96, LP16] for background, modern proofs, and further references.

Clearly, effective resistance can be computed in poly-time via the *matrix tree theorem*. In this paper, we study the *inverse problem*, of finding the smallest size graph with a given effective resistance. We restrict ourselves to planar graphs.

Theorem 1.1 (Main theorem)**.** *Let $t>c\geq 1$ be coprime integers. Then there exists a simple planar graph $G=(V,E)$ and an edge $e\in E$, such that*

$$\rho(G,e) \;=\; \frac{c}{t} \tag{1.2}$$

and

$$|V| \;\leq\; C\,\max\left\{\frac{t}{c}\,,\,\frac{t}{t-c}\,,\,\log t\right\}, \tag{1.3}$$

for some universal constant $C>0$.

In fact, the upper bound in the theorem is optimal in the following sense:

Proposition 1.2. *Let $G=(V,E)$ be connected simple planar graph, and let $e\in E$ be an edge that is not a bridge. Suppose that*

$$\rho(G,e) \;=\; \frac{c}{t} \tag{1.4}$$

Date: May 15, 2025.

is a reduced fraction. Then:

$$|V| \ \geq \ C' \max\left\{ \frac{t}{c}\,,\, \frac{t}{t-c}\,,\, \log t \right\}, \tag{1.5}$$

for some universal constant $C' > 0$.

Here the non-bridge condition is necessary to ensure that $\rho(G,e) < 1$. Proposition 1.2 follows easily from the results in the literature; we present a short proof in §3.3. See §4.3 for an explicit value of the constant C'.

The idea of inverse problems for combinatorial functions was developed independently in several different areas. Recently, it reemerged in the computational complexity context and systematically studied by the first and third author [CP24a]. This paper is a continuation of our previous study [CKP24] of *Sedláček's problem*, which is the inverse problem for the number of spanning trees in simple graphs (see below). There, we introduced a new approach based on continued fractions, and used the *Bourgain–Kontorovich technology* [BK14] to obtain the result. Here we follow a closely related approach.

Theorem 1.1 extends the earlier result [CP24c, Lemma 1.14], where a weaker bound $|V| = O(\log t\,(\log\log t)^2)$ was obtained for $\frac{c}{t} \in \left[\frac{1}{3}, \frac{2}{3}\right]$. Additionally, the graphs in [CP24c] were allowed to have multiple edges.

1.2. **Sedláček's problem.** Given a positive integer $t \geq 3$, let $\alpha(t)$ be the smallest number vertices of a simple graph with *exactly* t spanning trees. The study of the function $\alpha(t)$ was initiated by Sedláček in a series of papers [Sed66, Sed69, Sed70], and remains unresolved. The best known upper bound is $\alpha(t) = O((\log t)^{3/2}/(\log\log t))$ due to Stong [Sto22], while best known lower bound is $\alpha(t) = \Omega(\log t\,/\log\log t)$, which follows from *Cayley's formula* $\tau(K_n) = n^{n-2}$. Azarija and Škrekovski [AŠ12] conjectured that $\alpha(t) = o(\log t)$.

In [CKP24], the authors consider another, closely related problem by Sedláček, on the function $\alpha_{\mathrm{p}}(t)$ defined as the smallest number vertices of a simple *planar* graph with exactly t spanning trees. Clearly, we have $\alpha(t) \leq \alpha_{\mathrm{p}}(t)$ for all t. It is known and easy to see that $\alpha_{\mathrm{p}}(t) = \Omega(\log t)$, see §3.3. The main result in [CKP24] is that $\alpha_{\mathrm{p}}(t) = O(\log t)$ for a set of integers t of positive density. One can think of Theorem 1.1 as a tradeoff: the density condition is removed but the number of spanning trees is replaced with a ratio of two such numbers.

In a different direction, it follows from Theorem 1.1 that for all integers $t \geq 2$, there exists a connected simple planar graph $G = (V,E)$ such that

$$\tau(G) \equiv 0 \mod t \quad \text{and} \quad |V| = O(\log t). \tag{1.6}$$

This is also a special case of the following modular version of Theorem 1.1.

