# GROUPS OF OSCILLATING INTERMEDIATE GROWTH

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Abstract. We construct an uncountable family of finitely generated groups of intermediate growth, with *growth functions* of new type. These functions can have large oscillations between lower and upper bounds, both of which come from a wide class of functions. In particular, we can have growth oscillating between $e^{n^\alpha}$ and any prescribed function, growing as rapidly as desired. Our construction is built on top of any of the Grigorchuk groups of intermediate growth, and is a variation on the limit of permutational wreath product.

## 1. Introduction

The growth of finitely generated groups is a beautiful subject rapidly developing in the past few decades. With a pioneer invention of groups of *intermediate growth* by Grigorchuk [Gri2] about thirty years ago, it became clear that there are groups whose growth is given by difficult to analyze function. Even now, despite multiple improvements, much about their growth functions remains open (see [Gri5, Gri6]), with the sharp bounds constructed only this year in groups specifically designed for that purpose [BE].

In the other direction, the problem of characterizing growth functions of groups remains a major open problem, with only partial results known. Part of the problem is a relative lack of constructions of intermediate growth groups, many of which are natural subgroups of $\mathrm{Aut}(\mathbf{T}_k)$, similar to the original Grigorchuk groups in both the structure and analysis. In this paper we propose a new type of groups of intermediate growth, built by combining the action of Grigorchuk groups on $\mathbf{T}_k$ and its action on a product of copies of certain finite groups $H_i$. By carefully controlling groups $H_i$, and by utilizing delicate expansion results, we ensure that the growth oscillates between two given functions. Here the smaller function is controlled by a Grigorchuk group, and the larger function can be essentially *any* sufficiently rapidly growing function (up to some technical condition). This is the first result of this type, as even the simplest special cases could not be attained until now (see the corollaries below).

Our main result (Main Theorem 2.3) is somewhat technical and is postponed until the next section. Here in the introduction we give a rough outline of the theorem, state several corollaries, connections to other results, etc. For more on history of the subject, general background and further references see Section 11.

For a group $\Gamma$ with a generating set $S$, let $\gamma^S_\Gamma(n) = |B_{\Gamma,S}(n)|$, where $B_{\Gamma,S}(n)$ is the set of elements in $\Gamma$ with word length $\leq n$. Suppose $f_1, f_2, g_1, g_2 : \mathbb{N} \to \mathbb{N}$ monotone increasing subexponential integer functions which satisfy

$$(*) \qquad f_1 \succcurlyeq f_2 \succcurlyeq g_1 \succcurlyeq g_2 = \gamma^S_{\mathbf{G}_\omega}, \quad \text{where } \mathbf{G}_\omega = \langle S \rangle \text{ is a } \textit{Grigorchuk group } \text{[Gri3]}.$$

Roughly, the **Main Theorem** states that under further technical conditions strengthening $(*)$, there exists a finitely generated group $\Gamma$ and a generating set $S$, with growth function $h(n) = \gamma^S_\Gamma(n)$, such that $g_2(n) < h(n) < f_1(n)$ and $h(n)$ takes values in the intervals $[g_2(n), g_1(n)]$ and $[f_2(n), f_1(n)]$ infinitely often. We illustrate the theorem in Figure 1.

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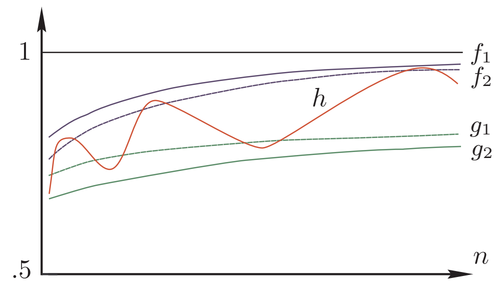



FIGURE 1. The graph of $\log_n \log$ of functions $f_1, f_2, g_1, g_2$ and $h$, as in the Main Theorem.

Stated differently, the Main Theorem implies that one can construct groups with specified growth which is oscillating within a certain range, between, $\exp\bigl(n^{\alpha(n)}\bigr)$ where $\alpha(n)$ is bounded from below by a constant, and a function which converges to $(1-)$ as rapidly as desired. Of course, it is conjectured that $\alpha(n) \geq 1/2$ for all groups of intermediate growth [Gri5] and $n$ large enough (cf. Subsection 11.8).

To get some measure of the level of complexity of this result, let us state a corollary of independent interest. Here we take the *first Grigorchuk group* $\mathbf{G}$ (cf. Subsection 11.2), and omit both $f_1$ and $g_2$, taking $g_1$ to be slightly greater that the best known upper bound for the growth of $\mathbf{G}$.

**Corollary 1.1** (Oscillating Growth Theorem)**.** *For every increasing function* $\mu : \mathbb{N} \to \mathbb{R}_+$*,* $\mu(n) = o(n)$*, there exists a finitely generated group* $\Gamma$ *and a generating set* $\langle S \rangle = \Gamma$*, such that*

$$\gamma^S_\Gamma(n) \,<\, \exp\bigl(n^{4/5}\bigr) \qquad \textit{for infinitely many } \ n \in \mathbb{N}, \quad \textit{and}$$
$$\gamma^S_\Gamma(n') \,>\, \exp\bigl(\mu(n')\bigr) \qquad \textit{for infinitely many } \ n' \in \mathbb{N}.$$

The corollary states that the growth of $\Gamma$ is both *intermediate* (i.e., super polynomial and subexponential), and *oscillating* between two growth functions which may have different asymptotic behavior. In fact, $\mu(n)$ can be as close to linear function as desired, so for example one can ensure that the ball sizes are $\leq \exp(n^\alpha)$ for some $n$, and $\geq \exp(n/\log\log\log n)$ for other $n$, both possibilities occurring infinite often. Let us now state a stronger version of the upper bound in the same setting.

**Corollary 1.2** (Oscillating growth with an upper bound)**.** *For* $\mu(n) = n^\alpha \log^\beta n$ *with* $5/6 < \alpha < 1$*, or* $\alpha = 1$ *and* $\beta < 0$*, there exist a finitely generated group* $\Gamma$ *and a generating set* $\langle S \rangle = \Gamma$*, such that*

$$\gamma^S_\Gamma(n) \,<\, \exp(n^{4/5}), \quad \gamma^S_\Gamma(n') \,>\, \exp(\mu(n')) \quad \textit{for infinitely many } \ n, \ n' \in \mathbb{N},$$
$$\textit{and} \quad \gamma^S_\Gamma(m) \,<\, \exp\bigl(\mu(9m)\bigr) \quad \textit{for all sufficiently large } \ m \in \mathbb{N}.$$

In other words, $\gamma_\Gamma(n_i)$ has the same asymptotics as $e^{\mu(n_i)}$, for a certain infinite subsequence $\{n_i\}$. Note a mild restriction on $\alpha$, which is a byproduct of our technique (see other examples in the next section).

Now, the Main Theorem and the corollaries are related to several other results. On the one hand, the growth of balls in the first Grigorchuk group $\mathbf{G}$ is bounded from above and below by

$$\exp(n^\beta) < \gamma^S_{\mathbf{G}}(n) < \exp(n^\alpha) \ \text{ for all generating sets } S,$$

integers $n$ large enough, and where $\alpha = 0.7675$ and $\beta = 0.5207$. Since this $\alpha$ is the smallest available upper bound for any known group of intermediate growth (see Subsection 11.2), this explain the lower bound in the corollaries (in fact, the power $4/5$ there can be lowered to any $\alpha' > \alpha$).

For the upper bound, a result by Erschler [Ers2] states that there is a group of intermediate growth, such that $\gamma^S_\Gamma(n) > f(n)$ for all $n$ large enough. This result does not specify exactly the asymptotic behavior of the growth function $\gamma^S_\Gamma(n)$, and is the opposite extreme when compared to the Oscillating Growth Theorem, as here *both* the upper and lower bounds can be as close to the exponential function as desired. This also underscores the major difference with our main theorem, as in this paper we emphasize the upper bounds on the growth, which can be essentially any subexponential function.

Combined with this Erschler's result, the Main Theorem states that one can get an oscillating growth phenomenon as close to the exponential function as desired. For example, Erschler showed in [Ers1], that a certain group Grigorchuk group $\mathbf{G}_\omega$ has growth between $g_2 = \exp(n/\log^{2+\epsilon} n)$ and $g_1 = \exp(n^{1-\epsilon}/\log n)$, for any $\epsilon > 0$. The following result is a special case of the Main Theorem applied to this group $\mathbf{G}_\omega$.

**Corollary 1.3.** *Fix $\epsilon > 0$. Define four functions:* $g_2(n) = e^{n/\log^{2+\epsilon} n}$, $g_1(n) = e^{n/\log^{1-\epsilon} n}$,

$$f_2(n) = e^{n/\log\log n}, \quad \text{and} \quad f_1(n) = e^{n\sqrt{\log\log\log n}/\log\log n}.$$

*Then there exists a finitely generated group $\Gamma$ and a generating set $\langle S\rangle = \Gamma$, with growth function $h(n) = \gamma_\Gamma^S(n)$, such that:*

$$g_2(n) < h(n) < f_1(n), \quad \text{for all } n \text{ large enough, and}$$

$$h(m) < g_1(m), \quad h(m') > f_2(m'), \quad \text{for infinitely many } m, m' \in \mathbb{N}.$$

Of course, here the functions $f_1, f_2$ are chosen somewhat arbitrarily, to illustrate the power of our Main Theorem.

Let us say now a few words about the *oscillation phenomenon*. In a recent paper [Bri3], Brieussel shows that there is a group $\Gamma$ of intermediate growth, such that

$$\liminf \frac{\log\log\gamma_\Gamma(n)}{\log n} = \alpha_- \quad \text{and} \quad \limsup \frac{\log\log\gamma_\Gamma(n)}{\log n} = \alpha_+,$$

for any fixed $\alpha = 0.7675 \leqslant \alpha_- \leqslant \alpha_+ \leqslant 1$. This result is somewhat weaker than our Main Theorem when it comes to the range of asymptotics of upper limits, but is stronger in a sense that the lower limits can be prescribed in advance, and $\alpha_+ - \alpha_-$ can be as small as desired (cf. Example 2.4). Since Brieussel uses groups different from $\mathbf{G}_\omega$, the Main Theorem cannot use them as an input. We postpone further discussion of this until Section 10 (see also Subsection 11.14).

A starting point for the construction in the proof of the Main Theorem (Theorem 2.3), is a sequence of finite groups $G_i$ with generating sets $S_i$, such that the growth of small balls in $X_i = \mathrm{Cayley}(G_i, S_i)$ is roughly $\gamma_{\mathbf{G}_\omega}$, but the diameter of Cayley graphs $X_i$ is close to logarithmic, i.e., the growth of large balls is almost exponential. These groups and generating sets can be combined into an infinite group $\Gamma$ and generating set $S$, such that for certain $n$, balls $\gamma_\Gamma^S(n)$ behave as small balls in $X_i$, while for other values of $n$, these balls behave as large balls in $X_i$, and thus have almost exponential size. This behavior implies that size of these balls oscillates as in the theorem.

The rest of the paper is structured as follows. We begin with the statement of the main theorem (Section 2), where also give examples and a very brief outline of the proof idea. We continue with basic definitions and notations in Section 3. We then explore graph and group limits in Section 4. We explore the Grigorchuk groups $\mathbf{G}_\omega$ in Section 5, giving some preliminary technical results, which we continue in Section 6. We then prove the Oscillating Growth Theorem (Corollary 1.1) in Section 7, as without functions $g_1$ and $f_2$ the result is technically easier to obtain. We then prove the Main Theorem 2.3 in Section 8. A key technical result (Main Lemma 8.1) is postponed until Section 9, while further generalizations are presented in Section 10. We conclude with final remarks and open problems in Section 11.

## 2. The Main theorem

2.1. **The statement.** We begin with two technical definitions.

