ABSTRACT. A closed PL-curve is called integral if it is comprised of unit intervals. Kenyon’s problem asks whether for every integral curve \( \gamma \) in \( \mathbb{R}^3 \), there is a dome over \( \gamma \), i.e. whether \( \gamma \) is a boundary of a polyhedral surface whose faces are equilateral triangles with unit edge lengths. First, we give an algebraic necessary condition when \( \gamma \) is a quadrilateral, thus giving a negative solution to Kenyon’s problem in full generality. We then prove that domes exist over a dense set of integral curves. Finally, we give an explicit construction of domes over all regular \( n \)-gons.

1. Introduction

The study of polyhedra with regular polygonal faces is a classical subject going back to ancient times. It was revived periodically when new tools and ideas have developed, most recently in connection to algebraic tools in rigidity theory. In this paper we study one of most basic problems in the subject – polyhedral surfaces in \( \mathbb{R}^3 \) whose faces are congruent equilateral triangles. We prove both positive and negative results on the types of boundaries these surfaces can have, suggesting a rich theory extending far beyond the current state of the art.

Formally, let \( \gamma \subset \mathbb{R}^3 \) be a closed piecewise linear (PL-) curve. We say that \( \gamma \) is integral if it is comprised of intervals of integer length. Now, let \( S \subset \mathbb{R}^3 \) be a PL-surface realized in \( \mathbb{R}^3 \) with the boundary \( \partial S = \gamma \), and with all facets comprised of unit equilateral triangles. In this case we say that \( S \) is a unit triangulation or dome over \( \gamma \), that \( \gamma \) is spanned by \( S \), and that \( \gamma \) can be domed.

**Question 1.1** (Kenyon, see §6.2). Is every integral closed curve \( \gamma \subset \mathbb{R}^3 \) spanned by a unit triangulation? In other words, can every such \( \gamma \) be domed?

For example, the unit square and the (unit sided) regular pentagon can be domed by a regular pyramid with triangular faces. Of course, there is no such simple construction for a regular heptagon. Perhaps, surprisingly, the answer to Kenyon’s question is negative in general.

A 3-dimensional unit rhombus is a closed curve \( \rho \subset \mathbb{R}^3 \) with four edges of unit length. This is a 2-parameter family of space quadrilaterals \( \rho(a,b) \) parameterized by the diagonals \( a \) and \( b \), defined as distances between pairs of opposite vertices.

**Theorem 1.2.** Let \( \rho(a,b) \subset \mathbb{R}^3 \) be a unit rhombus with diagonals \( a,b > 0 \). Suppose \( \rho(a,b) \) can be domed. Then there is a nonzero polynomial \( P \in \mathbb{Q}[x,y] \), such that \( P(a^2,b^2) = 0 \).

In other words, for \( a,b > 0 \) algebraically independent over \( \mathbb{Q} \), the corresponding unit rhombus cannot be domed, giving a negative answer to Kenyon’s question. In fact, our tools give further examples of a unit rhombi which cannot be domed, such as \( \rho(\frac{\sqrt{5}}{2}, \frac{\sqrt{3}}{2}) \), see Corollary 4.9.

The following result is a positive counterpart to the theorem. We show that the set of integral curves spanned by a unit triangulation is everywhere dense within the set of all integral curves.

Let \( \gamma, \gamma' \subset \mathbb{R}^3 \) be two integral closed curves of equal length. We assume the vertices of \( \gamma, \gamma' \) are similarly labeled \([v_1, \ldots, v_n]\) and \([v'_1, \ldots, v'_n]\), giving a parameterizations of the curves. The Fréchet distance \( |\gamma, \gamma'|_F \) in this case is given by

\[
|\gamma, \gamma'|_F = \max_{1 \leq i \leq n} |v_i, v'_i|.
\]
**Theorem 1.3.** For every integral curve \( \gamma \subset \mathbb{R}^3 \) and \( \varepsilon > 0 \), there is an integral curve \( \gamma' \subset \mathbb{R}^3 \) of equal length, such that \( |\gamma, \gamma'|_F < \varepsilon \) and \( \gamma' \) can be domed.

The theorem above does not give a concrete characterization of domed integral curves, and such a characterization seems difficult (see \([3]\)). We conclude with one interesting special case:

**Theorem 1.4.** Every regular integral \( n \)-gon in the plane can be domed.

This gives a new infinite class of regular polygon surfaces, comprised of one regular \( n \)-gon and many unit triangles. See Section 3 for the proof and some previously known special cases.

**Outline of the paper.** We begin with a technical proof of Theorem 1.3 in Section 2. Our proof is constructive and almost completely self-contained except for the Steinitz Lemma with Bergström constant, see \([25, \S 34]\). In Section 3, we follow with a (much shorter) constructive proof of Theorem 1.4 which is almost completely independent of the previous section, except for the earlier analysis of rhombi which can be domed, see \([2,1]\).

In Section 4, we prove Theorem 1.2 by extending the results of Gaifullin brothers \([12]\). We assume that the reader is familiar with the theory of places, see e.g. \([20, \text{Ch. 1}] \) and \([25, \S 41.7]\), which played a key role in the solution of the bellows conjecture, see \([7]\) (see also \([25, \S 34]\)). Shifting gears once again, Section 5 is independent of the rest of the paper. Here we make a number of interrelated conjectures on the integral curves which can be domed, which we then relate to the rigidity theory and the Euclidean Ramsey theory. Final remarks are given in Section 6.

In the Appendix A, we include a negative solution of the question in \([12]\) on the dimension of the flexes of doubly periodic surfaces. This counterexample arose upon careful inspection of our proof of Theorem 1.2 and is of independent interest (cf. \([28]\)).

**Notation.** Let \( |vw| \) denote the length between \( v, w \in \mathbb{R}^3 \). We use \([v_1 \ldots v_n], \ v_i \in \mathbb{R}^d \), to denote a closed polygonal curve \( \gamma \subset \mathbb{R}^d \). We use parentheses notation \((a_1, \ldots, a_n)\), \( a_i > 0 \), to denote the edge lengths of \( \gamma \), i.e. \( a_i = |v_iv_{i+1}| \), and \( a_n = |v_nv_1| \). Denote by \( |\gamma| = a_1 + \ldots + a_n \in \mathbb{N} \) the length of the integral curve \( \gamma \).

Throughout the paper we consider PL-curves modulo rigid motions. All curves will be in \( \mathbb{R}^3 \), integral and closed, unless stated otherwise. Similarly, all PL-surfaces \( S \) will have unit triangles, unless stated otherwise. They are realized in \( \mathbb{R}^3 \) by the vertex coordinates and such realizations have no additional extrinsic constraints (such as being embedding or immersion, cf. \([6,2]\)).

2. Integral curves which can be domed are dense

Denote by \( \mathcal{M}_n \) the space of all integral curves of length \( n \) in \( \mathbb{R}^3 \), modulo rigid motions, which is compact in the Fréchet topology. Let \( \mathcal{D}_n \subset \mathcal{M}_n \) denote the subset of integral curves which can be domed. The goal of this section is to prove Theorem 1.3 which states that \( \mathcal{D}_n \) is dense in \( \mathcal{M}_n \).

The proof goes through several stages of simplification of integral curves, along the following route:

\[
\text{integral curves} \rightarrow \text{generic curves} \rightarrow \text{near planar curves} \rightarrow \text{compact near planar curves}.
\]

Compact curves are curves which fit inside a ball or radius \( 3/2 \) and they are much simpler to analyze by induction. At each stage, the simplification of curves is made by a sequence of certain local transformations. Starting the second arrow, these transformations are called flips and are obtained by attaching unit rhombi which can be domed. Making these reductions rigorous is somewhat technical and will occupy much of this section. The rhombi \( \rho \in \mathcal{D}_4 \) will play a special role, so we consider them first.

2.1. Dense rhombi. Throughout the paper, a unit closed curve of length 4 is called a unit rhombus, or just a rhombus. Each unit rhombus is determined by the diagonals \( a \) and \( b \); we denote such unit rhombus by \( \rho(a,b) \). Observe that \( a^2 + b^2 \leq 4 \), with the equality achieved on plane rhombi.

**Lemma 2.1.** Fix the diagonal \( a \), and suppose \( 0 < a < 2 \), \( a \notin \mathbb{Q} \). Then the set of values of \( b \geq 0 \) for which \( \rho(a,b) \in \mathcal{D}_4 \) is dense in \([0,\sqrt{4-a^2}]\). In particular, for every \( \varepsilon > 0 \), there is a unit rhombus \( \rho = \rho(a,b) \in \mathcal{D}_4 \), such that \( |\rho, \rho_0|_F < \varepsilon \), where \( \rho_0 = \rho(a,\sqrt{4-a^2}) \) is a convex plane unit rhombus.
Proof. Let $c_1 = 1$. Consider a unit rhombus $p_1 = p(a, c_1)$, spanned by two unit triangles. Define $p_2 = p(a, c_2)$ to be a rhombus obtained by attaching two copies of $p_1$ together, with a common diagonal of length $a$. Similarly, defined $p_3 = p(a, c_3)$, etc. Clearly, every rhombus $p_m, m \geq 1$, can be domed by surface with $2m$ unit triangles. We have:

$$c_n = \sqrt{4 - a^2} \cdot |\sin n\alpha|, \text{ where } \alpha := \arcsin(4 - a^2)^{-\frac{1}{2}}.$$  

Thus, set $\{c_m, m \geq 1\}$ is dense in $[0, \sqrt{4 - a^2}]$, for all $\alpha \notin \pi \mathbb{Q}$. Finally, we have $\alpha \notin \pi \mathbb{Q}$, since otherwise $\sin \alpha = (4 - a^2)^{-\frac{1}{2}} \in \mathbb{Q}$, a contradiction with the assumption that $a \notin \mathbb{Q}$. □

The integer $m$ in the proof will be called a multiplier throughout this section. We can now prove Theorem 1.3 for $|\gamma| = 4$. Let $\mathcal{X} := \{x > 0 : \arcsin(4 - x^2)^{-\frac{1}{2}} \in \pi \mathbb{Q}\}$.