Theorem 1.3 (Alon–Bucić–Gishboliner [ABG25, Thm 3.1])**.** *Let $t \geq 2$ be a positive integer, and let $a, b \in \mathbb{N}$ be such that $(a,b) \neq (0,0) \mod t$. Then there exists a simple planar graph $G = (V,E)$ and an edge $e \in E$, such that*

$$\tau(G/e) \equiv a \mod t, \quad \tau(G) \equiv b \mod t, \quad \textit{and} \quad |V| = O(\log t).$$

The equation (1.6) follows from Theorem 1.3 by setting $a = 1$ and $b = 0$. The proof in [ABG25] involves the celebrated expander construction of $\mathrm{SL}(2,p)$ based on Selberg's theorem [Sel65]. Curiously, the same Selberg's theorem is the starting point of a long chain of results leading to [Bou12, BK14], which provided main tools for both [CKP24] and for this paper.

1.3. **Connections to continued fractions.** Given the integers $a_0 \geq 0$ and $a_1, \ldots, a_\ell \geq 1$, the corresponding *continued fraction* is defined as

$$[a_0; a_1, \ldots, a_\ell] \ := \ a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \frac{1}{a_\ell}}}$$

Integers a_i are called *partial quotients*, see e.g. [HW08, §10.1]. We use the notation $[a_1, \ldots, a_\ell]$ when $a_0 = 0$. Every positive rational number q admits two continued fraction representations since $[a_0; a_1, \ldots, a_\ell, 1] = [a_0; a_1, \ldots, a_\ell + 1]$. For $q \in \mathbb{Q}_{>0}$, we define

$$\mathrm{S}(q) \ := \ a_0 + a_1 + \ldots + a_\ell$$

and note that $\mathrm{S}(q)$ is the same for both representations. We also define $\mathrm{S}(0) := 0$.

We need the following remarkable result which was proved as a consequence of the Bourgain–Kontorovich technology [BK14].

Theorem 1.4 (Bourgain [Bou12, Prop. 1])**.** *Every rational number* $\frac{d}{c} \in [0, 1)$ *can be written as the following sum:*

$$\frac{d}{c} = q_1 + \ldots + q_k\,, \tag{1.7}$$

where $q_1, \ldots, q_k \in \mathbb{Q} \cap (-1, 1)$ *satisfy*

$$\mathrm{S}\big(|q_1|\big) + \ldots + \mathrm{S}\big(|q_k|\big) \ \leq \ C \log(c + d),$$

for some universal constant $C > 0$.

We apply Theorem 1.4 to our problem via the following theorem, which converts the problem of analyzing ratios of spanning trees into a problem on continued fractions. For an edge $e \in E$, we denote by $G - e$ the graph obtained by *deleting* the edge e on G.

Theorem 1.5. *Let* $q_1, \ldots, q_k \in \mathbb{Q}_{>0}$ *be positive rational numbers. Then there exists a simple planar graph* $G = (V, E)$ *and an edge* $e \in E$*, such that*

$$\frac{\tau(G - e)}{\tau(G/e)} \ = \ q_1 + \ldots + q_k$$

and

$$|E| \ = \ 4\big(\mathrm{S}(q_1) + \ldots + \mathrm{S}(q_k)\big) + 1.$$

Note that Theorem 1.4 and Theorem 1.5 immediately imply Theorem 1.1 when all rational numbers $q_1, \ldots, q_k$ in the representation of $\frac{t-c}{c}$ in (1.7) are nonnegative. The full version of Theorem 1.1 can then be obtained through a more involved argument that builds on both Theorem 1.4 and Theorem 1.5.

Remark 1.6. We note that Theorem 1.4 cannot be strengthened to require that all q_i are nonnegative. Indeed, if $q \in \mathbb{Q}_{>0}$ is less than $1/c$, then $S(q) \geq c$. Therefore, any expression of the unit fraction $1/c$ as $1/c = q_1 + \ldots + q_k$ with positive q_i must have $S(q_1) + \ldots + S(q_k) \geq kc$.

2. Preliminaries

In this section, we collect several graph theoretic lemmas and constructions that will be used in the proofs in Section 3. Several of these constructions are standard and have previously appeared in [CP24c, CKP24], but our *simplification* is new or at least used in a nonstandard way.