**Definition 2.1.** A function $f : \mathbb{N} \to \mathbb{R}$ is called *admissible* if $f(n)$ is increasing, subexponential, and the ratio

$$\frac{n}{\log f(n)} \quad \text{is increasing.}$$

**Definition 2.2.** If $f$ is an admissible function, define a function $f^* : \mathbb{R} \to \mathbb{R}$ as the following inverse function:

$$f^*(z) = \Phi^{-1}(z), \quad \text{where} \quad \Phi(x) = \frac{x}{\log f(x)}.$$

**Theorem 2.3** (Main theorem)**.** *Let $f_1, f_2, g_1, g_2 : \mathbb{N} \to \mathbb{N}$ be functions which satisfy the following conditions:*

(i) *$f_1$ and $f_2$ are admissible,*
(ii) *$f_2(n)^3 = o\big(f_1(n)\big)$,*
(iii) *$f_1/g_1$ is increasing,*
(iv) *$g_2(n) \le \gamma_{\mathbf{G}_\omega}(n)$, where $\mathbf{G}_\omega$ is a Grigorchuk group of intermediate growth,*
(v) *$\gamma_{\mathbf{G}_\omega}(n) = o\big(g_1(n)\big)$,*
(vi) $\exp\left[\frac{\log g_1(n)}{Cn^2}\, f_1^*\left(\frac{n}{C\log g_1(n)}\right)\right] > \frac{C\, f_2^*(Cn)}{n^2}$*, for all $C > 0$ and $n = n(C)$ sufficiently large.*

*Then there exists a finitely generated group $\Gamma$ and a generating set $\langle S\rangle = \Gamma$, with growth function $h(n) = \gamma_\Gamma^S(n)$, such that:*

(1) *$h(n) < f_1(n)$ for all $n \in \mathbb{N}$ large enough,*
(2) *$h(n) > f_2(n)$ for infinitely many $n \in \mathbb{N}$,*
(3) *$h(n) < g_1(n)$ for infinitely many $n \in \mathbb{N}$,*
(4) *$h(n) \ge g_2(n)$ for all $n \in \mathbb{N}$.*

Although the conditions are technical, they are mostly mild in a sense that many natural functions satisfy them. For example, condition (ii) may seem strong, but notice that our functions are greater than $\exp(n^\alpha)$, in which case $(f_2)^3 \preccurlyeq f_2$. Similarly, condition (iv) may seem restrictive, but in fact, due to Erschler's theorem [Ers2], the growth of such Grigorchuk groups can be as large as desired, even if we do not know anything else about these growth functions, and with the currently available tools cannot yet control their growth.

2.2. **Examples.** The condition (vi) implies that the growth of $f_1$ is somewhat faster than that of $g_1$. However if the growth if $g_1$ is close to exponential this condition also implies that $f_1$ is significantly larger then $f_1$, since $f_1^*(\cdot)$ is small in that case. To clarify this condition, we list some examples below.

*Example* 2.4. Let $\log g_1(n) \sim n^\alpha$ and $\log f_1(n) \sim n^\beta$, for some $0 < \alpha < \beta \le 1$. The condition (vi) in this case says that $\beta > 1/(2-\alpha) > \alpha$. This implies that the interval $(\alpha, \beta)$ cannot be arbitrary and thus, in particular, the Main theorem cannot imply Brieussel's theorem [Bri3] (see below).

*Example* 2.5. Let $g_1(n) \sim n/\log^\alpha n$, $\log f_2(n) \sim n/\log^\nu n$, and $\log f_1(n) \sim n/\log^\beta n$, where $0 < \beta \le \nu \le \alpha$. The condition (vi) in this case says that $\nu > 1/(\alpha/\beta - 2)$. For example, $\alpha = 5$, $\nu = 3$ and $\beta = 2$ works, but in order to have $\nu = \beta = 2$, one needs $\alpha > 5$.

*Example* 2.6. Let $g_1(n) \sim n/\log^\alpha n$ be as before, but now $\log f_2(n) \sim n/(\log\log n)^\nu$, and $\log f_1(n) \sim n/(\log\log n)^\nu$, where $0 < \beta \le \nu$ and $\alpha > 0$. The condition (vi) in this case says that $\nu > 1/(\alpha/\beta - 1)$. For example, $\alpha = 4$, $\nu = \beta = 2$ works fine. More generally, any $\nu = \beta > \alpha$ satisfy the condition, as well as $0 < \nu = \beta < \alpha - 1$.

2.3. **A sketch of the group construction and the proof.** The group $\Gamma$ is constructed from on a sequence of integers $\{m_i\}$ and a sequences of finite groups $\{H_i \subset \mathrm{Sym}(k_i)\}$ generated by 4 involutions. The group $\Gamma$ acts on a *decorated binary tree* $\widehat{\mathbf{T}}_2$ obtained from the (usual) infinite binary tree $\mathbf{T}_2$ as follows.[1] To each vertex on level $m_i$ we attach $k_i$ leaves, which are permuted by the group $H_i$. The group $\Gamma$, like the Grigorchuk group $\mathbf{G}_\omega$, is generated by 4 involutions, whose faithful

[1] The description of group $\Gamma$ is written in the language of [GP], which is different from the rest of this paper.

action is recursively defined. The definition is similar to the usual one; however, once we reach a vertex with leaves, one need to specify the action on these leaves, which is given by a generator of the group $H_i$. We illustrate the action of $\Gamma$ on $\widehat{\mathbf{T}}_2$ in Figure 2, where the set of leaves of size $H_i$ is decorating all vertices on $m_i$-th level.

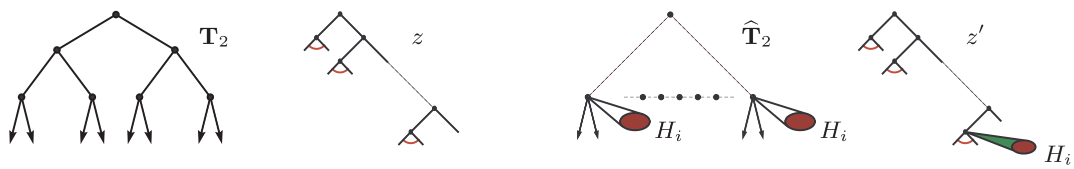



FIGURE 2. Binary tree $\mathbf{T}_2$, generator $z \in \{a, b, c, d\}$ of $\mathbf{G}_\omega$ acting by transposing selected branches in $\mathbf{T}_2$, the decorated binary tree $\widehat{\mathbf{T}}_2$ and generator $z'$ of $\Gamma$ acting on $\widehat{\mathbf{T}}_2$.

Now, the reasoning behind the proof of the Main Theorem is the following. Roughly speaking, the balls of radius $n$ does not see the group $H_i$ for $m_i > \log n$. Therefore if the levels $m_i$ grows sufficiently fast, the growth of $\Gamma$ is similar to the growth of $\mathbf{G}_\omega$ in the last final interval before the $i$-th level is reached. However, once we reach an element in $H_i$, the growth of $\Gamma$ is determined by the growth of $H_i$, and can be much more rapid in this period. If the groups $H_i$ have logarithmic diameter and their sizes increase sufficiently fast, we can ensure that the growth of $\Gamma$ is as close to exponential function as desired.

## 3. Basic definitions and notations

3.1. **Growth of groups.** Let $f, g : \mathbb{N} \to \mathbb{N}$ be two integer functions, such that $f(n)$, $g(n) \to \infty$ as $n \to \infty$. We write

$$f \ll g \quad \text{if } f(n) < g(cn), \qquad \text{for some } c > 0 \text{ and infinitely many } n \in \mathbb{N},$$
$$f \preccurlyeq g \quad \text{if } f(n) < g(cn), \qquad \text{for some } c > 0 \text{ and all sufficiently large } n \in \mathbb{N}.$$

For example, $n^{100} \preccurlyeq 3^n \preccurlyeq 2^n$ and $2^n \ll n^{n\,(n \bmod 2)} \ll n^2$. Note here that "$\ll$" is not transitive since $2^n$ is *not* $\ll n^2$. We write $f \sim g$, if $f \preccurlyeq g$ and $g \preccurlyeq f$.

Let $\Gamma$ be a finitely generated group and $S = S^{-1}$ a symmetric generating set, $\Gamma = \langle S \rangle$. Denote by $B_{\Gamma,S}(n)$ the set of elements $g \in \Gamma$ such that $\ell_S(g) \le n$, where $\ell_S$ is the *word length*, and let $\gamma^S_\Gamma(n) = |B_{\Gamma,S}(n)|$. Since for every other symmetric generating set $\langle S' \rangle = \Gamma$, we have $C_1 \ell_S(g) \le \ell_{S'}(g) \le C_2 \ell_S(g)$, which implies that $\gamma^S_\Gamma(n) \preccurlyeq \gamma^{S'}_\Gamma(n) \preccurlyeq \gamma^S_\Gamma(n)$. In other words, the asymptotics of $\gamma^S_\Gamma(n)$ are independent of the generating set $S$, so whenever possible we will write $\gamma_\Gamma(n)$ for simplicity.

Group $\Gamma$ has *exponential growth* if $\gamma_\Gamma(n) \succcurlyeq \exp(n)$, and *polynomial growth* if $\gamma_\Gamma(n) \preccurlyeq n^c$ for some $c > 0$. Similarly, $\Gamma$ has *intermediate growth* if $\gamma_\Gamma(n) \succcurlyeq n^c$ for all $c > 0$, and $\gamma_\Gamma(n) \preccurlyeq \exp f(n)$ for some $f(n)/n \to 0$, as $n \to \infty$.

*Remark* 3.1. Using this notation, the Main Theorem says that for any functions

$$f_1 \succcurlyeq f_2 \succcurlyeq g_1 \succcurlyeq g_2 \,=\, \gamma^S_{\mathbf{G}_\omega},$$

which satisfy additional technical assumptions, and where $\mathbf{G}_\omega$ is a *Grigorchuk group* of intermediate growth, there exists a group $\Gamma$, whose growth function satisfies

$$f_1 \succcurlyeq \gamma_\Gamma \succcurlyeq g_2, \quad \gamma_\Gamma \gg f_2, \quad \text{and} \quad \gamma_\Gamma \ll g_1.$$

In the special case when $f_1 \sim f_2$ and $g_1 \sim g_2$, this means that

$$f_1 \succcurlyeq \gamma_\Gamma \succcurlyeq g_2 \quad \text{and} \quad f_1 \ll \gamma_\Gamma \ll g_2.$$

For a group of intermediate growth, define

$$\alpha(\Gamma) \,=\, \lim_{n\to\infty} \frac{\log\log \gamma_\Gamma(n)}{\log n} \quad \text{if this limit exists,}$$

$$\alpha_+(\Gamma) \, = \, \limsup_{n\to\infty} \frac{\log\log \gamma_\Gamma(n)}{\log n} \quad \text{and} \quad \alpha_-(\Gamma) \, = \, \liminf_{n\to\infty} \frac{\log\log \gamma_\Gamma(n)}{\log n}\,.$$

Our Main theorem (and the result in [Bri3]), shows that $\alpha(\Gamma)$ is does not necessarily exist, the first construction of this kind. The result of Brieussel mentioned in the introduction, implies that there exist a group $\Gamma$ of intermediate growth with $\alpha(\Gamma) = \nu$, for any given $\alpha \le \nu \le 1$.

In addition to the growth function $\gamma_G^S$, we define a *normal growth function* $\widetilde{\gamma}_{G,X}^S$, as the number of elements in the group $G$ which can be expressed as words in the free group on length $n$, which also lie in the normal closure of elements $X \subset G$. In this paper we consider only the case $X = \{r\}$, which we denote $\widetilde{\gamma}_{G,r}^S$.

3.2. **Notation for groups and their products.** To simplify the notation, we use $\mathbb{Z}_m$ for $\mathbb{Z}/m\mathbb{Z}$ and trust this will not leave to any confusion. Let $\mathrm{PSL}_2(N)$ denotes group $\mathrm{PSL}_2(\mathbb{Z}/N\mathbb{Z})$, $\mathbf{F}_k$ denotes the free group on $k$ generators. By $G \twoheadrightarrow H$ we denote an *epimorphism* between the groups.

The group with presentation

$$\mathcal{G} = \langle a, b, c, d \mid a^2 = b^2 = c^2 = d^2 = bcd = 1\rangle$$

will play a central role throughout the paper, as all our groups and also all Grigorchuk groups are homomorphic images of $\mathcal{G}$. We call it a *free Grigorchuk group.*

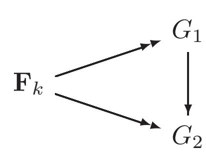
The *direct product* of groups $G$ and $H$ is denoted $G \oplus H$, rather than more standard $G \times H$. This notation allows us to write infinite product as $\bigoplus G_i$, where all but finitely many terms are trivial and we will typically omit the index of summation. We denote by $\prod G_i$ the (usually uncountable) group of sequences of group elements, without any finiteness conditions. Of course, groups $\bigoplus G_i$ nor $\prod G_i$ are not finitely generated.

Finally, let $H \wr G = G \ltimes H^\ell$ denotes the permutation wreath product of the groups, where $G \subset \Sigma_\ell$ is a permutation group.