Lemma 2.2. Let $\rho = p(a, b) \subset \mathbb{R}^3$ be a unit rhombus, and let $\varepsilon > 0$. Then there is a unit rhombus $\rho' = p(a', b') \in \mathcal{D}_4$, such that $|\rho, \rho'| \leq \varepsilon$. Moreover, if $a \notin \mathcal{X}$, one can take $a' = a$.

Proof. The second part follows from the above proof of Lemma 2.1. For the first part, choose $a \notin \mathcal{X}$, so that $|a - a'| < \varepsilon$. Then apply the construction as above. □

2.2. Reachable curves. Let us introduce some definitions and notation. Consider two integral curves $\gamma = [v_1 \ldots v_k \ldots v_n]$ and $\gamma' = [v_1' \ldots v_k' \ldots v_n']$, such that $[v_{k-1}v_kv_{k+1}v_k'] \in \mathcal{D}_4$. In this case we say that $\gamma$ and $\gamma'$ are $k$-flip connected, or just flip connected; write $\gamma \sim_k \gamma'$. Two integral curves $\gamma = [v_1 \ldots v_n]$ and $\gamma' = [v_1' \ldots v_n']$ are called flip equivalent, write $\gamma \sim \gamma'$, if

$$\gamma = \gamma_0 \to_{k_1} \gamma_1 \to_{k_2} \gamma_2 \to_{k_3} \cdots \to_{k_m} \gamma_m = \gamma',$$

for some integer sequence $k := (k_1, \ldots, k_N)$, where $1 \leq k_i \leq N$ for all $i = 1, \ldots, N$. See an example in Figure 1. Clearly, if $\gamma \sim \gamma'$ and $\gamma' \sim \gamma''$, then $\gamma \sim \gamma''$.

![Figure 1. Sequence of two flips $\gamma_1 \to \gamma_2 \to \gamma_3$, at $v$ and then at $w$. Here $\gamma_1 = [...vw...], \gamma_2 = [...v'w...], \text{ and } \gamma_3 = [...v'w'...]$](image)

We say that an integral curve $\gamma \subset \mathbb{R}^3$ of length $n$ is reachable, if for all $\varepsilon > 0$, there is an integral curve $\gamma' \subset \mathbb{R}^3$ of length $n$, such that $|\gamma, \gamma'| \leq \varepsilon$, and $\gamma' \in \mathcal{D}_n$. In this notation, Theorem 1.3 claims that all integral curves are reachable, while Lemma 2.2 proves this for curves of length 4.

Lemma 2.3. Let $\gamma \sim \gamma'$ are flip equivalent integral curves in $\mathbb{R}^3$. Suppose $\gamma$ is reachable. Then so is $\gamma'$.

In other words, the lemma says that if $\gamma \in \mathcal{M}_n$ is a limit point of $\mathcal{D}_n$, then so are all flip equivalent curves $\gamma \sim \gamma$.

Proof. Since $\gamma \sim \gamma'$, there is a flip sequence $k$ as in (2.1), and a sequence of multipliers $m = (m_1, \ldots, m_N) \in \mathbb{Z}^m$ counting how many pairs of unit triangles added at each flip. Use positive and negative integers $m_i$ to denote clockwise of counterclockwise direction of the flip $\gamma_{i-1} \to \gamma_i$. Thus, pair $(k, m)$ uniquely encodes the combinatorial structure of the flip equivalence. Let

$$\Phi_{k,m} : \mathcal{M}_n \to \mathcal{M}_n$$

be the map defining flip sequence as above. By construction, $\Phi_{k,m} : \mathcal{D}_n \to \mathcal{D}_n$, and $\Phi_{k,m}(\gamma) = \gamma'$.

Clearly, the map $\Phi_{k,m}$ is a composition of $|m_1| + \ldots + |m_N|$ continuous maps, and thus also continuous on $\mathcal{M}_n$. Since $\gamma$ is reachable, there is a sequence $\{\gamma(t) \to \gamma, t \in \mathbb{N}\}$ of converging curves $\gamma(t) \in \mathcal{D}_n$. Thus, we have another sequence of converging curves in $\mathcal{D}_n$:

$$\{\Phi_{k,m}(\gamma(t)) \to \gamma', t \in \mathbb{N}\},$$

which shows that $\gamma'$ is reachable. □
2.3. Generic curves. We say that an integral curve $\gamma = [v_1 \ldots v_n] \subset \mathbb{R}^3$ is generic, if all diagonals $|v_iv_j|$ are algebraically independent over $\mathbb{Q}$. Denote by $G_n \subset M_n$ the set of generic integral curves of length $n$.

**Lemma 2.4.** Let $\gamma \sim \gamma'$, where $\gamma \in G_n$ and $\gamma' \in M_n$. Then $\gamma' \in G_n$.

In other words, an integral curve that is flip equivalent to a generic integral curve, is also generic. The proof is straightforward and follows immediately from the algebraic formulas in the proof of Lemma 2.1.

**Lemma 2.5.** Set $G_n$ is dense in $M_n$.

**Proof.** Let $\gamma = [v_1 \ldots v_n] \in M_n$ be a integral curve in $\mathbb{R}^3$. Assume for now that no three adjacent vertices are collinear: $|v_i v_{(i+2) \mod n}| < 2$, for all $1 \leq i \leq n$. Consider a sequence of curves $\gamma = [v_1 v_2 \ldots v_n] \rightarrow [v_1' v_2 \ldots v_n] \rightarrow [v_1'' v_2' \ldots v_n] \rightarrow \ldots \rightarrow [v_1'' v_2' \ldots v_n] = \gamma'$, where at each step we perturb one vertex $v_k$ to a vertex $v_k'$, $|v_k v_k'| < \varepsilon$ such that, new coordinates of $v_k'$ are algebraically independent on all coordinates of $v_i'$, $i < k$ (but not of each other, of course). Observe that the resulting curve $\gamma'$ has now algebraically independent diagonals.

Assume now that $|v_{i-1} v_{i+1}| = 2$. Take a non-degenerate triangle $[v_i v_{i-1} v_{i+1}]$. Denote by $\zeta = [v_i \ldots v_j]$ a segment of $\gamma$. Rotate $\zeta \rightarrow \zeta' = [v_i' \ldots v_j']$ around $v_j$, so that $|v_{i-1} v_{i+1}^j| = 2 - \delta$, $|v_{i+1} v_{i+1}^j| < \delta$, and place $v_j'$ at unit distance from $v_{i-1}$ and $v_{i+1}$. This results in a new integral curve $\gamma' \in M_n$, s.t. $|\gamma',\gamma'|_F < \delta$. Proceed to do this for all $i$ as above. Taking $\delta > 0$ sufficiently small, gives an integral curve $\gamma'$ without collinearities, and s.t. $|\gamma',\gamma'|_F < \varepsilon$. The remaining cases when all vertices of $\gamma$ lie on a line are straightforward.

\[\square\]

2.4. Planar curves. An integral curve $\gamma \in M_n$ is called planar if it lies in a plane $H \subset \mathbb{R}^2$. Denote by $P_n \subset M_n$ the set of planar integral curves of length $n$.

**Lemma 2.6.** The flip equivalence class of every $\gamma \in G_n$ contains a planar curve $\xi \in P_n$ as its limit point.

In other words, for every $\varepsilon > 0$, and every generic integral curve $\gamma \in G_n$, there is a generic integral curve $\gamma' \in G_n$ and a planar integral curve $\xi \in P_n$, such that $\gamma \sim \gamma'$ and $|\gamma',\xi|_F < \varepsilon$. Note that the curve $\xi$ does not have to be generic itself, or be flip equivalent to $\gamma$.

**Proof.** The proof is based on the same idea of using flips to obtain a near-planar curve $\gamma'$. Let $\gamma = [v_1 \ldots v_n] \in G_n$, and let $\varphi : \mathbb{R}^3 \rightarrow \mathbb{R}$ be a generic linear function, i.e. defined by linear equations with coefficients that are algebraically independent to coordinates of $v_i$. Denote by $h_k := \varphi(v_k)$ the value to $\varphi$ on vertices of $\gamma$, $1 \leq k \leq n$. Cyclically, for all $k$ from $1$ to $n$, make $k$-flips:

\[\text{(2.3)}\]

$$\gamma = \gamma_0 \rightarrow \gamma_1 \rightarrow \gamma_2 \rightarrow \gamma_3 \rightarrow \ldots \rightarrow \gamma_n \rightarrow \gamma_{n+1} \rightarrow \gamma_{n+2} \rightarrow \gamma_3 \rightarrow \ldots$$

Choose integers $m$ (see the proof of Lemma 2.3) as follows. Consider a flip $\gamma_{jn+k-1} = [\ldots w_{k-1} w_k w_{k+1} \ldots] \rightarrow_k \gamma_{jn+k} = [\ldots w_{k-1} w'_k w_{k+1} \ldots]$.

By the proof of Lemma 2.1 we can always choose a multiplier $m_{jn+k}$ so that

\[\text{(2.4)}\]

$$\frac{2}{3} \alpha + \frac{1}{3} \beta < \varphi(w'_k) < \frac{1}{3} \alpha + \frac{2}{3} \beta,$$

where

$$\alpha := \min \{ \varphi(w_{k-1}), \varphi(w_{k+1}) \}, \quad \beta := \max \{ \varphi(w_{k-1}), \varphi(w_{k+1}) \}.$$

Note that we have $\alpha \neq \beta$, since $\varphi$ is generic, so there is always room to make such flip possible.