2.1. **Basic definitions.** Throughout this paper, all graphs are assumed to be planar and have no loops. Graphs *are* allowed to have multiple edges unless indicated otherwise. A graph with no loops is *simple* if it also does not have multiple edges. Let $G = (V, E)$ be a graph and let $e \in E$ be an edge in G. A pair (G, e) is called a *marked graph*. Our operations are defined on marked graphs.

2.2. **Subdivisions and duplications.** The *k-subdivision* of (G, e) is a marked graph (G', e') obtained by replacing edge e with a path of length $(k + 1)$, and marking one of the new edges as e'. Note that if G is connected, simple and planar, then so is G'. When $k = 1$, we omit the parameter and write *subdivision*.

The *k-duplication* of (G, e) is a marked graph (G'', e'') obtained by replacing edge e with $(k+1)$ parallel edges, and marking one of the new edges as e''. Note that if G is connected and planar, then so is G''. When $k = 1$, we omit the parameter and write *duplication*. Note also that these two operations are planar dual to each other (see below).

Proposition 2.1 ([CP24c, Thm 5.1])**.** *Let $c, d \geq 1$ be positive integers with $\gcd(c, d) = 1$. Suppose that*

$$\frac{d}{c} \;=\; [a_0\,; a_1, a_2, \ldots, a_\ell], \tag{2.1}$$

for some $a_0 \geq 0, a_1, \ldots, a_\ell \geq 1$. Then there exists a planar graph $G = (V, E)$ and an edge $e \in E$ satisfying

$$\tau(G - e) \;=\; d, \qquad \tau(G/e) \;=\; c,$$

and such that

$$|E| \;=\; a_0 \,+\, a_1 \,+\, \ldots \,+\, a_\ell \,+\, 1.$$

The proof of the proposition is given by a repeated subdivision and duplication of marked graphs starting with a single edge. Note that the resulting graph is not necessarily simple as this required additional constraints on a_i, see [CKP24, Lemma 2.3].

2.3. **Marked sum.** A marked graph (G, e) is called *proper* if $\tau(G - e) > 0$ and $\tau(G/e) > 0$. In other words, (G, e) is proper if G is connected and e is not a bridge. The *spanning tree ratio* of a proper marked graph is defined as

$$\zeta(G, e) \;:=\; \frac{\tau(G - e)}{\tau(G/e)} \;>\; 0.$$

It follows from (1.1) and the *deletion-contraction formula* $\tau(G - e) + \tau(G/e) = \tau(G)$, that

$$\rho(G, e) \;=\; \frac{1}{1 + \zeta(G, e)}\,. \tag{2.2}$$

Let (G, e) and (G', e') be marked graphs, where $G = (V, E)$, $G' = (V', E')$. By definition, we have $e \in E$ and $e' \in E'$. Define the *marked sum* $(G, e) \oplus (G', e')$ as the marked graph (G°, e°) obtained by taking the disjoint union of G and G' and identifying e with e' as a single edge e°. Note that if G and G' are proper, planar and simple, then so is $G^\circ = (V^\circ, E^\circ)$. Observe that $|V^\circ| = |V| + |V'| - 2$ and $|E^\circ| = |E| + |E'| - 1$.

Lemma 2.2 ([CP24c, Lem. 5.4])**.** *For proper marked graphs (G, e) and (G', e'), and the marked sum $(G^\circ, e^\circ) = (G, e) \oplus (G', e')$ defined above, we have:*

$$\zeta(G^\circ, e^\circ) \;=\; \zeta(G, e) \,+\, \zeta(G', e'). \tag{2.3}$$

2.4. **Plane duality.** Let $G = (V, E)$ be a proper planar graph, and let F be the set of faces of G. Denote by $G^* = (F, E)$ the *plane dual graph*. Here we identify edges in G and G^*. Note that G^* is also proper and planar. By *Euler's formula*, the number of vertices in G^* is given by $|F| = |E| - |V| + 2$. Recall also that $\tau(G^*) = \tau(G)$.

Lemma 2.3. *For a proper planar marked graph* (G, e)*, we have:*

$$\zeta(G^*, e) \, = \, \frac{1}{\zeta(G, e)} \quad \textit{and} \quad \rho(G^*, e) \, = \, 1 \, - \, \rho(G, e). \tag{2.4}$$

Proof. Note that deletion and contraction are dual operations. Thus we have:

$$\tau(G - e) \, = \, \tau(G^*/e^*) \quad \text{and} \quad \tau(G/e) \, = \, \tau(G^* - e^*). \tag{2.5}$$

This and (2.2) imply the result. □

2.5. **Simplification.** Let $G = (V, E)$ be a proper planar simple graph, and let $e \in E$. Define *doubling* of the marked graph (G, e) as a marked graph (G', e), where $G' = (V, E')$ is obtained by duplication of each edge other than e. Note that G' is also proper and planar.

