COVARIANTS OF THE SYMMETRIC GROUP AND ITS ANALOGS IN WEYL ALGEBRAS

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1. Many beautiful and important results of the theory of invariants admit odd and super analogs.

The principal object of attention in this theory is the structure of the graded G-module S(V) for a given G-module V. The odd analog of the symmetric algebra S(V) is the exterior algebra $\Lambda(V)$, and its superanalog is the algebra $E(V) = S(V) \otimes \Lambda(V)$. The important role of the latter in many algebraic and homological constructions was clarified by A. Weyl. We therefore call E(V) the Weyl algebra. It is bigraded: $E^{p,q}(V) = S^p(V) \otimes \Lambda^q(V)$, and the structure of its isotypical components relative to the group action is conveniently described by the Poincaré series:

$$P_\pi^G(t, s) = \sum_{p,q} m_{p,q}(\pi) t^p s^q, \tag{1}$$

where π is an irreducible representation of G, and $m_{p,q}(\pi)$ the multiplicity of the occurrence of this representation in the decomposition of $E^{p,q}(V)$. For a compact group G we have the formula

$$P_\pi^G(t, s) = \int_G \chi_\pi(g) \det \frac{1 + sT(g)}{1 - tT(g)} dg, \tag{2}$$

where $\chi_\pi(g)$ is the character of π, T the action of G in the space V and dg normalized Haar measure on G.

Unfortunately, this formula is rarely convenient for practical calculations, since it requires a knowledge of the character of the representation π and summation of a large number of terms. At the same time it is well known (see [1] or [2], Sec. V.5, Exercises) that, if G is the finite group generated by reflections in the space V, then the algebra of invariants $E(V)^G$ is itself isomorphic to an algebra of type $S(V_0) \otimes \Lambda(V_1)$ and its Poincaré series has the form

$$P_{\pi_0}^G(t, s) = \prod_{i=1}^{n} \frac{1 + st^{d_i - 1}}{1 - t^{d_i}}, \tag{3}$$

where $d_1, \ldots, d_n$ are the degrees of the basis invariants of G in S(V). It was further observed in [3-5] that when $V = \mathbf{R}^n$, and G is either the symmetric group S(n) acting by permuting coordinates; or the group $C(n) = S(n) \times \mathbf{Z}_2{}^n$, acting by permuting coordinates and changing their signs; or, finally the unique proper subgroup $D(n) = S(n) \times \mathbf{Z}_2{}^{n-1}$ of C(n) containing S(n) as a proper subgroup – the expression $P_\pi{}^G(t, 0)$ for all π can be given by a simple multiplicative formula. Formally speaking, this formula looks as if the isotypical component in question were a free module of rank 1 over some free commutative graded algebra.

In this paper we shall prove a superanalog of this assertion for the groups S(n) and C(n) (below, Sec. 2, Theorem 1), and also give a combinatorial interpretation of the multiplicities. The statement of the problem and the formulation of the results are due to the first author, the proof and the combinatorial interpretation to the second.

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2. We present the standard definitions necessary for the sequel (see [6, 7]).

A partition is a nonincreasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of nonnegative integers such that $|\lambda| = \Sigma\lambda_k < \infty$.

The diagram of a partition λ is the set of pairs $(i, j) \in \mathbf{Z}^2$, satisfying the inequalities $1 \le j \le \lambda_j$.

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We let $\lambda' = (\lambda_1', \ldots, \lambda_n', \ldots)$ denote the partition dual to λ, whose diagram is obtained from that of λ by transposition: $(i, j) \to (j, i)$.

The hook length of an element (i, j) of the diagram of λ is the number $h(i, j) = \lambda_i + \lambda_j' - i - j + 1$.

We recall, in addition, that the equivalence classes of the irreducible representations of $S(n)$ are enumerated by the partitions λ with $|\lambda| = n$, and those of the irreducible representations of $C(n)$ by the ordered pairs of partitions (λ, μ) with $|\lambda| + |\mu| = n$.

We shall write P_λ (resp., $P_{\lambda,\mu}$) for P_π if π is a representation of class λ (resp., of class (λ, μ)).

THEOREM 1. The following formulas are valid:

$$P_\lambda^{S(n)} = \prod_{(i,j)\in\lambda} \frac{t^i + st^j}{1 - t^{h(i,j)}} \text{ for } G = S(n), \tag{4}$$

$$P_{\lambda,\mu}^{C(n)} = \prod_{(i,j)\in\lambda} \frac{t^{2i} + st^{2j+1}}{1 - t^{2h(i,j)}} \prod_{(k,l)\in\mu} \frac{t^{2k+1} + st^{2l}}{1 - t^{2h(k,l)}} \text{ for } G = C(n). \tag{5}$$

Remark 1. Formulas (4) and (5) show that the isotypical components, as graded spaces, are isomorphic to the free module of rank 1 over some graded free supercommutative algebra.