Using (2.4), it is easy to see that there is a limit $(\varphi(w_1), \ldots, \varphi(w_n)) \rightarrow (h, \ldots, h)$, for some $h \in \mathbb{R}$. Here the limit is when $N \rightarrow \infty$, where $N$ is the number of flips in (2.3). The limit curve $\xi$ is integral and lies in the plane $H := \{ x \in \mathbb{R}^3 : \varphi(x) = h \}$. Therefore, for $N = N(\varepsilon)$ large enough, we obtain a curve $\gamma' := \gamma_N$, such that $\gamma' \sim \gamma$, and $|\gamma',\xi| < \varepsilon$. By Lemma 2.4 we have $\gamma' \in G_n$, which completes the proof.

\[\square\]
2.5. **Packing curves.** Let \( u_1, \ldots, u_n \in \mathbb{R}^d \) be unit vectors which satisfy \( u_1 + \cdots + u_n = 0 \). The *Steinitz Lemma* famously states, see e.g. [3], that there is always a permutation \( \sigma \in S_n \), s.t.

\[
\left| u_{\sigma(1)} + \cdots + u_{\sigma(k)} \right| \leq B_d \quad \text{for all } 1 \leq k \leq n,
\]

where \( B_d \leq 2d \) is a universal constant which depends only on the dimension \( d \). Bergström [4] found the optimal value \( B_2 = \sqrt{5}/4 \), see also §6.5.

Motivated by the Steinitz Lemma, we define a similar notion for integral curves. Let \( \gamma = [v_1 \ldots v_n] \in \mathcal{M}_n \) be an integral curve in \( \mathbb{R}^d \). We say that \( \gamma \) is \( B \)-packing, if \( |v_iv_i| \leq B \) for all \( 1 \leq i \leq n \).

**Lemma 2.7.** Every generic integral curve \( \gamma \in \mathcal{G}_n \) is flip equivalent to a generic integral curve \( \gamma' \in \mathcal{G}_n \) that is \( 3/2 \)-packing.

Here the constant \( B = 3/2 \) is chosen somewhat arbitrary. In fact, any constant \( \sqrt{5}/4 < B < \sqrt{3} \) will satisfy the lemma and suffice for our purposes.

**Proof.** Fix \( \varepsilon > 0 \) and \( \gamma \in \mathcal{G}_n \). Let \( \gamma' \sim \gamma \) and \( \xi = [w_1, \ldots, w_n] \in \mathcal{P}_n \) be as in the proof of Lemma 2.6, so \( |\gamma', \xi| < \varepsilon \). Define \( u_i = w_{i}w_{i+1}, 1 \leq i < n \), and \( u_n = w_nw_1^\top \). Clearly, \( u_i \) are unit vectors which satisfy \( u_1 + \cdots + u_n = 0 \). By the Steinitz Lemma, there is a permutation \( \sigma \in S_n \), s.t. (2.5) holds. Consider a reduced factorization of \( \sigma \) into adjacent transpositions \( (i, i+1) \in S_n \):

\[
\sigma = (k_1, k_1 + 1) \cdots (k_\ell, k_\ell + 1),
\]

where \( 1 \leq k_1, \ldots, k_\ell \leq n - 1 \), and \( \ell = \text{inv}(\sigma) \) is the number of inversions in \( \sigma \), see e.g. [31]. This reduced factorization may not be unique, of course.

Define a sequence of flips as in (2.1), according to this factorization:

\[
\gamma' = \gamma_0 \rightarrow k_1 \gamma_1 \rightarrow k_2 \gamma_2 \rightarrow k_3 \gamma_3 \ldots \rightarrow k_\ell \gamma_\ell = \gamma''.
\]

Recall that \( \gamma_j \) remain generic by Lemma 2.4. Thus, by the second part of Lemma 2.4 and induction, we can always choose the multipliers \( m_j \) so that \( |\gamma_j, \xi| < \varepsilon, 1 \leq j \leq \ell \).

Now let \( \varepsilon \to 0 \). As in the proof of Lemma 2.3 by continuity of \( \Phi_{k,m} \) in (2.2), we have the limit planar curve \( \gamma'' \to \varrho := [y_1, \ldots, y_n] \in \mathcal{P}_n \). By induction on the length \( \ell \) of the factorization, we have:

\[
y_i y_{i+1} = u_{\sigma(i)}, 1 \leq i < n, \quad \text{and} \quad y_n y_1 = u_{\sigma(n)}.
\]

By the Steinitz Lemma with Bergström constant \( B_2 = \sqrt{5}/4 \), we conclude that for sufficiently small \( \varepsilon > 0 \), the integral curve \( \gamma'' \) is \( (B_2 + \delta) \)-packing, for all \( \delta > 0 \). Taking \( \delta < (3/2 - B_2) \), we obtain the result. \( \square \)

**Remark 2.8.** In a special case of the proof above, in a convex centrally symmetric \( n \)-gon \( \xi \in \mathcal{P}_n \), \( n = 2k \), the sequence of unit vectors is \( u_1, \ldots, u_k, -u_1, \ldots, -u_k \). Take a permutation which gives the order \( u_1, \ldots, u_k, -u_k, \ldots, -u_1 \); the corresponding limit curve \( \varrho \in \mathcal{P}_n \) is then degenerate. For every reduced factorization as in the proof, the pattern of rhombi used in the flip sequence then defines a *zonotopal tilings*, see e.g. [25, Exc. 14.25].

2.6. **Proof of Theorem 1.3.** We prove the result by induction. First, closed integral curve of length 3 is a unit triangle, so Theorem 1.3 is trivially true in this case. The case of length 4 is resolved in Lemma 2.2. Note that by Lemma 2.5, it suffices to prove the theorem only for generic curves \( \gamma = [v_1 \ldots v_n] \in \mathcal{G}_n \). Formally, we will show for all \( n \geq 5 \), the set \( \mathcal{D}_n \cap \mathcal{G}_n \) is dense in \( \mathcal{G}_n \). In fact, to make the inductive argument work we will need a stronger assumption.

Let \( n = 5 \), and let \( \gamma \in \mathcal{G}_5 \) be a generic integral pentagon. By Lemma 2.7, there is \( \gamma' = [w_1, \ldots, w_5] \in \mathcal{G}_5 \), such that \( |\gamma' - \gamma| \leq 3/2 \) for all \( 1 \leq i \leq 5 \). Since \( w_1w_3, w_3w_4, w_1w_5 \leq \sqrt{3} \), it is easy to see that the triangle \( Q = [w_1w_3w_4] \) can be covered with a unit circle in the plane \( H \) spanned by \( Q \) (see e.g. [23, Cor. 1.8]). Then there is a point \( z \in \mathbb{R}^3 \), s.t. \( |w_1z| = |w_3z| = |w_4z| = 1 \), see Figure 2 (left).

Apply now Lemma 2.2 to rhombi \( \rho_1 = [w_1w_2w_3z] \) and \( \rho_2 = [w_1w_5w_4z] \), to obtain rhombi \( \rho'_1 = [w_1w_2w_3z] \in \mathcal{D}_4 \) and \( \rho'_2 = [w_1w_5w_4z] \in \mathcal{D}_4 \), which satisfy \( |w_2w_3|, |w_5w_4| < \varepsilon \). Attach unit triangle \([w_3w_2z]\) to rhombi \( \rho'_1 \) and \( \rho'_2 \). This gives the desired pentagon \( \eta = [w_1w_2w_3w_4w_5] \in \mathcal{D}_5 \), s.t. \( |\eta', \eta| < \varepsilon \). Thus, \( \gamma' \) is reachable. By Lemma 2.3 then so is \( \gamma \), as desired.

The argument above gives a continuous deformation \( \{\eta(t), t \in [0,1]\} \), where \( \eta^{(0)} = \gamma' \), and \( \eta^{(1)} \in \mathcal{D}_5 \cap \mathcal{G}_5 \) for all but countably many \( t > 0 \). Observe that the construction is flexible enough to allow convergence of angles on both sides: \( \angle w_1w_3w_2 \to \langle \angle w_1w_2w_3 \rangle + \), and \( \angle w_5w_2w_3 \to \langle \angle w_1w_2w_3 \rangle - \). This conclusion is used in the inductive step given below.
For \( n \geq 6 \), we employ a similar argument. Let \( \gamma \in \mathcal{G}_n \) be a generic integral curve as above. By Lemma 2.7, there is \( \gamma' = [w_1, \ldots, w_n] \in \mathcal{G}_n \), such that \( \gamma' \sim \gamma \), and \( |w_1 w_i| \leq 3/2 \) for all \( 1 \leq i \leq n \). Since \( |w_1 w_4| < 2 \), there is a generic point \( z \in \mathbb{R}^3 \), such that \( |w_1 z| = |w_4 z| = 1 \), see Figure 2 (right). Consider integral curves \( \eta = [w_1 w_2 w_3 z] \in \mathcal{G}_5 \) and \( \phi = [w_1 zw_4 w_5 \ldots w_n] \in \mathcal{G}_{n-1} \). By induction, there is a continuous deformation \( \{\phi(t), t \in [0, 1]\} \), where \( \phi(0) = \phi \) and \( \phi(t) \in \mathcal{D}_{n-1} \cap \mathcal{G}_{n-1} \) for all but countably many \( t \). Without loss of generality, assume that \( \angle w_1 z w_4 \to (\angle w_1 z w_4) \) as \( t \to 0 \).

Similarly, by the \( n = 5 \) case, there is a continuous deformation \( \{\eta(s), s \in [0, 1]\} \), where \( \eta(0) = \eta \), and \( \eta(s) \in \mathcal{D}_5 \cap \mathcal{G}_5 \) for all but countably many \( s > 0 \). From the observation above, we can assume that \( \angle w_1 z w_4 \to (\angle w_1 z w_4) \) as \( s \to 0 \).