Similarly, define *halving* of the marked graph (G, e) as a marked graph (G'', e), where $G'' = (V, E'')$ is obtained by subdivision of each edge other than e. Note that G'' is also proper, planar and simple. Finally, define *simplification* of (G, e) as a marked graph (G°, e) obtained by first doubling and then halving. Note that G° is again proper, planar and simple.

Lemma 2.4. *Let* $G = (V, E)$ *be a proper planar simple graph, and let* $e \in E$*. Then:*

$$\zeta(G', e) \, = \, 2\zeta(G, e) \, , \quad \zeta(G'', e) \, = \, \frac{\zeta(G, e)}{2} \quad \textit{and} \quad \zeta(G^\circ, e) \, = \, \zeta(G, e).$$

Proof. For the first equality, we have:

$$\tau(G' - e) \, = \, 2^{|V|-1} \, \tau(G - e) \quad \text{and} \quad \tau(G'/e) \, = \, 2^{|V|-2} \, \tau(G/e),$$

as desired. Now, note that $(G')^* = G''$. Thus the second equality follows from the first and (2.4). The third equality follows from the first two. □

3. Proofs of results

3.1. **Proof of Theorem 1.5.** Let $q_1, \ldots, q_k \in \mathbb{Q}_{>0}$ be positive rational numbers as in the theorem. It follows from Proposition 2.1, that there exists (not necessarily simple) planar marked graphs (G_i, e_i), where $G_i = (V_i, E_i)$, such that

$$\zeta(G_i, e_i) \, = \, q_i \quad \text{and} \quad |E_i| \, = \, \mathrm{S}(q_i) + 1,$$

for all $1 \le i \le k$. Let $(G, e) := (G_1, e_1) \oplus \cdots \oplus (G_k, e_k)$, where $G = (V, E)$. It follows from Lemma 2.2, that

$$\begin{aligned} \zeta(G, e) \, &= \, \zeta(G_1, e_1) \, + \, \ldots \, + \, \zeta(G_k, e_k) \, = \, q_1 \, + \, \ldots \, + \, q_k \qquad \text{and} \\ |E| \, &= \, |E_1| \, + \, \ldots \, + \, |E_k| \, - \, k + 1 \, = \, \mathrm{S}(q_1) \, + \, \ldots \, + \, \mathrm{S}(q_k) + 1. \end{aligned}$$

Since the marked graph (G, e) is not necessarily simple, take its simplification (G°, e), where $G^\circ = (V^\circ, E^\circ)$. It then follows from Lemma 2.4 that (G°, e) is a simple planar marked graph satisfying $\zeta(G^\circ, e) = \zeta(G, e)$ and

$$|E^\circ| \, = \, 4(|E| - 1) \, + \, 1 \, = \, 4\big(\mathrm{S}(q_1) + \ldots + \mathrm{S}(q_k)\big) + 1.$$

This completes the proof. □

3.2. **Proof of Theorem 1.1.** Let $d := t - c$. Using (2.2), it suffices to show that there exists a simple planar graph $G = (V, E)$ and edge $e \in E$, such that for sufficiently large t we have:

$$\zeta(G,e) \ = \ \frac{d}{c} \qquad \text{and} \qquad |E| \ \leq \ C \, \max\left\{ \frac{d}{c}\,, \, \frac{c}{d}\,, \, \log(c+d) \right\}, \tag{3.1}$$

for some universal constant $C > 0$. We prove (3.1) in the following series of lemmas. Denote by $C_0 > 0$ the universal constant in Theorem 1.4.

Lemma 3.1. *Condition* (3.1) *holds for* $d\,/\,c \ \geq \ \lceil C_0 \log(c+d) \rceil$.