3.3. **Marked Groups and their homomorphisms.** All groups we will consider will have ordered finite generating sets of the same size $k$. Whenever we talk mention a group $G$, we will mean a pair $(G, S)$ where $S = \{s_1, \ldots, s_k\}$ is a ordered generating set of $G$ of size $k$. Although typically described with Cayley graph, the order on the generators is crucial for our results. We call these *marked groups*, and $k$ will always denote the size of the generating set. By a slight abuse of notation, will often drop $S$ and refer to a *marked group* $G$, when $S$ is either clear from the context or not very relevant.

Throughout the paper, the homomorphisms between marked groups will send one generating set to the other. Formally, let $(G, S)$ and $(G', S')$ be marked groups, where $S = \{s_1, \ldots, s_k\}$ and $S' = \{s'_1, \ldots, s'_k\}$. Then $\phi : (G, S) \to (G', S')$ is a *marked group homomorphism* if $\phi(s_j) = s'_j$, and this map on generators extends to the (usual) homomorphism between groups: $\phi : G \to G'$.

An equivalent way to think of marked groups is to consider epimorphisms $\mathbf{F}_k \twoheadrightarrow G_1$, $\mathbf{F}_k \twoheadrightarrow G_2$, so that the map between groups correspond to commutative diagrams




3.4. **Direct sums of marked groups.** Let $\{G_i\}$ be a sequence of marked groups defined above, or more formally $\{(G_i, S_i)\}$. Denote by $G = \bigotimes G_i$ the subgroup of $\prod G_i$ generated by diagonally embedding the generating sets $S_i$, see Definition 4.1. Of course, group $G$ critically depends on the ordering of elements in $S_i$.

3.5. **Miscellanea.** With $\omega = (x_1, x_2, \ldots)$ will denote an infinite word in $\{0, 1, 2\}$ which will be used to construct the Grigorchuk group $\mathbf{G}_\omega$. Such word is call stabilizing if all $x_i$ are eventually the same. In this case the group $\mathbf{G}_\omega$ becomes virtually nilpotent.

We use $\log n$ to denote natural logarithms, but normally the base will be irrelevant. The radius of balls in the groups will be denoted with $n$. Finally, we use $\mathbb{N} = \{1, 2, \ldots\}$.

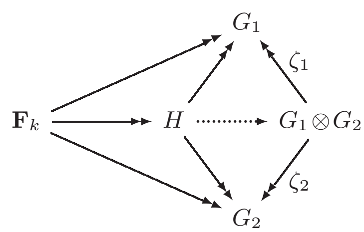
## 4. Limits of Groups

All marked groups we consider will have ordered finite generating sets of the same size $k$, and all maps between marked groups will send one generating set to the other.

**Definition 4.1.** Let $\{(G_i, S_i)\}, i \in I$ be a sequence of marked groups with generating sets $S_i = \{s_{i1}, \ldots, s_{ik}\}$. Define the $(\Gamma, S) = (\bigotimes G_i, S)$ to be the subgroup of $\prod G_i$ generated by diagonally embedding the generating sets of each $G_i$, i.e, $\bigotimes G_i = \langle s_1, \ldots, s_k \rangle$ where $s_j = \{s_{ij}\} \in \prod G_i$. Notice that $\Gamma$ comes with canonical epimorphisms $\zeta_i : \Gamma \twoheadrightarrow G_i$. Often the generating sets will be clear from the context and will simply use $\Gamma = \bigotimes G_i$. When the index set contains only 2 elements we denote the product by $G_1 \otimes G_2$.

**Remark 4.2.** The notations $\bigotimes G_i$ and $G_1 \otimes G_2$ are slightly misleading since these products depend not only on the groups but also on the generating sets. In this paper all groups are marked and come with a fixed generations set, which justifies this abuse of the notation.

**Remark 4.3.** The group $G_1 \otimes G_2$ satisfies the following universal property – for any marked group $H$ such that the left two triangles commute, there exits a homomorphism $H \to G_1 \otimes G_2$.




**Lemma 4.4.** (i) *If $G_i$ is any sequence of marked groups then growth function of $\Gamma = \bigotimes G_i$ is larger than the growth functions of each $G_i$, i.e.,*

$$\gamma_\Gamma(n) \geq \gamma_{G_i}(n) \quad \textit{for all} \;\; i\,.$$

(ii) *If $G_1$ and $G_2$ are two marked groups then the growth function of $G_1 \otimes G_2$ is bounded by the product of the growth functions for $G_i$*

$$\gamma_{G_1 \otimes G_2}(n) \leq \gamma_{G_1}(n) \cdot \gamma_{G_2}(n)\,.$$

*Proof.* By definition of the product of marked groups the map $\zeta_i : \Gamma \to G_i$ is not only surjective, but also satisfies $\zeta_i(B_{\Gamma,S}(n)) = B_{G_i,S_i}(n)$, which implies the first part. The injectivity of the product of the projections $\zeta_1$ and $\zeta_2$ and the observation

$$\zeta_1 \times \zeta_2 : B_{\Gamma,S}(n) \hookrightarrow B_{G_1,S_1}(n) \times B_{G_2,S_2}(n),$$

imply the second part. □

**Definition 4.5.** We say that the sequence of marked groups $\{(G_i, S_i)\}$ *converge* (in the the so-called *Chabauty topology*) to a group $(G, S)$ if for any $n$ there exists $m = m(n)$ such that such that for any $i > m$ the ball of radius $n$ in $G_i$ is the same as the ball of radius $n$ in $G$. We write $\lim G_i = G$.

Equivalently, this can be stated as follows: if $R_i = \ker(\mathbf{F}_k \twoheadrightarrow G_i)$ and $R = \ker(\mathbf{F}_k \twoheadrightarrow G)$ then

$$\lim_{i \to \infty} R_i \cap B_{\mathbf{F}_k}(n) \;=\; R \cap B_{\mathbf{F}_k}(n),$$

i.e., for a fixed $n$ and sufficiently large $i$ the sets $R_i \cap B_{\mathbf{F}_k}(n)$ and $R \cap B_{\mathbf{F}_k}(n)$ coincide.

**Lemma 4.6.** *Let $\{G_i\}$ be a sequence of marked groups which converge to a marked group $G$. Define the $\Gamma = \bigotimes G_i$, then there is an epimorphism $\pi : \Gamma \twoheadrightarrow G$. Moreover, the kernel of $\pi$ is equal to the intersection $\Gamma \cap \bigoplus G_i$.*

*Proof.* There is an obvious map $\pi$ which sends the generators of $\Gamma$ to the generators of $G$. A word $w$ represents the trivial element in $\Gamma$ if and only if $w$ is trivial in all $G_i$. Therefore this word is trivial in infinitely many of $G_i$ and is as well trivial in the limit $G$, i.e., the map $\pi$ extends to a group homomorphism.

The convergence of $\{G_i\} \to G$ implies that if a word $w \in \mathbf{F}$ of length $n$ which evaluates to $\{g_i\} \in \prod G_i$ is in the kernel of $\pi$ then the components $g_i$ have to be trivial for large $i$ (otherwise the ball of radius $n$ in the Cayley graph of $G_i$ will be different form the one in $G$). Therefore, the word $w \in \bigoplus G_i$. The other inclusion is obvious. □

Lemma 4.6 allows us to think of $G$ as the *group at infinity* for $\Gamma$. We will be interested in sequences of groups which satisfy the additional property that

$$\text{(spliting)} \qquad \lim_{i\to\infty} G_i = G \quad \text{and} \quad \Gamma = \Big[\bigotimes G_i\Big] \cap \Big[\bigoplus G_i\Big] = \bigoplus N_i,$$

where $N_i$ are normal subgroups of $G_i$.

**Lemma 4.7.** *If the groups $G_i$ satisfy the condition (spliting), then there exists a group homomorphism $\pi_i : G \to G_i/N_i$ which makes the following diagram commute:*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \ker\pi & \hookrightarrow & \Gamma & \overset{\pi}{\twoheadrightarrow} & G & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \zeta_i} & & \downarrow{\scriptstyle \zeta_i} & & \vdots{\scriptstyle \pi_i} & & \\
1 & \longrightarrow & N_i & \hookrightarrow & G_i & \twoheadrightarrow & G_i/N_i & \longrightarrow & 1
\end{array}$$

*Proof.* The existence and uniqueness of the homomorphism $\pi$ follows from the exactness of the rows in the diagram above. □

Using these maps one can obtain estimates for the size of the ball in the groups $\Gamma$:

**Lemma 4.8.** *If the groups $\{G_i\}$ satisfy the condition (spliting) and $B_{G_i}(n)$ is the same as $B_G(n)$ for $i > m$ then*

$$\gamma_\Gamma(n) \;\le\; \gamma_G(n) \prod_{j\le m} |N_j|.$$

*Proof.* If two elements $g, h \in B_\Gamma(n)$ are send to the same element elements in $B_G(n)$ then are also the same in $B_{G_j}(n)$ for all $j > m$, i.e., their difference $g^{-1}h$ is inside

$$\Gamma \cap \bigoplus_{j\le m} G_i = \bigoplus_{j\le m} N_j.$$

Therefore, the fibers of the restriction of $\pi$ to $B_\Gamma(n)$ have size at most $\prod_{j\le m} |N_j|$, which implies the inequality in the lemma. □

## 5. The Grigorchuk group

5.1. **Basic results.** In this section we present variations standard results on the Grigorchuk groups $\mathbf{G}_\omega$ (cf. Subsection 11.1). Rather than give standard definitions as a subgroup of $\mathrm{Aut}(\mathbf{T}_2)$, we define $\mathbf{G}$ via its properties. We refer to [GP, Har1] for a more traditional introduction and most results in this subsection.

**Definition 5.1.** Let $\varphi : \mathcal{G} \twoheadrightarrow \mathcal{G}$ denote the automorphism of order 3 of the group $G$ which cyclicly permutes the generators $b$, $c$ and $d$, i.e.,

$$\varphi(a) = a, \quad \varphi(b) = c, \quad \varphi(c) = d, \quad \varphi(d) = b.$$

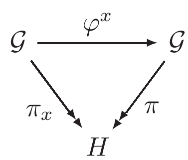
**Definition 5.2.** Let $\pi : \mathcal{G} \twoheadrightarrow H$ be an epimorphism, i.e., suppose group $H$ comes with generating set consisting of 4 involutions $\{a, b, c, d\}$ which satisfy $bcd = 1$. By $F(H)$ we define the subgroups of $H \wr \mathbb{Z}_2 = \mathbb{Z}_2 \ltimes (H \oplus H)$ generated by the elements $A, B, C, D$ defined as

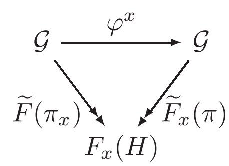
$$A = (\xi; 1, 1), \quad B = (1; a, b), \quad C = (1; a, c) \quad \text{and} \quad D = (1; 1, d),$$

where $\xi^2 = 1$ is the generator of $\mathbb{Z}_2$. It is easy to verify that $A, B, C, D$ are involutions which satisfy $BCD = 1$, which allows us to define an epimorphism $\widetilde{F}(\pi) : \mathcal{G} \to F(H)$.

The construction can be twisted by the powers automorphism $\varphi$

$$\widetilde{F}_x(\pi) := \widetilde{F}(\pi \circ \varphi^{-x}) \circ \varphi^x.$$



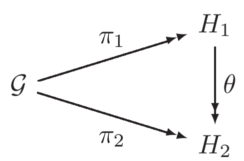


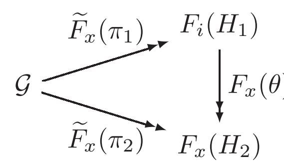

An equivalent way of defining the group $F_x(H)$ is as the subgroups generated by

$$\begin{array}{llll} A_0 = (\xi; 1, 1), & B_0 = (1; a, b), & C_0 = (1; a, c) & D_0 = (1; 1, d), \\ A_1 = (\xi; 1, 1), & B_1 = (1; a, b), & C_1 = (1; 1, c) & D_1 = (1; a, d), \\ A_2 = (\xi; 1, 1), & B_2 = (1; 1, b), & C_2 = (1; a, c) & D_2 = (1; a, d), \end{array}$$

**Remark 5.3.** Strictly speaking, the notation $F_i(H)$ is not precise since in order to define this group we need to specify a generating set, thus the correct notation should be $\widetilde{F}_i(\pi)$. However since all groups $H$ are marked, i.e., come with an epimorphism $\mathcal{G} \twoheadrightarrow H$, this allows us to slightly simplify the notation.