Attaching \( \phi(t) \) to \( \eta(s) \) with the same angle as above, we obtain a continuous deformation \( \{\gamma(t), t \in [0, 1]\} \), where \( \gamma(0) = \gamma' \) and \( \gamma(t) \in \mathcal{D}_{n-1} \cap \mathcal{G}_{n-1} \) for all but countably many \( t \). In particular, the curve \( \gamma' \in \mathcal{G}_n \) is reachable. By Lemma 2.3, we conclude that \( \gamma \) is also reachable, as desired. This completes the proof of the induction step and finishes the proof of the theorem. \( \square \)

### 3. Regular polygons

#### 3.1. Classical domes

Denote by \( Q_n \subset \mathbb{R}^2 \) the regular \( n \)-gon with unit sides in the \( xy \)-plane with the center at the origin \( O \). From the introduction, there is a trivial dome over \( Q_3 \) and \( Q_6 \), and domes over \( Q_4, Q_5 \) are given by regular pyramids. Less obviously, a tiling of \( Q_{12} \) given in Figure 3 (left), gives a natural dome over \( Q_{12} \), when square pyramids are added. Similarly, recall that the regular octagon \( Q_8 \) and decagon \( Q_{10} \) are spanned by the surfaces of Johnson solids square cupola and pentagonal cupola, respectively, see Figure 3 (right) and [16] for details.\(^1\) In fact, both are cuts of the Archimedean solids, see e.g. [9], p. 88. The faces of both surfaces are regular triangles, squares or pentagons. Adding a pyramid to each face we obtain domes over \( Q_8 \) and \( Q_{10} \).

![Figure 3. Left: Tiling giving a dome over \( Q_{12} \). Right: Pentagonal cupola giving a dome over \( Q_{10} \).](image)

\(^1\)The image on the right is available from the Wikimedia Commons, and is free to use with attribution.
3.2. Proof of Theorem 1.4. We follow notation in the proof of Lemma 2.1 and employ the symmetry of \( Q_n \) at every step.

First, attach a unit triangle to each side of \( Q_n \) at angle \( \theta > 0 \) to the plane. Make the angle \( \theta \) very small, to be chosen at a later point. Denote by \( a_1 \) the distances between vertices of adjacent unit triangles and assume that \( a_1 \notin \mathbb{Q} \). Note that \( a_1 > 0 \) is well defined for \( n \geq 7 \).

Moving along the boundary of \( Q_n \), attach to adjacent unit edges \( n \) rhombi \( R_1 = \rho_{m_1}(a_1,*) \). To simplify the notation, we use \((*)\) for the second diagonal, since it is completely determined by the multiplier \( m_1 \) and \( a \), see the proof of Lemma 2.1. Take \( m_1 \) large enough and chosen so that \( R_1 \) is nearly planar, at an angle \( \theta > \theta_1 \) with the plane. Such \( m_1 \) exists by Lemma 2.1 if we assume further that \( a_1 \notin \mathbb{Q} \).

Next, moving along the boundary, denote by \( a_2 \) the distances between vertices of adjacent rhombi \( R_1 \), and observe that \( a_2 \notin \mathbb{Q} \). Now attach to the adjacent unit edges rhombi \( R_2 = \rho_{m_2}(a_2,*) \), where the multiplier \( m_2 \) is large enough and chosen so that \( R_2 \) is nearly planar, at an angle \( \theta_2 > \theta_1 \), see Figure 4. Again, such \( m_2 \) exists by Lemma 2.1 if we assume further that \( a_2 \notin \mathbb{Q} \).

Repeat this procedure for \( k \) iterations, until the distance \( \beta \) to the vertical \( z \)-axis from new rhombi vertices satisfies \( \beta < \sqrt{1 - \alpha^2}/4 \). Here \( \alpha := a_{k+1} \) denotes the distance between vertices of adjacent rhombi \( R_k = \rho_{m_k}(a_k,*) \), and we assume that that \( \alpha \notin \mathbb{Q} \). The above bound on \( \beta \) corresponds to having the projection of the nearly planar rhombus \( R_k+1 \) cover the origin \( O \), see Figure 4 (center).

At this stage, attach to the adjacent unit edges new unit rhombi \( R = \rho_M(\alpha,*) \) in such a way that the new vertices are at distance \( \delta > 0 \) from the \( z \)-axis, see Figure 4 (center). By Lemma 2.1 distance \( \delta > 0 \) can be made as small as necessary.

Now, the construction above is uniquely determined by the angle \( \theta > 0 \) and the integer sequence

\[
m := (m_1, m_2, \ldots, m_k, M).
\]

Since the number of vectors \( m \) is countable, the assumptions \( a_i, \alpha \notin \mathbb{Q} \) over all \( m \) represent countably many inequalities on \( \theta \), so for some \( \theta > 0 \) the above construction is well defined.

The resulting (partial) surface \( S \) is continuously deformed with \( \theta \) for every fixed \( m \) (cf. the proof of Lemma 2.3). Continuously decreasing \( \theta > 0 \) and using the symmetry, we can place all the remaining free rhombi vertices onto the \( z \)-axis. This completes the construction of a dome over \( Q_n \) for all \( n \geq 7 \), and the smaller cases \( 3 \leq n \leq 6 \) are discussed above.

Finally, for a regular \( n \)-gon \( rQ_n \), replace unit triangles \( Q_3 \) with their scaled version \( rQ_3 \) and proceed as above. Now triangulate every copy \( rQ_3 \) with \( r^2 \) unit triangles \( Q_3 \), completing the construction of a dome over \( rQ_n \). □

**Figure 4.** Nearly planar tiling of a portion of \( Q_n \) with rhombi and its vertical slice.

**Remark 3.1.** Also, one can ask if a version of the arm lemma (see e.g. [25, §23]), holds in this case. We believe this to be true for every fixed \( m \), on a sufficiently small interval \( \theta \in (\theta_0 - \epsilon, \theta_0 + \epsilon) \), but this result is not necessary for the continuity argument in the proof.
4. The algebra of squared diagonals

4.1. Contractible domes. As a warm-up to the proof of Theorem 1.2, we first present a short argument for the case when the spanning surface $S$ is homeomorphic to a disc.

Proposition 4.1. Let $\gamma \subset \mathbb{R}^3$ be a unit rhombus $\gamma = \rho(s,t)$, with diagonal lengths $s$ and $t$. Suppose $\gamma$ can be domed by a surface homeomorphic to a disc. Then there exists a polynomial $P \in \mathbb{Q}[x,y]$, such that $P(s^2,t^2) = 0$.

For the proof of the proposition, we need to consider doubly periodic surfaces homeomorphic to the plane. Let $K$ be a simplicial connected pure 2-dimensional complex with a free action of the group $G = \mathbb{Z} \oplus \mathbb{Z}$ with generators $a$ and $b$. Assume that $G$ acts as a linear bijection on each simplex of $K$, and that the number of orbits of triangles under the action of $G$ is finite. Consider a mapping $\theta : K \to \mathbb{R}^3$, linear on each simplex of $K$, and equivariant with respect to the action of $\mathbb{Z} \oplus \mathbb{Z}$, such that $a$ and $b$ act by translations with vectors $\alpha$ and $\beta$, respectively. Then the pair $(K, \theta)$ is called a doubly periodic triangular surface. Sometimes, with a slight abuse of notation, we call the surface $K$ as well.

Now, let us construct a doubly periodic surface comprised of unit triangles for every unit triangulation of a unit rhombus.

Lemma 4.2. For a unit rhombus $\gamma = \rho(a,b)$ with diagonals $s$ and $t$, there is a doubly periodic surface of unit triangles with two orthogonal periodicity vectors of length $s$ and $t$, respectively. Moreover, there is such a surface homeomorphic to the plane if the triangulation of the rhombus spans a surface homeomorphic to a disc.

Proof. First we construct a doubly periodic surface whose cells are either parallel translates of $\gamma$ or $-\gamma$ (see Figure 6). This surface is combinatorially equivalent to a tiling of the plane with unit squares. A chessboard coloring of such a tiling makes white squares correspond to parallel translates of $\gamma$ and black squares correspond to parallel translates of $-\gamma$. The periodicity vectors of this surface are the vectors of the diagonals of $\gamma$.

Attaching a spanning unit triangulation to each translate of $\gamma$ and $-\gamma$ we obtain a doubly periodic polyhedral surface comprised of unit triangles with required periodicity vectors. Clearly, if a spanning triangulation of $\gamma$ is homeomorphic to a disk, the resulting doubly periodic surface is homeomorphic to the plane.

![Figure 5. Doubly periodic surface from translates of $\gamma$ and $-\gamma$.](image)

At this point we consider only doubly periodic triangular surfaces $(K, \theta)$ with unit triangular faces. For a fixed complex $K$, let $\mathcal{G}(K)$ be the set of all possible Gram matrices formed by vectors $\alpha$ and $\beta$ for all doubly periodic triangular surfaces $(K, \theta)$. For a Gram matrix $G \in \mathcal{G}(K)$, we denote its entries by $g_{11}$, $g_{12} = g_{21}$, and $g_{22}$.

Theorem 4.3 (Gaifullin–Gaifullin [12]). Let $K$ be a simplicial pure 2-dimensional complex homeomorphic to $\mathbb{R}^2$ with a free action of the group $\mathbb{Z} \oplus \mathbb{Z}$. Then there is a one-dimensional real affine algebraic subvariety of $\mathbb{R}^3$ containing $\mathcal{G}(K)$.

In particular, the entries of each Gram matrix $G$ from $\mathcal{G}(K)$ satisfy a one-dimensional system of two non-trivial polynomial equations with integer coefficients:

\[
\begin{align*}
p(g_{11}, g_{12}, g_{22}) &= 0 \\
q(g_{11}, g_{12}, g_{22}) &= 0.
\end{align*}
\]
Remark 4.4. In fact, the result in [12] is more general, as the authors consider all polygonal doubly periodic surfaces homeomorphic to the plane with arbitrary sets of side lengths. In this setting, the coefficients of polynomials $p$ and $q$ are obtained from the ideal generated by squares of all side lengths of polygons in the polygonal surface.