Proof. Let c', d' be coprime positive integers defined as

$$\frac{d'}{c'} \ = \ \frac{d}{c} \mod 1,$$

so we have $d'/c' \in [0,1)$. Note that $c' + d' \leq c + d$. Now, by Theorem 1.4, there exist rational numbers $q_1', \ldots, q_k' \in (-1,1)$ such that

$$\frac{d'}{c'} \ = \ q_1' \, + \, \ldots \, + \, q_k' \qquad \text{and}$$
$$\mathrm{S}\big(|q_1'|\big) \, + \, \ldots \, + \, \mathrm{S}\big(|q_k'|\big) \ \leq \ C_0 \log(c'+d') \ \leq \ C_0 \log(c+d). \tag{3.2}$$

Note that it follows from the assumed lower bound on d/c and (3.2) that

$$k \ \leq \ C_0 \log(c+d) \ \leq \ \left\lfloor \frac{d}{c} \right\rfloor. \tag{3.3}$$

We now define $q_1, \ldots, q_k$ as

$$q_1 \ := \ \left\lfloor \frac{d}{c} \right\rfloor - k + 1 + q_1' \qquad \text{and} \qquad q_i \ := \ 1 + q_i' \quad \text{for all} \quad i \geq 2.$$

It follows that $q_1 > 0$ by (3.3), and $q_2, \ldots, q_k > 0$ by definition. Also note that

$$\frac{d}{c} \ = \ q_1 \, + \, \ldots \, + \, q_k \, .$$

Observe that for a rational number $q = [a_1, \ldots, a_\ell] \in \mathbb{Q} \cap (0,1)$, we have:

$$1 - q \ = \ \begin{cases} [1, a_1 - 1, a_2, \ldots, a_\ell] & \text{if } a_1 > 1, \\ [a_2 + 1, \ldots, a_\ell] & \text{if } a_1 = 1. \end{cases}$$

This implies that

$$\mathrm{S}(1 + q_i') \ \leq \ 1 + \mathrm{S}(|q_i|') \quad \text{for all} \quad 1 \leq i \leq k.$$

Therefore, we have:

$$\mathrm{S}\big(q_1\big) \, + \, \ldots \, + \, \mathrm{S}\big(q_k\big) \ = \ \left\lfloor \frac{d}{c} \right\rfloor \, - \, k \, + \, \mathrm{S}(1 + q_1') \, + \, \ldots \, + \, \mathrm{S}(1 + q_k') \ \leq \ \frac{d}{c} \, + \, C_0 \log(c+d),$$

where the inequality is due to (3.2).

By Theorem 1.5, there exists a simple planar graph $G = (V, E)$ and edge $e \in E$, such that

$$\zeta(G,e) \ = \ q_1 \, + \, \ldots \, + \, q_k \ = \ \frac{d}{c} \qquad \text{and}$$
$$|E| \ = \ 4\big(\mathrm{S}(q_1) \, + \, \ldots \, + \, \mathrm{S}(q_k)\big) \, + \, 1 \ \leq \ 4\frac{d}{c} \, + \, 4\,C_0 \log(c+d) \, + \, 1.$$

Thus (G, e) satisfies (3.1) with a constant

$$C_1 \ := \ 4\,C_0 \, + \, 5.$$

This completes the proof of the lemma. $\square$

Lemma 3.2. *Condition* (3.1) *holds for* $d\,/\,c \geq 1$.

Proof. Let $K := 4\lceil C_1 \log(c+d)\rceil$. Let $L \in \{0,\ldots,K-1\}$ be the unique integer such that

$$\frac{d}{c} - \frac{L}{K} \in \big[1/K, 2/K\big) \mod 1.$$

Let c', d' be coprime positive integers defined as

$$\frac{d'}{c'} = \frac{d}{c} - \frac{L}{K} \mod 1, \tag{3.4}$$

so we have

$$\frac{d'}{c'} \in \big[1/K, 2/K\big). \tag{3.5}$$

Note that

$$\log(c'+d') \leq \log(c+d) + \log K \leq 2\log(c+d),$$

for sufficiently large $(c+d)$. Note also that

$$\frac{c'}{d'} \geq_{(3.5)} \frac{K}{2} = 2\lceil C_1 \log(c+d)\rceil \geq C_1 \lceil \log(c'+d')\rceil.$$