Remark 2. The irreducible representations of the group $D(n)$ are obtained by restriction of the irreducible representations $\pi_{\lambda,\mu}$ of $C(n)$ with $\lambda = \mu$, with the exception of the special representations $\pi_\lambda^{\pm}$, which are obtained by restriction of $\pi_{\lambda,\lambda}$.

This means that there are generally no formulas of type (4) and (5) for the irreducible representations of $D(n)$.

3. We shall show that the first assertion of Theorem 1 follows from the second.

LEMMA 1. The following equality is valid:

$$P_{\lambda,\mu}^{C(n)}(t, s) = P_\lambda^{S(n)}(t^2, st)\, t^{|\mu|} P_\mu^{S(n)}(t^2, st^{-1}). \tag{6}$$

To prove the lemma we shall need an explicit description of the representation π of type (λ, μ). As is known, this is given by

$$\pi_{\lambda,\mu} = \operatorname{Ind}_{C(|\lambda|)\times C(|\mu|)}^{C(n)} (\pi_\lambda' \times \pi_\mu''), \tag{7}$$

where π_λ' denotes the representation of the group $C(|\lambda|)$ which coincides on $S(|\lambda|)$ with π_λ and is trivial on the normal subgroup $\mathbf{Z}_2^{|\lambda|}$; π_μ'' denotes the representation of $C(|\mu|)$ which coincides on $S(|\mu|)$ with π_μ and on the normal subgroup $\mathbf{Z}_2^\mu$ with the sign representation.

It follows from (7), (2) and Frobenius' formula for the character of the induced representation (see, e.g., [8]) that equality (6) need be proved only in the "extreme" cases, when either $\mu = \varnothing$ or $\lambda = \varnothing$. In the first case this follows from the fact that the invariants of the Weyl algebra $E(\mathbf{R}^n)$ relative to the group $\mathbf{Z}_2^n$, acting by reflections in the coordinate hyperplanes, themselves form a Weyl algebra with even generators $x_1^2, \ldots, x_n^2$ and odd generators $x_1 dx_1, \ldots, x_n dx_n$. In the second case we are interested in the elements of $E(\mathbf{R}^n)$ which are antisymmetric relative to $\mathbf{Z}_2^n$. It is easy to see that they form a free module of rank 1 with generator $x_1 x_2 \ldots x_n$ over the algebra of symmetric elements generated by the even generators $x_1^2, \ldots, x_n^2$ and the odd generators $x_1^{-1}dx_1, \ldots, x_n^{-1}dx_n$. This completes the proof of the lemma

4. We now proceed to the combinatorial interpretation of the coefficients $m_{p,q}(\pi)$. We recall two more standard definitions.

A table of type λ is a function f on a diagram λ, taking nonnegative integer values, such that the sets $\lambda^k = \{(i, j) \in \lambda \mid f(i, j) \le k\}$ are also diagrams of partitions.

It is convenient to represent the elements of a diagram not by points (i, j) but by square cells with centers at these points; the table is obtained by entering the values of f in the cells.

The weight of a sequence $k = (k_1, k_2, \ldots)$ of nonnegative integers is the sequence $\mu(k) = (\mu_0(k), \mu_1(k), \ldots)$, where $\mu_i(k) = \#\{j \mid k_j = i\}$. Weights remain unchanged when the terms of the original sequence are permuted.

We now define the corresponding superanalogs.

A <u>supertable of type</u> λ is a function $f = (f_0, f_1)$ on a diagram λ, taking values in $\mathbf{Z}_+\times\mathbf{Z}_2$ and such that f_0, $f_0 + f_1$ are ordinary tables. It is convenient to represent a supertable by inserting in each cell (i, j) the number $f_0(i, j)$ and calling it white if $f_1(i, j) = 0$ and black if $f_1(i, j) = 1$.

A supertable f of type λ is said to be <u>regular</u> if the set $A_{k,\varepsilon} = \{(i, j) \in \lambda | f_0(i, j) = k, f_1(i, j) = \varepsilon\}$ contains at most one cell in each column when $\varepsilon = 0$ and at most one cell in each row when $\varepsilon = 1$.

The <u>superweight</u> of a sequence $k = (k_1, k_2, \ldots)$ of black and white numbers is the pair of sequences $\tilde{\mu}(k) = (\mu'(k), \mu''(k))$, where $\mu'(k)$ is the weight of the subsequence of white numbers and $u''(k)$ that of the subsequence of black numbers.