Proof of Proposition 4.1. If a unit rhombus with diagonals $s$ and $t$ can be spanned by unit triangles then, by Lemma 4.2 there is a doubly periodic triangular surface with orthogonal periodicity vectors of length $s$ and $t$. The entries of the Gram matrix of periodicity vectors are $g_{11} = s^2$, $g_{12} = 0$, $g_{22} = t^2$. By Theorem 4.3 there are two polynomials $p$, $q$ with integer coefficients vanishing on the entries of the Gram matrix. Thus, at least one of the equations $p(s^2, 0, t^2) = 0$ and $q(s^2, 0, t^2) = 0$ is non-trivial, and can be used for the polynomial $P$. □

4.2. Theory of places. There is no result generalizing Theorem 4.3 for doubly periodic surfaces of non-trivial topology and, moreover, we will show in Theorem A.2 that such a generalization is not true. However, for our purposes, we do not need two polynomials $p$ and $q$ as in Theorem 4.3. It is sufficient to find at least one polynomial that is non-trivial whenever $g_{12} = 0$. The machinery developed in [12] is based on the proof of the bellows conjecture for orientable 2-dimensional surfaces [7], and is also the basis for our approach. We use places of fields as the main algebraic instrument of the proof.

Let $F$ be a field and $\hat{F}$ be $F$ extended by $\infty$, i.e. $\hat{F} = F \cup \{\infty\}$ with arithmetic operations extended to $\hat{F}$ by

$$a \pm \infty = \infty \quad \text{and} \quad \frac{a}{\infty} = 0, \quad \text{for all} \quad a \in F,$$

$$a \cdot \infty = \frac{a}{0} = \infty \quad \text{for all} \quad a \in \hat{F} \setminus \{0\}.$$

The expressions

$$0 \div \infty, \quad \frac{0}{\infty}, \quad 0 \cdot \infty \quad \text{and} \quad \infty \pm \infty$$

are not defined.

Let $L$ be a field. A map $\phi : L \to \hat{F}$ is called a place if $\phi(1) = 1$ and

$$\phi(a \pm b) = \phi(a) \pm \phi(b), \quad \phi(a \cdot b) = \phi(a) \cdot \phi(b) \quad \text{for all} \quad a, b \in L,$$

whenever the right-hand side expressions are defined.

As a direct consequence of the definition, we have $\phi(0) = 0$ for all places, and $\phi(x) = \infty$ for $x \neq 0$ if and only if $\phi(-x) = \infty$ and if and only if $\phi(1/x) = 0$. It is also clear that whenever $\operatorname{char} F = 0$, we must also have $\operatorname{char} L = 0$. Similarly, we have $\phi(kx) = \infty$ for a non-zero $k \in \mathbb{Z}$, if and only if $\phi(x) = \infty$.

We will use the following basic fact on extensions of places.

Lemma 4.5 (see e.g. [20 Ch. 1, Thm 1]). Let $L$ be a field containing a ring $R$. Let $\phi$ be a homomorphism of $R$ in an algebraically closed field $\Omega$, and suppose $\phi(1) = 1$. Then $\phi$ can be extended to a place $L \to \Omega \cup \{\infty\}$.

4.3. General domes. For a doubly periodic triangular surface $(K, \theta)$, let $\alpha$ and $\beta$ be the periodicity vectors of the surface. The number of orbits under the action of $G = \mathbb{Z} \oplus \mathbb{Z}$ is finite, so we choose a representative of each orbit. Let $(x_1, y_1, z_1), \ldots, (x_N, y_N, z_N)$ be their coordinates in $\mathbb{R}^3$, and let $(x_\alpha, y_\alpha, z_\alpha), \ (x_\beta, y_\beta, z_\beta)$ be the coordinates of the periodicity vectors. Define field $L$ as follows:

$$L := \mathbb{Q}\langle x_1, y_1, z_1, \ldots, x_N, y_N, z_N, x_\alpha, y_\alpha, z_\alpha, x_\beta, y_\beta, z_\beta \rangle.$$

Note that $L$ does not depend on the choice of representatives of orbits and the choice of the basis for the lattice $\Lambda$.

When vertices $a, b$ on the surface $(K, \theta)$ form an edge, denote by $\ell_{ab}$ the squared distance between them:

$$\ell_{ab} := (x_a - x_b)^2 + (y_a - y_b)^2 + (z_a - z_b)^2.$$

Clearly, $\ell_{ab} \in L$. For each surface the set of all possible $\ell_{ab}$ is finite. Let $R$ be the $\mathbb{Q}$-subalgebra of $L$ generated by all $\ell_{ab}$ of the surface.

Let $\Lambda$ be the lattice generated by $\alpha$ and $\beta$. All vectors in $\Lambda$ can be written as integer linear combinations of the periodicity vectors, $\lambda = k\alpha + m\beta$. In case $k$ and $m$ are relatively prime, a vector $\lambda$ is called primitive. Denote by $\Lambda^*$ the set of primitive vectors $\lambda \in \Lambda$. 
As the first step in the proof of Theorem 4.7, we prove the lemma on finite elements of places.

**Lemma 4.6** (Main lemma). For a doubly periodic triangular surface \((K, \theta)\) obtained by the construction in Lemma 4.2, let \(\phi : \mathbb{R}^3 \to F \cup \{\infty\}\) be a place that is finite on all \(\ell_{ab}\) defined by the surface and let \(\text{char} F = 0\). Then there is a vector \(\lambda \in \Lambda, \lambda \neq 0\), such that \(\phi\) is finite on \((\lambda, \lambda)\).

For the proof, we use the following technical result of Connelly, Sabitov, and Walz [7, Lemma 4], see also [25, §34.3].

**Theorem 4.7** (Connelly–Sabitov–Walz). Let \(u\) be a vertex of a triangular surface in \(\mathbb{R}^3\) and \(v_1, \ldots, v_d, d \geq 4\), be adjacent to it in this cyclic order, denote also \(v_{d+1} = v_1\) and \(v_{d+2} = v_2\). Let \(\phi\) be a place that is defined on \(Q(x_u, y_u, z_u, x_{v_1}, y_{v_1}, z_{v_1}, \ldots, x_{v_d}, y_{v_d}, z_{v_d})\) and is finite on all \(\ell_{uv}, \ell_{v_{i+1}v_i}, 1 \leq i \leq d\). Then \(\phi\) is finite on at least one of the squared diagonal lengths \(\ell_{v_{i+1}v_i}, 1 \leq i \leq d\).

### 4.4. Proof of the Main Lemma 4.6

The statement of the lemma is true if one of the edges of the surface forms a vector from \(\Lambda\). We define the **complexity** as a partial ordering of doubly periodic triangular surfaces with the same periodicity lattice \(\Lambda\). Surfaces with edges from \(\Lambda\) are called the **least complex** (an example is given in Figure 6). For surfaces without edges from \(\Lambda\), the ordering is defined as follows.

A surface \(K_1\) is said to be **less complex** than \(K_2\), if the Euler characteristic of \(K_1/\Lambda\) is greater than the Euler characteristic of \(K_2/\Lambda\). The surface \(K_1\) is less complex than \(K_2\) if \(\chi(K_1/\Lambda) = \chi(K_2/\Lambda)\), and \(K_1/\Lambda\) has fewer vertices than \(K_2/\Lambda\). The surface \(K_1\) is less complex than \(K_2\) if \(K_1/\Lambda\) and \(K_2/\Lambda\) have the same Euler characteristic and the same number of vertices, but the smallest vertex degree of \(K_1\) is less than the smallest vertex degree of \(K_2\). The proof will proceed by induction on complexity.

**First case.** Suppose the surface contains the edges \(ab, bc, ca\), but does not contain a triangle \([abc]\). The closed curve formed by the edges \(ab\), \(bc\) and \(ca\), divides its neighborhood into two components. Then we define the surgery along \([abc]\) by removing vertices \(a, b, c\), edges \(ab, bc, ca\), and adding two copies of \([abc]\), which we call \([a'b'c']\) and \([a''b''c'']\). We do this in such a way that \([a'b'c']\) and \([a''b''c'']\) retain the incidences of \(a, b, c\) in the first and the second component of the neighborhood, respectively. If this surgery keeps the surface connected, then it increases the Euler characteristic of \(K/\Lambda\). If the surgery splits the surface into two new surfaces then the Euler characteristic for each of them is not smaller than the initial Euler characteristic, for both of them there are fewer vertices than for the initial surface, and at least one of them is a connected doubly periodic triangular surface with the periodicity lattice \(\Lambda\). We call the latter the **connectivity property**, see Figure 6.

![Figure 6. Connectivity property in the First Case of a doubly periodic surface.](image-url)
making its corresponding surface connected (cf. Remark \ref{rem} below). Since the set of $\ell_{ab}$ for either surface is a subset of the initial set, we can use the inductive step.

**Second case.** Suppose there are no triples of vertices $a, b, c$ as in the first case. Consider a vertex $u$ of the surface with the smallest degree $d$ adjacent to vertices $v_1, \ldots, v_d$. The smallest degree must be at least 4, because the first case holds otherwise. We use Theorem \ref{thm} but we have to be careful with applying it as the field in Theorem \ref{thm} is not a subfield of the field $L$ defined earlier. The issue is that some vertices $v_i$ and $v_j$ may belong to the same orbit under the action of the lattice $\Lambda$.

Let $R$ be a $Q$-subalgebra of $K = \mathbb{Q}(x_u, y_u, z_u, v_1, y_{v_1}, \ldots, x_{v_d}, y_{v_d}, z_{v_d})$ generated by all $\ell_{uv_i}$, $\ell_{v_iv_{i+1}}$, $1 \leq i \leq d$. There is a natural homomorphism $\psi$ from $R$ to $L$ mapping all elements of $R$ to their corresponding expressions in $L$. Note that this homomorphism is not necessarily defined on all elements of $L$. For example, when $v_3 = v_1 + \alpha$ and $v_6 = v_4 + \alpha$, the image of $1/(x_{v_3} + x_{v_4} - x_{v_1} - x_{v_6})$ is not defined.