**Proposition 5.4.** *Each $F_x$ is a functor form the category of homomorphic images of $\mathcal{G}$ to itself, i.e., a group homomorphism $H_1 \to H_2$ which preserves the generators induces, a group homomorphism $F_x(H_1) \to F_x(H_2)$.*






**Proposition 5.5.** *The functors $F_x$ commutes with the products of marked groups, i.e.,*

$$F_x\left(\bigotimes H_j\right) = \bigotimes F_x(H_j).$$

*Proof.* This is immediate consequence of the functoriality of $F_i$ and the universal property of the products of marked groups. Equivalently one can check directly from the definitions. □

**Definition 5.6.** One can define the functor $F_\omega$ for any finite word $\omega \in \{0, 1, 2\}^*$ as follows

$$F_{x_1 x_2 \dots x_i}(H) := F_{x_1}(F_{x_2}(\dots F_{x_i}(H) \dots))$$

If $\omega$ is an infinite word on the letters $\{0, 1, 2\}$ by $F^i_\omega$ we will denote the functor $F_{\omega_i}$ where $\omega_i$ is the prefix of $\omega$ of length $i$.

**Theorem 5.7** (cf. [Gri3])**.** *The Grigorchuk group $\mathbf{G}$ is the unique group such that $\mathbf{G} = F_{012}(\mathbf{G})$.*

**Remark 5.8.** In [Gri3], Grigorchuk defined a group $\mathbf{G}_\omega$ for any infinite word $\omega$. One way to define these groups is by $\mathbf{G}_{x\omega} = F_x(\mathbf{G}_\omega)$, where $x$ is any letter in $\{0, 1, 2\}$. The *first Grigorchuk group* is denoted $\mathbf{G} = \mathbf{G}_{(012)^\infty}$, which corresponds to a periodic infinite word. If the word $\omega$ stabilize then the group $\mathbf{G}_\omega$ is virtually nilpotent and has polynomial growth.

Although we will not use Theorem 5.7, the following constructions gives the idea of the connection. Let $\mathbf{G}_{\omega,i} = F^i_\omega(\mathbf{1})$, where $\mathbf{1}$ denotes the trivial group with one element (with the trivial map $\mathcal{G} \twoheadrightarrow \mathbf{1}$).

**Proposition 5.9.** *There is a canonical epimorphism $\mathbf{G}_\omega \twoheadrightarrow \mathbf{G}_{\omega,i}$. The groups $\mathbf{G}_{\omega,i}$ naturally act on finite binary rooted tree of depth $i$ and this action comes from the standard action of the Grigorchuk group on the infinite binary tree $\mathbf{T}_2$.* □

**Remark 5.10.** The group $F^i_\omega(H)$ is a subgroup of the permutational wreath product $H \wr_{X_i} \mathbf{G}_{\omega,i}$, where $X_i$ is the set of leaves of the binary tree of depth $i$ (cf. Subsection 2.3).

### 5.2. **Contraction in Grigorchuk groups.**

**Lemma 5.11.** *Let $\pi : \mathcal{G} \twoheadrightarrow H$ be an epimorphism, i.e., group $H$ is generated by* 4 nontrivial *involutions which satisfy $bcd = 1$. If the word $\omega$ does not stabilize, then the balls of radius $n \le \vartheta(m)$ in the groups $F^m_\omega(H)$ and $\mathbf{G}_\omega$ coincide, where $\vartheta(m) = 2^m - 1$ is strictly increasing function $\vartheta : \mathbb{N} \to \mathbb{N}$.*

*Proof.* It is enough to show that the set of words of length $2\vartheta(m)$ which are trivial in $F^m_\omega(H)$ is the same as the ones which are trivial in $\mathbf{G}_\omega$.

Observe that every word $w \in \mathcal{G}$ can be expanded to $(\xi^{a_w}; w', w'') \in F(\bar{H})$ where $w'$ and $w''$ are words of length $\le (|w| + 1)/2$. If $a_w \ne 0$ then $w$ is not zero in $F(\bar{H})$, for any group $\bar{H}$. Iterating this $m$ times, shows that any word $w$ of length $< 2\vartheta(m)$ is either nontrivial in both $F^m_\omega(H)$ and $\mathbf{G}_\omega$; or evaluates to many words of length at most 1 acting on the copies of $H$. If one of these words in nontrivial then $w$ otherwise it is trivial.

Here we are using that the non-stabilization on $\omega$ implies that the elementals $a, b, c, d$ are nontrivial in $\mathbf{G}_{\bar{\omega}}$ for any suffix $\bar{\omega}$ of $\omega$. □

**Remark 5.12.** One can show that a stronger result holds if $\omega$ does not contain $0^k$, $1^k$ and $2^k$ as subwords. Indeed, then the balls is $F^m_\omega(H)$ and $\mathbf{G}_\omega$ of radius $\vartheta(m)$ are the same as the balls in $\mathbf{G}_{\omega,m+k+1}$. The last group groups is of the form $F^m_\omega(H')$ where $H' = F_{\omega'}(\mathbf{1})$ where $\omega'$ is a subword of $\omega$ of length $k+1$ and the conation on $\omega$ implies that the generators $a, b, c$ and $d$ are nontrivial in $H'$.

**Remark 5.13.** Here we use that the length of each word $w'$ and $w''$ is shorter than $w$. In many cases one can also show that the sum of the lengths (or some suitably defined norm) of these words is less that that of $w$. Such contracting property is used to obtain upper bounds for the growth of $\mathbf{G}_\omega$, see [Bar1, BGS, Gri3, MP].

We conclude with an immediate corollary of the Proposition 5.9 and Lemma 5.11, which can also be found in [Gri6].

**Corollary 5.14.** *Let $\{\mathcal{G} \twoheadrightarrow H_i\}$ be any sequence of groups generated by $k = 4$ nontrivial involutions and let $\{m_i\}$ be an increasing sequence. Then the sequence of groups $\{F^{m_i}_\omega(H_i)\}$ converge (in the Chabauty topology) to $\mathbf{G}_\omega$.*

**Remark 5.15.** This can be used as an alternative definition of the groups $\mathbf{G}_\omega$, which shows that there exists a canonical epimorphism $\mathcal{G} \twoheadrightarrow \mathbf{G}_\omega$.

### 5.3. **Growth lemmas.**

Let $r$ denote the element $[c, [d, [b, (ad)^4]]] \in \mathcal{G}$ and let $r_x = \varphi^x(r)$ be its twists by the automorphism $\varphi$ described in Definition 5.1.

**Lemma 5.16.** *Let $\mathcal{G} \twoheadrightarrow H$ be a finite image of $\mathcal{G}$ which normally generated by element $r_{x_{k+1}}$ defined above. Then the kernel of the map $F^k_\omega(H) \twoheadrightarrow \mathbf{G}_{\omega,k}$ induced by $F^k_\omega$ from the trivial homomorphism $H \twoheadrightarrow \mathbf{1}$, is isomorphic to $H^{\oplus 2^k}$. Moreover, there exists a word $\eta_{\omega,k} \in F$ of length $\le \mathrm{K} \cdot 2^k$, such that such the image of $\eta_{\omega,k}$ in $F^k_\omega(H)$ normally generated this kernel and $\eta_{\omega,k}$ is trivial in $F^{k+1}_\omega(H')$, for every $\mathcal{G} \twoheadrightarrow H'$.*

*Proof.* Consider the substitutions $\sigma, \tau$ (endomorphisms $\mathcal{G} \to \mathcal{G}$), defined as follows:

- $\sigma(a) = aca$ and $\sigma(s) = s$, for $s \in \{b, c, d\}$,
- $\tau(a) = c$, $\tau(b) = \tau(c) = a$ and $\tau(d) = 1$.

It is easy to see that for any word $\eta$, the evaluation of $\sigma(\eta)$ in $F(H)$ is equal to

$$\big(1; \tau(\eta), \eta\big) \in \{1\} \times H \times H \subset H \wr \mathbb{Z}_2 .$$

Define words $\{w_i\}$ for $i = 0, \dots, k$ as follows: $w_0 = r_{x_{k+1}}$ and $w_{i+1} = \sigma_{x_{k-i}}(w_i)$ where $\sigma_{x_i} = \varphi^{x_i}\sigma\varphi^{-x_i}$ the the twist of the substitution $\sigma$. Notice that all these words have the form $[c, [d, [b, *]]]$ because $\sigma_{x_i}$ fixes $b$, $c$ and $d$. Therefore $\tau_x(w_i) = 1$

By construction the word $\eta_{\omega,k} = w_k$ evaluates in $F^k_\omega(\bar H)$ to $r_{x_{k+1}}$ in one of the copies of $\bar H$, for any group $\bar H$. The expression $(ad)^4$ inside $r$ ensures that $r_{x_{k+1}}$ is trivial if $\bar H$ is of the form $F_{x_{k+1}}(\bar H')$, which proves the last claim.

The first claim follows form the transitivity of the action of $\mathbf{G}_\omega$ (and $F^k_\omega(H)$) on the $m$-th level of the binary tree and the assumption that $H$ is normally generated by $r_{x_{k+1}}$. □

**Remark 5.17.** The lemma says that if $H$ is normally generated by the element $r$, then the inclusion in Remark 5.10 is an equality.

**Corollary 5.18.** *For $H$ as in Lemma 5.16 and every integer $n \geq 1$, we have:*

$$\gamma_H(n) \leq \gamma_{F^k_\omega(H)}(c_k n), \quad \textit{where} \quad c_k = 2^{k+1} - 1.$$

*Proof.* Use that $\sigma_{k,\omega}(B_H(n)) \subset B_{F^k_\omega(S)}(c_k n)$ because the composition $\sigma_{k,\omega}$ of $k$ substitutions $\sigma_{x_j}$ increases the lengths of the words at most $2^{k+1} - 1$ times. □

**Corollary 5.19.** *For $H$ as in Lemma 5.16 and every integer $n \geq 1$ and any $t < 2^k$, we have:*

$$\widetilde{\gamma}_{H,r_{x_{k+1}}}(n)^t \leq \gamma_{F^k_\omega(H)}(c_k t n),$$

*where $\widetilde{\gamma}_H(n)$ is the* normal growth function*, i.e. the number of elements in $H$ which can be expressed as words of length less then $n$ in the normal subgroup $X = \langle r_{x_{k+1}}\rangle^{\mathbf{F}_4}$ of the free group $\mathbf{F}_4 = \langle a, b, c, d\rangle$.*

*Proof.* As before, but use the fact that there are many copies of $H$. □

## 6. Growth in $\mathrm{PSL}_2(\mathbb{Z}_N)$

For the proof of Theorem 2.3, we need the following technical result:

**Lemma 6.1.** *Let $N$ such that $-1$ is a square in $\mathbb{Z}_N$ and $2 \not| N$, i.e., the only prime factors which appear in the prime decomposition of $N$ are of the form $p = 1 \mod 4$. Then there exist a generating set $S_N = \{a, b, c, d\}$ of the group $H_N = \mathrm{PSL}_2(\mathbb{Z}_N)$ such that*

(1) *there is an epimorphism of marked groups $\mathcal{G} \twoheadrightarrow H_N = \mathrm{PSL}_2(\mathbb{Z}_N)$,*
(2) *the group $H_N$ is normally generated by the image of element $r = [c, [d, [b, (ad)^4]]]$,*
(3) *$\gamma_{H_N}(n) > \exp(n/\mathrm{K})$, for $n < \mathrm{K} \log |H_N| < 3\mathrm{K} \log N$, and $\mathrm{K} > 0$ is an absolute constant,*
(4) *$\widetilde{\gamma}_{H_N,r}(n) > \exp(n/\mathrm{K}')$, for $n < \mathrm{K}' \log |H_N| < 3\mathrm{K}' \log N$, and $\mathrm{K}' > 0$ is an absolute constant.*

Here property (3) means that the size of balls in the Cayley graphs of $\mathrm{PSL}_2(\mathbb{Z}_N)$ grow exponentially. For the proof of Corollary 1.1 we do not really need the exact form of these groups nor property (4), only the fact the their sizes go to infinity. However the proof of Theorem 2.3 uses that these groups are related to $\mathrm{PSL}_2(\mathbb{Z})$.