To each superweight $\tilde{\mu} = (\mu', \mu'')$ there corresponds an $S(|\mu|)$-module $M^{\tilde{\mu}}$ which is constructed by the following rule:

$$M^{\tilde{\mu}} = \mathrm{Ind}_{S(|\mu'|)\times S(|\mu''|)}^{S(|\mu|)} (1 \otimes \mathrm{sgn}). \tag{8}$$

Now, in the Weyl algebra $E(\mathbf{R}^n)$, we consider a monomial of the form $x_1^{k_1}\ldots x_n^{k_n} dx_1^{\varepsilon_1} \wedge \ldots \wedge dx_n^{\varepsilon_n}$. The superweight of this monomial is defined as the superweight of the sequence $(k_1, \ldots, k_n)$, where k_j is considered white or black according as $\varepsilon_j = 0$ or 1.

The following result is obvious:

<u>LEMMA 2.</u> The set of monomials of superweight $\tilde{\mu}$ in the Weyl algebra generates a submodule isomorphic to $M^{\tilde{\mu}}$.

The proof follows directly from the definition of induced module (see [8]).

Let $K_{\lambda,\tilde{\mu}}$ denote the multiplicity with which the irreducible representation λ occurs in the module $M^{\tilde{\mu}}$. If $\mu'' = 0$, this is the classical Kostka number, for which several combinatorial interpretations have been known for some time. The quantity in which we are interested may be expressed in terms of a superanalog of the Kostka number, as follows:

$$P_\lambda(t, s) = \sum_{k,\varepsilon} K_{\lambda, \tilde{\mu}(k,\varepsilon)} t^{|\mu'(k,\varepsilon)|} s^{|\mu''(k,\varepsilon)|}, \tag{9}$$

as is directly evident from the decomposition of the Weyl algebra into submodules generated by monomials of the given superweight.

<u>5.</u> The problem is thus reduced to looking for a generating function for superanalogs of the Kostka numbers. To that end we must first understand the combinatorial interpretation of the numbers $K_{\lambda,\tilde{\mu}(k,\varepsilon)}$.

Let us define the superweight of a supertable as the superweight of the sequence $|A_{k,\varepsilon}|_{k\geq 0,\varepsilon\in\{0,1\}}$. Then $K_{\lambda,\tilde{\mu}(k,\varepsilon)}$ is equal to the number of regular supertables of superweight $\tilde{\mu}$ and type λ (see [10]). We shall give a new combinatorial interpretation of the superanalogs of the Kostka number, equivalent to the classical one but for which the generating function is easily calculated.

A <u>colored table</u> of type λ is a function $f = (f_0, f_1)$ on a diagram λ, taking values in $\mathbf{Z}_+ \times \mathbf{Z}_2$ and such that $g(i, j) := f_0 + f_1(i - j) - i$ is an ordinary table. As before, we shall call the number entered in a cell $(i, j) \in \lambda$ white or black according as $f_1(i, j) = 0$ or $f_1(i, j) = 1$.

The superweight of a colored table is defined like that of a supertable.

In our new terms, the number $K_{\lambda,\tilde{\mu}(k,\varepsilon)}$ is equal to the number of colored tables of superweight $\tilde{u}$ and type λ. To prove this assertion we need the following combinatorial

<u>LEMMA 3.</u> The number of supertables of type λ and superweight $\tilde{\mu}$ is equal to the number of colored tables of type λ and superweight $\tilde{\mu}$.

The idea of the proof consists in the explicit construction of a bijection between the two sets: given a supertable, we shall construct a colored table. To that end we successively fill up the sets $(i, j) \in A_{(k,\varepsilon)}$ with numbers, increasing k, ε until the whole diagram λ has been filled. With this done, we obtain at each step a colored table of type $A_{k,\varepsilon}$ and super weight $\tilde{\mu}_{(k,\varepsilon)}$, obtained by restricting the length of the sequence μ' and μ'' to (k) and $(k-1+\varepsilon)$, respectively. Thus, we must demonstrate the passage from a colored table of type $A_{k,0}$ to $A_{k,1}$ or from a table of type $A_{k-1,1}$ to $A_{k,0}$. We shall discuss the first case in detail.

Let us fill $A_{k,1}\backslash A_{k,0}$ with black numbers just as if we were filling up a standard table. By the definition of a supertable and the set $A_{k,\varepsilon}$, all these numbers equal k and in each row there is at most one cell filled with a black number k. We now "advance" these cells along the rows as much as possible, i.e., exchange the values of a cell x containing a black number k and of the neighboring cell y (i, j), which contains a white one, if

$$f_0(y) - i > k - j - 1. \tag{*}$$

It is easy to see that if condition (*) is fulfilled only the increment to f changes within a row, while the order in the columns is preserved, i.e., after (*) has been satisfied in succession for all the rows containing black k's, we obtain a colored table of type $A_{k,1}$ with the same superweight as the corresponding supertable of type $A_{k,1}$. For the second case the procedure is analogous: rows are replaced with columns by "advancing" white cells containing k whenever

$$f(y) + j > k - i - 1. \tag{**}$$

Thus, it remains to prove that the correspondence thus constructed is bijective. This is obvious if one proceeds as follows. The inverse correspondence can be constructed inductively, by singling out horizontal and vertical strips, operating in the reverse order. It is then obvious that the correspondence is invertible, hence also bijective.