At this point, we use Lemma \ref{lem} and extend $\psi$ to $\overline{\psi} : K \to \overline{L} \cup \{\infty\}$. The place $\phi$ can be also extended to a place $\overline{\phi} : \overline{L} \to \overline{F} \cup \{\infty\}$. In order to construct this extension, we apply Lemma \ref{lem} to a subring of all elements of $L$ whose images under $\phi$ are finite. For the constructed mapping, $\overline{\phi}(x) = 0$ if $\phi(x) = 0$. Subsequently, if $\phi(x)$ is $\infty$, $\overline{\phi}(x)$ must be $\infty$ as well and $\overline{\phi}$ extends the whole place $\phi$.

Using $\overline{\phi}(\infty) = \infty$, we can define the composition $\overline{\phi} \circ \overline{\psi}$. This composition is the place from $K$ to $\overline{F} \cup \{\infty\}$. Applying Theorem \ref{thm} to $\overline{\phi} \circ \overline{\psi}$ we conclude that there is $i$ such that the composition and, subsequently, $\phi$ is finite on $\ell_{v_iv_{i+2}}$.

For the next step, we substitute two triangles of the surface, $uv_i v_{i+1}$ and $uv_{i+1} v_{i+2}$ with $uv_i v_{i+2}$ and $v_i v_{i+1} v_{i+2}$ simultaneously deleting the edge $uv_{i+1}$ and adding the edge $v_i v_{i+2}$. There was no edge $v_i v_{i+2}$ prior to this operation because otherwise the triangle $uv_i v_{i+2}$ would satisfy the case considered above. At the same time we make the same operations for all triangles that are the images of $uv_i v_{i+1}$ and $uv_{i+1} v_{i+2}$ under the action of $\Lambda$. As the result we obtain another surface $K'$ such that $K'/\Lambda$ is topologically the same as $K/\Lambda$ and has the same number of vertices but the minimum vertex degree of $K'$ is smaller. The place $\phi$ is still finite on all $\ell_{ab}$ for edges $ab$, so all conditions of the lemma still hold.

Observe that the operations in both cases decrease the complexity of the surface. Note that this cannot continue indefinitely since the Euler characteristic is at most 2, and the number of edges and vertex degrees are positive. Thus, at some point we reach the least complex surface for which the statement of the lemma is true. \hfill $\square$

**Remark 4.8.** In the proof of the First Case, the connectivity property fails for general doubly periodic surfaces. In particular, if the elements of the fundamental group of $K/\Lambda$ corresponding to periodicity vectors do not commute, two new surfaces may be both disconnected unions of one-periodic pieces. An example is given in Figure \ref{fig}. Here we show only the bottom half of the surface, which has connected components periodic along $\alpha$. The top half is attached to the bottom along red triangles and has similar structure, but with connected component periodic along $\beta$. This observation will prove crucial in the proof of Theorem \ref{thm} in the Appendix.

![Figure 7. Non-example to the connectivity property in the First Case for general doubly periodic surfaces.](image-url)
4.5. Proof of Theorem 1.2. Let \( R' \) be the \( \mathbb{Q} \)-subalgebra of \( \mathbb{L} \) obtained by adding all \( (\lambda, \lambda)^{-1}, \lambda \in \Lambda^* \), to the subalgebra \( R \): \[
 R' = R \left[ (\lambda, \lambda)^{-1} \mid \lambda \in \Lambda^* \right].
\]

Let \( I' \) be the following ideal in \( R' \): \[
 I' = \left( (\lambda, \lambda)^{-1} \mid \lambda \in \Lambda^* \right) \triangleleft R'.
\]

Assume that \( I' \neq R' \). Then, by Krull's theorem (see e.g. [2]), there exits a maximal ideal \( I \), such that \( I' \subset I \). Let \( F = R'/I \). Since \( R' \) contains \( \mathbb{Q} \), field \( F \) must contain \( \mathbb{Q} \) as well and char \( F = 0 \). Let \( \bar{F} \) be an algebraic closure of \( F \). The quotient homomorphism \( R' \to F \) satisfies the conditions of Lemma 4.5 for \( \Omega = \bar{F} \) so it can be extended to the place \( \phi : R' \to \bar{F} \cup \{ \infty \} \). The quotient homomorphism is equal to 0 on all \( (\lambda, \lambda)^{-1}, \lambda \in \Lambda^* \). Therefore, the place \( \phi \) is infinite on \( (\lambda, \lambda) \) for all \( \lambda \in \Lambda^* \). This implies that the same holds for all non-zero \( \lambda \in \Lambda \). On the other hand, the quotient homomorphism is finite on \( R' \). Therefore, we get a contradiction with Lemma 4.6. We conclude that the assumption that \( I' \neq R' \) is false.

From above, we have that \( I' = R' \). In particular, this implies that \( 1 \in I' \):

\[
 (4.1) \quad 1 = \sum_{i=1}^{M} \frac{r_i}{(\lambda_{i1}, \lambda_{i1})(\lambda_{i2}, \lambda_{i2}) \cdots (\lambda_{ip_i}, \lambda_{ip_i})},
\]

where all \( \lambda_{ij} \in \Lambda^* \), and all \( r_i \in R \). After multiplying by the least common multiple of all denominators, the left hand side of (4.1) becomes

\[
 Z := \prod_{j=1}^{N} (\lambda_j, \lambda_j) = \prod_{j=1}^{N} (k_j \alpha + m_j \beta, k_j \alpha + m_j \beta) = \prod_{j=1}^{N} (k_j^2 \alpha^2 + 2k_j m_j \alpha \beta + m_j^2 \beta^2),
\]

where \( \lambda_j = k_j \alpha + m_j \beta \).

In the same manner we can write down the products in the right hand side of (4.1) times \( Z \). We rewrite the equation via the entries of the Gram matrix of the lattice \( \Lambda \), which are equal to \( g_{11} = (\alpha, \alpha) = s^2 \), \( g_{22} = (\beta, \beta) = t^2 \), and \( g_{12} = 0 \). We also use the fact that polynomial functions \( r_i \in R \) take only rational values on doubly periodic unit triangular surfaces, and denote by \( q_i \in \mathbb{Q} \) the value of \( r_i \) on the surface \((K, \theta)\). We then have:

\[
 (4.2) \quad \prod_{j=1}^{N} (k_j^2 s^2 + m_j^2 t^2) - \sum_{i=1}^{M} q_i \prod_{j=1}^{N} (k_j^2 s^2 + m_j^2 t^2) = 0,
\]

where all \( N_i < N \). Note that this is the only time in the proof we use the fact that we have unit triangles, and that the periodicity vectors are orthogonal.

We conclude that the polynomial \( P(s^2, t^2) \) formed by the equation (4.2) has rational coefficients. Let \( x \leftarrow s^2 \) and \( y \leftarrow t^2 \). From above, the polynomial \( P(x, y) \) is nonzero since

\[
 \deg \prod_{j=1}^{N} (k_j^2 x + m_j^2 y) = N,
\]

and the degree of all other terms in (4.2) have degrees \( N_i < N \). This completes the proof. \( \square \)

4.6. Further applications. Note that polynomials \( P \) found in the proof of Theorem 1.2 are quite special. In some cases, with a more careful analysis, one can conclude non-existence of domes for some rhombi whose diagonals are algebraically dependent over \( \mathbb{Q} \). For example, consider the rhombi whose ratio of diagonal lengths is algebraic:

**Corollary 4.9.** Let \( s \notin \overline{\mathbb{Q}} \) and \( t/s \in \overline{\mathbb{Q}} \). Then the unit rhombus \( \rho(s, t) \) cannot be domed.

For example, the corollary implies that \( \rho(\frac{1}{2}, \frac{1}{2}) \) and \( \rho(\frac{\sqrt{2}}{\sqrt{3}}, \frac{\sqrt{3}}{\sqrt{2}}) \) cannot be domed.

**Proof.** Suppose \( \rho(s, t) \in \mathcal{D}_4 \), and let \( c := t/s \in \overline{\mathbb{Q}} \). Consider a polynomial \( P(s^2, t^2) \), as in the proof of Theorem 1.2. Viewed as a polynomial in \( x = s^2 \) over \( \overline{\mathbb{Q}} \) the leading degree term of \( P \) becomes

\[
 \prod_{j=1}^{N} (k_j^2 x + m_j^2 c^2 x),
\]
so it still has a higher degree than all other terms. Therefore, \( s \in \overline{\mathbb{Q}} \), a contradiction.

The following is a generalization of the previous corollary:

**Corollary 4.10.** Let \( s \notin \overline{\mathbb{Q}} \), and let \( s^2 \) and \( t^2 \) be algebraically dependent with the minimal polynomial \( P(s^2, t^2) = 0 \). Suppose \( P \in \overline{\mathbb{Q}}[x, y] \) is given by

\[
P(x, y) = x^k y^{m-k} + \sum_{i+j<m} c_{ij} x^i y^j,
\]

for some \( 0 \leq k \leq m \). Then the unit rhombus \( \rho(s, t) \) cannot be domed.

More generally, the same conclusion holds for

\[
P(x, y) = x^k y^{m-k} + \sum_{i=0}^{k-1} b_i (m-i) x^i y^{m-i} + \sum_{i+j<m} c_{ij} x^i y^j,
\]

such that \( b_{ij} \leq 0 \) for all \( i + j = m \), \( 0 \leq i < k \).

For example, the corollary implies that \( \rho(\frac{1}{\sqrt{2}}, \sqrt{3}) \) and \( \rho(\frac{1}{\sqrt{3}} - \frac{\sqrt{3}}{2}, \sqrt{3}) \) cannot be domed.

**Proof.** The proof follows the previous pattern. In the first case, note that the maximal degree condition (4.3) is chosen in such a way that the leading term in (4.2) remains non-zero. This implies the result. In the second, more general case, maximal degree condition (4.3) is replaced with (4.4) which also remains nonzero since the signs do not allow cancellations. The details are straightforward.