By Lemma 3.1 applied to $c'\,/\,d'$, there exists a simple planar graph $G = (V,E)$ and an edge $e \in E$, satisfying

$$\zeta(G,e) = \frac{c'}{d'} \qquad \text{and}$$

$$|E| \leq C_1\left(\frac{c'}{d'} + \log(c'+d')\right) \leq_{(3.5)} C_1\big(K + \log(c'+d')\big) \leq C_1\,(4C_1+2)\log(c+d).$$

Let $G^* = (F,E)$ be the plane dual of G. The marked graph (G^*,e) satisfies

$$\zeta(G^*,e) = \frac{d'}{c'} \qquad \text{and} \qquad |E| \leq C_1\,(4C_1+2)\log(c+d).$$

Note that G^* is not necessarily simple.

In a different direction, we have $d\,/\,c \geq 1 \geq d'\,/\,c'$ by assumption in the lemma. Also note that $K(d/c - d'/c')$ is an integer by (3.4). By Theorem 1.5 applied to $q_1 \leftarrow (d/c - d'/c')$ and $k=1$, there exists a planar graph $G' = (V',E')$ and an edge $e' \in E'$, such that

$$\zeta(G',e') = \frac{d}{c} - \frac{d'}{c'} \quad \text{and} \quad |E'| \leq 4\,\mathrm{S}\left(\frac{d}{c} - \frac{d'}{c'}\right) + 1.$$

Note that

$$|E'| \leq 4\,\mathrm{S}\left(\frac{d}{c} - \frac{d'}{c'}\right) + 1 \leq 4\left(\frac{d}{c} + K\right) + 1 \leq 4\left(\frac{d}{c} + 4C_1\log(c+d)\right) + 17,$$

where the second inequality is because $K(d\,/\,c - d'\,/\,c')$ is an integer.

Now, let (G°, f), where $G^\circ = (V^\circ, E^\circ)$ and $f \in E^\circ$, be the marked graph obtained by taking the marked sum $(G^*,e) \oplus (G',e')$ followed by simplification. By construction, H is a simple planar graph. It follows from Lemma 2.2 and Lemma 2.4, that

$$\zeta(G^\circ,f) = \zeta(G^*,e) + \zeta(G',e') = \frac{d}{c} \qquad \text{and}$$

$$|E^\circ| \leq 4\big(|E| + |E'| - 2\big) + 1 \leq 16\,\frac{d}{c} + 8C_1(2C_1+9)\log(c+d) + 61.$$

Thus (G°,f) satisfies (3.1) with a constant

$$C_2 := 16C_1^2 + 72C_1 + 77.$$

This completes the proof of the lemma. □

Lemma 3.3. *Condition* (3.1) *holds for* $d\,/\,c\, <\, 1$.

Proof. Since $c\,/\,d > 1$, applying Lemma 3.2 to $c\,/\,d$ gives a simple planar marked graph (G, E) satisfying

$$\zeta(G,e) \ = \ \frac{c}{d} \quad \text{and} \quad |E| \ \le \ C_2 \max\left\{\frac{c}{d}\,,\, \log(c+d)\right\}.$$

Apply the planar dual and then the simplification operations to the marked graph (G,e) to obtain a simple planar marked graph (G°, e), where $G^\circ = (V^\circ, E^\circ)$ and $e \in E^\circ$. We have:

$$\begin{aligned} \zeta(G^\circ, e) \ &= \ \frac{1}{\zeta(G,e)} \ = \ \frac{d}{c} \qquad \text{and} \\ |E^\circ| \ &= \ 4\big(|E|-1\big) + 1 \ \le \ 4|E| \ \le \ 4\,C_2 \max\left\{\frac{c}{d}\,,\, \log(c+d)\right\}. \end{aligned}$$

Thus (G°, e) satisfies (3.1) with a constant $4C_2$. This completes the proof of the lemma and finishes the proof of the theorem. □

3.3. **Proof of Proposition 1.2.** The proposition consists of three inequalities which we prove separately. For the first and second inequality, we need the following result:

Lemma 3.4. *For every connected graph* $G = (V, E)$ *and edge* $e = (x, y) \in E$*, we have:*

$$\rho(G,e) \ \ge \ \frac{1}{2}\left(\frac{1}{\deg(x)} \ + \ \frac{1}{\deg(y)}\right),$$

where $\deg(v)$ *denotes the degree of vertex* $v \in V$.