*Proof.* Consider the following matrices in $\mathrm{PSL}_2\big(\mathbb{Z}[\boldsymbol{i}, 1/2]\big)$, where $\boldsymbol{i}^2 = -1$,

$$a = \begin{pmatrix} \boldsymbol{i} & \boldsymbol{i}/4 \\ 0 & -\boldsymbol{i} \end{pmatrix}, \quad b = \begin{pmatrix} 0 & \boldsymbol{i} \\ \boldsymbol{i} & 0 \end{pmatrix}, \quad c = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad d = \begin{pmatrix} \boldsymbol{i} & 0 \\ 0 & -\boldsymbol{i} \end{pmatrix}.$$

A direct computation shows that these elements are of order 2 and $bcd = 1$, i.e., there is a (non-surjective[2]) homomorphism $\mathcal{G} \to \mathrm{PSL}_2\big(\mathbb{Z}[\boldsymbol{i}, 1/2]\big)$. Moreover, we have

$$(ad)^4 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \qquad [c, [d, [b, (ad)^4]]] = \begin{pmatrix} -1 & 2 \\ 2 & -5 \end{pmatrix}.$$

[2]The images contains $\mathrm{PSL}_2\big(\mathbb{Z}[1/2]\big)$ as a subgroup of index 2.

This implies that the image of $\{a,b,c,d\}$ in $\mathrm{PSL}_2(\mathbb{Z}_N)$ satisfies properties (1) and (2), because $r$ is not contained in any proper finite index normal subgroup of the image.

Property (3) is satisfied because the standard expander generators of $\mathrm{PSL}_2(\mathbb{Z}_N)$ can be expressed as short words in the generators (see e.g. [HLW, Lub1]) :

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = (da)^4 \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = c(ad)^4 c.$$

Property (4) follows since $r$ and $r^a$ generate a non-solvable subgroup of $\mathrm{PSL}_2(\mathbb{Z})$. Now the recent expansion results [BG1], imply that the Cayley graphs of $\mathrm{PSL}_2(\mathbb{Z}_N)$ with respect to the images of $r$ and $r^a$ are expanders. This completes the proof (cf. Subsection 11.12). □

**Remark 6.2.** Twisting by the automorphism $\varphi^x$ one sees that the lemma remains valid if $r$ is replaced by $r_x$.

**Remark 6.3.** If $\{p_j\}$ is a finite sequence of different primes which are $3 \mod 4$ then the product of $\mathrm{PSL}_2(p_j)$ as marked groups (with resect to the generating sets constructed above) is

$$\bigotimes \mathrm{PSL}_2(p_j) = \mathrm{PSL}_2(\mathbb{Z}_N) \quad \text{where } N = \prod p_i.$$

The following is an immediate consequence of Corollaries 5.18 and 5.19:

**Corollary 6.4.** *Let $N$ satisfies the conditions of Lemma 6.1. The size of a ball of radius $n < \mathrm{D}_i \log |\mathrm{PSL}_2(\mathbb{Z}_N)|$ inside $F^i_\omega(\mathrm{PSL}_2(\mathbb{Z}_N))$ is more than $e^{n/\mathrm{D}_i}$ where $\mathrm{D}_i = 2^{i+1}\mathrm{K}$.*

**Corollary 6.5.** *Let $N$ satisfies the conditions of Lemma 6.1. The size of the intersection of a ball of radius $n < 2^i \mathrm{D}'_i \log |\mathrm{PSL}_2(\mathbb{Z}_N)|$ inside $F^i_\omega(\mathrm{PSL}_2(\mathbb{Z}_N))$, with the subgroup $\mathrm{PSL}_2(\mathbb{Z}_N)^{2^i} = \ker\{F^i_\omega(\mathrm{PSL}_2(\mathbb{Z}_N)) \to \mathbf{G}_{\omega,i}\}$ is more than $e^{n/\mathrm{D}'_i}$ where $\mathrm{D}'_i = 2^{i+1}\mathrm{K}'$.*

## 7. Proof of the Oscillating Growth Theorem

We are going to prove the following result, which implies the Oscillating Growth Theorem (see Corollary 1.1), and is a stepping stone to the proof of the Main Theorem.

**Theorem 7.1.** *For every admissible integer functions $f, g : \mathbb{N} \to \mathbb{N}$, such that $\lim g(n)/\gamma_{\mathbf{G}_\omega}(n) = \infty$ and any sequences of integers $\{a_i\} \to \infty$ and $\{b_i\} \to \infty$, there exists a finitely generated group $\Gamma$ and a generating set $\langle S \rangle = \Gamma$, such that*

$$\gamma^S_\Gamma(n) < g(n) \ \text{ for infinitely many } \ n \in \{b_i\},$$

*and*

$$\gamma^S_\Gamma(n) > f(n) \ \text{ for infinitely many } \ n \in \{a_i\}.$$

**Remark 7.2.** The reason for including the subsequences $a_i$ and $b_i$ is to be able to ensure that for any $f \gg f'$ and $g \ll g'$ then we have $\gamma_\Gamma \gg f'$ and $\gamma_\Gamma \ll g'$. In particular we can guarantee that

$$\liminf_{n\to\infty} \log_n \log \gamma_\Gamma(n) = \liminf_{n\to\infty} \log_n \log \gamma_{\mathbf{G}_\omega}(n).$$

*Proof of Theorem 7.1.* The group $\Gamma$ will be the product of marked groups $G_i = F^{m_i}_\omega(H_i)$ where $H_i = \mathrm{PSL}_2(p_i)$ and $\{m_i\}$ and $\{p_i\}$ are sequences which grow sufficiently fast constructed using the functions $f$ and $g$. Lemmas 5.11 and 5.16 imply that the sequence of groups $G_i$ converge to $\mathbf{G}$ and satisfies the condition (spliting) with $N_i = H_i^{\oplus 2^{m_i}}$.

By Corollary 6.5, the growth of $\gamma_\Gamma$ is faster that the each $\gamma_{G_i}$. When $p_i$ is sufficiently large one can find $n_i \in \{a_j\}$ such that

$$\text{(lower)} \qquad \gamma_\Gamma(n_i) \geq \gamma_{G_i}(n_i) > f(n_i),$$

which guarantees that $\gamma_\Gamma \gg f$.

Also, if $m_i$ grows sufficiently fast then the $G_i$ converge very quickly to $\mathbf{G}_\omega$ and by Lemma 4.8 there exists $n'_i \in \{b_j\}$ such that

$$\gamma_\Gamma(n'_i) \leq \gamma_{\mathbf{G}}(n'_i) \prod_{j<i} |N_i| < g(n'_i). \tag{upper}$$

This guarantees that $\gamma_\Gamma \ll g$.

The only thing which is left is to determine how fast the sequences $\{m_i\}$ and $\{p_i\}$ have to grow in order to ensure the above inequalities. Define the the sequences $m_i$ and $p_i$ as follows:

- $m_1 = 1$ and $H_1 = \mathbf{1}$.
- Let $n_i \in \{b_j\}$ be in integer such that $\frac{g(n_i)}{\gamma_{\mathbf{G}_\omega}(n_i)} > \prod_{j<i} |H_j|^{2^{m_j}}$ (such integer exists since $\{b_s\} \to \infty$ and $\frac{g(n)}{\gamma_{\mathbf{G}_\omega}(n)} \to \infty$). Define $m_i$ such that $\vartheta(m_i) > n_i$ and that $m_i > m_{i-1}$. This choice of $m_i$ ensures that the inequalities (upper) are satisfied, because the kernel $N_i$ of the map $G_i \to \mathbf{G}_{\omega,m_i}$ is isomorphic to the direct sum of $2^{m_i}$ copies of $H_i$ by Lemma 5.16.
- Let $n'_i \in \{a_j\}$ be in integer such that

$$\frac{n'_i}{\log f(n'_i)} \geq \mathrm{D}_{m_i} = \mathrm{K} \cdot 2^{m_i},$$

  where $\mathrm{D}_i$ is the constant from Corollary 6.4. Such integer exists since $\{a_i\} \to \infty$ and $n/\log f(n) \to \infty$. Define $H_i$ to be a group together with generating set (twisted by $\varphi^{x_{m_i+1}}$) from Lemma 6.1 of size more than $e^{n'_i}$. Again this choice of $n'_i$ ensures that the inequalities (lower) are satisfied.

These two (rather crude) estimates for the size of the balls in $\Gamma$ shows that conditions (lower) and (upper) are satisfied. Therefore the growth function of $\Gamma$ is infinitely often larger than $f$ and infinitely often smaller than $g$. □

## 8. Control of the upper bound

**8.1.** Roughly speaking, we obtain a very good control over the upper bound by using finite groups $H_i$ of the carefully chosen size. We observe that Lemma 6.1 gives us an "almost continuous" family of finite groups which can be plugged into the construction.

Unfortunately, the growth estimates we have so far are too crude for such results. If the sequence $m_i$ grows sufficiently fast, then the growth of the group $\Gamma$ (in certain range), is very well approximated by the growth of the group $\Gamma_i = G_i \otimes \mathbf{G}_\omega$. This is because

$$\begin{aligned} \gamma_\Gamma(n) &\geq \gamma_{\Gamma_i}(n) \quad \text{for all } n, \\ \gamma_\Gamma(n) &< L_i \gamma_{\Gamma_i}(n) \quad \text{for all } n \leq \vartheta(m_{i+1}) \text{ and } L_i = \prod_{j<i} |N_i|. \end{aligned}$$

The first condition follows form the observation that there are maps from the marked group $\Gamma$ to both $G_i$ and $\mathbf{G}_\omega$. By Remark 4.3 this gives a is a map onto their product $\Gamma_i$. Therefore the growth in the image is slower that the growth of $\Gamma$.

The second condition is a consequence of the fact that the ball of radius $\vartheta(m_{i+1})$ in $\Gamma$ is the same as the ball in the product $\bigotimes_{j\leq i+1} G_i$ or in $\mathbf{G}_\omega \otimes \left[\bigotimes_{j\leq i} G_i\right]$, and that

$$\left|\ker\left(\mathbf{G}_\omega \otimes \left[\bigotimes_{j\leq i} G_i\right] \twoheadrightarrow \Gamma_i\right)\right| = L_i.$$

If $m_{i+1}$ is very large if suffice to find $H_i$ such that the growth of $\Gamma_i$ is always bellow $f_1$ but sometimes it is above $f_2$.

**8.2.** Below we present much better bounds on the growth in the following marked group:

$$\Lambda^i_\omega(H) = F^i_\omega(H) \otimes \mathbf{G}_\omega \subset F^i_\omega(H) \oplus \mathbf{G}_\omega,$$

which is closely related to the group $\Gamma_i$ mentioned above. The growth of the balls in $\Lambda$ is in 3 different regimes depending on the scale. For small radius $n < t_i$ the balls are the same as the ball in $G_\omega$ and grow sub-exponentially.

For big radius $n > T_i = 2^i D_i \operatorname{diam}|H|$ the finite group $F^i_\omega(H)$ has been exhausted and the size of the ball of radius $n$ is very close to $|H|^{2^i}$ time the size of the ball in $\mathbf{G}_\omega$ and again is sub-exponential.

In the intermediate range $t_i < n < T_i$ the growth is more complicated – it is similar to the growth in the finite group $H$ and therefore is "locally" is very close to exponential. However, the proof of the next results requires to obtain some bounds for this intermediate range. As usual in such situations, understanding the exact growth in the intermediate range is extremely difficult and our bounds are far from optimal. Improving these bounds will result in weakening the technical conditions (vi) of Theorem 2.3 and (v) in Lemma 8.1.

**8.3.** The following technical lemma ensures that we can find the group $H$ such that the growth of the group $\Lambda^i_\omega(H)$ is between $f_1$ and $f_2$. We postpone the proof until the next section.

**Lemma 8.1** (Main lemma)**.** *Let $f_1, f_2, g : \mathbb{N} \to \mathbb{N}$ be admissible functions which satisfy the conditions*

(i) $f_1(n)/g(n)$ *is increasing function,*
(ii) $f_1(n) > g(n)^3$ *for all sufficiently large $n$,*
(iii) $f_1(n) > f_2(n)^3$ *for all sufficiently large $n$,*
(iv) $g(n) \geq \gamma_{\mathbf{G}_\omega}(n)$, *where $\mathbf{G}_\omega$ is a Grigorchuk group of intermediate growth,*
(v) $\exp\left[\dfrac{\log g(n)}{Cn^2} f_1^*\left(\dfrac{n}{C \log g(n)}\right)\right] > \dfrac{C f_2^*(Cn)}{n^2}$, *for any $C > 0$ and sufficiently large $n = n(C)$.*

*Then, for every $L > 0$ and all sufficiently large $i$, one can find a finite marked group $H_i$, such that:*

(1) *$H_i$ is normally generated by $r_{x_{i+1}}$,*
(2) *there exists $n$ such that $\gamma_\triangle(n) > f_2(n)$,*
(3) *$f_1(n) > L\gamma_\triangle(n)$ for all $n > \vartheta(i)$,*

*where $\gamma_\triangle = \gamma_{\Lambda^i_\omega(H_i)}$.*

*Proof of Theorem 2.3.* The proof is almost the same as the proof of the Theorem 7.1, but one needs to pick the groups $H_i$ of the correct size. First we pick $m_1$ such that for $n > \vartheta(m_1)$ we have

$$f_1(n) > f_2(n)^3 \quad \text{and} \quad f_1(n) > g_1(n) > \gamma_{\mathbf{G}_\omega}(n)$$

which is possible because the functions satisfy conditions (i-v).