**Remark 4.11.** The approach in the corollaries fails in two notable cases we cover in the next section. First, by Proposition 5.2 below, a triangle \( \Delta \) with side lengths \( (2, 2, 1) \) can be domed if and only if a unit rhombus \( \rho(\frac{1}{2}, \sqrt{3}) \) can be domed. Since the argument above is not applicable for \( \rho(s, t) \) for which \( s^2, t^2 \in \mathbb{Q} \), we cannot conclude that \( \Delta \) cannot be domed, cf. Conjecture 5.1.

The second example where the above approach is applicable is the case of planar rhombi \( \rho(s, t) \), where \( s^2 + t^2 = 4 \), see §5.3. In fact, one of the product terms \( k_j^2 s^2 + m_j^2 t^2 = (k_j^2 - m_j^2) d^2 + 4m_j^2 \) of the leading degree term in \( P \) can be equal to 4 when \( k_j = \pm 1 \) and \( m_j = \pm 1 \).

5. Big picture

### 5.1. Integer-sided triangles

It may seem from the proof of Theorem 1.2 that only integral curves with non-algebraic diagonals cannot be domed. In fact, we believe that only very few curves can be domed.

**Conjecture 5.1.** An isosceles triangle \( \Delta \) with side lengths \( (2, 2, 1) \) cannot be domed.

As many other domes on curves problems, this conjecture turned out to be equivalent to that over a certain unit rhombus.

**Proposition 5.2.** Let \( \Delta \) be an isosceles triangle with side lengths \( (2, 2, 1) \), and let \( \rho_o = \rho(\frac{1}{2}, \sqrt{3}) \). Then \( \Delta \) can be domed if and only if \( \rho_o \) can be domed.

**Proof.** Attach three unit triangles to \( \Delta \) as in Figure 8. Observe that the boundary of the resulting surface is exactly \( \rho_o \).

![Figure 8](image-url)

**Figure 8.** Proof of Proposition 5.2: \( \Delta \in \mathcal{D}_5 \) if and only if \( \rho_o \in \mathcal{D}_4 \).

In a contrapositive fashion, let us show that if \( \Delta \in \mathcal{D}_5 \), then all integral triangles can be domed.
Proposition 5.3. Let $\Delta$ be an isosceles triangle with side lengths $(2, 2, 1)$. If $\Delta$ can be domed, then so can every integer-sided triangle.

Proof. Whenever clear, we denote polygons with their edge length sequence. Observe that all triangles $(k, k, k)$ and all trapezoids $(1, \ell, 1, \ell + 1)$ can be domed by a plane triangulation. To construct domes over all integer-sided triangles, we use the following rules:

\begin{enumerate}
\item for integer $k > 1$, $1 \leq \ell < \sqrt{3}k$, two copies of $(k, k, 1)$, one $(k, k, \ell)$, and a $(1, \ell, 1, \ell + 1)$ trapezoid, give a triangle $(k, k, \ell + 1)$ via a construction as above,
\item for integer $k > 1$, $\ell < \sqrt{3}k$, two copies of $(k, k, \ell)$, and one $(k, k, 1)$, give a triangle $(\ell, \ell, 1)$ via a tetrahedron.
\end{enumerate}

We now construct all triangles $(k, k, 1)$ one by one, alternating the rules above in the following order:

$$\Delta = (2, 2, 1) \rightarrow (1) (2, 2, 3) \rightarrow (2) (3, 3, 1) \rightarrow (1) (3, 3, 4) \rightarrow (2) (4, 4, 1) \rightarrow \ldots$$

Next, we construct domes over general isosceles triangles $(k, k, \ell)$, for all $1 \leq \ell < k$, as follows:

$$(k, k, 1) \rightarrow (1) (k, k, 2) \rightarrow (1) (k, k, 3) \rightarrow (1) \ldots$$

Finally, we can span $(a, b, c)$ using triangles $(k, k, a)$, $(k, k, b)$ and $(k, k, c)$, for $k \geq \max\{a, b, c\}$ large enough. This completes the proof. \[\square\]

Remark 5.4. Suppose, contrary to Conjecture 5.1, that a triangle $(2, 2, 1)$ can be domed. That would easily imply Theorem 1.4. Indeed, let $r > r_{Q_n}$ be an integer greater than the radius of $r_{Q_n}$. By Proposition 5.3, triangle $(r, r, r)$ can also be domed. Symmetrically attach these triangles to all edges in $r_{Q_n}$, to form a pyramid over $r_{Q_n}$.

5.2. Flexible surfaces. Let $S \subset \mathbb{R}^3$ be a PL-surface homeomorphic to a sphere, and whose faces are unit triangles. We say that $S$ is a closed dome. Such $S$ is called flexible, if there is a continuous family $\{S_t, t \in [0, 1]\}$ of (intrinsically) isometric but non-congruent closed domes; closed dome $S$ is called rigid otherwise.

Conjecture 5.5. Every closed dome $S \subset \mathbb{R}^3$ is rigid.

Curiously, this general conjecture implies Conjecture 5.1 which at first glance might seem unrelated.

Proposition 5.6. Conjecture 5.5 implies Conjecture 5.1.

Proof. By contradiction, suppose Conjecture 5.1 is false. In other words, suppose triangle $\Delta$ with side lengths $(2, 2, 1)$ can be domed. By Proposition 5.3, then so can every integer-sided triangle, including triangles with sides $(3, 7, 7)$ and $(4, 7, 7)$, respectively. Four copies of each triangle can be attached to form a flexible Bricard octahedron (see e.g. [25, §30.4]), refuting Conjecture 5.5. \[\square\]

Remark 5.7. We believe that the rigidity claim in the conjecture can be replaced with infinitesimal rigidity, see e.g. [25, §33]. This is a weaker notion, thus giving a stronger conjecture.

5.3. Planar unit rhombi. Denote by $A$ the set of all $a \geq 0$, such that the planar rhombus $\rho(a, \sqrt{4-a^2})$ can be domed. It follows from Lemma 2.1 that $X \subseteq A$, so $A$ is infinite.

Conjecture 5.8. Set $A$ is countable.

The following result is our only evidence in favor of this conjecture.

Proposition 5.9. Conjecture 5.5 implies Conjecture 5.8.
There are unit triangles
Conjecture 5.14. if the theory of places can be applied to the following problem:
two triangles is especially important in view of the
Steinhaus problem
\[\gamma\]
Let
Conjecture 5.13. following conjecture:
Domes over multi-curves.
5.5.
\[\ldots\]
Consider a closed 2-manifold
\[M\]
is a realization, the space of surfaces with this combinatorial type (modulo rigid motions, as always). Since the limit
Conjecture 5.5.
\[\Box\]
We have:
Conjecture 5.10.
If the planar integral rhombus
Proposition 5.11.
 cannot be domed.

Proof. Observe that 2
Proof. In the spirit of the proof Theorem 1.3, there is a natural way to split Conjecture 5.15 into two parts.
Conjecture 5.16. For every integral curve $\gamma \in \mathcal{M}_n$, there is a finite set of unit rhombi $\rho_1, \ldots, \rho_k \in \mathcal{M}_4$, and a dome over $\gamma \cup \rho_1 \cup \cdots \cup \rho_k$.

Conjecture 5.16 is of independent interest. If true, it reduces Conjecture 5.15 to the following claim:

Conjecture 5.17. For every two unit rhombi $\rho_1, \rho_2 \in \mathcal{M}_4$, there is a unit rhombus $\rho_3 \in \mathcal{M}_4$ and a dome over $\rho_1 \cup \rho_2 \cup \rho_3$.

5.6. General algebraic dependence. While much of the paper and earlier conjectures are largely concerned with reducing the problem to domes over rhombi, there is another direction one can explore. Namely, one can ask if Theorem 1.2 can be generalized to all integral curves.

Let $\gamma = [v_1 \ldots v_n] \in \mathcal{M}_n$ be an integral curve. Denote by $\mathbb{L}_n = \mathbb{Q}[x_{1,3}, x_{1,4}, \ldots, x_{n-2,n}]$ the ring of rational polynomials with variables corresponding to diagonals of $\gamma$. Let $\text{CM}_n \subset \mathbb{L}_n$ be the ideal spanned by all Cayley–Menger determinants on vertices $\{v_1, \ldots, v_n\}$, see [7] and [25, §41.6]. We can now formulate the conjecture.

Conjecture 5.18. Let $\gamma = [v_1 \ldots v_n] \in \mathcal{D}_n$ be an integral curve which can be domed, where $n \geq 5$. Denote by $d_{ij} = |v_iv_j|$ the diagonals of $\gamma$, where $1 \leq i < j \leq n$. Then there is a nonzero polynomial $P \in \mathbb{L}_n$, $P \notin \text{CM}_n$, such that $P(d^2_{1,3}, d^2_{1,4}, \ldots, d^2_{n-2,n}) = 0$.

This conjecture can be viewed as a direct analogue of Sabitov’s theory of volume being algebraic over squared diagonal lengths, see §6.6. It would be interesting to see if this result can be obtained by expending our argument in Section 4. Perhaps, Conjecture 5.18 could be used to deduce Conjecture 5.15 from Theorem 1.2.

6. Final remarks

6.1. Our choice of terminology “dome over curve $\gamma$” owes much to the architectural style of the iconic geodesic domes popularized by Buckminster Fuller, and his ill-fated 1960 proposal of a Dome over Manhattan, see e.g. [5, pp. 321–324].

6.2. Kenyon formulated Question 1.1 in [19] Problem 2, in an undated webpage going back to at least April 2005. It is best understood in the context of regular polygonal surfaces (see e.g. [11, 8]). While we study only the weaker notion (realizations), both the immersed and the embedded surfaces can be considered, as they add further constraints to the domes. Note also that combinatorially, a dome is a unit distance complex of dimension two [17], a notion generalizeing the unit distance graphs in §5.4.