We now go back to the quantity (9), for which we wish to obtain an explicit expression. If $K_{\alpha,\tilde{\mu}(k,\varepsilon)}$ is summed over all μ in our new terms, we see that the coefficient of $t^p s^q$ in (9) is exactly equal to the number of colored tables in which the sum of all numbers is p and the number of black numbers q. It is now not difficult to write down the generating function. Indeed, recall that every colored table of type λ is a function $\tilde{f} = (f_0, f_1)$ on the diagram λ such that $\tilde{f} = (f_0', 0) + (f_0'', f_1)$, where f_0' is an ordinary table and

$f_0''(i, j) = \begin{cases} i, f_1(i,j) = 0, \\ j, f_1(i,j) = 1. \end{cases}$ Hence it is clear that the generating function for the number of

colored tables is the product of the generating functions for ordinary tables and for the "coordinate" colored tables. But the first generating function is well known (see [6, 9])

and is equal to $\prod\limits_{(i,j)\in\lambda} \frac{1}{1-t^{h(i,j)}}$, where the coefficient of $t^{p'}$ is the number of tables with

p'. As to the second generating function, it is obviously equal to $\prod\limits_{(i,j)\in\lambda} (t^i + st^j)$, where

the coefficient of $t^{p''}s^q$ is the number of "coordinate" colored tables in which the sum of all numbers is p" and the sum of black numbers q.

We have thus proved

<u>THEOREM 2.</u> The following formula is valid:

$$\prod_{(i,j)\in\lambda} \frac{t^i + st^j}{1-t^{h(i,j)}} = \sum_{p,q} m_{p,q}(\lambda)\, t^p s^q, \tag{10}$$

where $m_{p,q}$ is the number of colored tables in which the sum of all numbers is p and the number of black numbers is q.

We thus obtain from (9) and (10):

$$P_\lambda(t, s) = \prod_{(i,j)\in\lambda} \frac{t^i + st^j}{1-t^{h(i,j)}}, \tag{11}$$

which proves Theorem 1.

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ONE-DIMENSIONAL SCHRÖDINGER OPERATOR WITH UNBOUNDED POTENTIAL: THE PURE POINT SPECTRUM

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1. INTRODUCTION

In a recent article [1], Simon and Spencer established a simple general criterion for the absence of an absolutely continuous spectrum for the one-dimensional Schrödinger operator with unbounded potential. This criterion has definitive and especially simple form in the discrete case of an operator h acting on $l_2(\mathbf{Z})$ defined by the operation

$$(\hat{h}\psi)(x) = -\psi(x+1) - \psi(x-1) + q(x)\psi(x), \ x \in \mathbf{Z}. \tag{1.1}$$

That is, according to [1] the operator h does not have an absolutely continuous component in its spectrum if $\overline{\lim}_{x_j} |q(x)| = \overline{\lim}_{x\to-\infty} |q(x)| = \infty$, i.e., if the potential is unbounded for $x \to \pm\infty$.

An analogous result is true for an operator $h(\theta)$, defined on $l_2(\mathbf{Z}_+)$ by operation (1.1) and boundary conditions

$$\psi(-1)\cos\theta - \psi(0)\sin\theta = 0, \tag{1.2}$$

where $\theta \in [0, \pi)$ is the boundary period. The period can be arbitrary. It is clear that the condition of unboundedness of the potential in this case is introduced only for $x \to +\infty$.

Idea of the Proof. Let the sequence of points x_j, $j = 0, \pm 1, \ldots$ (coordinates of the peaks) be chosen such that $\sum_j |q(x_j)|^{-1} < \infty$. We consider the block operator $\bar{h}$ obtained by by the imposition of conditions $\psi(x_j) = 0$ at the peak points. From the resolvent identity for pair h and $\bar{h}$, it is not difficult to see that for nonreal z, the difference $(h - z)^{-1} - (\bar{h} - z)^{-1}$ is the sum over j of rank 2 operators, the norm of each of which does not exceed $\mathrm{const}\cdot|q(x_j)|^{-1}$. Therefore the nuclear norm of this difference is finite, and then the required assertion follows from the point spectrum of $\bar{h}$ and the stability of the singular

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