When choosing the depths $m_i$ one need to satisfy three conditions: the first one is $m_i > m_{i-1}$ ensures that the groups grows; the second one as in Theorem 7.1 is that $\frac{g_1(n_i)}{\mathbf{G}_\omega(n_i)} \geq L$ for some $n_i \leq \vartheta(m_i)$, where $L = \prod_{j<i} |H_j|^{2^{m_j}}$ which guarantees that the growth of $\Gamma$ will be sometimes smaller than $g_1$. The last one is that $m_i$ is larger than the bound for $i$ in Lemma 8.1 which depends on $L$.

If $m_i$ is chosen as above then we can apply the Lemma 8.1 (with $g_1$ instead of $g$) and obtain the group $H_i$. The second property of $H_i$ implies that the growth of $\Gamma$ is larger than $f_2$ for some $n > \vartheta(m_i)$, and the third implies that it is bellow $f_1$ for $\vartheta(m_i) < n < \vartheta(m_{i+1})$.

As a result we have that the growth of the group $\gamma_\Gamma$ is between $f_1$ and $g_2$ for all sufficiently large $n > \vartheta(m_1)$, and is above $f_2$ and below $g_1$ at least once in each interval $\vartheta(m_i) < n < \vartheta(m_{i+1})$, which completes the proof. $\square$

## 9. Proof of Main Lemma 8.1

**9.1. Outline.** The following is a rough outline of the proof. We start with some estimates of the growth of $\Lambda^i_\omega$ in the intermediate range: the upper bound is coming from the submultiplicativity of the growth functions, and the lower is based on the growth inside $H$. It is clear that the lower bound is far from being optimal, but we suspect that the upper on is relatively close the the optima bound.

These bounds give that (Corollary 9.3) that if the group $H$ is small then the growth of $\Lambda^i_\omega(H)$ is slower than $f_1$ and using Corollary 6.5 this growth is faster that $f_2$ if the group $H$ is big (Corollary 9.4). If the gap between $f_1$ and $f_2$ is sufficiently large then these sets have a nontrivial intersection which implies the existence of $H$ satisfying the requirements of the lemma.

Unfortunately, for this strategy to work one need that the gap between $f_1$ and $f_2$ to be very big. The reason for that, is that we are using very crude estimates for the sizes of balls, which does not allow us to obtain better estimates for the growth of the group $\Lambda^i_\omega(H)$. In order to obtain results where the functions $f_1$ and $f_2$ we argue by contradiction. As a result, we only show the existence of the group $H$, but not an algorithm to construct it.

**9.2. Three classes of marked finite groups.** First we divide the finite groups $H$ into 3 classes: $\mathcal{D}_-$, $\mathcal{D}_+$ and $\mathcal{D}_\circ$, depending how the growth of $\Lambda^M_\omega(H)$ compares with $f_1$ and $f_2$. If one assumes that the class $\mathcal{D}_\circ$ is empty (and the gap between $f_1$ and $f_2$ is not too small) then $\mathcal{D}_-$ is closed under products of marked groups (Corollary 9.7) which allows us to construct a group in $\mathcal{D}_-$ which is much larger than the bound in Corollary 9.3. Finally one obtains a contradiction if the size of this group is larger than the estimate from Corollary 6.5.

**Definition 9.1.** Given a marked group $H$, to simplify the notation denote by $\gamma_\triangle$ the growth function of $\Lambda^i_\omega(H)$. Let $\mathcal{D}^i_-$ denote the set of marked groups $H$ such that $f_2(n) > \gamma_\triangle(n)$ for all $n > \vartheta(i)$. Similarly, let $\mathcal{D}^i_+$ denote the set of marked groups $H$ such that $f_1(n)^{2/3} \leq \gamma_\triangle(n)$ for some $n > \vartheta(i)$. Finally, by $\mathcal{D}^i_\circ$ denote the set of marked groups $H$ such that $f_1(n)^{2/3} > \gamma_\triangle(n)$ for all $n > \vartheta(i)$, but $f_2(n) \leq \gamma_\triangle(n)$ for some $n$.

The conclusion of Lemma 8.1 is equivalent to saying that $\mathcal{D}^i_\circ$ is not empty when $i$ is sufficiently large, since we can guarantee that $f_1(n)^{1/3} > L$ for $n > \vartheta(i)$.

**9.3. Details: large gap.**

**Lemma 9.2.** *Fix the group $H$. The growth of the function $\gamma_\triangle$ is bounded above by the function $\Upsilon_T$ defined as follows*

$$\Upsilon_T(n) = \begin{cases} g(n) & \text{for } n \leq \vartheta(i) \\ \exp(n/\phi_i) & \text{for } \vartheta(i) \leq n \leq T \\ |H|^{2^i} g(n) & \text{for } n \geq T \end{cases}$$

*where* $\phi_i = \min\left\{ \frac{n}{\log g(n)} \mid \frac{\vartheta(i)}{2} \leq n \leq \vartheta(i) \right\}$.

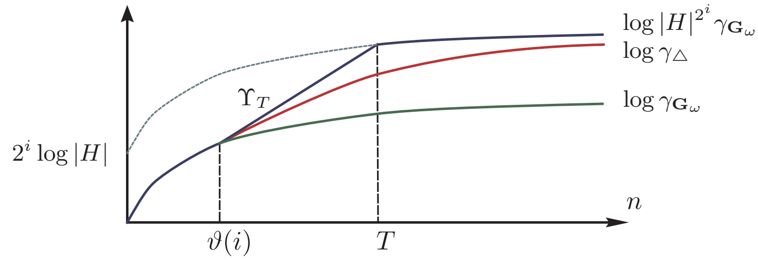



Figure 3. The graph of functions as in Lemma 9.2.

*Proof.* The ball of radius less than $\vartheta(i)$ in $\Lambda_\omega^i(H)$ is the same as in the group $\mathbf{G}_\omega$ which gives the bound for small $n$. The kernel of $\Lambda_\omega^i(H) \to \mathbf{G}_\omega$ has size $|H|^{2^i}$ which easily implies the bound for large $n$.

In the middle range one uses sub multiplicativity of growth functions $\gamma_\triangle(a+b) \leq \gamma_\triangle(a) \cdot \gamma_\triangle(b)$ for any $a, b$. This implies that if $n \geq m$ then

$$\frac{\log \gamma_\triangle(n)}{n} \leq \max_{m/2 \leq s \leq m} \frac{\log \gamma_\triangle(s)}{s}.$$

This inequality for $m = \vartheta(i)$ is equivalent to the bound in the middle range (see Figure 3). □

The lemma implies that if the size of $H$ is very small, then the growth of $\Lambda_\omega^i(H)$ is smaller then $f_1$.

**Corollary 9.3.** (i) *Let $f$ be an admissible function, such that $f(n)/g(n)$ for some $g(n) > \gamma_{\mathbf{G}_\omega}(n)$ is increasing. If*

$$|H| < \exp\left[\frac{f^*(\phi_i)}{2^i\,\phi_i} - \frac{\log g(f^*(\phi_i))}{2^i}\right],$$

*then $f(n) > \gamma_\triangle(n)$ for all integer $n$.*

(ii) *Moreover, if $f$ also satisfies $f(n) \geq g(n)^3$ for $n \geq \vartheta(i)$ then for*

$$|H| \,<\, \mathrm{U}_i(f) := \exp\left[\frac{f^*(\frac{2}{3}\phi_i)}{2^{i+1}\,\phi_i}\right].$$

*we have that $f(n)^{2/3} > \gamma_\triangle(n)$ for all integers $n \geq \vartheta(i)$.*

*Proof.* For the first part, compute the point $T$ where the graph of $(f_1)^{2/3}$ intersects with $\exp(n/\phi)$.

By Lemma 9.2, if $|H|^{2^i} \leq \frac{f_1(T)}{g(T)}$, then the growth of $\Lambda_\omega^i(H)$ is slower than $\Upsilon_T$, which is less than $f$. The second part uses the estimate $\frac{f_1(T)}{g(T)} \geq \big[f_1(T)\big]^{2/3}$. □

This following result is a strengthening of Corollary 6.5.

**Corollary 9.4.** *If $H = \mathrm{PSL}_2(\mathbb{Z}_N)$ with the generating set from Lemma 6.1 and*

$$|H| \,>\, \mathrm{L}_i(f) \,=\, \exp\left[\frac{f^*(\mathrm{D}_i')}{2^i \mathrm{D}_i'}\right],$$

*then $f(n) < \gamma_\triangle(n)$ for some $n$.*

*Proof.* By Corollary 6.5 $\gamma_\triangle(n) \geq \exp(n/\mathrm{D}_i')$ for $n \leq 2^i \mathrm{D}_i' \log |H|$. For $n = f^*(\mathrm{D}_i)$ the bound is the same as $f(n)$, but we can apply the estimate only if $|H| > \exp(n/2^i\mathrm{D}_i')$. □

*Proof of Lemma 8.1 for large gap.* Let the functions $f_1$ and $f_2$ satisfy

$$\text{(big gap)} \qquad \frac{\log g(n)}{n^2} f_1^*\left(\frac{n}{C \log g(n)}\right) > \frac{C}{n^2} f_2^*(Cn),$$

for any constant $C$ and any sufficiently large $n$. Notice that, up to a constants, both $\vartheta(i)$ and $\mathrm{D}_i'$ are equal to $2^i$. Substituting $\varrho = 2^i$ one gets

$$\mathrm{L}_i(f_2) \approx \exp\left[\frac{C_1}{\varrho^2} f_2^*(C_2\varrho)\right]$$

where $C_1$ and $C_2$ are universal constants. Similarly,

$$\mathrm{U}_i(f_1) \approx \exp\left[\frac{\log g(\varrho)}{C_3 \varrho^2} f^*\left(\frac{\varrho}{C_4 \log g(\varrho)}\right)\right].$$

If the functions $f_1$ and $f_2$ satisfy the equation (big gap) then both $\mathrm{U}_i(f_1)/\mathrm{L}_i(f_2)$ and $\mathrm{U}_i(f_1)$ then to $\infty$ as $i$ increases. Therefore, there exists $i_0$, such that for $i > i_0$ we have:

$$\big[f_1(\vartheta(i))\big]^{1/3} > L, \quad \mathrm{U}_i(f_1)/\mathrm{L}_i(f_2) > 10, \quad \mathrm{U}_i(f_1) > 1000.$$

Under these conditions there exists a prime $p_i = 1 \mod 4$ such that $\mathrm{U}_i(f_1) > \mathrm{PSL}_2(p_i) > \mathrm{L}_i(f_2)$ because the above conditions translate to

$$A_i > p_i > B_i$$

where $A_i/B_i > 2$ and $A_i > 13$, which allows us to apply Bertrand's postulate.

Corollary 9.3 implies that the growth $\gamma_i = \gamma_{\Lambda^i_\omega(H_i)}$ of group $\Lambda^i_\omega(H_i)$, where $H_i = \mathrm{PSL}_2(p_i)$, is slower than $f_1$. Therefore

$$L\gamma_i(n) \, < \, Lf_1(n)^{2/3} \, < \, f_1(n) \quad \text{for all} \ \ n \geq \vartheta(i).$$

Also Corollary 9.4 that the growth of is not slower than $\Lambda^i_\omega(H_i)$, i.e.,

$$\gamma_i(n) \, > \, f_2(n) \quad \text{for some} \ \ n \geq \vartheta(i).$$

Therefore the group $H_i = \mathrm{PSL}_2(p_i)$ has all necessary properties. □

**9.4. Details: small gap.** Unfortunately, if $f_1 \sim f_2$, then the gap between the functions $f_1$ and $f_2$ is not sufficiently big for the above argument to work. From this point on, we assume that the functions $f_1, f_2 : \mathbb{N} \to \mathbb{N}$ satisfy the following conditions:

- $f_1(n) > (f_2(n))^3$
- $f_2(n) \geq \gamma_{\mathbf{G}_\omega}(n)$

**Remark 9.5.** Corollary 9.3 gives that (for a fixed $i$) if the size of $H$ is small then $H \in \mathcal{D}^i_-$. In particular by Lemma 6.1 the class $\mathcal{D}^i_-$ contains the groups $\mathrm{PSL}_2(p)$ for $p^3/2 < \mathrm{U}_i(f_2)$ and $p = 3 \mod 4$.