The lemma is well known via connection to the random walks on G and the *commute time* interpretation

$$\kappa(G,e) \ = \ 2|E|\,\rho(G,e)$$

given in [C+96, Thm 2.1], combined with the inequality

$$\kappa(G,e) \ \ge \ |E| \left(\frac{1}{\deg(x)} \ + \ \frac{1}{\deg(y)}\right),$$

see e.g. [Lov96, Cor. 3.3]. Since G is simple, we have $\deg(x), \deg(y) < |V|$. This implies the first inequality:

$$|V| \ > \ \frac{1}{\rho(G,e)} \ = \ \frac{t}{c}\,. \tag{3.6}$$

For the second inequality, denote by F the set of faces in G. Since G is simple and planar, it follows from Euler's formula that $|E| \le 3|V| - 6$ and $|F| \le 2|V| - 4$. Applying Lemma 3.4 to the dual graph $G^* = (F, E)$, we conclude:

$$|V| \ > \ \frac{|F|}{2} \ >_{(3.6)} \ \frac{1}{2\,\rho(G^*,e)} \ =_{(2.4)} \ \frac{1}{2\big(1-\rho(G,e)\big)} \ = \ \frac{1}{2}\left(\frac{t}{t-c}\right). \tag{3.7}$$

Finally, for the third inequality, we have:

$$t \ \le \ \tau(G) \ < \ 2^{|E|} \ < \ 2^{3\,|V|}\,, \tag{3.8}$$

as desired. □

4. Final remarks and open problems

4.1. Bourgain's Theorem 1.4 was partially motivated by Hall's classic result that every number in the interval $(\sqrt{2}-1, 4\sqrt{2}-4)$ can be presented as the sum of two continued fractions whose partial quotients do not exceed four [Hall47, Thm 3.1]. Hančl and Turek [HT23] gave a version of this result for partial fractions of the type $[a_1, 1, a_2, 1, a_3, 1, \ldots]$. In our previous paper [CKP24], we used partial fractions of this type to study Sedláček's problem, see §1.2. While initially, we intended to obtain Theorem 1.1 via a finite version of the Hančl–Turek result, this turned out to be unnecessary due to the simplification operation given in §2.5.

4.2. For a finite poset $P = (X, \prec)$, the *relative number of linear extensions* $e(P-x)/e(P)$ plays roughly the same role as the effective resistance for graphs. In [CP24b], we used recent analytic estimates on sums of quotients of continued fractions to show that for all $d/3 \geq c \geq 1$ there exists a poset $P = (X, \prec)$ and an element $x \in X$, such that

$$\frac{e(P-x)}{e(P)} = \frac{c}{d} \quad \text{and} \quad |X| \leq C \max\left\{\frac{d}{c}, \log d \log\log d\right\}.$$

By analogy with Theorem 1.1, one can hope to remove the $\log\log d$ factor; this was in fact conjectured in [CP24b, Eq. (1.7)]. Unfortunately, the approach in this paper is not applicable since the analogue of the marked sum for posets called the *flip-flop construction* defined in [CP24b, §3.3] cannot be used more than once.

4.3. Throughout the paper we made no effort to compute or optimize the constants. This is in part because the constant C in Bourgain's Theorem 1.4 is not specified. We note, however, that the constant C' in Proposition 1.2 can be easily computed. In fact, a more careful argument shows that one can remove the constant $1/2$ in the inequality (3.7). Similarly, one can improve (3.8) by using the $\tau(G) < 5.23^{|V|}$ bound in [BS10]. After these improvements, one can take $C' := 0.6$.

Acknowledgements. We are grateful to Noga Alon, Matija Bucić, Peter Sarnak, Ilya Shkredov and Nikita Shulga for helpful comments and remarks on the literature. SHC was supported by NSF grant DMS-2246845. AK was supported by NSF grant DMS-2302641, BSF grant 2020119 and a Simons Fellowship. IP was supported by NSF grant CCF-2302173. This paper was written when AK was visiting the Institute for Advanced Study. We are grateful for the hospitality.

References

[ABG25] Noga Alon, Matija Bucić and Lior Gishboliner, The spanning tree spectrum: improved bounds and simple proofs, preprint (2025), 8 pp.; `arXiv:2503.23648`.