6.3. One can view the proof of Theorem 1.2 as a rigidity result for 2-surfaces with unit triangular faces and a single boundary. There are several related rigidity results for surfaces with square and regular pentagonal faces, see e.g. [11, 10].

6.4. It is quite possible that Conjecture 5.13 is false while Conjecture 5.10 is true, since the former seems much stronger. This conjecture is partly motivated by our early attempts to use Monsky’s valuation approach [22, 23], to obtain a negative answer to Kenyon’s Question 1.1.

6.5. The Steinitz Lemma mentioned in §2.5 is a special case of the remarkable 1913 result by Steinitz, motivated by Riemann’s study of conditionally convergent series of real numbers. Bergström’s lower bound $B_2 \geq \sqrt{5}/4$ comes from taking unit edge vectors in the $(k, k, 1)$ triangle, while the matching upper bound is based on elementary arguments in plane geometry. For general $d$, the best known bound $B_d \leq d$ is due to Grinberg and Sevast’janov [14]. Bárány and others conjecture that $B_d = O(\sqrt{d})$, which would match the Bergström–type lower bound $B_d \geq \sqrt{(d+3)/4}$. We refer to an interesting survey [3] for these results and further references.

6.6. Building on his earlier work and on [7], Sabitov in [26] and [27, §14], proved that a small diagonal in a closed orientable simplicial polyhedron (of any genus), depends algebraically on the lengths of edges of the polyhedron and this dependence is generically non-trivial. Following the proof of Theorem 1.2, we can extend this result to non-orientable polyhedra.
6.7. While Theorem 1.3 is technical, it is natural in view of the existing recreational literature. Notably, in the Scottish book, Steinhaus introduced the tetrahedral chains, which are polyhedra with a chain-like partition into regular unit tetrahedra. They can be viewed as special types of domes over two triangles, see §5.5. Steinhaus's 1957 problem asks if tetrahedral chains can be closed, and if they are dense in $\mathbb{R}^3$.

While the former was given a negative answer in 1959 by ´Swierczkowski, the latter was partially resolved only recently by Elgersma and Wagon [11]. A somewhat stronger version was later proved by Stewart [32].

Stewart’s paper is especially notable. He uses the ergodic theory of non-amenable group actions, and reproves the (previously known) fact that as a subgroup of $O(3, \mathbb{R})$, the group $G$ of face reflections of a regular tetrahedron is isomorphic to a free product: $G \cong \mathbb{Z}_2 \ast \mathbb{Z}_2 \ast \mathbb{Z}_2 \ast \mathbb{Z}_2$. From there, Stewart showed that $G$ is dense in $O(3, \mathbb{R})$. One can view this result as an advanced generalization of our Lemma 2.2. The original Steinhaus problem about the group of reflections being dense in the full group $O(3, \mathbb{R}) \ltimes \mathbb{R}^3$ of rigid motions remains open.

Acknowledgements. We are grateful to Arseniy Akopyan, Nikolay Dolbilin, Oleg Musin, Bruce Rothschild and Ian Stewart, for interesting comments and help with the references. We are especially thankful to Rick Kenyon for encouragement. The authors were partially supported by the NSF.

References

Appendix A. Doubly periodic surface with a three-dimensional flex

A.1. Flex dimension. In [12], Gaifullin and Gaifullin studied the case of doubly periodic surfaces homeomorphic to the plane. In this case they proved a stronger result than Theorem 4.3, that there are two primitive vectors $\lambda, \mu \in \Lambda^*$, such that the place $\phi$ is finite both on $(\lambda, \lambda)$ and $(\lambda, \mu)$. They concluded the following result:

**Theorem A.1** ([12, Thm 1.4]). Every embedded doubly periodic triangular surface homeomorphic to a plane has at most one-dimensional doubly periodic flex.

By a *doubly periodic flex* of the triangular surface $S$ we mean a continuous rigid deformation $\{S_t, t \in [0, \delta)\}$ for some $\delta > 0$, which preserves double periodicity, i.e. invariant under the action of $G = \mathbb{Z} \oplus \mathbb{Z}$ (the action of $G$ can also depend on $t$). The continuity of $S$ is meant with respect to all dihedral angles.

Here we identify deformations modulo changes of parameter $t$ and ask for the dimension of the space of flexing at $t = 0$, i.e. when $S_0 = S$. For example, the surface in Figure 6 when triangulated along the shadow lines has only one doubly periodic flex along these lines.

Let us mention that flexible doubly periodic surfaces is an important phenomenon in Rigidity Theory, with *Kokotsakis surfaces* introduced in 1933, giving classical examples, see e.g. [15, 18]. Note that there are doubly periodic polyhedral surfaces whose flexes are not doubly periodic, see [30]. We refer to [28, §25.5] for a recent short survey on rigidity of periodic frameworks, and further references.

A.2. New construction. In [12, Question 1.5], the authors asked if Theorem A.1 can be extended to surfaces which are not homeomorphic to a plane. In this section we give a negative answer to this question by an explicit construction.

**Theorem A.2.** There is a doubly periodic triangular surface whose doubly periodic flex is three-dimensional.

**Proof.** First, consider a flexible polyhedron $F$ satisfying the following conditions. Polyhedron $F$ must have two faces $f_1$ and $f_2$, both of them centrally symmetric, such that the distance between their centers changes during the flexing and all other faces of $F$ are triangles. To construct such $F$, take, for example, one of Bricard’s flexible octahedra $B$ (see e.g. [21, §2.3] and [25, §30]), and attach two square pyramids to either two of its triangular faces. We illustrate this step in Figure 9 where $B$ is replaced with the usual octahedron for clarity.

![Figure 9. Illustration of the first step of the construction.](image)

We call the *axis of $F$*, the line segment connecting the centers of $f_1$ and $f_2$ (see Figure 9). Let $H$ be a flexible polyhedron that has two faces congruent to one of the triangular faces of $F$. We attach

![Figure 10. Illustration of the second step of the construction.](image)
two copies of $F$, which we call $F_1$ and $F_2$, to each of these two faces of $H$ (see Figure 10). For our construction we are interested in such $H$ that, when flexing $H$, the angle $\sigma$ between the axes of $F_1$ and $F_2$ changes and is never zero. Again, a suitable Bricard’s octahedron satisfies this condition. We then have a three-dimensional flexing of the whole structure: flexing of $F_1$ changes the length of the axis of $F_1$, flexing of $F_2$ changes the length of the axis of $F_2$, flexing of $H$ changes the angle $\sigma$ between the axes.

For the next step of the construction we consider $F'$, the image of $F$ under the central symmetry with respect to the center of $f_2$. By $\hat{F}$ we denote the union of $F$ and $F'$ attached by $f_2$ (see Figure 11). From now on, we consider only flexes of $\hat{F}$ such that it stays centrally symmetric with respect to the center of $f_2$. Note that during the flex, face $f_1$ and its counterpart in $F'$, face $f'_1$, are translates of each other and the distance between their centers changes during the flex. One can think of $\hat{F}$ as a polyhedral version of accordion with bellows such that the sturdy parts of the accordion always stay parallel but the distance between them may change.

![Figure 11. Illustration of the pieces of the infinite accordion.](image)

Using infinitely many copies of $\hat{F}$, we construct a periodic flexible surface $S$ (infinite accordion) by attaching $f_1$ of one copy of $\hat{F}$ to $f'_1$ of the next copy of $\hat{F}$. See Figure 12 for an illustration (cf. Figure 6). The space of periodic flexes of $S$ is one-dimensional.

Consider a flex $\{S_t\}$ of $S = S_0$ with periodicity vector $\alpha$. At one of the flexed copies of $\hat{F}$ we attach $H$ along $g_1$. Then attach to $H$ along $g_2$ to another copy of $\hat{F}'$ which forms its own copy $S'$ of the infinite accordion. Assume the vector of periodicity of $S'$ is $\beta$. Now we attach all translates $H + k\alpha$, $k \in \mathbb{Z}$, to surface $S$, and attach $S' + k\alpha$ to each of them. Then attach $H + k\alpha + m\beta$, $m \in \mathbb{Z}$, to all $S' + k\alpha$, and attach all translates $S + m\beta$ to all translates of $H$. The resulting surface $\mathcal{F}$ is doubly periodic. It can be flexed in the following two ways:

- by changing lengths of both $\alpha$ and $\beta$ when flexing $\{S_t\}$ and $\{S'_t\}$, respectively, and
- by changing the angle $\sigma$ between the axes of $S$ and $S'$, by simultaneously flexing all copies of $H$.

Therefore, the space of doubly periodic flexes of $\mathcal{F}$ is three-dimensional.

![Figure 12. Illustration of the infinite accordion surface $S$ in the proof, and the three other copies $S + \alpha$, $S'$ and $S' + \beta$, of the doubly periodic surface $\mathcal{F}$.](image)
Remark A.3. Note that the surface $S$ in the proof is not necessarily embedded. One can similarly construct the analogous embedded surface, by a more careful choice of a flexible polyhedron, cf. [6]. It would be interesting to see if such constructions can have engineering applications. We refer to a recent thesis [21], which reviews several new constructions of embedded flexible polyhedra with larger flex dimensions, and discusses various applications.

Remark A.4. This surface $F$ is a counterexample to a natural generalization of the Main Lemma 4.6. Let us mention why the proof of the Main Lemma fails for $F$. Note that when the initial polyhedra $F$ and $H$ are homeomorphic to a sphere, we have $K/\Lambda$ is a surface of genus 2 (in the notation of the proof of the Main Lemma), where the elements of the fundamental group corresponding to $\alpha$ and $\beta$ do not commute, i.e. stand for two different handles of the surface. In particular, the inductive step in the First Case of the proof of the Main Lemma would not work for surgeries since cutting along $t$ would disconnect all translates of $S_1$ and all translates of $S_2$, see Remark 4.8.