**Lemma 9.6.** *If $H_1, H_2 \in \mathcal{D}^i_-$, then $H_1 \otimes H_2$ is not in $\mathcal{D}^i_+$.*

*Proof.* Proposition 5.5 implies that

$$\Lambda^i_\omega(H_1 \otimes H_2) \, = \, \Lambda^i_\omega(H_1) \otimes \Lambda^i_\omega(H_2).$$

By Lemma 4.6

$$\gamma_{\Lambda^i_\omega(H_1 \otimes H_2)}(n) \, \leq \, \gamma_{\Lambda^i_\omega(H_1)}(n) \cdot \gamma_{\Lambda^i_\omega(H_2)}(n) \, \leq \, f_2(n) \cdot f_2(n) \, < \, f_1(n),$$

i.e., the growth of $\Lambda^i_\omega(H_1 \otimes H_2)$ is slower than $f_1$ and the group $H_1 \otimes H_2$ is not in $\mathcal{D}^i_+$. □

**Corollary 9.7.** *If the set $\mathcal{D}^i_\circ$ is empty, then $\mathcal{D}^i_-$ is closed under $\bigotimes$.*

**Corollary 9.8.** *If the set $\mathcal{D}^i_\circ$ is empty, then $\mathcal{D}^i_-$ contain $\mathrm{PSL}_2(\mathbb{Z}_N)$, where $N$ is the product of all primes $p = 3$ mod 4, such that $p^3/2 < \mathrm{U}_i(f_1)$.*

*Proof.* Corollary 9.3 says that if $\mathrm{PSL}_2(p)$ is not inside $\mathcal{D}^i_+$ if $|\mathrm{PSL}_2(p)| = p^3/2(1 + o(1)) < \mathrm{U}_i(f_1)$. The previous Corollary and Remark 6.3 finish the proof. □

**Corollary 9.9.** *If the set $\mathcal{D}^i_\circ$ is empty, then $\mathcal{D}^i_-$ contain $\mathrm{PSL}_2(\mathbb{Z}_N)$*

$$\log N \, \approx \, \mathrm{N}_i(f_1) = \frac{1}{2} \left[ 2\mathrm{U}_i(f_1) \right]^{1/3}.$$

*Proof.* This follows easily from the Dirichlet theorem on the distribution of primes (mod 4), and a calculation of the product of primes given in [Ruiz]. We omit the (easy) details. □

*Proof of Lemma 8.1.* The idea is the same as in the case of large gap between $f_1$ and $f_2$. If one assumes that $\mathcal{D}^i_\circ$ is empty the one can use Corollary 9.9 to construct groups in $\mathcal{D}^i_-$.

Again, up to a constants, both $\vartheta(i)$ and $\mathrm{D}'_i$ are equal to $2^i$. Substituting $\varrho = 2^i$ one gets

$$\mathrm{L}_i(f_2) \approx \exp\left[ \frac{C_1}{\varrho^2} \, f_2^*(C_2 \varrho) \right],$$

where $C_1$ and $C_2$ are universal constants. Similarly

$$\mathrm{U}_i(f_1) \approx \exp\left[ \frac{\log g(\varrho)}{C_3 \varrho^2} f^* \left( \frac{\varrho}{C_4 \log g(\varrho)} \right) \right] \quad \text{and}$$

$$\mathrm{N}_i(f_1) \approx \exp\left(\exp\left[\frac{\log g(\varrho)}{C_5\varrho^2} f^*\left(\frac{\varrho}{C_4 \log g(\varrho)}\right)\right]\right).$$

The condition (v) implies that

$$\mathrm{N}_i(f_1)/\mathrm{L}_i(f_2) \to \infty \quad \text{and} \quad \mathrm{U}_i(f_1) \to \infty, \quad \text{as} \quad i \to \infty.$$

Therefore, there exists $i_0$, such that for $i > i_0$ we have:

$$\left[f_1\big(\vartheta(i)\big)\right]^{1/3} > L, \quad \mathrm{N}_i(f_1)/\mathrm{L}_i(f_2) > 2, \quad \mathrm{U}_i(f_1) > 1000.$$

However, Corollaries 9.4 and 9.9 imply that if $\mathcal{D}^i_\circ = \varnothing$, then the group $\mathrm{PSL}_2(N)$ is neither in $\mathcal{D}^i_-$ nor in $\mathcal{D}^i_+$, a contradiction. This implies that $\mathcal{D}^i_\circ$ is not empty, and therefore there exits a group $H$ with the desired properties. □

## 10. Generalizations of the Construction

**10.1.** Suppose $G$ acts on a set $X$ by $H \wr_X G$; we denote the restricted wreath product $G \ltimes \bigoplus_{i\in X} H$. One easy modification of our construction is to use the permutation wreath product $P \wr_X \mathbf{G}_\omega$ as groups at infinity, where $P$ is a finite group and $X$ is an orbit in the action of $\mathbf{G}_\omega$ on the boundary of the binary tree $\mathbf{T}_2$. The advantage of using these groups is that (unlike the the groups $\mathbf{G}_\omega$) their growth rate is known in some cases, see [BE].

**Theorem 10.1.** *The same as Theorem 2.3 but we use the growth of the group $\mathbb{Z}_2 \wr_X \mathbf{G}_\omega$ instead the group $\mathbf{G}_\omega$. Here $X$ is the boundary of the binary tree $\mathbf{T}_2$.*

*Outline of the proof.* Here is the list of changes we need in the construction.
- instead of the group $\mathcal{G}$, consider the free product $\mathcal{G} * \mathbb{Z}_2$,
- modify the functors $F_i$ to include the extra generators, by adding $G = (1; 1, g)$ for every $g \in \mathbb{Z}_2$,
- use $\mathbb{Z}_2$ in place of the trivial group $\mathbf{1}$,
- use the limit of the groups $F^i_\omega(\mathbb{Z}_2)$ is $\mathbb{Z}_2 \wr_X \mathbf{G}_\omega$,
- change groups $H_i$ constructed in Lemma 6.1, to contain the group $\mathbb{Z}_2$.

The rest of the proof follows verbatim. We omit the details. □

**10.2.** It is easy to see that the growth types of the groups $P \wr_X G$ for fixed $G$ and different $P$ are the same if $P$ is finite and nontrivial. Thus in the theorem one can replace $\mathbb{Z}_2 \wr_X \mathbf{G}_\omega$ with $P \wr_X \mathbf{G}_\omega$ for any finite group $P$.

It seems possible to extend this result to wreath products of the forms $P \wr_X \mathbf{G}_\omega$ where the group $P$ is not finite, but one needs a sofic approximation (a sequence of finite groups $\{P_i\}$ which converge to $P$) of the group $P$ instead. The above outline need to be modified to by replacing $\mathbb{Z}_2$ with $P_i$.

**10.3.** Another easy generalization direction is to use the groups constructed in [Seg] instead of the Grigorchuk groups $\mathbf{G}_\omega$. However this will make the words needed in Lemma 5.16 and 6.1 not so explicit, but it is clear that such words exist. In fact, as far as we are aware the growth of these groups has not been studied and it is not clear if there any examples of this type which are of intermediate growth.

**10.4.** It would be interesting to analyze for which groups $\mathbf{G}$ and subexponential functions $f$, there exists a sequence of finite groups $G_i$ which converge to $\mathbf{G}$ and the growth of $\bigotimes G_i$ oscillates between $\gamma_{\mathbf{G}}$ and the function $f$. Theorem 2.3 shows that this is possible if $\mathbf{G}$ is the Grigorchuk group $\mathbf{G}_\omega$ and Theorem 10.1 if $\mathbf{G}$ is a wreath product of $\mathbf{G}_\omega$ with a finite group. We believe that for any group $\mathbf{G}$ of intermediate growth such sequence exists, provided that the gap between the growth of $\mathbf{G}$ and $f$ is sufficiently large:

**Conjecture 10.2.** *For every group $G$ of intermediate growth and a subexponential function $f(n)$ which grows sufficiently fast (depending on $\gamma_G$), there exists a sequence of finite groups $\{G_i\}$, such that*

(1) $\lim G_i = G$, *and*
(2) *the growth of* $\Gamma = \bigotimes G_i$ *oscillates between* $\gamma_G$ *and* $f$.

## 11. Historical remarks and open problems

**11.1.** We refer to [GP, Har1] for the introduction to groups of intermediate growth, and to [BGS, Gri5, Gri6, GH, GNS] for the surveys on the subject and open problems.

Although $\mathbf{G}$ is historically first group of intermediate growth [Gri1, Gri2], there is now a large number of constructions of intermediate growth *branch groups* (see [BGS]). Groups $\mathbf{G}_\omega$ corresponding to infinite words $\omega \in \{0,1,2\}^\infty$, were introduced by Grigorchuk in [Gri3]. They form a continuum family of intermediate growth groups. In this setting, the Grigorchuk group $\mathbf{G} = \mathbf{G}_{(012)^\infty}$ corresponds to a periodic word sequence, and is sometime called the *first Grigorchuk group* [BGS].

**11.2.** Let us mention that Grigorchuk's original bounds for $\mathbf{G}$ were $\alpha_-(\mathbf{G}) \geq 0.5$ and $\alpha_+(\mathbf{G}) \leq 0.991$. These bounds were successively improved, with the current records being

$$\alpha_-(\mathbf{G}) \geq 0.5207, \quad \alpha_+(\mathbf{G}) \leq 0.7675,$$

where both constants correspond to solutions of certain algebraic equations. The bound for $\alpha_-$ is in [Bri1] (see also [Bar2, Leo]), and for $\alpha_+(\mathbf{G})$ in [Bar1, MP]. Whether the limit $\alpha(\mathbf{G})$ exists remains an open problem. However, Grigorchuk conjectures that $\alpha_-(\Gamma) \geq \alpha_-(\mathbf{G})$ for *every* group of intermediate growth [Gri5].

**11.3.** One of the few examples of groups of intermediate growth where the type of the growth function is known precisely are permutational wreath products $\Gamma = P \wr_X \mathbf{G}_\omega$ where $\omega = (012)^\infty$, where $X$ is the boundary of the binary tree. If $P$ is a finite group then $\gamma_\Gamma \sim \exp(n^\alpha)$ for $\alpha = 0.7675$, if $P = \mathbb{Z}$ then the growth is $\gamma_\Gamma \sim \exp(n^\alpha \log n)$, see [BE].

**11.4.** Free Grigorchuk group $\mathcal{G}$ defined in Section 3, is clearly isomorphic to a free product $\mathbb{Z}_2^2 * \mathbb{Z}_2$, and thus *non-amenable*. It should not be confused with the *universal Grigorchuk group*

$$\bigotimes_\omega \mathbf{G}_\omega \;=\; \mathcal{G} / \bigcap_\omega \ker(\mathcal{G} \twoheadrightarrow \mathbf{G}_\omega),$$

which is known to have exponential growth, and is conjectured to be amenable [Gri7, §8].

**11.5.** It is well known and easy to see [Har2], that groups of exponential growth cannot have oscillations:

$$\liminf_{n\to\infty} \frac{\log \gamma_\Gamma^S(n)}{n} \;=\; \limsup_{n\to\infty} \frac{\log \gamma_\Gamma^S(n)}{n} \quad \text{for all } \langle S \rangle = \Gamma.$$

Denote this limit by $\varkappa(\Gamma, S) > 1$. It was recently discovered by Wilson [Wil] that there exits groups with $\inf_S \varkappa(\Gamma, S) = 1$ (see also [Bri2]).

**11.6.** We conjecture that condition (vi) in the Main Theorem can be weakened to

$$\text{(vi}'\text{)} \quad \frac{\log g_1(n)}{n^2} \cdot f_1^*\left(\frac{n}{\log g_1(n)}\right) \to \infty$$

If true, this would significantly weaker the conditions on the growth of $f_1$ and $f_2$, allowing further values of parameters in the examples from Subsection 2.2.