[AŠ12] Jernej Azarija and Riste Škrekovski, Minimal graphs with a prescribed number of spanning trees, in *Open problem garden*, 2012; available online at tinyurl.com/2tc6v469

[Bou12] Jean Bourgain, Partial quotients and representation of rational numbers, *C. R. Math.* **350** (2012), 727–730.

[BK14] Jean Bourgain and Alex Kontorovich, On Zaremba's conjecture, *Annals of Math.* **180** (2014), 137–196.

[BS10] Kevin Buchin and André Schulz, On the number of spanning trees a planar graph can have, in *Proc. 18th ESA*, Springer, Berlin, 2010, 110–121.

[CKP24] Swee Hong Chan, Alex Kontorovich and Igor Pak, Spanning trees and continued fractions, preprint (2024), 20 pp.; `arXiv:2411.18782`.

[CP24a] Swee Hong Chan and Igor Pak, Computational complexity of counting coincidences, *Theor. Comp. Sci.* **1015** (2024), Paper No. 114776, 19 pp.

[CP24b] Swee Hong Chan and Igor Pak, Linear extensions and continued fractions, *European J. Combin.* **122** (2024), 104018, 14 pp.

[CP24c] Swee Hong Chan and Igor Pak, Equality cases of the Stanley–Yan log-concave matroid inequality, preprint (2024), 35 pp.; `arXiv:2407.19608`.

[C+96] Ashok K. Chandra, Prabhakar Raghavan, Walter L. Ruzzo, Roman Smolensky and Prasoon Tiwari, The electrical resistance of a graph captures its commute and cover times, *Comput. Complexity* **6** (1996), 312–340.

[Hall47] Marshall Hall, On the sum and product of continued fractions, *Annals of Math.* **48** (1947), 966–993.

[DS84] Peter G. Doyle and Laurie J. Snell, *Random walks and electric networks*, MAA, Washington, DC, 1984, 159 pp.

[HT23] Jaroslav Hančl and Ondřej Turek, Continued fractions with bounded even-order partial quotients, *Ramanujan J.* **62** (2023), 69–110.

[HW08] Godfrey H. Hardy and Edward M. Wright, *An introduction to the theory of numbers* (sixth ed., revised), Oxford Univ. Press, Oxford, UK, 2008, 621 pp.

[KNW17] Alex Kontorovich, Peter McNamara and Geordie Williamson, Appendix to: Schubert calculus and torsion explosion (by Geordie Williamson), *Jour. AMS* **30** (2017), 1023–1046.

[Lov96] László Lovász, Random walks on graphs: a survey, in *Combinatorics, Paul Erdős is eighty*, Vol. 2, Bolyai Math. Soc., Budapest, 1996, 353–397.

[LP16] Russell Lyons and Yuval Peres, *Probability on trees and networks*, Cambridge Univ. Press, New York, 2016, 699 pp.

[Nas59] Crispin St J. A. Nash-Williams, Random walk and electric currents in networks, *Proc. Cambridge Philos. Soc.* **55** (1959), 181–194.

[Sed66] Jiří Sedláček, On the spanning trees of finite graphs (in Czech), *Časopis Pěst. Mat.* **91** (1966), 221–227.

[Sed69] Jiří Sedláček, On the number of spanning trees of finite graphs, *Časopis Pěst. Mat.* **94** (1969), 217–222.

[Sed70] Jiří Sedláček, On the minimal graph with a given number of spanning trees, *Canad. Math. Bull.* **13** (1970), 515–517.

[Sel65] Atle Selberg, On the estimation of Fourier coefficients of modular forms, in *Proc. Sympos. Pure Math.*, Vol. VIII, AMS, Providence, RI, 1965, 1–15.

[Sto22] Richard Stong, Minimal graphs with a prescribed number of spanning trees, *Australas. J. Combin.* **82** (2022), 182–196.

(Swee Hong Chan) Department of Mathematics, Rutgers University, Piscataway, NJ 08854.
Email address: `sweehong.chan@rutgers.edu`

(Alex Kontorovich) Department of Mathematics, Rutgers University, Piscataway, NJ 08854.
Email address: `alex.kontorovich@rutgers.edu`

(Igor Pak) Department of Mathematics, UCLA, Los Angeles, CA 90095.
Email address: `pak@math.ucla.edu`