Heuristically, one expect that the growth of $\Lambda_\omega^i(\mathrm{PSL}_2(\mathbb{Z}_N)))$ behaves reasonably with $N$. This implies that if $\mathcal{D}_-^i$ contains enough groups then $\mathcal{D}_\circ^i$ is not empty. It is possible to prove such statement for a fixed $i$ using that the group $\Lambda_\omega^i(\mathrm{PSL}_2(\mathbb{Z})))$ which is reasonably close to a nice arithmetic group. However it is for from clear how to do this for all $i$ large enough.

**11.7.** In the context of Subsection 2.3, in order for this strategy to work, infinitely many groups $H_i$ need to be nontrivial. However, one can show that taking the limits in Section 4 cannot possibly work if the group is not finitely presented. The following lemma clarifies our reasoning.

**Lemma 11.1.** *In the context of Section 4, if the limit group* $G$ *is finitely presented, then almost all groups* $N_i$ *are trivial. Consequently,* $\Gamma = G \ltimes N$ *for some finite group* $N$.

*Proof.* Suppose that $G$ has a presentation where all realtors have length at most $k$. The if the ball of radius $k$ in the group $H$ coincides with the ball of radius $k$ in $G$ then $H$ is a homomorphic image of $G$ since all defining relations of $G$ are satisfied in $H$. Therefore group $G_i$ are images of $G$ for big $i$ which implies that $N_i$ are trivial (again for big $i$). $\square$

Of course, in particular, the lemma shows that the Grigorchuk groups $\mathbf{G}_\omega$ of intermediate growth are *not* finitely presented, a well known result in the field [Gri6, Gri7]. Similarly the lemma shows that Conjecture 10.2 implies that all groups of intermediate growth are not finitely presented a classical old open problem [Gri5, Gri6].

**11.8.** It follows from Shalom and Tao's recent extension [ST] of the Gromov's theorem [Gro], that every group of growth $n^{(\log\log n)^{o(1)}}$ must be virtually nilpotent, and thus have polynomial growth. It is a major open problem whether this result can be extended to groups of growth $e^{o(\sqrt{n})}$. Only partial results have been obtained in this direction [Gri5] (see also [BGS, Gri4, Gri6]).

**11.9.** The growth of groups is in many ways parallel to the study of *subgroup growth* (see [Lub2, LS]). In this case, a celebrated construction of Segal [Seg] (see also [Neu2]), showed that the group can have nearly polynomial growth without being virtually solvable of finite rank. In other words, the Shalom-Tao extension of Gromov theorem does not have a subgroup growth analogue. Interestingly, Segal's construction also uses the Grigorchuk type groups, and takes the iterated permutational wreath product of permutation groups; it is one of the motivations behind our construction.

Let us mention here that Pyber completely resolved the "gap problem" by describing groups with subgroup growth given by any prescribed increasing function (within a certain range). His proof relies on sequences of finite alternating groups of pairwise different degrees also [Pyb2], generalizing a classical construction of B. H. Neumann [Neu1] (see also [Pyb1]).

On the other end of the spectrum, let us mention that for subgroup growth there is no strict lower bound, i.e., for any function $f(n)$ there is a f.g. group where the number of subgroups of index $n$ is less than $f(n)$ infinitely often [KN, Seg].

**11.10.** In another variation on the group growth is the *representation growth*, defined via the number $r_n(G)$ of irreducible complex representations of dimension $n$, whose kernel has finite index. In this case there is again no upper bound for the growth of $r_n(G)$ (this follows from [KN]). We refer to [LL] for the introduction to the subject, and to [Cra] for recent lower bound on representation growth. See also [Jai] and [Voll] for the zeta-function approach.

**11.11.** The growth of algebras (rather than group) is well understood, and much more flexibility is possible (see e.g. [Ufn]). The results in this paper and [Bri3], it seems, suggest that the growth of groups can be much less rigid than previously believed.

**11.12.** In the proof of Lemma 6.1, using the length $\leq 10$ of standard generators in $\{a, b, c, d\}$, one can get an explicit bound $\mathrm{K} < 10 \cdot 2000$ (see [Lub1, §8] and [HLW, §11.2]). Bounds in [BG1], giving $\mathrm{K}'$, can also potentially be made explicit.

Most recently, it was shown that for primes $p = 1 \bmod 4$ of positive density, *all* Cayley graphs of $\mathrm{PSL}_2(p)$ have universal expansion, a result conjectured for all primes [BG2]. These most general bounds have yet to be made explicit, however.

**11.13.** A finitely generated group $G$ called *sofic* if it is a limit of some sequence of finite groups. The existence of finitely generated non-sofic groups is a well known open problem [Pes, §3]. Let us also mention that convergence of Grigorchuk groups was also studied in [Gri3], and a related notion of *Benjamini-Schramm convergence* for graph sequences [BS].

**11.14.** As a minor but potentially important difference, let us mention that the oscillating growth established by Brieussel [Bri3] does not give explicit bounds on the "oscillation times", while in this paper we compute them explicitly, up to some global constants. Since our result mostly do not overlap with those in [Bri3], it would be useful to quantify the former.

Interestingly, both this paper and [Bri3] have been obtained independently and using different tools, they were both originally motivated by probabilistic applications (see [KP]), to the analysis of the *return probability* and the *rate of escape* of a random walk on groups. We refer to [Woe] for a general introduction to the subject.

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## References


[Bar1] L. Bartholdi, The growth of Grigorchuk's torsion group, *Int. Math. Res. Not.* **20** (1998), 1049–1054.

[Bar2] L. Bartholdi, Lower bounds on the growth of a group acting on the binary rooted tree, *Int. J. Algebra Comput.* **11** (2001), 73–88.

[BE] L. Bartholdi and A. G. Erschler, Growth of permutational extensions, `arXiv:1011.5266`.

[BGS] L. Bartholdi, R. I. Grigorchuk and Z. Sunik, Branch groups, in *Handbook of Algebra*, vol. 3, North-Holland, Amsterdam, 2003, 989–1112.

[BS] I. Benjamini and O. Schramm, Recurrence of distributional limits of finite planar graphs, *El. J. Probab.* **6** (2001), Paper 23.

[BG1] J. Bourgain and A. Gamburd, Uniform expansion bounds for Cayley graphs of $\mathrm{SL}_2(\mathbb{F}_p)$, *Ann. Math.* **167** (2008), 625–642.

[BG2] E. Breuillard and A. Gamburd, Strong uniform expansion in $\mathrm{SL}(2,p)$, *Geom. Funct. Anal.* **20** (2010), 1201–1209.

[Bri1] J. Brieussel, *Growth of certain groups of automorphisms of rooted trees* (in French), Doctoral Dissertation, University of Paris 7, 2008; available at `http://www.institut.math.jussieu.fr/ theses/2008/brieussel/`

[Bri2] J. Brieussel, Amenability and non-uniform growth of some directed automorphism groups of a rooted tree, *Math. Z.* **263** (2009), 265–293.

[Bri3] J. Brieussel, Growth behaviors in the range $e^{r^\alpha}$, `arXiv:1107.1632`.

[Cra] D. A. Craven, Lower bounds for representation growth, *J. Group Theory* **13** (2010), 873–890.

[Ers1] A. Erschler, Boundary behavior for groups of subexponential growth, *Ann. Math.* **160** (2004), 1183–1210.

[Ers2] A. Erschler, On the degrees of growth of finitely generated groups, *Funct. Anal. Appl.* **39** (2005), 317–320.

[Gri1] R. I. Grigorchuk, On Burnside's problem on periodic groups, *Funct. Anal. Appl.* **14** (1980), 41–43.

[Gri2] R. I. Grigorchuk, On the Milnor problem of group growth, *Soviet Math. Dokl.* **28** (1983), 23–26.

[Gri3] R. I. Grigorchuk, Degrees of growth of finitely generated groups and the theory of invariant means, *Math. USSR-Izv.* **25** (1985), 259–300.

[Gri4] R. I. Grigorchuk, Degrees of growth of $p$-groups and torsion-free groups, *Math. USSR-Sb.* **54** (1986), 185–205.

[Gri5] R. Grigorchik, Solved and unsolved problems around one group, in *Infinite groups: geometric, combinatorial and dynamical aspects*, Birkhäuser, Basel, 2005, 117–218.

[Gri6] R. I. Grigorchik, Some topics of dynamics of group actions on rooted trees (in Russian), *Proc. Steklov Inst. Math.* **273** (2011), 1–118.

[Gri7] R. I. Grigorchik, Problem of Milnor on group growth and its consequences, preprint (2011).

[GH] R. Grigorchuk and P. de la Harpe, On problems related to growth, entropy, and spectrum in group theory, *J. Dynam. Control Systems* **3** (1997), 51–89.

[GNS] R. I. Grigorchuk, V. V. Nekrashevych and V. I. Sushchanskiĭ, Automata, dynamical systems, and groups, *Proc. Steklov Inst. Math.* **231** (2000), 128–203.

[GP] R. Grigorchuk and I. Pak, Groups of intermediate growth: an introduction, *Enseign. Math.* **54** (2008), 251–272.

[Gro] M. Gromov, Hyperbolic groups, in *Essays in group theory*, Springer, New York, 1987, 75–263.

[Har1] P. de la Harpe, *Topics on Geometric Group Theory*, University of Chicago Press, Chicago, IL, 2000.

[Har2] P. de la Harpe, Uniform growth in groups of exponential growth, *Geom. Dedicata* **95** (2002), 1–17.

[HLW] S. Hoory, N. Linial and A. Wigderson, Expander graphs and their applications, *Bull. Amer. Math. Soc.* **43** (2006), 439–561.

[Jai] A. Jaikin-Zapirain, Zeta function of representations of compact p-adic analytic groups, *J. Amer. Math. Soc.* **19** (2006), 91–118.

[KN] M. Kassabov and N. Nikolov, Cartesian products as profinite completions, *Int. Math. Res. Not.* **2006**, Art. ID 72947, 17 pp.

[KP] M. Kassabov and I. Pak, The oscillating rate of escape of random walks on groups, in preparation (2011).

[LL] M. Larsen and A. Lubotzky, Representation growth of linear groups, *J. Europ. Math. Soc.* **10** (2008), 351–390.

[Leo] Yu. G. Leonov, On a lower bound for the growth function of the Grigorchuk group, *Math. Notes* **67** (2000), 403–405.

[Lub1] A. Lubotzky, *Discrete groups, expanding graphs and invariant measures*, Birkhäuser, Basel, 1994.

[Lub2] A. Lubotzky, Subgroup growth, in *Proc. ICM Zürich*, Birkhäuser, Basel, 1995, 309–317.

[LS] A. Lubotzky and D. Segal, Subgroup growth, Birkhäuser, Basel, 2003.

[MP] R. Muchnik and I. Pak, On growth of Grigorchuk groups, *Int. J. Algebra Comput.* **11** (2001), 1–17.

[Nek] V. Nekrashevych, *Self-similar groups*, AMS, Providence, RI, 2005.

[Neu1] B. H. Neumann, Some remarks on infinite groups, *J. London Math. Soc.* **12** (1937), 120–127.

[Neu2] P. M. Neumann, Some questions of Edjvet and Pride about infinite groups, *Illinois J. Math.* **30** (1986), 301–316

[Pes] V. G. Pestov, Hyperlinear and sofic groups: a brief guide, *Bull. Symbolic Logic* **14** (2008), 449–480.

[Pyb1] L. Pyber, Old groups can learn new tricks, in *Groups, combinatorics & geometry*, World Sci., River Edge, NJ, 2003, 243–255.

[Pyb2] L. Pyber, Groups of intermediate subgroup growth and a problem of Grothendieck, *Duke Math. J.* **121** (2004), 169–188.

[Ruiz] S. M. Ruiz, A result on prime numbers, *Math. Gazette* **81** (1997), 269–270.

[Seg] D. Segal, The finite images of finitely generated groups, *Proc. London Math. Soc.* **82** (2001), 597–613.

[ST] Y. Shalom and T. Tao, A finitary version of Gromov's polynomial growth theorem, *Geom. Funct. Anal.* **20** (2010), 1502–1547.

[Ufn] V. A. Ufnarovskij, Combinatorial and asymptotic methods in algebra, in *Algebra VI*, Springer, Berlin, 1995, 1–196.

[Voll] C. Voll, Functional equations for zeta functions of groups and rings, *Ann. Math.* **172** (2010), 1181–1218.

[Wil] J. S. Wilson, On exponential growth and uniformly exponential growth for groups, *Invent. Math.* **155** (2004), 287–303.

[Woe] W. Woess, *Random walks on infinite graphs and groups*, Cambridge University Press, Cambridge, 2000.